MtG

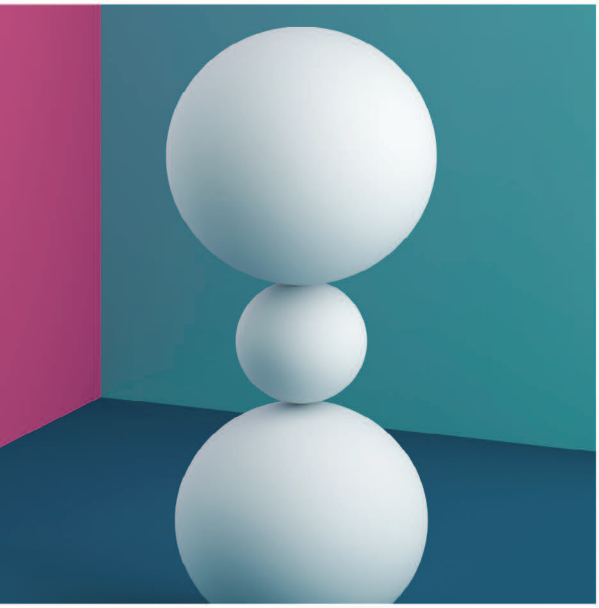
CBSE CHAMPION

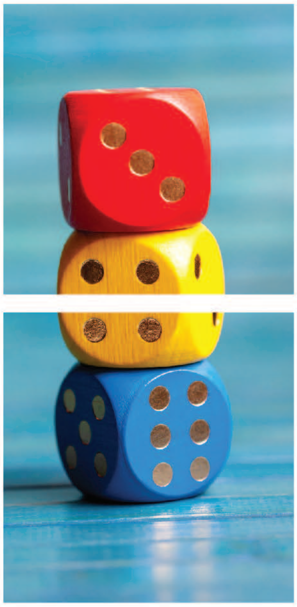
FOR 2024 EXAM

CBSE 10 SOLVED PAPERS

2023-2014 YEARS

CHAPTERWISE-TOPICWISE





- TERM I, TERM II, DELHI, ALL INDIA, FOREIGN & COMPARTMENT PAPERS
- LAST 3 YEARS CBSE OFFICIAL SAMPLE QUESTIONS
- ANSWERS AS PER CBSE MARKING SCHEME WITH PREPARATORY TOOLS AND TOPPER'S ANSWERS

CLASS 12

- Topicwise Graphical Analysis • Comprehensive Theory with Brain Map
- Questions Labelled as per CBSE Cognitive Levels • Chapterwise Self Assessments
- 5 Practice Papers • CBSE Sample Paper
- **Latest CBSE Sample Paper (Issued on 31st March 2023)**

SCORE MORE 100% MTG GUARANTEE

MATHEMATICS

CBSE 10 SOLVED PAPERS

2023-2014 YEARS

CHAPTERWISE-TOPICWISE

MATHEMATICS

CLASS 12

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Preface

We feel pleased and delighted in presenting the revised edition of the book **"CBSE Champion Chapterwise-Topicwise Mathematics"** as per the CBSE Curriculum for the current academic year. As we know, CBSE Board has taken multiple steps towards the implementation of Competency Based Education (CBE) in schools, range from aligning assessment to CBE, development of exemplar resources for teachers and students on CBE pedagogy and assessment and continued teacher capacity building so, special efforts have been put to align this book to CBE. It will give students comprehensive knowledge of subject according to the latest CBSE syllabus and pattern.

The salient features of the book are as follows :

- **Topicwise Graphical Analysis :** Bar Graph Analysis of Previous 10 Years' CBSE Board Papers' Questions (MCQ, 1 mark, 2 marks, 3 marks, 4 marks and 6 marks) is provided. The Graph and 'Weightage Xtract' will help students to figure out the chapters and topics which are to be revised frequently and how much is the weightage of these topics.
- **Comprehensive and Lucid Theory :** Well explained theory with important formulae, flowcharts and tables for quick recap and summarise the chapter with Brain Map.
- **Chapterwise-Topicwise Questions and Solutions :** Previous 10 years' CBSE-Term I, Term II, DELHI, ALL INDIA and FOREIGN papers' questions are segregated chapterwise-topicwise. Detailed solutions of these questions are given according to the CBSE Marking Scheme.
- **Strictly Based on CBSE Syllabus & NCERT Topics List :** Topicwise questions are arranged in descending chronological (2023-2014) order, so that latest years' questions come first in practice and revision.
- **CBSE Cognitive Levels :** Questions are labelled as per the cognitive levels issued by CBSE for assessment and evaluation to provide evidence of students' knowledge and ability.

Cognitive Levels :

Remembering	Understanding	Applying	Analysing	Evaluating	Creating
R	U	Ap	An	Ev	Cr

- **Key Concepts Highlight :** Key concepts have been highlighted in the solutions for their reinforcement.
- **Exam Preparatory Tools :** Detailed solutions are supplementary with Exam Preparatory Tools like **Key Points, Answer Tips, Concept Applied, Commonly Made Mistakes, Alternative Methods, Shortcuts,** etc. to improve aspirant's score.
- **Topper's Answers :** Topper's Answers of few selected CBSE-2022 Questions are given which will help aspirants to write perfect answers in the upcoming CBSE Board Exam.
- **Chapterwise Self Assessments (Solved) :** To master a chapter, self assessment test is given at the end of each chapter, covering all typology of questions.
- **CBSE Sample Questions :** Last three years' official CBSE Sample Questions are segregated chapterwise-topicwise for which solutions are given as per the CBSE Marking Scheme.
- **Practice Papers as per CBSE Blue Print :** 5 Practice Papers (Solved) strictly based on the latest syllabus and latest design and blue print of CBSE Sample Paper issued by CBSE Board. Various types of questions like Case/Passage/Source Based Questions, MCQs, Assertion-Reason, SA and LA type, are included.
- **Solved CBSE Sample Paper :** Latest solved CBSE sample paper is included.

We are sure that the value addition done in this revised book will be helpful to students in achieving success in the coming board examinations. Every possible effort has been made to make this book error free. Useful suggestions by our readers for the rectification and improvement of the book content would be gracefully acknowledged and incorporated in further editions.

Readers are welcome to send their suggestions and feedback at **editor@mtg.in**.

All the Best

MTG Editorial Board

Glimpse of MTG CBSE Champion

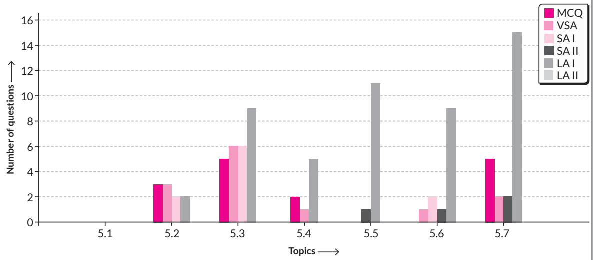
CHAPTER 5

Continuity and Differentiability

TOPICS

5.1 Introduction
5.2 Continuity
5.3 Differentiability
5.4 Exponential and Logarithmic Functions
5.5 Logarithmic Differentiation
5.6 Derivatives of Functions in Parametric Forms
5.7 Second Order Derivative

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

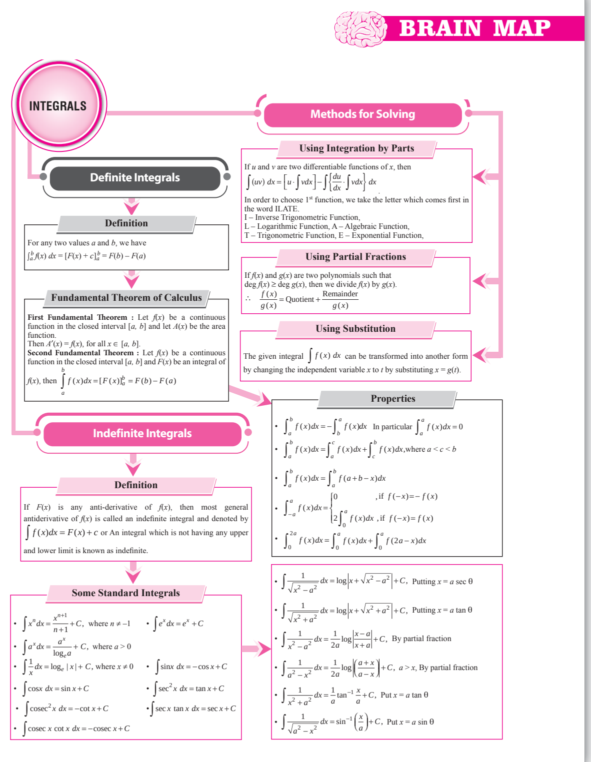
- Topic 5.3 is highly scoring topic.
- Maximum weightage is of Topic 5.3 *Differentiability.*

Helps you to figure out the topics which carry maximum weightage!

QUICK RECAP

Indefinite Integral

- Integration is the inverse process of differentiation.
  i.e., $\frac{d}{dx}F(x)=f(x) \Rightarrow \int f(x)\,dx = F(x)+C$, where C is the constant of integration.
  Integrals are also known as antiderivatives.



Quickly revise chapter from 'Quick Recap' and summarise through 'Brain Map'!

Previous Years' CBSE Board Questions

13.1 Introduction

MCQ

1. Five fair coins are tossed simultaneously. The probability of the events that atleast one head comes up is
   (a) $\frac{27}{32}$ (b) $\frac{5}{32}$ (c) $\frac{31}{32}$ (d) $\frac{1}{32}$ *(2023)*
2. A die is thrown once. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is
   (a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) 0 (d) 1 *(2020)*

VSA (1 mark)

3. A coin is tossed once. If head comes up, a die is thrown, but if tail comes up, the coin is tossed again. Find the probability of obtaining head and number 6. *(2021 C)*
4. Two cards are drawn at random and one-by-one without replacement from a well-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black. *(2020)*
5. From a pack of 52 cards, 3 cards are drawn at random (without replacement). The probability that they are two red cards and one black card, is ________. *(2020C)*
6. A bag contains 3 black, 4 red and 2 green balls. If three balls are drawn simultaneously at random, then the probability that the balls are of different colours is ________. *(2020)* Ap

SA I (2 marks)

7. A box B_1 contains 1 white ball and 3 red balls. Another box B_2 contains 2 white balls and 3 red balls. If one ball is drawn at random from each of the boxes B_1 and B_2, then find the probability that the two balls drawn are of the same colour. *(Term II, 2021-22)*
8. A bag contains cards numbered 1 to 25. Two cards are drawn at random, one after the other, without replacement. Find the probability that the number on each card is a multiple of 7.

LA I (4 marks)

11. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black. *(Delhi 2015)* Cr

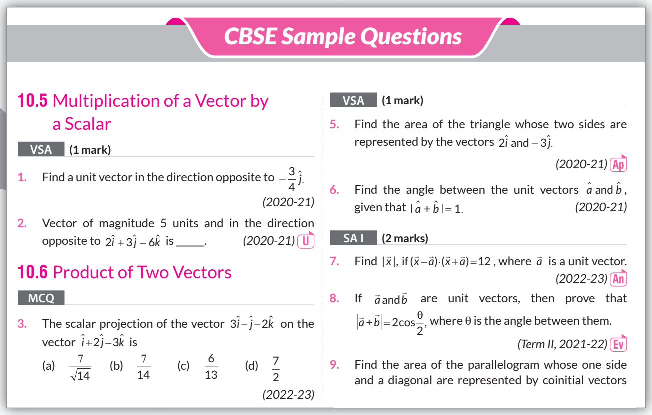
13.2 Conditional Probability

MCQ

12. For two events A and B, if $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$, then $P(A \cup B)$ is
    (a) 0.24 (b) 0.3 (c) 0.48 (d) 0.96 *(2023)*
13. If for any two events A and B,
    $P(A)=\frac{4}{5}$ and $P(A \cap B)=\frac{7}{10}$, then $P(B/A)$ is
    (a) $\frac{1}{10}$ (b) $\frac{1}{8}$ (c) $\frac{7}{8}$ (d) $\frac{17}{20}$ *(2023)*
14. **In the following questions, a statements are Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices:**
    Assertion (A) : Two coins are tossed simultaneously. The probability of getting two heads, if it is known that at least one head comes up, is $\frac{1}{3}$.
    Reason (R) : Let E and F be two events with a random experiment, then $P(F/E)=\frac{P(E \cap F)}{P(E)}$.
    (a) Both (A) and (R) are true and (R) is the correct explanation of (A)
    (b) Both (A) and (R) are true, but (R) is not the correct explanation of the (A)
    (c) (A) is true, and (R) is False.
    (d) (A) is false, but (R) is true. *(2023)*
15. A card is picked at random from a pack of 52 playing cards. Given that the picked card is a queen, the probability of this card to be a card of spade is

Chapterwise-Topicwise last 10 years' question papers with CBSE cognitive levels labelling!

Last 3 years' CBSE sample questions are segregated chapterwise-topicwise with CBSE cognitive levels labelling!


CBSE Sample Questions

10.5 Multiplication of a Vector by a Scalar

VSA (1 mark)

1. Find a unit vector in the direction opposite to $-\frac{3}{4}\hat{j}$. (2020-21)

2. Vector of magnitude 5 units and in the direction opposite to $2\hat{i}+3\hat{j}-6\hat{k}$ is ____. (2020-21) U

10.6 Product of Two Vectors

MCQ

3. The scalar projection of the vector $3\hat{i}-\hat{j}-2\hat{k}$ on the vector $\hat{i}+2\hat{j}-3\hat{k}$ is
(a) $\frac{7}{\sqrt{14}}$ (b) $\frac{7}{14}$ (c) $\frac{6}{13}$ (d) $\frac{7}{2}$ (2022-23)

VSA (1 mark)

5. Find the area of the triangle whose two sides are represented by the vectors $2\hat{i}$ and $-3\hat{j}$. (2020-21) Ap

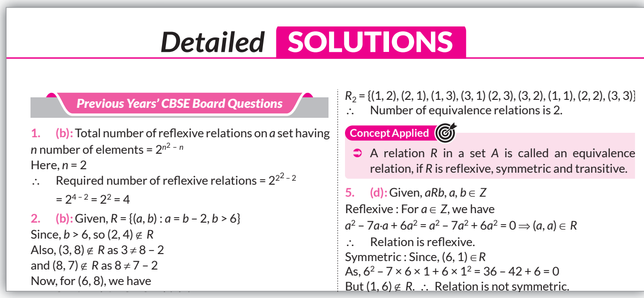
6. Find the angle between the unit vectors $\hat{a}$ and $\hat{b}$, given that $|\hat{a}+\hat{b}|=1$. (2020-21)

SA I (2 marks)

7. Find $|\vec{x}|$, if $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$, where $\hat{a}$ is a unit vector. (2022-23) An

8. If $\vec{a}$ and $\vec{b}$ are unit vectors, then prove that $|\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$, where θ is the angle between them. (Term II, 2021-22) Ev

9. Find the area of the parallelogram whose one side and a diagonal are represented by coinitial vectors


Detailed SOLUTIONS

Previous Years' CBSE Board Questions

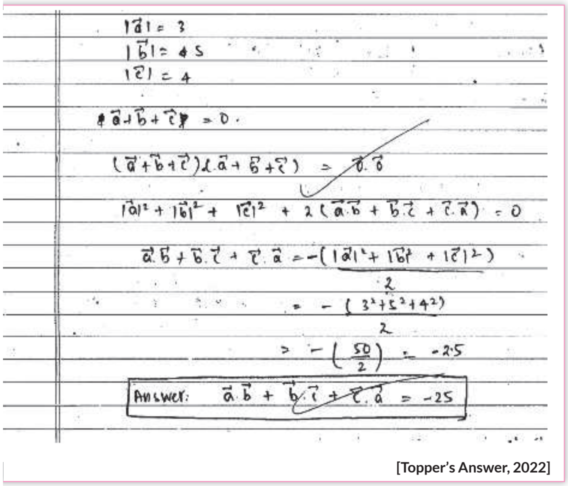
1. (b): Total number of reflexive relations on a set having n number of elements $= 2^{n^2-n}$
Here, $n = 2$
∴ Required number of reflexive relations $= 2^{2^2-2}$
$= 2^{4-2} = 2^2 = 4$

2. (b): Given, $R = \{(a, b) : a = b - 2, b > 6\}$
Since, $b > 6$, so $(2, 4) \notin R$
Also, $(3, 8) \notin R$ as $3 \neq 8 - 2$
and $(8, 7) \notin R$ as $8 \neq 7 - 2$
Now, for (6, 8), we have

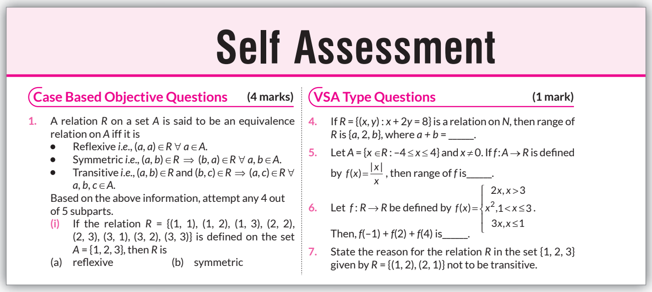
$R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1) (2, 3), (3, 2), (1, 1), (2, 2), (3, 3)\}$
∴ Number of equivalence relations is 2.

Concept Applied
A relation R in a set A is called an equivalence relation, if R is reflexive, symmetric and transitive.

5. (d): Given, aRb, $a, b \in Z$
Reflexive : For $a \in Z$, we have
$a^2 - 7a \cdot a + 6a^2 = a^2 - 7a^2 + 6a^2 = 0 \Rightarrow (a, a) \in R$
∴ Relation is reflexive.
Symmetric : Since, $(6, 1) \in R$
As, $6^2 - 7 \times 6 \times 1 + 6 \times 1^2 = 36 - 42 + 6 = 0$
But $(1, 6) \notin R$. ∴ Relation is not symmetric.


[Topper's Answer, 2022]

Detailed solutions are supplemented with Exam Preparatory Tools like Key Points, Answer Tips, Concept Applied, Commonly Made Mistakes, Alternative Methods, Shortcuts, Topper's Answers, etc. to improve aspirants score!


Self Assessment

Case Based Objective Questions (4 marks)

1. A relation R on a set A is said to be an equivalence relation on A iff it is
- Reflexive i.e., $(a, a) \in R\ \forall\ a \in A$.
- Symmetric i.e., $(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A$.
- Transitive i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$.

Based on the above information, attempt any 4 out of 5 subparts.
(i) If the relation $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ is defined on the set $A = \{1, 2, 3\}$, then R is
(a) reflexive (b) symmetric

VSA Type Questions (1 mark)

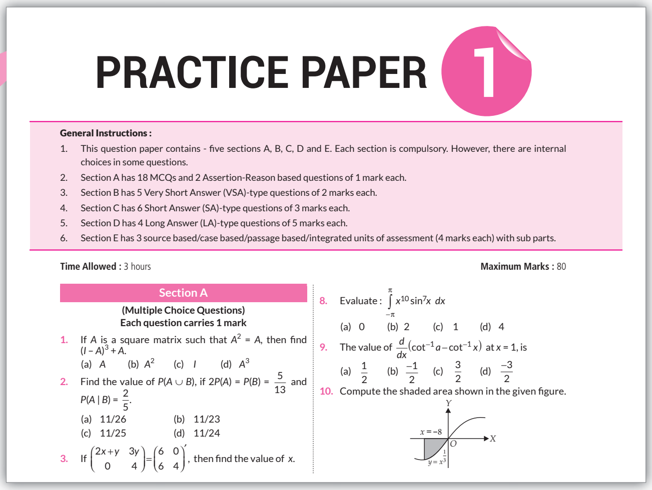
4. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, then range of R is $\{a, 2, b\}$, where $a + b$ = ____.

5. Let $A = \{x \in R : -4 \le x \le 4\}$ and $x \neq 0$. If $f : A \to R$ is defined by $f(x) = \frac{|x|}{x}$, then range of f is ____.

6. Let $f : R \to R$ be defined by $f(x) = \begin{cases} 2x, & x > 3 \\ x^2, & 1 < x \le 3 \\ 3x, & x \le 1 \end{cases}$.
Then, $f(-1) + f(2) + f(4)$ is ____.

7. State the reason for the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ not to be transitive.

Completely solved chapterwise self assessments and practice papers (based on the latest CBSE Sample Paper), covering all typology of questions which help to decode the CBSE exam pattern!


PRACTICE PAPER 1

General Instructions :
1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub parts.

Time Allowed : 3 hours **Maximum Marks :** 80

Section A
(Multiple Choice Questions)
Each question carries 1 mark

1. If A is a square matrix such that $A^2 = A$, then find $(I - A)^3 + A$.
(a) A (b) A^2 (c) I (d) A^3

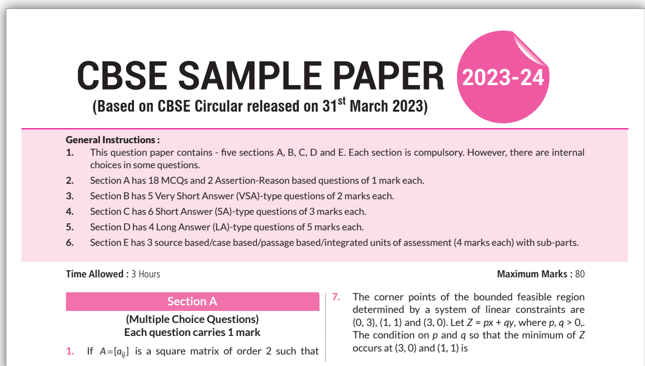
2. Find the value of $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$.
(a) 11/26 (b) 11/23
(c) 11/25 (d) 11/24

3. If $\begin{pmatrix} 2x+y & 3y \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 6 & 4 \end{pmatrix}$, then find the value of x.

8. Evaluate : $\int_{-\pi}^{\pi} x^{10}\sin^7 x\, dx$
(a) 0 (b) 2 (c) 1 (d) 4

9. The value of $\frac{d}{dx}(\cot^{-1}a - \cot^{-1}x)$ at $x = 1$, is
(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

10. Compute the shaded area shown in the given figure.


CBSE SAMPLE PAPER 2023-24
(Based on CBSE Circular released on 31st March 2023)

General Instructions :
1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub-parts.

Time Allowed : 3 Hours **Maximum Marks :** 80

Section A
(Multiple Choice Questions)
Each question carries 1 mark

1. If $A = [a_{ij}]$ is a square matrix of order 2 such that

7. The corner points of the bounded feasible region determined by a system of linear constraints are (0, 3), (1, 1) and (3, 0). Let $Z = px + qy$, where $p, q > 0$. The condition on p and q so that the minimum of Z occurs at (3, 0) and (1, 1) is

Latest CBSE Sample Paper has been included.

Syllabus*

CLASS - XII (2023-24)

S. No.	Units	Marks
1	Relations and Functions	08
2	Algebra	10
3	Calculus	35
4	Vectors and Three - Dimensional Geometry	14
5	Linear Programming	05
6	Probability	08
	Total	**80**

Unit I : Relations and Functions

1. **Relations and Functions** **(15) Periods**
   Types of relations: reflexive, symmetric, transitive and equivalence relations. One to one and onto functions.
2. **Inverse Trigonometric Functions** **(15) Periods**
   Definition, range, domain, principal value branch. Graphs of inverse trigonometric functions.

Unit II : Algebra

1. **Matrices** **(25) Periods**
   Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operations on matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Non-commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).
2. **Determinants** **(25) Periods**
   Determinant of a square matrix (up to 3 × 3 matrices), minors, co-factors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

Unit III : Calculus

1. **Continuity and Differentiability** **(20) Periods**
   Continuity and differentiability, chain rule, derivative of inverse trigonometric functions, like $\sin^{-1} x$, $\cos^{-1}x$ and $\tan^{-1}x$, derivative of implicit functions. Concept of exponential and logarithmic functions.
   Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives.
2. **Applications of Derivatives** **(10) Periods**
   Applications of derivatives: rate of change of quantities, increasing/decreasing functions, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).
3. **Integrals** **(20) Periods**
   Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, Evaluation of simple integrals of the following types and problems based on them.

$$\int\frac{dx}{x^2\pm a^2},\int\frac{dx}{\sqrt{x^2\pm a^2}},\int\frac{dx}{\sqrt{a^2-x^2}},\int\frac{dx}{ax^2+bx+c},$$

$$\int\frac{dx}{\sqrt{ax^2+bx+c}},\ \int\frac{px+q}{ax^2+bx+c}dx,\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx,$$

$$\int\sqrt{a^2\pm x^2}dx,\ \int\sqrt{x^2-a^2}dx,\ \int\sqrt{ax^2+bx+c}\,dx$$

   Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.
4. **Applications of the Integrals** **(15) Periods**
   Applications in finding the area under simple curves, especially lines, circles/ parabolas/ellipses (in standard form only)
5. **Differential Equations** **(15) Periods**
   Definition, order and degree, general and particular solutions of a differential equation. Solution of differential equations by method of separation of

*For latest details refer www.cbse.gov.in variables, solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

$\frac{dy}{dx}+py=q$, where p and q are functions of x or constants.

$\frac{dx}{dy}+px=q$, where p and q are functions of y or constants.

Unit IV : Vectors and Three-Dimensional Geometry

1. **Vectors** **(15) Periods**
   Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and application of scalar (dot) product of vectors, vector (cross) product of vectors.

2. **Three - dimensional Geometry** **(15) Periods**
   Direction cosines and direction ratios of a line joining two points. Cartesian equation and vector equation of a line, skew lines, shortest distance between two lines. Angle between two lines.

Unit V : Linear Programming

1. **Linear Programming** **(20) Periods**
   Introduction, related terminology such as constraints, objective function, optimization, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded or unbounded), feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

Unit VI : Probability

1. **Probability** **(30) Periods**

   Conditional probability, multiplication theorem on probability, independent events, total probability, Bayes' theorem, Random variable and its probability distribution, mean of random variable.

QUESTION PAPER DESIGN

Class - XII

Mathematics

Time : 3 Hours **Max. Marks :** 80

S. No.	Typology of Questions	Total Marks	% Weightage (approx.)
1.	**Remembering :** Exhibit memory of previously learned material by recalling facts, terms, basic concepts, and answers. **Understanding :** Demonstrate understanding of facts and ideas by organizing, comparing, translating, interpreting, giving descriptions, and stating main ideas.	44	55
2.	**Applying :** Solve problems to new situations by applying acquired knowledge, facts, techniques and rules in a different way.	20	25
3.	**Analysing :** Examine and break information into parts by identifying motives or causes. Make inferences and find evidence to support generalizations **Evaluating :** Present and defend opinions by making judgments about information, validity of ideas, or quality of work based on a set of criteria. **Creating :** Compile information together in a different way by combining elements in a new pattern or proposing alternative solutions.	16	20
	Total	**80**	**100**

1. No chapter wise weightage. Care to be taken to cover all the chapters.
2. Suitable internal variations may be made for generating various templates keeping the overall weightage to different form of questions and typology of questions same.

Choice(s):

There will be no overall choice in the question paper.

However, 33% internal choices will be given in all the sections.

Internal Assessment **20 Marks**

- Periodic Tests (Best 2 out of 3 tests conducted) 10 Marks
- Mathematics Activities 10 Marks

Note: For activities NCERT Lab Manual may be referred.

Contents

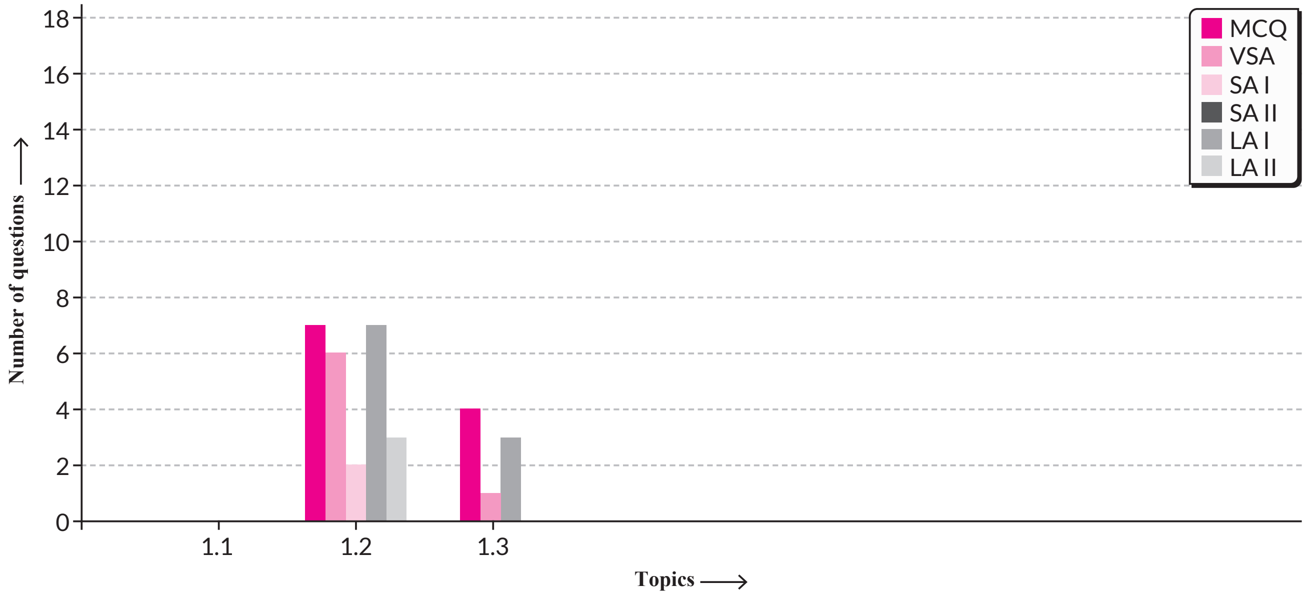
CHAPTER 1

Relations and Functions

TOPICS

1.1 Introduction
1.2 Types of Relations
1.3 Types of Functions

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 1.2 is highly scoring topic.
- Maximum weightage is of Topic 1.2 *Types of Relations.*
- Maximum MCQ, VSA, LA I & LA II type questions were asked from Topic 1.2 *Types of Relations.*

QUICK RECAP

Relation

- A relation R from a set A to a set B is a subset of $A \times B$. So, we say $R \subseteq A \times B$. A relation from a set A to itself is called a relation in A.

Empty Relation

- If no element of A is related to any element of A, then relation R in A is called an empty relation *i.e.*, $R = \phi \subset A \times A$.

Universal Relation

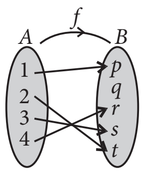
- If each element of A is related to every element of A, then relation R in A is called universal relation *i.e.*, $R = A \times A$.

- A relation R in a set A is called
  (i) reflexive, if $(a, a) \in R$, for all $a \in A$
  (ii) symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$
  (iii) transitive, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, for all $a, b, c \in A$
  - A relation R in a set A is called an **equivalence relation**, if R is reflexive, symmetric and transitive.
  - In a relation R in a set A, the set of all elements related to any element $a \in A$ is denoted by $[a]$ *i.e.*, $[a] = \{x \in A : (x, a) \in R\}$

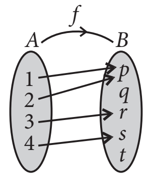
  Here, $[a]$ is called an equivalence class of $a \in A$.

Function

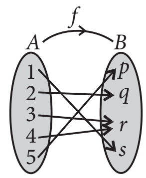
- A relation f from a set A to a set B is called a function if
  (i) for each $a \in A$, there exists some $b \in B$ such that $(a, b) \in f$ *i.e.*, $f(a) = b$
  (ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$
  - A function $f : A \to B$ is called
  (i) **one-one or injective function,** if distinct elements of A have distinct images in B *i.e.*, for $a, b \in A$, $f(a) = f(b) \Rightarrow a = b$



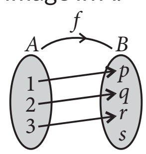
  - **Many-one function :** If two or more elements of A have same image in B.
  i.e.; for $a, b \in A$ such that $a \neq b$, $f(a) = f(b)$.



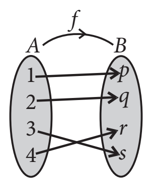
  (ii) **Onto or surjective function**, if for every element $b \in B$, there exists some $a \in A$ such that $f(a) = b$.



  - **Into function :** If there exists an element in B having no pre image in A.



  - A function $f : A \to B$ is called **bijective function**, if it is both one-one and onto function.



BRAIN MAP

RELATIONS AND FUNCTIONS

Relations

- R is a relation from A to B (where $A, B \neq \phi$) if $R \subseteq A \times B$ *i.e.* $R \subseteq \{(a, b) : a \in A, b \in B\}$
- **Inverse Relation :** R^{-1} is the inverse relation of R if $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.
- **Note :** Domain (R) = Range (R^{-1})
  Range (R) = Domain (R^{-1})

Types of Relations

- **Empty (Void) Relation :** $R = \phi \Rightarrow R$ is void.
- **Universal Relation :** $R = A \times B \Rightarrow R$ is universal.
- **Reflexive Relation :** Every element is related to itself. *i.e.,* R is reflexive in $A \Leftrightarrow (a, a) \in R\ \forall\ a \in A$.
- **Symmetric Relation :** R is symmetric in A if $(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A$.
- **Transitive Relation :** R is transitive in A if $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$.
- **Equivalence Relation :** If R is reflexive, symmetric and transitive, then R is equivalence.
- **Antisymmetric Relation :** R is antisymmetric if $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$.
- **Identity Relation :** $R = \{(a, a)\ \forall\ a \in A\}$ is an identity relation in A.

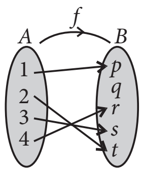
Functions

- A relation $f : A \to B$, where every element of set A has only one image in set B.

Types of Functions

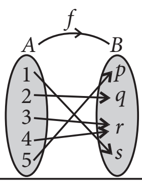
One-one (Injective) Function

- A function $f : A \to B$ is one one, if no two elements of A have same image in B.
  (i) $n(A) \leq n(B)$
  (ii) If for every $x_1, x_2 \in A$
  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$



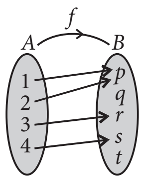
Onto (Surjective) Function

- A function $f : A \to B$ is onto, if every element of B has atleast one pre-image in A.
  (i) $n(A) \geq n(B)$
  (ii) Range = Codomain



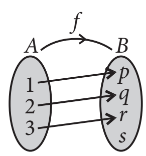
Many-one Function

- A function $f : A \to B$ is many-one, if two or more than two elements of A have the same image in B.



Into Function

- A function $f : A \to B$ is into, if there exists atleast single element in B having no pre-image in A.



Bijective Function

- A function which is both one-one & onto.
  (i) $n(A) = n(B)$
  (ii) Range = Codomain

Previous Years' CBSE Board Questions

1.2 Types of Relations

MCQ

1. Let $A = \{3, 5\}$. Then number of reflexive relations on A is
(a) 2 (b) 4
(c) 0 (d) 8 *(2023)*

2. Let R be a relation in the set N given by
$R = \{(a, b) : a = b - 2, b > 6\}$. Then
(a) $(8, 7) \in R$ (b) $(6, 8) \in R$
(c) $(3, 8) \in R$ (d) $(2, 4) \in R$ *(2023)*

3. A relation R is defined on N. Which of the following is the reflexive relation?
(a) $R = \{(x, y): x > y, x, y \in N\}$
(b) $R = \{(x, y): x + y = 10, x, y \in N\}$
(c) $R = \{(x, y): xy$ is the square number, $x, y \in N\}$
(d) $R = \{(x, y): x + 4y = 10, x, y \in N\}$
(Term I, 2021-22) An

4. The number of equivalence relations in the set $\{1, 2, 3\}$ containing the elements $(1, 2)$ and $(2, 1)$ is
(a) 0 (b) 1
(c) 2 (d) 3 *(Term I, 2021-22)*

5. A relation R is defined on Z as aRb if and only if $a^2 - 7ab + 6b^2 = 0$. Then, R is
(a) reflexive and symmetric
(b) symmetric but not reflexive
(c) transitive but not reflexive
(d) reflexive but not symmetric
(Term I, 2021-22) Ap

6. Let $A = \{1, 3, 5\}$. Then the number of equivalence relations in A containing $(1, 3)$ is
(a) 1 (b) 2
(c) 3 (d) 4 *(2020)*

7. The relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1), (1, 1)\}$ is
(a) symmetric and transitive, but not reflexive
(b) reflexive and symmetric, but not transitive
(c) symmetric, but neither reflexive nor transitive
(d) an equivalence relation *(2020)*

VSA (1 mark)

8. Write the smallest reflexive relation on set $A = \{a, b, c\}$. *(2021 C)*

9. A relation R in a set A is called ________, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$. *(2020)* R

10. A relation in a set A is called ________ relation, if each element of A is related to itself. *(2020)* R

11. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, write the range of R. *(AI 2014)*

12. Let $R = \{(a, a^3) : a$ is a prime number less than 5$\}$ be a relation. Find the range of R. *(Foreign 2014)*

13. Let R be the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2$ divides $(a - b)\}$. Write the equivalence class $[0]$. *(Delhi 2014 C)*

SA I (2 marks)

14. Check if the relation R in the set $\mathbb{R}$ of real numbers defined as $R = \{(a, b) : a < b\}$ is (i) symmetric, (ii) transitive. *(2020)*

15. Let W denote the set of words in the English dictionary. Define the relation R by
$R = \{(x, y) \in W \times W$ such that x and y have at least one letter in common$\}$.
Show that this relation R is reflexive and symmetric, but not transitive. *(2020)*

LA I (4 marks)

16. Show that the relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ given by $R = \{(a, b) : |a - b|$ is divisible by 2$\}$ is an equivalence relation. *(2020)*

17. Check whether the relation R defined on the set $A = \{1, 2, 3, 4, 5, 6\}$ as
$R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive. *(2019)*

18. Show that the relation R on the set Z of all integers, given by
$R = \{(a, b) : 2$ divides $(a - b)\}$ is an equivalence relation. *(2019)*

19. Show that the relation R on $\mathbb{R}$ defined as $R = \{(a,b) : a \le b\}$, is reflexive and transitive but not symmetric. *(NCERT, Delhi 2019)*

20. Show that the relation S in the set $A=\{x \in Z : 0 \le x \le 12\}$ given by $S = \{(a, b) : a, b \in Z, |a - b|$ is divisible by 3$\}$ is an equivalence relation. *(AI 2019)* Ap

21. Let $A = \{1, 2, 3, ..., 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class $[(2, 5)]$. *(Delhi 2014)*

22. Let R be a relation defined on the set of natural numbers N as follow :
$R = \{(x, y) \mid x \in N, y \in N$ and $2x + y = 24\}$
Find the domain and range of the relation R.
Also, find if R is an equivalence relation or not. *(Delhi 2014 C)* An

LA II (5/6 marks)

23. If N denotes the set of all natural numbers and R is the relation on $N \times N$ defined by $(a, b)\ R\ (c, d)$, if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation. *(2023, Delhi 2015)*

24. Let $A = \{x \in Z : 0 \le x \le 12\}$. Show that $R = \{(a, b) : a, b \in A, |a - b|$ is divisible by 4$\}$, is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2]. *(2018)*

25. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b|$ is divisible by 2$\}$ is an equivalence relation. Write all the equivalence classes of R. *(AI 2015 C)*

1.3 Types of Functions

MCQ

26. The function $f : R \to R$ defined by $f(x) = 4 + 3 \cos x$ is
(a) bijective (b) one-one but not onto
(c) onto but not one-one
(d) neither one-one nor onto *(Term I, 2021-22)* An

27. The number of functions defined from $\{1, 2, 3, 4, 5\} \to \{a, b\}$ which are one-one is
(a) 5 (b) 3
(c) 2 (d) 0 *(Term I, 2021-22)*

28. Let $f : R \to R$ be defined by $f(x) = 1/x$, for all $x \in R$, Then, f is
(a) one-one (b) onto
(c) bijective (d) not defined *(Term I, 2021-22)*

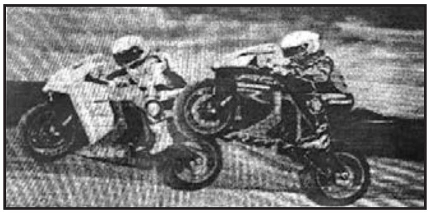
29. The function $f : N \to N$ is defined by
$$f(n) = \begin{cases} \frac{n+1}{2}, \text{if } n \text{ is odd} \\ \frac{n}{2}, \quad \text{if } n \text{ is even} \end{cases}$$
The function f is
(a) bijective
(b) one-one but not onto
(c) onto but not one-one
(d) neither one-one nor onto *(Term I, 2021-22)* Ev

VSA (1 mark)

30. If $f = \{(1, 2), (2, 4), (3, 1), (4, k)\}$ is a one-one function from set A to A, where $A = \{1, 2, 3, 4\}$, then find the value of k. *(2021 C)*

LA I (4 marks)

31. **Case Study :** An organization conducted bike race under two different categories - Boys and girls. There were 28 participants in all. Among all of them, finally three from category 1 and two from category 2 were selected for the final race. Ravi forms two sets B and G with these participants for his college project.
Let $B = \{b_1, b_2, b_3\}$ and $G = \{g_1, g_2\}$, where B represents the set of Boys selected and G the set of Girls selected for the final race.



Based on the above information, answer the following questions.
(i) How many relations are possible from B to G?
(ii) Among all the possible relations from B to G, how many functions can be formed from B to G?
(iii) Let $R : B \to B$ be defined by $R = \{(x, y) : x$ and y are students of the same sex$\}$. Check if R is an equivalence relation.

OR

A function $f : B \to G$ be defined by $f = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$. Check if f is bijective, justify your answer. *(2023)* Ap

32. Let $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \to \mathbb{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that f is a one-one function. Also, check whether f is an onto function or not. *(2023)*

33. Show that the function $f : (-\infty, 0) \to (-1, 0)$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in (-\infty, 0)$ is one-one and onto. *(2020)*

CBSE Sample Questions

1.2 Types of Relations

MCQ

1. A relation R in set $A = \{1, 2, 3\}$ is defined as $R = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$. Which of the following ordered pair in R shall be removed to make it an equivalence relation in A?
(a) (1, 1) (b) (1, 2) (c) (2, 2) (d) (3, 3) *(Term I, 2021-22)* An

2. Let the relation R in the set $A = \{x \in Z : 0 \le x \le 12\}$, given by $R = \{(a, b) : |a - b|$ is a multiple of 4.$\}$ Then [1], the equivalence class containing 1, is
(a) $\{1, 5, 9\}$ (b) $\{0, 1, 2, 5\}$
(c) ϕ (d) A *(Term I, 2021-22)* Ev

VSA (1 mark)

3. How many reflexive relations are possible in a set A whose $n(A) = 3$? *(2020-21)* Ap

4. A relation R in $S = \{1, 2, 3\}$ is defined as $R = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$. Which element(s) of relation R be removed to make R an equivalence relation? *(2020-21)*

5. An equivalence relation R in A divides it into equivalence classes A_1, A_2, A_3. What is the value of $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$. *(2020-21)*

SA I (2 marks)

6. Let R be the relation in the set Z of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Show that the relation R is transitive? Write the equivalence class of 0. *(2020-21)* Ap

SA II (3 marks)

7. Check whether the relation R in the set Z of integers defined as $R = \{(a, b) : a + b$ is "divisible by 2"$\}$ is reflexive, symmetric or transitive. Write the equivalence class containing 0, *i.e.*, [0]. *(2020-21)*

LA II (5/6 marks)

8. Given a non-empty set X, define the relation R on $P(X)$ as :
For $A, B \in P(X)$, $(A, B) \in R$ iff $A \subset B$. Prove that R is reflexive, transitive, and not symmetric. *(2022-23)*

9. Define the relation R in the set $N \times N$ as follows:
For $(a, b), (c, d) \in N \times N$, $(a, b)\, R\, (c, d)$ iff $ad = bc$. Prove that R is an equivalence relation in $N \times N$. *(2022-23)*

1.3 Types of Functions

MCQ

10. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Based on the given information, f is best defined as
(a) Surjective function (b) Injective function
(c) Bijective function (d) Function
(Term I, 2021-22) Ev

11. The function $f : R \to R$ defined as $f(x) = x^3$ is
(a) One-one but not onto
(b) Not one-one but onto
(c) Neither one-one nor onto
(d) One-one and onto *(Term I, 2021-22)*

VSA (1 mark)

12. Check whether the function $f : R \to R$ defined as $f(x) = x^3$ is one-one or not. *(2020-21)*

13. A relation R in the set of real numbers **R** defined as $R = \{(a, b) : \sqrt{a} = b\}$ is a function or not. Justify *(2020-21)*

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. **(b):** Total number of reflexive relations on a set having n number of elements $= 2^{n^2 - n}$
Here, $n = 2$
$\therefore$ Required number of reflexive relations $= 2^{2^2 - 2}$
$= 2^{4-2} = 2^2 = 4$

2. **(b):** Given, $R = \{(a, b) : a = b - 2, b > 6\}$
Since, $b > 6$, so $(2, 4) \notin R$
Also, $(3, 8) \notin R$ as $3 \neq 8 - 2$
and $(8, 7) \notin R$ as $8 \neq 7 - 2$
Now, for $(6, 8)$, we have
$8 > 6$ and $6 = 8 - 2$, which is true
$\therefore$ $(6, 8) \in R$

3. **(c) :** Consider, $R = \{(x, y) : xy$ is the square number, $x, y \in N\}$
As, $xx = x^2$, which is the square of natural number x.
$\Rightarrow$ $(x, x) \in R$. So, R is reflexive.

Concept Applied
- A relation R in a set A is called reflexive, if $(a, a) \in R$, for all $a \in A$.

4. **(c) :** Equivalence relations in the set $\{1, 2, 3\}$ containing the elements $(1, 2)$ and $(2, 1)$ are
$R_1 = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3)\}$
$R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1) (2, 3), (3, 2), (1, 1), (2, 2), (3, 3)\}$
$\therefore$ Number of equivalence relations is 2.

Concept Applied
- A relation R in a set A is called an equivalence relation, if R is reflexive, symmetric and transitive.

5. **(d):** Given, aRb, $a, b \in Z$
Reflexive : For $a \in Z$, we have
$a^2 - 7a{\cdot}a + 6a^2 = a^2 - 7a^2 + 6a^2 = 0 \Rightarrow (a, a) \in R$
$\therefore$ Relation is reflexive.
Symmetric : Since, $(6, 1) \in R$
As, $6^2 - 7 \times 6 \times 1 + 6 \times 1^2 = 36 - 42 + 6 = 0$
But $(1, 6) \notin R$. $\therefore$ Relation is not symmetric.

6. **(b):** Equivalence relations in the set containing the element $(1, 3)$ are
$R_1 = \{(1, 1), (3, 3), (1, 3), (3, 1), (5, 5)\}$
$R_2 = \{(1, 1),(3, 3),(5, 5),(1, 5),(5, 1),(3, 5),(5, 3), (1, 3), (3, 1)\}$
$\therefore$ There are 2 possible equivalence relations.

7. **(c) :** Given $R = \{(1, 2), (2, 1), (1, 1)\}$ is a relation on set $\{1, 2, 3\}$
Reflexive : Clearly $(2, 2), (3, 3) \notin R$
$\therefore$ R is not a reflexive relation.
Symmetric : Now, $(1, 2) \in R$ and $(2, 1) \in R$ $\therefore$ R is symmetric.
Transitive : Now, $(2, 1) \in R$ and $(1, 2) \in R$ but $(2, 2) \notin R$
$\therefore$ R is not transitive relation.
R is symmetric, but neither reflexive nor transitive.

8. We have, $A = \{a, b, c\}$
A relation R on the set A is said to be reflexive if $(a, a) \in R$, $\forall\, a \in A$
$\therefore$ $R = \{(a, a), (b, b), (c, c)\}$ is the required smallest reflexive relation on A.

9. A relation R in a set A is called symmetric, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

10. A relation in a set A is called reflexive relation, if each element of A is related to itself.

11. Here, $R = \{(x, y) : x + 2y = 8x, y \in N\}$.
For $x = 1, 3, 5, ...$
$x + 2y = 8$ has no solution in N.
For $x = 2$, we have $2 + 2y = 8 \Rightarrow y = 3$
For $x = 4$, we have $4 + 2y = 8 \Rightarrow y = 2$
For $x = 6$, we have $6 + 2y = 8 \Rightarrow y = 1$
For $x = 8, 10, ...$
$x + 2y = 8$ has no solution in N.
$\therefore$ Range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$

12. Given relation is
$R = \{(a, a^3) : a$ is a prime number less than $5\}$.
$\therefore$ $R = \{(2, 8), (3, 27)\}$. So, the range of R is $\{8, 27\}$.

13. Here, $R = \{(a, b) : 2$ divides $(a - b)\}$
$\therefore$ Equivalence class of $[0] = \{a \in A : (a, 0) \in R\}$.
$\Rightarrow$ $(a - 0)$ is divisible by 2 and $a \in A \Rightarrow a = 0, 2, 4$
Thus $[0] = \{0, 2, 4\}$.

14. We have, $R = \{(a, b) : a < b\}$, where $a, b \in \mathbb{R}$
(i) Symmetric : Let $(x, y) \in R$, *i.e.*, $x\,R\,y \Rightarrow x < y$
But $y \not< x$, so $(x, y) \in R \Rightarrow (y, x) \notin R$
Thus, R is not symmetric.
(ii) Transitive : Let $(x, y), (y, z) \in R$
$\Rightarrow$ $x < y$ and $y < z \Rightarrow x < z$
$\Rightarrow$ $(x, z) \in R$. Thus, R is transitive.

15. We have, $R = \{(x, y) \in W \times W : x$ and y have at least one letter in common$\}$
Reflexive : Clearly $(x, x) \in R$, because same words will contains all common letters.
$\Rightarrow$ R is reflexive.
Symmetric : Let $(x, y) \in R$ *i.e.*, x and y have at least one letter in common.
$\Rightarrow$ y and x will also have at least one letter in common.
$\Rightarrow$ $(y, x) \in R$
$\Rightarrow$ R is symmetric.
Transitive : Let, x = LAND, y = NOT and z = HOT
Clearly $(x, y) \in R$ as x and y have a common letter and $(y, z) \in R$ as y and z have 2 common letters.
but $(x, z) \notin R$ as x and z have no letter in common.
Hence, R is not transitive.

Concept Applied

- A relation R in a set A is not transitive if for $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$

16. We have, $A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(a, b) : |a - b|$ is divisible by $2\}$
(i) Reflexive : For any $a \in A$
$|a - a| = 0$, which is divisible by 2.
Thus, $(a, a) \in R$. So, R is reflexive.
(ii) Symmetric : For any $a, b \in A$
Let $(a, b) \in R$
$\Rightarrow$ $|a - b|$ is divisible by $2 \Rightarrow |b - a|$ is divisible by 2
$\Rightarrow$ $(b, a) \in R$ $\therefore$ $(a, b) \in R \Rightarrow (b, a) \in R$ $\therefore$ R is symmetric.
(iii) Transitive : For any $a, b, c \in A$
Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $|a - b|$ is divisible by 2 and $|b - c|$ is divisible by 2.
$\Rightarrow$ $a - b = \pm 2\,k_1$ and $b - c = \pm\, 2k_2\ \forall\, k_1, k_2 \in N$
$\Rightarrow$ $a - b + b - c = \pm 2(k_1 + k_2) \Rightarrow a - c = \pm 2\,k_3\ \forall\, k_3 \in N$
$\Rightarrow$ $|a - c|$ is divisible by $2 \Rightarrow (a, c) \in R$ $\therefore$ R is transitive.
Hence, R is an equivalence relation.

17. We have, $A = \{1, 2, 3, 4, 5, 6\}$ and a relation R on A defined as $R = \{(a, b) : b = a + 1\}$
Reflexive : Let $(a, a) \in R$
$\Rightarrow$ $a = a + 1 \Rightarrow a - a = 1 \Rightarrow 0 = 1$, which is not possible.
$\therefore$ $(a, a) \notin R \Rightarrow R$ is not reflexive.
Symmetric : Let $(a, b) \in R \Rightarrow b = a + 1$ ...(i)
Now, if $(b, a) \in R$
$\Rightarrow$ $a = b + 1 \Rightarrow b = b + 1 + 1$ (using (i))
$\Rightarrow$ $b = b + 2 \Rightarrow b - b = 2 \Rightarrow 0 = 2$, which is not possible
$\Rightarrow$ $(b, a) \notin R \Rightarrow R$ is not symmetric.
Transitive : Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $b = a + 1$ and $c = b + 1 \Rightarrow c = a + 1 + 1$
$\Rightarrow$ $c = a + 2 \neq a + 1 \Rightarrow (a, c) \notin R \Rightarrow R$ is not transitive.

18. We have, $R = \{(a, b) : 2$ divides $(a - b)\}$
Reflexive : For any $a \in Z$, $a - a = 0$ and 2 divides 0.
$\Rightarrow$ $(a, a) \in R$ for every $a \in Z$ $\therefore$ R is a reflexive.
Symmetric : Let $(a, b) \in R$
$\Rightarrow$ 2 divides $(a - b)$
$\Rightarrow$ $a - b = 2m$, for some $m \in Z$
$\Rightarrow$ $b - a = 2m$
$\Rightarrow$ 2 divides $b - a$
$\Rightarrow$ $(b, a) \in R$
$\therefore$ R is symmetric.
Transitive : Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ 2 divides $(a - b)$ and 2 divides $(b - c)$
$\Rightarrow$ $a - b = 2m$ and $b - c = 2n$ for some $m, n \in Z$
$\Rightarrow$ $a - b + b - c = 2m + 2n$
$\Rightarrow$ $a - c = 2(m + n)$
$\Rightarrow$ 2 divides $a - c$
$\Rightarrow$ $(a, c) \in R$
$\therefore$ R is transitive.
Hence, R is an equivalence relation.

19. We have, $R = \{(a, b) : a \le b,\ a, b \in \mathbb{R}\}$
(i) Reflexive : Since $a \le a$ $\therefore$ $aRa\ \forall\, a \in R$
Hence, R is reflexive.
(ii) Symmetric : $(a, b) \in R$ such that $aRb \Rightarrow a \le b \not\Rightarrow b \le a$
So, $(b, a) \notin R$.
Hence, R is not symmetric.
(iii) Transitive : Let $a, b, c \in R$ such that aRb and bRc
Now, $aRb \Rightarrow a \le b$ ... (i) and $bRc \Rightarrow b \le c$ ... (ii)
From (i) and (ii), we have $a \le b \le c \Rightarrow a \le c$ $\therefore$ aRc
Hence, relation R is transitive.

20. We have, $A = \{x \in Z : 0 \le x \le 12\}$
$\therefore$ $A = \{0, 1, 2, 3, ..., 12\}$
Also, $S = \{(a, b) : a, b \in Z, |a - b|$ is divisible by $3\}$

(i) Reflexive : For any $a \in A$,
$|a - a| = 0$, which is divisible by 3
Thus, $(a, a) \in S \therefore S$ is reflexive.
(ii) Symmetric : Let $(a, b) \in S$
$\Rightarrow |a - b|$ is divisible by 3.
$\Rightarrow |b - a|$ is divisible by 3 $\Rightarrow (b, a) \in S$ i.e. $(a, b) \in S \Rightarrow (b, a) \in S$
$\therefore$ S is symmetric.
(iii) Transitive :
Let $(a, b) \in S$ and $(b, c) \in S$
$\Rightarrow |a - b|$ is divisible by 3 and $|b - c|$ is divisible by 3.
$\Rightarrow (a - b) = \pm 3k_1$ and $(b - c) = \pm 3k_2$; $\forall k_1, k_2 \in N$
$\Rightarrow (a - b) + (b - c) = \pm 3(k_1 + k_2)$
$\Rightarrow (a - c) = \pm 3(k_1 + k_2)$; $\forall k_1, k_2 \in N$
$\Rightarrow |a - c|$ is divisible by 3 $\Rightarrow (a, c) \in S \therefore S$ is Transitive.
Hence, S is an equivalence relation.

Concept Applied

- A relation R in a set A is called
  (i) reflexive, if $(a, a) \in R$, for all $a \in A$
  (ii) symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$
  (iii) transitive, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, for all $a, b, c \in A$

21. Given $A = \{1, 2, 3, 4, ..., 9\}$
To show : R is an equivalence relation.
(i) Reflexive : Let (a, b) be an arbitrary element of $A \times A$. Then, we have $(a, b) \in A \times A \Rightarrow a, b \in A$
$\Rightarrow a + b = b + a$ (by commutativity of addition on $A \subset N$)
$\Rightarrow (a, b) R (a, b)$
Thus, $(a, b) R (a, b)$ for all $(a, b) \in A \times A$. So, R is reflexive.
(ii) Symmetric : Let $(a, b), (c, d) \in A \times A$ such that $(a, b) R (c, d)$
$\Rightarrow a + d = b + c \Rightarrow b + c = a + d$
$\Rightarrow c + b = d + a$ (by commutativity of addition on $A \subset N$)
$\Rightarrow (c, d) R (a, b)$.
Thus, $(a, b) R (c, d) \Rightarrow (c, d) R (a, b)$ for all $(a, b), (c, d) \in A \times A$.
So, R is symmetric.
(iii) Transitive : Let $(a, b), (c, d), (e, f) \in A \times A$ such that $(a, b) R (c, d)$ and $(c, d) R (e, f)$
Now, $(a, b) R (c, d) \Rightarrow a + d = b + c$ ...(i)
and $(c, d) R (e, f) \Rightarrow c + f = d + e$ ...(ii)
Adding (i) and (ii), we get $(a + d) + (c + f) = (b + c) + (d + e)$
$\Rightarrow a + f = b + e \Rightarrow (a, b) R (e, f)$
Thus, $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow (a, b) R (e, f)$.
So, R is transitive. $\therefore R$ is an equivalence relation.
Equivalence class of $[(2, 5)] = \{(x, y) \in N \times N : (x, y) R (2, 5)\}$
$= \{(x, y) \in N \times N : x + 5 = y + 2\}$
$= \{(x, y) \in N \times N : y = x + 3\} = \{(x, x + 3) : x \in A\}$
$= \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

Answer Tips

- First, prove the given relation is an equivalence relation and then find the equivalence class by using the given relation.

22. Here, $R = \{(x, y) \mid x \in N, y \in N$ and $2x + y = 24\}$
$R = \{(1, 22), (2, 20), (3, 18), ..., (11, 2)\}$
Domain of $R = \{1, 2, 3, 4, ..., 11\}$
Range of $R = \{2, 4, 6, 8, 10, 12, ..., 22\}$
R is not reflexive as if $(2, 2) \in R \Rightarrow 2 \times 2 + 2 = 6 \neq 24$
In fact R is neither symmetric nor transitive.
$\Rightarrow$ R is not an equivalence relation.

23. (i) Reflexive : Let (a, b) be an arbitrary element of $N \times N$. Then, $(a, b) \in N \times N$
$\Rightarrow ab(b + a) = ba(a + b)$
[by commutativity of addition and multiplication on N]
$\Rightarrow (a, b) R (a, b)$
So, R is reflexive on $N \times N$.
(ii) Symmetric : Let $(a, b), (c, d) \in N \times N$ such that $(a, b) R (c, d)$.
$\Rightarrow ad(b + c) = bc(a + d) \Rightarrow cb(d + a) = da(c + b)$
[by commutativity of addition and multiplication on N]
Thus, $(a, b) R (c, d) \Rightarrow (c, d) R (a, b)$ for all $(a, b), (c, d) \in N \times N$.
So, R is symmetric on $N \times N$.
(iii) Transitive : Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b) R (c, d)$ and $(c, d) R (e, f)$. Then,
$(a, b) R (c, d) \Rightarrow ad(b + c) = bc(a + d)$

$$\Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad} \Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \quad ...(i)$$

and $(c, d) R (e, f) \Rightarrow cf(d + e) = de(c + f)$

$$\Rightarrow \frac{d+e}{de} = \frac{c+f}{cf} \Rightarrow \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f} \quad ...(ii)$$

Adding (i) and (ii), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$$\Rightarrow \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f} \Rightarrow \frac{b+e}{be} = \frac{a+f}{af}$$

$\Rightarrow af(b + e) = be(a + f) \Rightarrow (a, b) R (e, f)$
So, R is transitive on $N \times N$.
Hence, R is an equivalence relation.

24. We have, $A = \{x \in Z : 0 \le x \le 12\}$
$\therefore A = \{0, 1, 2, 3, ..., 12\}$
and $S = \{(a, b) : |a - b|$ is divisible by $4\}$
(i) Reflexive : For any $a \in A$, $|a - a| = 0$, which is divisible by 4. Thus, $(a, a) \in R \therefore R$ is reflexive.
(ii) Symmetric : Let $(a, b) \in R$
$\Rightarrow |a - b|$ is divisible by 4
$\Rightarrow |b - a|$ is divisible by 4 $\Rightarrow (b, a) \in R$
i.e., $(a, b) \in R \Rightarrow (b, a) \in R \therefore R$ is symmetric.
(iii) Transitive : Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow |a - b|$ is divisible by 4 and $|b - c|$ is divisible by 4
$\Rightarrow a - b = \pm 4k_1$ and $b - c = \pm 4k_2$; $\forall k_1, k_2 \in N$
$\Rightarrow (a - b) + (b - c) = \pm 4(k_1 + k_2)$; $\forall k_1, k_2 \in N$
$\Rightarrow a - c = \pm 4(k_1 + k_2)$; $\forall k_1, k_2 \in N$
$\Rightarrow |a - c|$ is divisible by $4 \Rightarrow (a, c) \in R \therefore R$ is transitive.
Hence, R is an equivalence relation.
The set of elements related to 1 is $\{1, 5, 9\}$.
Equivalence class for $[2]$ is $\{2, 6, 10\}$.

Concept Applied

- In a relation R in a set A, the set of all elements related to any element $a \in A$ is denoted by $[a]$
  i.e., $[a] = \{x \in A : (x, a) \in R\}$
  Here, $[a]$ is called an equivalence class of $a \in A$.

25. We have, $A = \{1, 2, 3, 4, 5\}$
and $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$
(i) Reflexive : For any $a \in A$,
$|a - a| = 0$, which is divisible by 2
Thus, $(a, a) \in R$ $\therefore$ R is reflexive.
(ii) Symmetric : Let $(a, b) \in R$
$\Rightarrow$ $|a - b|$ is divisible by 2
$\Rightarrow$ $|b - a|$ is divisible by 2 $\Rightarrow$ $(b, a) \in R$
i.e., $(a, b) \in R \Rightarrow (b, a) \in R$ $\therefore$ R is symmetric.
(iii) Transitive : Let $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow$ $|a - b|$ is divisible by 2 and $|b - c|$ is divisible by 2
$\Rightarrow$ $a - b = \pm 2k_1$ and $b - c = \pm 2k_2$; $\forall\ k_1, k_2 \in N$
$\Rightarrow$ $(a - b) + (b - c) = \pm 2(k_1 + k_2)$; $\forall\ k_1, k_2 \in N$
$\Rightarrow$ $(a - c) = \pm 2(k_1 + k_2)$; $\forall\ k_1, k_2 \in N$
$\Rightarrow$ $|a - c|$ is divisible by 2 $\Rightarrow (a, c) \in R$ $\therefore$ R is transitive.
Hence, R is an equivalence relation.
Further R has only two equivalence classes, namely $[1] = [3] = [5] = \{1, 3, 5\}$ and $[2] = [4] = \{2, 4\}$.

26. (d): We have, $f(x) = 4 + 3\cos x$, $\forall\ x \in R$

At $x = \frac{\pi}{2}, f\left(\frac{\pi}{2}\right) = 4 + 3\cos\frac{\pi}{2} = 4 \Rightarrow f\left(-\frac{\pi}{2}\right) = 4 + 3\cos\left(-\frac{\pi}{2}\right) = 4$

Since, $f\left(\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right)$, But $\frac{\pi}{2} \neq -\frac{\pi}{2}$

Therefore, f is not one-one.
As $-1 \le \cos x \le 1$, $\forall\ x \in R \Rightarrow 1 \le 4 + 3\cos x \le 7$, $\forall\ x \in R$
$\Rightarrow$ $f(x) \in [1, 7]$, where $[1, 7]$ is subset of R. $\therefore$ f is not onto.

Concept Applied
- Range of $\cos x$ is $[-1, 1]$.

27. (d): $\because$ $f : X \to Y$ is one-one, if different element of X have different image in Y under f. But here, no such situation is possible.

28. (d): Given $f(x) = \frac{1}{x}$, for all $x \in R$

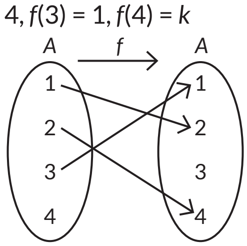
At $x = 0 \in R$, $f(x)$ is not defined.

29. (c): Given, $f(x) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

Now, $f(1) = \frac{1+1}{2} = 1, f(2) = \frac{2}{2} = 1$

$\Rightarrow$ $f(1) = f(2)$ but $1 \neq 2$ $\therefore$ f is not one-one.
But f is onto ($\because$ range of f is N.)

30. We have, $A = \{1, 2, 3, 4\}$ function $f : A \to A$ is one-one and $f(1) = 2, f(2) = 4, f(3) = 1, f(4) = k$



As f is one-one, so no two element of A has same image in A.
$\therefore$ $f(4) = 3 \Rightarrow k = 3$

Concept Applied
- For a function to be one-one, no two elements should have the same image in A.

31. (i) Here $n(B) = 3$ and $n(G) = 2$
$\therefore$ Number of relation from B to $G = 2^{3 \times 2} = 2^6$
(ii) Number of functions formed from B to $G = 2^3 = 8$
(iii) We have, $R = \{(x, y) = x$ and y are students of the same sex$\}$
$\therefore$ R is reflexive as $(x, x) \in R$.
R is symmetric as $(x, y) \in R \Rightarrow (y, x) \in R$.
Since, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$
Hence, R is an equivalence relations.

OR

We have $f : B \to G$ be defined by $f = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$
Since, elements b_1 and b_3 have the same image, therefore, the functions is not one-one but it is many one functions.
Since, every element in G has its pre-image in B, so the functions is onto.
For bijection, function should be one-one and onto both.
Hence, the function is surjective but not injective.

32. The function $f : R - \left\{-\frac{4}{3}\right\} \to R$ is given by $f(x) = \frac{4x}{3x+4}$.

One-one : Let $x, y \in R - \left\{-\frac{4}{3}\right\}$ such that $f(x) = f(y)$

$\Rightarrow$ $\frac{4x}{3x+4} = \frac{4y}{3y+4}$

$\Rightarrow$ $4x(3y + 4) = 4y(3x + 4) \Rightarrow 12xy + 16x = 12xy + 16y$
$\Rightarrow$ $16x = 16y \Rightarrow x = y$
$\therefore$ f is one-one.
Onto : Let y be an arbitrary element of R. Then $f(x) = y$

$\Rightarrow \frac{4x}{3x+4} = y \Rightarrow 4x = 3xy + 4y \Rightarrow 4x - 3xy = 4y \Rightarrow x = \frac{4y}{4-3y}$

As $y \in R - \left\{\frac{4}{3}\right\}$, $\frac{4y}{4-3y} \in R$

Also, $\frac{4y}{4-3y} \neq \frac{-4}{3}$ as if

$\frac{4y}{4-3y} = -\frac{4}{3} \Rightarrow 12y = 12y - 16$, which is not possible.

Thus, $x = \frac{4y}{4-3y} \in R - \left\{-\frac{4}{3}\right\}$ such that

$$f(x) = f\left(\frac{4x}{3x+4}\right) = \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

So, every element in $R - \left\{\frac{4}{3}\right\}$ has pre-image in $R - \left\{-\frac{4}{3}\right\}$

$\therefore$ f is not onto.

33. Given, $f(x) = \frac{x}{1+|x|}, x \in (-\infty, 0)$

$= \frac{x}{1-x}$ $\quad (\because\ x \in (-\infty, 0), |x| = -x)$

For one-one : Let $f(x_1) = f(x_2)$, $x_1, x_2 \in (-\infty, 0)$

$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1(1 - x_2) = x_2(1 - x_1)$

$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2 \Rightarrow x_1 = x_2$

Thus, $f(x_1) = f(x_2)$, $\Rightarrow x_1 = x_2$

$\therefore$ f is one-one

For onto : Let $f(x) = y$

$\Rightarrow y = \frac{x}{1-x} \Rightarrow y(1 - x) = x \Rightarrow y - xy = x$

$\Rightarrow x + xy = y \Rightarrow x(1 + y) = y \Rightarrow x = \frac{y}{1+y}$

Here, $y \in (-1, 0)$

So, x is defined for all values of y in codomain. $\therefore$ f is onto.

Concept Applied

- A function $f : A \to B$ is called
  - (i) one-one or injective function, if distinct elements of A have distinct images in B. *i.e.*, for $a, b \in A$, $f(a) = f(b) \Rightarrow a = b$
  - (ii) onto or surjective function, if for every element $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

CBSE Sample Questions

1. **(b):** We have, $(1, 2) \in R$ but $(2, 1) \notin R$
So, (1, 2) should be removed from R to make it an equivalence relation. **(1)**

2. **(a):** We have, $R = \{(a, b) : |a - b|$ is a multiple of $4\}$
$\therefore$ The set of elements related to 1 is {1, 5, 9}.
So, equivalence class for [1] is {1, 5, 9} **(1)**

3. Number of reflexive relations on a set having n elements $= 2^{n(n-1)}$
So, required number of reflexive relations $= 2^{3(3-1)} = 2^6$ **(1)**

4. We have, $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$
which is reflexive and transitive.
For R to be symmetric (1, 2) should be removed from R. **(1)**

5. As we know that, union of all equivalence classes of a set is the set itself.
$\therefore$ $A_1 \cup A_2 \cup A_3 = A$
Also, $A_1 \cap A_2 \cap A_3 = \phi$
[$\because$ Equivalence classes are either equal or disjoint] **(1)**

6. Let $(a, b) \in R$ and $(b, c) \in R$. Then, 2 divides $(a - b)$ and 2 divides $(b - c)$: where $a, b, c \in Z$
So, 2 divides $[(a - b) + (b - c)]$
$\Rightarrow$ 2 divides $(a - c) \Rightarrow (a, c) \in R$. So, relation R is transitive. **(1)**
Equivalence class of $0 = \{0, \pm 2, \pm 4, \pm 6, ...\}$ **(1)**

7. (i) Reflexive : Since, $a + a = 2a$ which is even.
$\therefore$ $(a, a) \in R \ \forall \ a \in Z$
Hence, R is reflexive. **(1/2)**
(ii) Symmetric : If $(a, b) \in R$, then $a + b = 2\lambda \Rightarrow b + a = 2\lambda$
$\Rightarrow$ $(b, a) \in R$. Hence, R is symmetric. **(1)**
(iii) Transitive : If $(a, b) \in R$ and $(b, c) \in R$
then $a + b = 2\lambda$ .... (i) and $b + c = 2\mu$ ... (ii)
Adding (i) and (ii), we get
$a + 2b + c = 2(\lambda + \mu) \Rightarrow a + c = 2(\lambda + \mu - b)$
$\Rightarrow$ $a + c = 2k$, where $k = \lambda + \mu - b \Rightarrow (a, c) \in R$
Hence, R is transitive. **(1)**
Equivalence class containing 0 *i.e.*,
$[0] = \{..., -4, -2, 0, 2, 4, ...\}$ **(1/2)**

8. We have, a relation R on X such that, $(A, B) \in R$ iff $A \subset B$ for $A, B \in P(X)$. **(1/2)**
Reflexive : Clearly every set is a subset of itself.
$\Rightarrow$ $(A, A) \in R$
$\therefore$ R is reflexive. **(1)**
Symmetric : Let $(A, B) \in R$
$\Rightarrow$ $A \subset B$
$\Rightarrow$ B is a super set of A. **(1/2)**
$\Rightarrow$ $B \not\subset A \Rightarrow (B, A) \notin R$
$\therefore$ R is not symmetric. **(1)**
Transitive : Let $(A, B) \in R$ and $(B, C) \in R$, for all $A, B, C \in P(X)$
$\Rightarrow$ $A \subset B$ and $B \subset C \Rightarrow A \subset B \subset C$ **(1/2)**
$\Rightarrow$ $A \subset C \Rightarrow (A, C) \in R$
$\therefore$ R is transitive. **(1)**
Hence, R is reflexive and transitive but not symmetric. **(1/2)**

9. Reflexive : Let $(a, b) \in N \times N$. Then $ab = ba$
(By commutativity of multiplication of natural number)
$\Rightarrow$ $(a, b) R (b, a)$
Thus, $(a, b) R (b, a)$ for all $(a, b) \in N \times N$
So, R is reflexive. **(1)**
Symmetric : Let $(a, b), (c, d) \in N \times N$ such that $(a, b) R (c, d)$
$\Rightarrow ad = bc \Rightarrow bc = ad \Rightarrow cb = da$
(By commutativity of multiplication of natural numbers)
$\Rightarrow$ $(c, d) R (a, b)$
Thus, $(a, b) R (c, d) = (c, d) R (a, b)$ for $(a, b), (c, d) \in N \times N$
So, R is symmetric. **(1)**
Transitive : Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b) R (c, d)$ and $(c, d) R (e, f)$
Now, $(a, b) R (c, d) \Rightarrow ad = bc$ ...(i)
and $(c, d) R (e, f) \Rightarrow cf = de$ ...(ii)
Multiplying (i) and (ii), we get $ad \cdot cf = bc \cdot de$ **(1)**
$\Rightarrow$ $af = be \Rightarrow (a, b) R (e, f)$
Thus, $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow (a, b) R (e, f)$ **(1)**
So, R is transitive.
$\therefore$ R is an equivalence relation. **(1)**

10. **(b):** As every pre-image $x \in A$, has a unique image $y \in B$.
$\Rightarrow$ f is injective function. **(1)**

11. **(d):** Let $x_1, x_2 \in R$ be such that $f(x_1) = f(x_2)$
$\Rightarrow$ $x_1^3 = x_2^3 \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one.
Let $f(x) = x^3 = y$ for some arbitrary element $y \in R \Rightarrow x = y^{1/3}$
$\Rightarrow$ $f(y^{1/3}) = y$
Every image $y \in R$ has a unique pre-image in R.
$\Rightarrow$ f is onto
$\therefore$ f is one-one and onto. **(1)**

12. Let $f(x_1) = f(x_2)$ for some $x_1, x_2 \in R$.
$\Rightarrow$ $(x_1)^3 = (x_2)^3$
$\Rightarrow$ $x_1 = x_2$, hence $f(x)$ is one-one. **(1)**

13. Since $\sqrt{a}$ is not defined for $a \in (-\infty, 0)$
$\therefore$ $R = \{(a, b) : \sqrt{a} = b\}$ is not a function. **(1)**

Self Assessment

Case Based Objective Questions (4 marks)

1. A relation R on a set A is said to be an equivalence relation on A iff it is
   - Reflexive *i.e.*, $(a, a) \in R \ \forall \ a \in A$.
   - Symmetric *i.e.*, $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b \in A$.
   - Transitive *i.e.*, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \ \forall \ a, b, c \in A$.

   Based on the above information, attempt any 4 out of 5 subparts.

   (i) If the relation R = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} is defined on the set A = {1, 2, 3}, then R is
   (a) reflexive (b) symmetric
   (c) transitive (d) equivalence

   (ii) If the relation R = {(1, 2), (2, 1), (1, 3), (3, 1)} is defined on the set A = {1, 2, 3}, then R is
   (a) reflexive (b) symmetric
   (c) transitive (d) equivalence

   (iii) If the relation R on the set N of all natural numbers defined as $R = \{(x, y) : y = x + 5$ and $x < 4\}$, then R is
   (a) reflexive (b) symmetric
   (c) transitive (d) equivalence

   (iv) If the relation R on the set A = {1, 2, 3, ..., 13, 14} defined as $R = \{(x, y) : 3x - y = 0\}$, then R is
   (a) reflexive (b) symmetric
   (c) transitive (d) None of these

   (v) If the relation R on the set A = {1, 2, 3} defined as R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}, then R is
   (a) reflexive only (b) symmetric only
   (c) transitive only (d) equivalence

Multiple Choice Questions (1 mark)

2. If $f : R \to R$ be a function defined by $f(x) = \frac{x^2 - 8}{x^2 + 2}$, then f is
   (a) one-one but not onto
   (b) one-one and onto
   (c) onto but not one-one
   (d) neither one-one nor onto

   OR

   Let A = {2, 3, 4, 5,......, 17, 18}. Let '$\simeq$' be the equivalence relation on $A \times A$, cartesian product of A with itself defined by $(a, b) \simeq (c, d)$ iff $ad = bc$. Then number of ordered pairs of the equivalence class (3, 2) is
   (a) 4 (b) 5 (c) 6 (d) 7

3. Let A = {1, 2, 3} and consider the relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}. Then R is
   (a) reflexive but not symmetric
   (b) reflexive but not transitive
   (c) symmetric and transitive
   (d) neither symmetric nor transitive

VSA Type Questions (1 mark)

4. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, then range of R is $\{a, 2, b\}$, where $a + b$ = _____.

5. Let $A = \{x \in R : -4 \le x \le 4\}$ and $x \ne 0$. If $f : A \to R$ is defined by $f(x) = \frac{|x|}{x}$, then range of f is_____.

6. Let $f : R \to R$ be defined by $f(x) = \begin{cases} 2x, x > 3 \\ x^2, 1 < x \le 3 \\ 3x, x \le 1 \end{cases}$.
   Then, $f(-1) + f(2) + f(4)$ is_____.

7. State the reason for the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} not to be transitive.

   OR

   Let A = {1, 2, 3, 4} and B = {5, 6, 7, 8, 9} and set f = {(1, 5), (2, 6), (3, 7), (4, 8)} be a function from A to B. State whether f is one-one or not.

SA I Type Questions (2 marks)

8. Let $f : R \to R$ be defined by (i) $f(x) = x + |x|$ (ii) $f(x) = x + 1$. Determine whether f is onto or not.

9. Let A be a finite set. If $f : A \to A$ is an onto function, then show that f is one-one also.

10. Write the domain of the relation R defined on the set Z of integers as follows :
    $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$

11. Let the function $f : R \to R$ be defined by $f(x) = \cos x$, $\forall \ x \in R$. Show that f is neither one-one nor onto.

    OR

    Let $f : R \to R$ be defined by $f(x) = x^2 + 1$. Find the pre-image of 17 and (–3).

SA II Type Questions (3 marks)

12. Show that $f : R^+ \to R^+$ defined by $f(x) = \frac{1}{2x}$ is bijective, where R^+ is the set of all non zero positive real number.

    OR

    Let N be the set of natural numbers and relation R on set N be defined by $R = \{(x, y) : x, y \in N, x + 4y = 10\}$. Check whether R is reflexive, symmetric and transitive.

13. Let $A = \left\{x : x \in R, -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$ and $B = \{y : y \in R, -1 \le y \le 1\}$. Show that the function $f : A \to B$ such that $f(x) = \sin x$ is bijective.

14. Show that the function $f : R \to R$ defined by $x^3 + x$ is a bijection.

15. Let N be the set of all natural numbers and R be a relation in N defined by $R = \{(a, b) : a$ is a factor of $b\}$, then show that R is reflexive and transitive but not symmetric.

Case Based Questions (4 marks)

16. Consider the mapping $f : A \to B$ is defined by $f(x) = \frac{x-1}{x-2}$ such that f is a bijection.
Based on the above information, answer the following questions.
(i) Find range of f.
(ii) If $g : R - \{2\} \to R - \{1\}$ is defined by $g(x) = 2f(x) - 1$, then find the range of $g(x)$.

LA Type Questions (4/6 marks)

17. m is said to be related to n if m and n are integers and $m - n$ is divisible by 13. Does this define an equivalence relation?

18. Classify the following functions as injective, surjective or bijective.
(i) $f : R \to R$ defined by $f(x) = \sin x$
(ii) $f : R \to R$ defined by $f(x) = \sin^2 x + \cos^2 x$

19. If $A = \{1, 2, 3, 4\}$, define relations on A which have properties of being :
(i) reflexive, transitive but not symmetric.
(ii) symmetric but neither reflexive nor transitive.
(iii) reflexive, symmetric and transitive.

20. Show that the function $f : R \to R$ such that

$$f(x) = \begin{cases} 1, \text{if } x \text{ is rational,} \\ -1, \text{if } x \text{ is irrational} \end{cases}$$

is many one and not onto.

Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f(\sqrt{2})$ (iii) $f(\pi)$ (iv) $f(2+\sqrt{3})$

OR

Show that :
(i) the exponential function $f : R \to R$ defined by $f(x) = e^x$ is one-one but not onto.
(ii) the logarithmic function $f : R^+ \to R$ defined by $f(x) = \log_a x$, $a > 0$, $a \neq 1$ is a bijective function.

Detailed SOLUTIONS

1. (i) (a) : Clearly, $(1, 1), (2, 2), (3, 3) \in R$.
So, R is reflexive on A.
Since, $(1, 2) \in R$ but $(2, 1) \notin R$. So, R is not symmetric on A.
Since, $(2, 3) \in R$ and $(3, 1) \in R$ but $(2, 1) \notin R$. So, R is not transitive on A.

(ii) (b): Since, (1, 1), (2, 2) and (3, 3) are not in R.
So, R is not reflexive on A.
Now, $(1, 2) \in R \Rightarrow (2, 1) \in R$ and $(1, 3) \in R \Rightarrow (3, 1) \in R$.
So, R is symmetric.
Clearly, $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$.
So, R is not transitive on A.

(iii) (c): We have, $R = \{(x, y) : y = x + 5$ and $x < 4\}$, where $x, y \in N$. $\therefore$ $R = \{(1, 6), (2, 7), (3, 8)\}$
Clearly, (1, 1), (2, 2) etc. are not in R. So, R is not reflexive.
Since, $(1, 6) \in R$ but $(6, 1) \notin R$. So, R is not symmetric.
Since, $(1, 6) \in R$ and there is no ordered pair in R which has 6 as the first element. Same is the case for (2, 7) and (3, 8).
So, R is transitive.

(iv) (d): We have, $A = \{1, 2, ..., 14\}$, $R = \{(x, y) : 3x - y = 0\}$, where $x, y \in A$
$\therefore$ $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$
Clearly, $(1, 1) \notin R$. So, R is not reflexive on A.
Since, $(1, 3) \in R$ but $(3, 1) \notin R$. So, R is not symmetric on A.
Since, $(1, 3) \in R$ and $(3, 9) \in R$ but $(1, 9) \notin R$. So, R is not transitive on A.

(v) (d): Clearly, $(1, 1), (2, 2), (3, 3) \in R$. So, R is reflexive on A.
We find that the ordered pairs obtained by interchanging the components of ordered pairs in R are also in R. So, R is symmetric on A.
For $1, 2, 3 \in A$ such that (1, 2) and (2, 3) are in R implies that (1, 3) is also in R.
So, R is transitive on A.
Thus, R is an equivalence relation.

2. (d): Given function $f : R \to R$ defined by $f(x) = \frac{x^2-8}{x^2+2}$ for $1, -1 \in R$

$$f(-1) = \frac{(-1)^2-8}{(-1)^2+2} = \frac{-7}{3}$$

$$f(1) = \frac{1-8}{1+2} = \frac{-7}{3}$$

$\therefore$ $-1 \neq 1$ but $f(-1) = f(1)$
Hence, $f(x)$ is not one-one.
Also, $y = f(x) \Rightarrow y = \frac{x^2-8}{x^2+2} \Rightarrow yx^2 + 2y = x^2 - 8 \Rightarrow x = \sqrt{\frac{2y+8}{1-y}}$
x is not defined for $y = 1$, so $f(1)$ has no pre image in R, hence f is not onto.

OR

(c) : To find the equivalence class of (3, 2), we will take ordered pairs (a, b) with element a as multiple of 3 and element b as multiple of 2.
Then, the ordered pairs are
{(3, 2), (6, 4), (9, 6), (12, 8), (15, 10), (18, 12)}
Hence, required number of ordered pairs are 6.

3. (a) : Given $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$
$\because$ $(1, 1), (2, 2), (3, 3) \in R$ $\therefore$ R is reflexive.
Now, $(1, 2) \in R$ but $(2, 1) \notin R$. So, R is not symmetric.
Also, for all $x, y, z \in A$, $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow$ $(x, z) \in R$ $\therefore$ R is transitive.

4. Given, $R = \{(x, y) : x + 2y = 8\} \Rightarrow R = \{(2, 3), (4, 2), (6, 1)\}$
$\therefore$ Range of $R = \{1, 2, 3\} = \{a, 2, b\}$. Thus, $a + b = 1 + 3 = 4$

5. We have, $A = \{x \in R : -4 \le x \le 4\}$ and $x \ne 0$

Now, $f : A \to R$ defined by $f(x) = \frac{|x|}{x}$

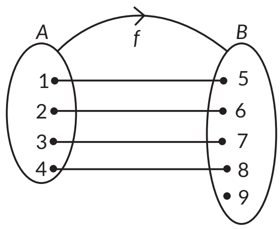
Clearly, $f(x) = \begin{cases} 1, \text{if } x > 0 \\ -1, \text{if } x < 0 \end{cases}$ $\therefore$ Range of $f = \{-1, 1\}$

6. We have given, $f(x) = \begin{cases} 2x, x > 3 \\ x^2, 1 < x \le 3 \\ 3x, x \le 1 \end{cases}$

Now, $f(-1) + f(2) + f(4) = 3(-1) + 2^2 + (2 \times 4) = -3 + 4 + 8 = 9$

7. We have, $R = \{(1, 2), (2, 1)\}$ defined on set $\{1, 2, 3\}$.
As, $(1, 2) \in R$, $(2, 1) \in R$ but $(1, 1) \notin R$. $\therefore$ R is not transitive.

OR



Since, every element of A has its distinct image in B.
$\therefore$ f is one-one.

8. (i) $f(x) = x + |x|$

$= \begin{cases} x + x, \text{if } x \ge 0 \\ x - x, \text{if } x < 0 \end{cases} = \begin{cases} 2x, \text{if } x \ge 0 \\ 0, \text{if } x < 0 \end{cases}$

Thus, $f(x) = 2x \ge 0 \ \forall \ x \ge 0$ and $f(x) = 0 \ \forall \ x < 0$
$\therefore$ $f(x)$ can't be negative for any $x \in R$
$\Rightarrow$ negative values do not have preimage in R.
Thus, f is not onto.
(ii) We have $f(x) = x + 1 \ \forall \ x \in R$
For any $y \in R$, $y = f(x) \Rightarrow y = x + 1 \Rightarrow x = y - 1$
$\therefore$ $f(y - 1) = y - 1 + 1 = y$. Hence, f is onto.

9. Let $A = \{a_1, a_2, ..., a_n\}$ be a finite set.
In order to prove that f is one-one function, we have to show that $f(a_1), f(a_2), ..., f(a_n)$ are distinct elements of A.
Clearly, range of $f = \{f(a_1), f(a_2), ..., f(a_n)\}$
Since, $f : A \to A$ is an onto function, then range of $f = A$
$\Rightarrow$ $\{f(a_1), f(a_2), ..., f(a_n)\} = A$
But A is finite set containing n elements.
Thus, $f(a_1), f(a_2), f(a_3), ..., f(a_n)$ are distinct elements of A.
Hence, $f : A \to A$ is one-one.

10. $R = \{(a, b) : a^2 + b^2 = 25\}$ be a relation on Z.
The domain of R is the value of $a \in Z$ that satisfy the relation
$a^2 + b^2 = 25 \Rightarrow a^2 = 25 - b^2 \Rightarrow a = \pm\sqrt{25 - b^2}$
$\therefore$ Domain of $R = \{0, \pm 3, \pm 4, \pm 5\}$

11. We have, $f(x) = \cos x \ \forall \ x \in R$

Now, $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

Also, $f\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

Since, $f\left(\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right)$. But $\frac{\pi}{2} \ne -\frac{\pi}{2}$ $\therefore$ f is not one-one.

Since, $-1 \le \cos x \le 1, \forall \ x \in R$. So there is no pre-image for real numbers, which belongs to the interval $[-1, 1]$
$\therefore$ f is not onto.

OR

If $f : A \to B$ is such that $y \in B$, then
$f^{-1}(y) = \{x \in A : f(x) = y\}$
In other words, $f^{-1}(y)$ is the set of pre-images of y.
Let $f^{-1}(17) = x$
$\Rightarrow$ $f(x) = 17 \Rightarrow x^2 + 1 = 17 \Rightarrow x^2 = 17 - 1 = 16 \Rightarrow x = \pm 4$
$\therefore$ $f^{-1}(17) = \{-4, 4\}$
Again, let $f^{-1}(-3) = x$, then
$f(x) = -3 \Rightarrow x^2 + 1 = -3 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4}$
Clearly, there is no solution.
So, $f^{-1}(-3)$ has no pre image.

12. Here, $f : R^+ \to R^+$ defined by $f(x) = \frac{1}{2x}$
One-One : Let $x_1, x_2 \in R^+$ (domain)
Now, $f(x_1) = f(x_2) \Rightarrow \frac{1}{2x_1} = \frac{1}{2x_2} \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$
$\therefore$ f is one-one.
Onto : Let $y \in R^+$ (co-domain) be any arbitrary element then $y \ne 0$
Let $y = f(x)$
$\Rightarrow$ $y = \frac{1}{2x} \Rightarrow x = \frac{1}{2y} \in R^+$ $\therefore$ f is onto.
Hence, f is bijective where $\frac{1}{2y}$ is non zero real number.
Hence, each element of co-domain (R^+) is the image of some element of domain (R^+).

OR

We have, $R = \{(x, y) : x, y \in N, x + 4y = 10\}$
$\therefore$ $R = \{(2, 2), (6, 1)\}$
Reflexive : Let $x \in N$ be any element.
Since $(x, x) \notin R$ $\therefore$ R is not reflexive.
Symmetric : Since $(6, 1) \in R$ but $(1, 6) \notin R$
$\therefore$ R is not symmetric.
Transitive : Let $x, y, z \in N$, then $(x, y) \in R$ and $(y, z) \in R$
$(x, y) \in R \Rightarrow x + 4y = 10$ ... (i)
and $(y, z) \in R \Rightarrow y + 4z = 10$ ... (ii)
From (i) and (ii), $x + 4(10 - 4z) = 10$
$\Rightarrow$ $x + 40 - 16z = 10 \Rightarrow x - 16z = -30$ $\therefore$ $(x, z) \notin R$
Thus, R is none of reflexive, symmetric and transitive.

13. Here, $A = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $B = [-1, 1]$
$f : A \to B$ is given by $f(x) = \sin x$
To show f is one-one.
Let x_1, x_2 A be such that $f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2$
$\Rightarrow$ $x_1 = x_2$ $\left[\because x_1, x_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$
Thus, f is one-one.
Also, range $(f) = [-1, 1] = B$. So, f is onto.
Thus, f is one-one and onto and hence bijective.

14. We have $f : R \to R$ defined by $f(x) = x^3 + x$
Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
$\Rightarrow$ $x_1^3 + x_1 = x_2^3 + x_2 \Rightarrow x_1^3 - x_2^3 + x_1 - x_2 = 0$
$\Rightarrow$ $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0 \Rightarrow x_1 - x_2 = 0$

$$\left[\because x_1^2 + x_1x_2 + x_2^2 \ge x_1^2 + 2x_1\left(\frac{x_2}{2}\right) + \left(\frac{x_2}{2}\right)^2 - \left(\frac{x_2}{2}\right)^2 + x_2^2\right.$$

$$= \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3x_2^2}{4} \ge 0 \text{ for all } x_1, x_2 \in R.$$

$\therefore$ $x_1^2 + x_1x_2 + x_2^2 + 1 \ge 1$ for all $x_1, x_2 \in R$

$\Rightarrow \quad x_1 = x_2 \Rightarrow f$ is one-one.

Let y be any arbitrary element of R, then there exists $x \in R$ such that $f(x) = y \Rightarrow x^3 + x = y \Rightarrow x^3 + x - y = 0$

Since, odd degree equation has atleast one real root.

Thus, for every value of y, the equation

$x^3 + x - y = 0$ has a real root α, such that $\alpha^3 + \alpha - y = 0 \Rightarrow f(\alpha) = y$

Thus, for every $y \in R, \exists\, \alpha \in R$ such that $f(\alpha) = y$

So, f is onto function. Hence, $f : R \to R$ is a bijection.

15. Here, $R = \{(a, b) : a$ is a factor of b for $a, b \in N\}$

Reflexive : Let a be an arbitrary element of N then, clearly, a is a factor of a.

$\therefore \quad (a, a) \in R\ \forall\ a \in N\ \therefore\ R$ is reflexive.

Symmetric : Clearly 2 and 6 are natural numbers and 2 is a factor of 6.

$\therefore \quad (2, 6) \in R$ but 6 is not a factor of 2 $\Rightarrow (6, 2) \notin R$

Thus, $(2, 6) \in R$ but $(6, 2) \notin R\ \therefore\ R$ is not symmetric.

Transitive : Let $a, b, c \in N$

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad$ (a is a factor of b) and (b is a factor of c)

$\Rightarrow \quad b = ad$ and $c = be$ for some $d, e \in N$

$\Rightarrow \quad c = (ad)e = a(de)$ [By associative law]

$\Rightarrow \quad a$ is a factor of $c \Rightarrow (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R\ \therefore\ R$ is transitive.

16. **(i)** Let $y = f(x)$, then $y = \dfrac{x-1}{x-2}$

$\Rightarrow \quad xy - 2y = x - 1 \Rightarrow xy - x = 2y - 1 \Rightarrow x = \dfrac{2y-1}{y-1}$

Since, $x \in R - \{2\}$, therefore $y \neq 1$

Hence, range of $f = R - \{1\}$

(ii) We have, $g(x) = 2f(x) - 1$

$$= 2\left(\frac{x-1}{x-2}\right) - 1 = \frac{2x-2-x+2}{x-2} = \frac{x}{x-2}$$

Let $y = g(x) = \dfrac{x}{x-2} \Rightarrow yx - 2y = x \Rightarrow yx - x = 2y$

$\Rightarrow \quad x(y - 1) = 2y \Rightarrow x = \dfrac{2y}{y-1}$

Since, $x \in R - \{2\}\ \therefore\ y \neq 1$

Hence, range of $g(x) = R - \{1\}$

17. Z be the set of all integers.

$R = \{(m, n) : m - n$ is divisible by 13$\}$

Reflexive : Let $m \in Z$

$m - m = 0 \Rightarrow m - m$ is divisible by 13

$\Rightarrow \quad (m, m) \in R. \quad \therefore\ R$ is reflexive.

Symmetric : Let $m, n \in Z$ and $(m, n) \in R$

$\Rightarrow \quad m - n = 13p$, for some $p \in Z$

$\Rightarrow \quad n - m = 13(-p)$, where $(-p) \in Z$

$\Rightarrow \quad n - m$ is divisible by 13 $\Rightarrow (n, m) \in R$

$\therefore \quad R$ is symmetric.

Transitive : Let $(m, n) \in R$ and $(n, q) \in R$, for some $m, n, q \in Z$

$\Rightarrow \quad m - n = 13p$ and $n - q = 13s$, for some $p, s \in Z$

$\Rightarrow \quad m - q = 13(p + s)$ $[\because p, s \in Z \Rightarrow p + s \in Z]$

$\Rightarrow \quad m - q$ is divisible by 13 $\Rightarrow (m, q) \in R\ \therefore\ R$ is transitive.

Hence, R is an equivalence relation on Z.

18. (i) Here, $f : R \to R$ given by $f(x) = \sin x$

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad \sin x_1 = \sin x_2 \Rightarrow x_1 = n\pi + (-1)^n x_2 \not\Rightarrow x_1 = x_2$

$\therefore \quad f$ is not one-one.

Let $y \in R$ be any arbitrary element, then there exists $x \in R$ such that $f(x) = y$

$\Rightarrow \quad \sin x = y \Rightarrow x = \sin^{-1} y \qquad [\because$ For $y > 1, x \notin R$ (domain)$]$

$\therefore \quad f$ is not onto.

Hence, f is not a bijective function.

(ii) $f : R \to R$ defined by $f(x) = \sin^2 x + \cos^2 x$

Since, $f(x) = \sin^2 x + \cos^2 x = 1$

Now, $f(x) = 1$ is a constant function and we know that constant function is neither injective nor surjective.

$\therefore \quad f$ is neither one-one nor onto.

19. We have given, $A = \{1, 2, 3, 4\}$

(i) Consider, $R_1 = \{(1, 1), (1, 2), (2, 3), (2, 2), (1, 3), (3, 3)\}$

As, $(1, 1), (2, 2), (3, 3) \in R_1$

$\therefore \quad R_1$ is reflexive.

Also, $(1, 2) \in R_1, (2, 3) \in R_1 \Rightarrow (1, 3) \in R_1$

So, R_1 is also transitive.

Since, $(2, 3) \in R_1$ but $(3, 2) \notin R_1$.

So, it is not symmetric.

(ii) Consider, $R_2 = \{(1, 2), (2, 1)\}$

As, $(1, 2) \in R_2$ and $(2, 1) \in R_2$

So, it is symmetric but it is neither reflexive nor transitive.

(iii) Consider, $R_3 = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (2, 3), (3, 2)\}$, universal relation

Hence, R_3 is reflexive, symmetric and transitive.

20. (i) Here, $x = \dfrac{1}{2}$, which is rational $\therefore\ f\left(\dfrac{1}{2}\right) = 1$

(ii) Here, $x = \sqrt{2}$, which is irrational $\therefore\ f(\sqrt{2}) = -1$

(iii) Here, $x = \pi$, which is irrational $\therefore\ f(\pi) = -1$

(iv) Here, $x = 2 + \sqrt{3}$, which is irrational $\therefore\ f(2 + \sqrt{3}) = -1$

Clearly, $f(x)$ is many one as $f(x) = -1$ for $x = \sqrt{2}$ and $2 + \sqrt{3}$.

And $f(x)$ takes values only 1 and -1.

Range of $f(x) \subset$ co-domain.

Here, $f(x)$ does not take all the values of the co-domain.

$\therefore \quad f(x)$ is not onto.

OR

(i) We have, $f : R \to R$ given by $f(x) = e^x$

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow \dfrac{e^{x_1}}{e^{x_2}} = 1$

$\Rightarrow \quad e^{x_1 - x_2} = 1 = e^0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$\therefore \quad f$ is one-one.

Clearly, range of $e^x = (0, \infty) = R^+$ but co-domain $= R$.

Thus, Range $\neq$ codomain

$\therefore\ f$ is not onto.

(ii) We have, $f : R^+ \to R$ given by $f(x) = \log_a x : a > 0$ and $a \neq 1$

Let $x_1, x_2 \in R^+$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad \log_a x_1 = \log_a x_2 \Rightarrow \dfrac{x_1}{x_2} = 1 \Rightarrow x_1 = x_2 \quad \therefore\ f$ is one-one.

Let $y \in R$ be arbitrary element, then there exists $x \in R^+$ such that $f(x) = y$

$\Rightarrow \quad \log_a x = y \Rightarrow x = a^y \in R^+ \quad [\because a > 0]$

$\Rightarrow \quad a^y > 0$

Thus, for all $y \in R$, there exist $x = a^y \in R^+$ such that $f(x) = y$.

$\therefore \quad f$ is onto.

Hence, f is one-one and onto *i.e.*, f is bijective.

CHAPTER 2

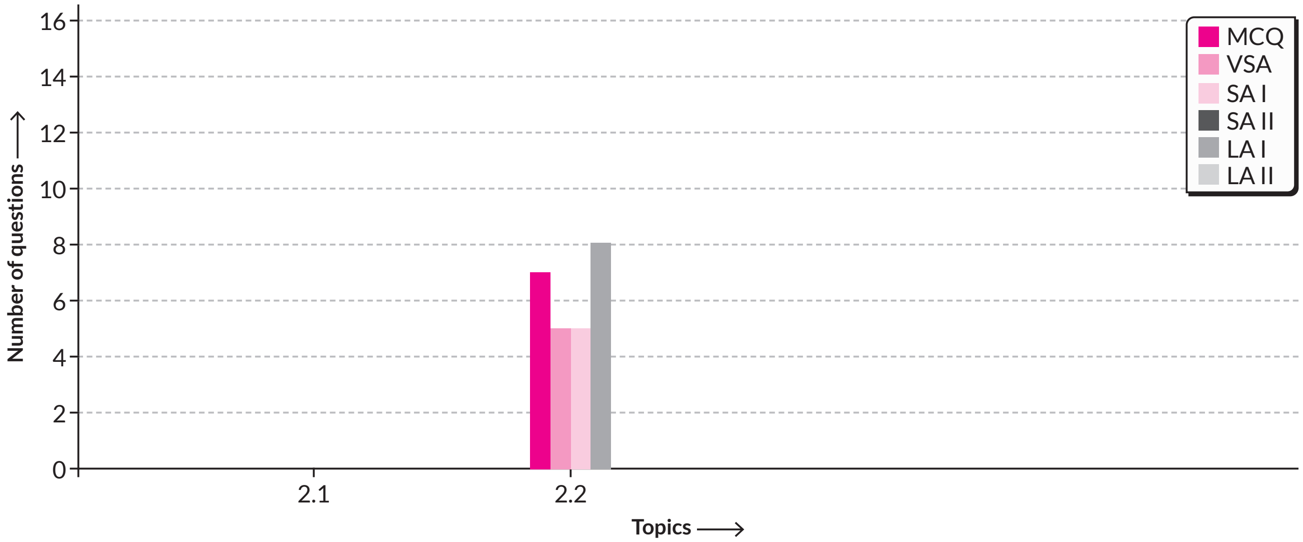
Inverse Trigonometric Functions

TOPICS

2.1 Introduction

2.2 Basic Concepts

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Maximum weightage is of Topic 2.2 *Basic Concepts.*
- Maximum LA I type questions were asked from Topic 2.2 *Basic Concepts.*
- No SA II and LA II type questions were asked till now.

QUICK RECAP

Inverse Trigonometric Functions

- Trigonometric functions are not one-one and onto over their natural domains and ranges *i.e.*, *R*(real numbers). But some restrictions on domains and ranges of trigonometric function ensures the existence of their inverses.
  Let $y = f(x) = \cos x$, then its inverse is $x = \cos^{-1} y$
- The domains and ranges (principal value branches) of inverse trigonometric functions are as follows :

Functions	Domain	Range
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$(0, \pi)$
$y = \text{cosec}^{-1} x$	$R-(-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
$y = \sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$

- The value of the inverse trigonometric functions which lies in its principal value branch is called the principal value of inverse trigonometric function.

Properties of Inverse Trigonometric Functions

- $\sin^{-1}(\sin x) = x \quad , \quad \forall -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

  $\cos^{-1}(\cos x) = x \quad , \quad \forall\ 0 \le x \le \pi$

  $\tan^{-1}(\tan x) = x \quad , \quad \forall -\frac{\pi}{2} < x < \frac{\pi}{2}$

  $\cot^{-1}(\cot x) = x \quad , \quad \forall\ 0 < x < \pi$

  $\sec^{-1}(\sec x) = x \quad , \quad \forall\ x \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$

  $\text{cosec}^{-1}(\text{cosec}\, x) = x \quad , \quad \forall\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

- $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x \quad , \quad \forall\ x \ge 1 \text{ or } x \le -1$

  $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \quad , \quad \forall\ x \ge 1 \text{ or } x \le -1$

  $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \forall x > 0 \\ -\pi + \cot^{-1} x, & \forall x < 0 \end{cases}$

- $\sin(\sin^{-1} x) = x \quad , \quad \forall -1 \le x \le 1$

  $\cos(\cos^{-1} x) = x \quad , \quad \forall -1 \le x \le 1$

  $\tan(\tan^{-1} x) = x \quad , \quad \forall\ x \in R, (-\infty < x < \infty)$

  $\cot(\cot^{-1} x) = x \quad , \quad \forall\ x \in R, (-\infty < x < \infty)$

  $\sec(\sec^{-1} x) = x \quad , \quad \forall\ x \le -1 \text{ or } x \ge 1$

  $\text{cosec}(\text{cosec}^{-1} x) = x \quad , \quad \forall\ x \le -1 \text{ or } x \ge 1$

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad , \quad \forall -1 \le x \le 1$

  $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad , \quad \forall\ x \in R$

  $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2} \quad , \quad \forall\ x \le -1 \text{ or } x \ge 1$

- $\sin^{-1}(-x) = -\sin^{-1} x \quad , \quad \forall -1 \le x \le 1$

  $\cos^{-1}(-x) = \pi - \cos^{-1} x \quad , \quad \forall -1 \le x \le 1$

  $\tan^{-1}(-x) = -\tan^{-1} x \quad , \quad \forall -\infty < x < \infty$

  $\cot^{-1}(-x) = \pi - \cot^{-1} x \quad , \quad \forall -\infty < x < \infty$

  $\sec^{-1}(-x) = \pi - \sec^{-1} x \quad , \quad \forall\ |x| \ge 1$

  $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x \quad , \quad \forall\ |x| \ge 1$

BRAIN MAP

INVERSE TRIGONOMETRIC FUNCTIONS

Function

- $y = \sin^{-1} x$
- $y = \cos^{-1} x$
- $y = \tan^{-1} x$
- $y = \cot^{-1} x$
- $y = \text{cosec}^{-1} x$
- $y = \sec^{-1} x$

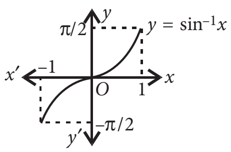
Domain

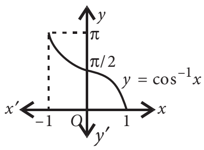
- $[-1, 1]$
- $[-1, 1]$
- R
- R
- $R - (-1, 1)$
- $R - (-1, 1)$

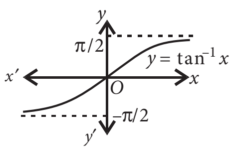
Range

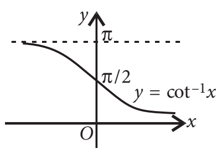
- $[-\pi/2, \pi/2]$
- $[0, \pi]$
- $(-\pi/2, \pi/2)$
- $(0, \pi)$
- $[-\pi/2, \pi/2] - \{0\}$
- $[0, \pi] - \{\pi/2\}$

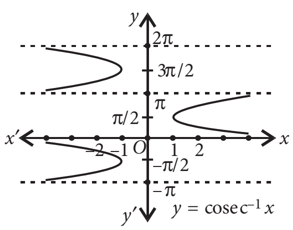
Graph

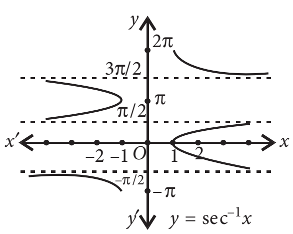
- 

- 

- 

- 

- 

- 

Previous Years' CBSE Board Questions

2.2 Basic Concepts

MCQ

1. $\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]$ is equal to
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$ *(2023)*

2. If $f(x) = |\cos x|$, then $f\left(\frac{3\pi}{4}\right)$ is
(a) 1 (b) −1 (c) $\frac{-1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$ *(2023)*

3. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (a), (b), (c) and (d) as given below.
Assertion (A) : All trigonometric functions have their inverses over their respective domains.
Reason (R) : The inverse of $\tan^{-1}x$ exists for some $x \in \mathbb{R}$.
(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).
(b) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of Assertion (A).
(c) Assertion (A) is true but Reason (R) is false.
(d) Assertion (A) is false but Reason (R) is true. *(2023)*

4. The value of $\sin^{-1}\left(\cos\frac{13\pi}{5}\right)$ is
(a) $-\frac{3\pi}{5}$ (b) $-\frac{\pi}{10}$ (c) $\frac{3\pi}{5}$ (d) $\frac{\pi}{10}$ *(Term I, 2021-22)* U

5. The principal value of $\cot^{-1}(-\sqrt{3})$ is
(a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$ *(2020)*

6. $\tan^{-1}3 + \tan^{-1}\lambda = \tan^{-1}\left(\frac{3+\lambda}{1-3\lambda}\right)$ is valid for what values of λ?
(a) $\lambda \in \left(-\frac{1}{3}, \frac{1}{3}\right)$ (b) $\lambda > \frac{1}{3}$
(c) $\lambda < \frac{1}{3}$ (d) All real values of λ *(2020)*

7. The principal value of $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$ is
(a) $\frac{2\pi}{5}$ (b) $\frac{-2\pi}{5}$ (c) $\frac{3\pi}{5}$ (d) $\frac{-3\pi}{5}$ *(2020)* U

VSA (1 mark)

8. The range of the principal value branch of the function $y = \sec^{-1}x$ is ________. *(2020)*

9. The principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is ________. *(2020)* U

10. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$. *(Foreign 2014)* Ap

11. Write the principal value of $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$. *(AI 2014C)*

12. Find the value of the following :
$\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$ *(AI 2014C)*

SA I (2 marks)

13. Write the domain and range (principle value branch) of the following functions:
$f(x) = \tan^{-1}x$. *(2023)*

14. Evaluate : $\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]$ *(2023)*

15. Simplify $\sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$. *(2021C)*

16. Prove that : $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ *(2020)* Ev

17. Prove that : $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\cos^{-1}x, \frac{1}{\sqrt{2}}\le x\le 1$. *(2020)* Ap

LA I (4 marks)

18. Solve for x : $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ *(2020C)*

19. Prove that : $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$. *(2020C)* Ap

20. Prove that : $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in[0,1]$ *(2020, 2019C)*

21. Prove that :
$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$ *(AI 2019)* Ev

22. Prove that : $\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{63}{65}=\frac{\pi}{2}$ *(2019)*

23. If $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $x > 0$, find the value of x and hence find the value of $\sec^{-1}\left(\frac{2}{x}\right)$. *(2019)* Ev

24. Find the value of $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$. *(2019)*

25. Prove that :
$\tan^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x; -\frac{1}{\sqrt{2}}\le x\le 1$ *(2019C)* Ap

CBSE Sample Questions

2.2 Basic Concepts

MCQ

1. In the given question, a statement of assertion (A) is followed by a statement of reason (R). Choose the correct answer out of the following choices.
Assertion (A) : The domain of the function $\sec^{-1} 2x$ is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.
Reason (R) : $\sec^{-1}(-2) = -\frac{\pi}{4}$
(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true. *(2022-23)*

2. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) -1 (d) 1
(Term I, 2021-22) Ap

3. $\sin(\tan^{-1}x)$, where $|x| < 1$, is equal to
(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x}{\sqrt{1+x^2}}$
(Term I, 2021-22)

4. Simplest form of
$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right), \pi < x < \frac{3\pi}{2}$ is
(a) $\frac{\pi}{4} - \frac{x}{2}$ (b) $\frac{3\pi}{2} - \frac{x}{2}$ (c) $-\frac{x}{2}$ (d) $\pi - \frac{x}{2}$
(Term I, 2021-22) Cr

5. If $\tan^{-1}x = y$, then
(a) $-1 < y < 1$ (b) $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$
(c) $\frac{-\pi}{2} < y < \frac{\pi}{2}$ (d) $y \in \left\{\frac{-\pi}{2}, \frac{\pi}{2}\right\}$
(Term I, 2021-22)

SA I (2 marks)

6. Find the value of $\sin^{-1}\left[\sin\left(\frac{13\pi}{7}\right)\right]$. *(2022-23)*

7. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), \frac{-3\pi}{2} < x < \frac{\pi}{2}$ in the simplest form. *(2020-21)* Ev

Detailed SOLUTIONS

1. **(a) :** We have,
$$\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]$$
$$=\sin\left(\frac{\pi}{2}\right)=1$$

2. **(d) :** $f(x) = |\cos x|$
At $\frac{\pi}{2} < x < \pi$, $\cos x < 0$ $\quad \therefore |\cos x| = -\cos x \Rightarrow f(x) = -\cos x$
$$\therefore \quad f\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\pi - \frac{\pi}{4}\right)$$
$$= \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad [\because \cos(\pi - \theta) = -\cos\theta]$$

3. **(d) :** All trigonometric functions are periodic and hence not invertible over their respective domains but all trigonometric functions have inverse over their restricted domains.
Inverse of $\tan^{-1}x$ is $\tan x$ which is defined for
$$x \in R - (2n+1)\frac{\pi}{2},\ n \in Z$$
$\therefore$ Assertion is false and reason is true.

4. **(b) :** We have, $\sin^{-1}\left(\cos\frac{13\pi}{5}\right) = \sin^{-1}\left[\cos\left(2\pi + \frac{3\pi}{5}\right)\right]$
$$= \sin^{-1}\left[\cos\frac{3\pi}{5}\right] = \sin^{-1}\left[\cos\left(\frac{\pi}{2}+\frac{\pi}{10}\right)\right]$$
$$= \sin^{-1}\left(-\sin\frac{\pi}{10}\right) = -\sin^{-1}\left(\sin\frac{\pi}{10}\right) = -\frac{\pi}{10}$$

Answer Tips

- $\cos(2\pi + \theta) = \cos\theta$, $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

5. **(d) :** We know that $\cot^{-1}(x) \in (0, \pi)$
$$\cot^{-1}(-\sqrt{3}) = \cot^{-1}\left(-\cot\frac{\pi}{6}\right)$$
$$= \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{6}\right)\right] \qquad [\because \cot(\pi-\theta) = -\cot\theta]$$
$$= \cot^{-1}\left[\cot\left(\frac{5\pi}{6}\right)\right] = \frac{5\pi}{6} \qquad [\because \cot^{-1}[\cot\theta] = \theta]$$
Thus, the principal value of $\cot^{-1}(-\sqrt{3})$ is $\frac{5\pi}{6}$.

6. (c) : Given, $\tan^{-1}3+\tan^{-1}\lambda=\tan^{-1}\left(\frac{3+\lambda}{1-3\lambda}\right)$

$\tan^{-1}3+\tan^{-1}\lambda=\tan^{-1}\left(\frac{3+\lambda}{1-3\lambda}\right)$ for $3\lambda<1$ $\therefore$ $3\lambda<1\Rightarrow\lambda<\frac{1}{3}$

Concept Applied

➲ $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy<1$

7. (b) : We have, $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$

We know that the range of $\tan^{-1}x$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

$\therefore$ $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)=\tan^{-1}\left(\tan\left(\pi-\frac{2\pi}{5}\right)\right)$

$=\tan^{-1}\left[-\tan\left(\frac{2\pi}{5}\right)\right]$ [$\because$ $\tan(\pi-\theta)=\tan\theta$]

$=-\tan^{-1}\left[\tan\left(\frac{2\pi}{5}\right)\right]=-\frac{2\pi}{5}$ [$\because$ $\tan^{-1}(\tan\theta)=\theta$]

8. The range of the principal value branch of the function $y=\sec^{-1}x$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$.

9. Let $y=\cos^{-1}\left(\frac{-1}{2}\right)\Rightarrow\cos y=\frac{-1}{2}\Rightarrow\cos y=-\cos\left(\frac{\pi}{3}\right)$

Since, the range of $\cos^{-1}x$ is $[0,\pi]$

$\Rightarrow$ $\cos y=-\cos\left(\pi-\frac{\pi}{3}\right)$ [$\because$ $\cos(\pi-\theta)=-\cos\theta$]

$\Rightarrow$ $y=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

Hence, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

10. Given $\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

$=\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)=\frac{2\pi}{3}+2\times\frac{\pi}{6}=\pi$

[$\because$ Range of $\cos^{-1}x$ is $[0,\pi]$ & of $\sin^{-1}x$ is $[-\pi/2,\pi/2]$]

Commonly Made Mistake

➲ Remember the domain and range of inverse trigonometric functions.

11. Here, $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]=\tan^{-1}(-1)=-\frac{\pi}{4}$

This is the required principal value as it is lie in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Key Points

➲ The range of $y=\tan^{-1}x$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

12. $\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$

$=\cot\left(\frac{\pi}{2}-2\cot^{-1}\left(\cot\frac{\pi}{6}\right)\right)=\cot\left(\frac{\pi}{2}-2\cdot\frac{\pi}{6}\right)$

$=\cot\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=\cot\frac{\pi}{6}=\sqrt{3}$

13. Domain of $\tan^{-1}x=(-\infty,\infty)$ and range of $\tan^{-1}x$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

14. $\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]=\cos^{-1}\left[\cos\left(\frac{7\pi}{3}\right)\right]$ ($\because$ $\cos(-\theta)=\cos\theta$)

$=\cos^{-1}\left(\cos\left(2\pi+\frac{\pi}{3}\right)\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$

$=\frac{\pi}{3}$ ($\because$ $\cos^{-1}(\cos x)=x\ \forall\ 0\le x\le\pi$)

15. Let $x=\cos\theta$

$\sec^{-1}\left(\frac{1}{2x^2-1}\right)=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$=\sec^{-1}(\sec 2\theta)=2\theta$

Hence, $\sec^{-1}\left(\frac{1}{2x^2-1}\right)=2\cos^{-1}x$

Concept Applied

➲ $\cos 2\theta=2\cos^2\theta-1$, $\sec\theta=\frac{1}{\cos\theta}$

16. Consider L.H.S. $=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{1}{3}\right)\right]$

$=\frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right)$ ...(i) $\left[\because\ \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

Let $a=\cos^{-1}\left(\frac{1}{3}\right)$

$\Rightarrow$ $\cos a=\frac{1}{3}\Rightarrow\sin a=\sqrt{1-\cos^2 a}$ [$\because$ $\sin^2\theta+\cos^2\theta=1$]

$\Rightarrow$ $\sin a=\sqrt{1-\left(\frac{1}{3}\right)^2}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}\Rightarrow a=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

So, L.H.S. $=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)=$ R.H.S.

17. Consider L.H.S. $=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $x=\cos\theta$, we get

$=\sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because$ $\sin^2\theta+\cos^2\theta=1$]

$=\sin^{-1}(2\cos\theta\sin\theta)=\sin^{-1}(\sin 2\theta)$ [$\because$ $\sin^{-1}(\sin\theta)=\theta$]

$=2\theta$

Since, $x=\cos\theta$

$\Rightarrow$ $\theta=\cos^{-1}x$

$\therefore$ $2\theta=2\cos^{-1}x=$ R.H.S.

18. Given, $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

Put $x=\sin y$

$\Rightarrow$ $\sin^{-1}(1-\sin y)-2\sin^{-1}(\sin y)=\frac{\pi}{2}$

$\Rightarrow$ $\sin^{-1}(1-\sin y)-2y=\frac{\pi}{2}$ [$\sin^{-1}(\sin\theta)=\theta$]

$\Rightarrow \quad \sin^{-1}(1-\sin y)=\frac{\pi}{2}+2y \Rightarrow 1-\sin y=\sin\left(\frac{\pi}{2}+2y\right)$

$\Rightarrow \quad 1-\sin y=\cos 2y$ $\quad$ $[\sin(\pi/2+\theta)=\cos\theta]$

$\Rightarrow \quad 1-\sin y = 1-2\sin^2 y$ $\quad$ $[\because \cos 2\theta = 1-2\sin^2\theta]$

$\Rightarrow \quad 2\sin^2 y - \sin y = 0$

Replace $\sin y = x$

$\Rightarrow \quad 2x^2-x=0 \Rightarrow x(2x-1)=0 \Rightarrow x=0, \frac{1}{2}$

19. Consider, L.H.S. = $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

$=\tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7} \quad =\tan^{-1}\frac{1}{\frac{3}{4}}+\tan^{-1}\frac{1}{7}$

$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7} \quad =\tan^{-1}\frac{\left(\frac{4}{3}+\frac{1}{7}\right)}{1-\frac{4}{21}}=\tan^{-1}\frac{\frac{31}{21}}{\frac{17}{21}}$

$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$=\tan^{-1}\frac{31}{17}=$ R.H.S.

Hence proved.

Concept Applied

➲ $2\tan^{-1}\theta=\tan^{-1}\left(\frac{2\theta}{1-\theta^2}\right)$

20. Consider, R.H.S. = $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

Put $x=\tan^2\theta$ ...(i)

$\therefore \quad$ R.H.S. = $\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\frac{1}{2}(2\theta)=\theta$

From equation (i), we get

$\tan\theta=\sqrt{x} \Rightarrow \theta=\tan^{-1}\sqrt{x}$ = L.H.S.

$\therefore \quad$ L.H.S. = R.H.S.

Hence proved.

21. Let $x=\cos^{-1}\left(\frac{12}{13}\right)$ and $y=\sin^{-1}\left(\frac{3}{5}\right)$

or $\quad \cos x=\frac{12}{13}$ and $\sin y=\frac{3}{5}$

Now, $\sin x=\sqrt{1-\cos^2 x}$ and $\cos y=\sqrt{1-\sin^2 y}$

$\Rightarrow \quad \sin x=\sqrt{1-\frac{144}{169}}$ and $\cos y=\sqrt{1-\frac{9}{25}}$

$\Rightarrow \quad \sin x=\frac{5}{13}$ and $\cos y=\frac{4}{5}$

We know that,

$\sin(x+y)=\frac{5}{13}\times\frac{4}{5}+\frac{12}{13}\times\frac{3}{5}=\frac{20}{65}+\frac{36}{65}=\frac{56}{65} \Rightarrow x+y=\sin^{-1}\left(\frac{56}{65}\right)$

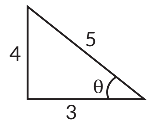
or, $\quad \cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$

Concept Applied

➲ $\sin(x+y)=\sin x\cos y+\cos x\sin y$

22. Consider L.H.S. = $\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{63}{65}$

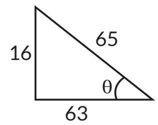
$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{63}{65}$



$=\tan^{-1}\left[\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\times\frac{5}{12}}\right]+\cos^{-1}\frac{63}{65}$

$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$=\tan\left[\frac{63}{16}\right]+\tan^{-1}\left[\frac{16}{63}\right]$



$=\tan^{-1}\left[\frac{\frac{63}{16}+\frac{16}{63}}{1-\frac{63}{16}\times\frac{16}{63}}\right] = \tan^{-1}(\infty) = \frac{\pi}{2}=$ R.H.S.

Hence proved.

23. Given, $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \quad \tan^{-1}x-\tan^{-1}\left(\frac{1}{x}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\left[\because \tan^{-1}\left(\frac{1}{x}\right)=\cot^{-1}x, x>0\right]$

$\Rightarrow \quad \tan^{-1}\left(\frac{x-\frac{1}{x}}{1+x\cdot\frac{1}{x}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\left[\because \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left[\frac{x-y}{1+xy}\right]\right]$

$\Rightarrow \quad \frac{x^2-1}{2x}=\frac{1}{\sqrt{3}} \Rightarrow \sqrt{3}x^2-\sqrt{3}=2x \Rightarrow \sqrt{3}x^2-2x-\sqrt{3}=0$

$\Rightarrow \quad \sqrt{3}x^2-3x+x-\sqrt{3}=0 \Rightarrow \sqrt{3}x(x-\sqrt{3})+1(x-\sqrt{3})=0$

$\Rightarrow \quad (\sqrt{3}x+1)(x-\sqrt{3})=0 \Rightarrow \sqrt{3}x+1=0$ or $x-\sqrt{3}=0$

$\Rightarrow \quad x=\frac{-1}{\sqrt{3}}$ or $x=\sqrt{3}$

Since, $x > 0$

So, $x=\frac{-1}{\sqrt{3}}$ is rejected. $\therefore \quad \sec^{-1}\left(\frac{2}{x}\right)=\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=\frac{\pi}{6}$

24. We have, $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$

Let $\cos^{-1}\frac{4}{5}=A \Rightarrow \cos A=\frac{4}{5}$

We know that $\sin A=\sqrt{1-\cos^2 A}$

$\Rightarrow \quad \sin A=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$

Let $\tan^{-1}\frac{2}{3}=B\Rightarrow \tan B=\frac{2}{3}$

We know that, $1+\tan^2\theta=\sec^2\theta$

$\Rightarrow \quad \sec B=\sqrt{1+\tan^2 B}=\sqrt{1+\frac{4}{9}}=\frac{\sqrt{13}}{3}$

$\Rightarrow \quad \cos B=\frac{3}{\sqrt{13}} \qquad \left[\because \sec\theta=\frac{1}{\cos\theta}\right]$

$\Rightarrow \quad \sin B=\sqrt{1-\cos^2 B}=\sqrt{1-\frac{9}{13}}=\frac{2}{\sqrt{13}}$

Now, $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)=\sin(A+B)$

$=\sin A\cos B+\cos A\sin B$

[$\because \sin(x+y)=\sin x\cos y+\cos x\sin y$]

$=\frac{3}{5}\times\frac{3}{\sqrt{13}}+\frac{4}{5}\times\frac{2}{\sqrt{13}}=\frac{9}{5\sqrt{13}}+\frac{8}{5\sqrt{13}}=\frac{17}{5\sqrt{13}}$

25. Consider L.H.S. $=\tan^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right)$

Put $x=\cos\theta$, we get

L.H.S. $=\tan^{-1}\left[\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right]$

$=\tan^{-1}\left[\frac{\left|\sqrt{2}\cos\frac{\theta}{2}\right|+\left|\sqrt{2}\sin\frac{\theta}{2}\right|}{\left|\sqrt{2}\cos\frac{\theta}{2}\right|-\left|\sqrt{2}\sin\frac{\theta}{2}\right|}\right]$

[$\because 1+\cos^2\theta=2\cos^2\theta,\ 1-\cos^2\theta=2\sin^2\theta$]

$=\tan^{-1}\left[\frac{-\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}{-\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}\right]=\tan^{-1}\left[\frac{-\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{-\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right]$

$=\tan^{-1}\left[\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right]=\tan^{-1}\left[\frac{\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}\right]=\tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$

$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]=\frac{\pi}{4}-\frac{\theta}{2} \qquad [\because \tan^{-1}(\tan\theta)=\theta]$

Since, $x=\cos\theta \Rightarrow \theta=\cos^{-1}x$

$\therefore \quad$ L.H.S. $=\frac{\pi}{4}-\frac{\cos^{-1}x}{2}=$ R.H.S.

CBSE Sample Questions

1. (c) : $\sec^{-1}x$ is defined if $x\le -1$ or $x\ge 1$.

Hence, $\sec^{-1}2x$ will be defined if $x\le-\frac{1}{2}$ or $x\ge\frac{1}{2}$

The range of the function $\sec^{-1}x$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ **(1)**

Hence, A is true and R is false.

2. (d) : We have,

$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]$

$=\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]=\sin\left(\frac{\pi}{2}\right)=1$ **(1)**

3. (d) : We have, $\sin(\tan^{-1}x)$

Let $\tan^{-1}x=\theta \Rightarrow x=\tan\theta \Rightarrow \sin\theta=\frac{x}{\sqrt{x^2+1}}$

$\therefore \quad \sin(\tan^{-1}x)=\sin\theta=\frac{x}{\sqrt{x^2+1}}$ **(1)**

4. (a) : We have,

$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right),\ \pi<x<\frac{3\pi}{2}$

$=\tan^{-1}\left(\frac{\left|\sqrt{2}\cos\frac{x}{2}\right|+\left|\sqrt{2}\sin\frac{x}{2}\right|}{\left|\sqrt{2}\cos\frac{x}{2}\right|-\left|\sqrt{2}\sin\frac{x}{2}\right|}\right)$

$=\tan^{-1}\left(\frac{-\sqrt{2}\cos\frac{x}{2}+\sqrt{2}\sin\frac{x}{2}}{-\sqrt{2}\cos\frac{x}{2}-\sqrt{2}\sin\frac{x}{2}}\right) \qquad \left(\because \frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}\right)$

$=\tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)=\tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)=\frac{\pi}{4}-\frac{x}{2}$ **(1)**

5. (c) : Range of $\tan^{-1}x=\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

$\therefore \quad \frac{-\pi}{2}<y<\frac{\pi}{2}$ **(1)**

6. Given, $\sin^{-1}\left[\sin\left(\frac{13\pi}{7}\right)\right]=\sin^{-1}\left[\sin\left(2\pi-\frac{\pi}{7}\right)\right]$ **(1)**

$=\sin^{-1}\left[\sin\left(-\frac{\pi}{7}\right)\right]=-\frac{\pi}{7}$ **(1)**

7. We have,

$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)=\tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right]$ **(1/2)**

$=\tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$

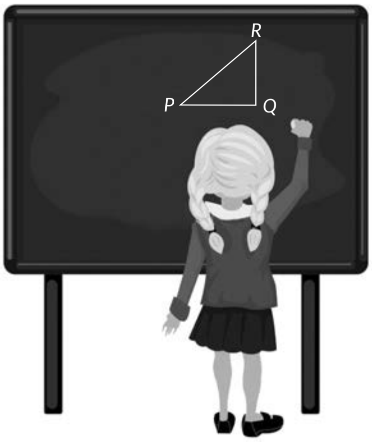
$=\tan^{-1}\left[\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]=\tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]$ **(1)**

$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\frac{\pi}{4}+\frac{x}{2}$ **(1/2)**

Self Assessment

Case Based Objective Questions (4 marks)

1. In the math class, the teacher asked a student to construct a triangle on a black board and name it as *PQR*. Two angles *P* and *Q* were given to be equal to $\tan^{-1}\left(\frac{1}{3}\right)$ and $\tan^{-1}\left(\frac{1}{2}\right)$ respectively. Based on ΔPQR, the teacher give some questions to the students.



Based on above information, attempt any 4 out of 5 subparts.

(i) The value of third angle is ________.

(a) $\tan^{-1}\left(\frac{1}{3}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$

(c) $90° - \tan^{-1}\left(\frac{1}{3}\right)$

(d) $180° - \left(\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right)\right)$

(ii) The value of $\cos P + \sin P$ is ________.

(a) $\frac{2}{\sqrt{10}}$ (b) $\frac{3}{\sqrt{10}}$ (c) $\frac{4}{\sqrt{10}}$ (d) $\frac{7}{\sqrt{10}}$

(iii) The value of $\cos(P + Q + R)$ = ________.

(a) 1 (b) −1 (c) 0 (d) $\frac{1}{2}$

(iv) If $P = \cos^{-1} x$, then the value of $10x^2$ = ________.

(a) 7 (b) 6 (c) 8 (d) 9

(v) If $Q = \sin^{-1} x$ and $P = \cos^{-1} y$, then value of $x^2 + y^2$ = ________.

(a) $\frac{13}{10}$ (b) $\frac{17}{10}$ (c) $\frac{11}{10}$ (d) $\frac{19}{10}$

Multiple Choice Questions (1 mark)

2. If $\tan^{-1}(\cot\theta) = 2\theta$, then θ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) None of these

3. Find the principal value of $\cot^{-1}(-\sqrt{3})$.

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

OR

Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

4. If $6\sin^{-1}(x^2 - 6x + 8.5) = \pi$, then x is equal to

(a) 1 (b) 2 (c) 3 (d) 8

VSA Type Questions (1 mark)

5. Find the set of values of $\sec^{-1}\left(\frac{1}{2}\right)$.

OR

What is the domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$?

6. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

OR

Find the domain of the function $\cos^{-1}(2x - 1)$.

7. Find the value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$.

SA I Type Questions (2 marks)

8. Evaluate : $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

9. Find the value of $\sec^2(\tan^{-1}2) + \mathrm{cosec}^2(\cot^{-1}3)$.

OR

If $\sin^{-1}(x^2 - 7x + 12) = n\pi$, $\forall\, n \in I$, then find the value of x.

SA II Type Questions (3 marks)

10. Find the range of $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$.

11. If $\mathrm{cosec}^{-1} x + \mathrm{cosec}^{-1} y + \mathrm{cosec}^{-1} z = -\frac{3\pi}{2}$, find the value of $\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

12. Find the value of $\tan(\cos^{-1} x)$ and hence evaluate $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$.

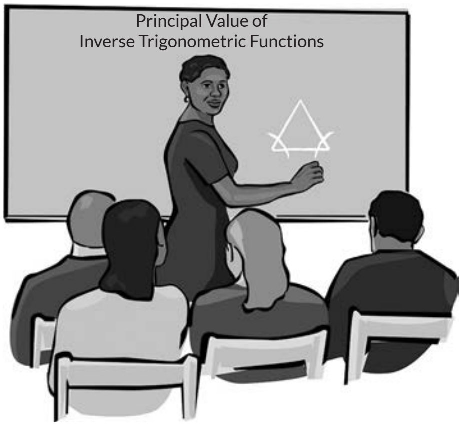
OR

Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Case Based Questions (4 marks)

13. A math teacher explained to the students about topic "Principal Value of Inverse Trigonometric Functions". He told that the value of an inverse trigonometric

functions which lies in the range of principal branch is called the principal value of that inverse.



Based on given information, answer the following questions.

(i) Find the principal value of $\sin^{-1}(1)+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

(ii) What will be the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$?

LA Type Questions (4 / 6 marks)

14. Evaluate : $\frac{1}{\pi}\left\{216\sin^{-1}\left(\sin\frac{7\pi}{6}\right)+27\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+28\tan^{-1}\left(\tan\frac{5\pi}{4}\right)+200\cot^{-1}\left(\cot\frac{-\pi}{4}\right)\right\}$

OR

If $x, y, z \in [-1, 1]$ such that $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then find the value of $xy + yz + zx$.

15. Show that $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$.

OR

Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

Detailed SOLUTIONS

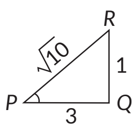
1. (i) (d) : Given, $\angle P=\tan^{-1}\left(\frac{1}{3}\right)$ and $\angle Q=\tan^{-1}\left(\frac{1}{2}\right)$

In ΔPQR, $\angle P + \angle Q + \angle R = 180°$ [By angle sum property]

$\therefore\ \angle R = 180° - (\angle P + \angle Q)$

$=180°-\left(\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right)$

(ii) (c) : Since, $\angle P=\tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan P=\frac{1}{3}$



$\Rightarrow \sin P=\frac{1}{\sqrt{10}}$ and $\cos P=\frac{3}{\sqrt{10}}$

$\therefore$ Value of $\cos P+\sin P=\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{10}}=\frac{4}{\sqrt{10}}$

(iii) (b) : Since, PQR is a triangle

$\therefore\ \angle P + \angle Q + \angle R = 180°$

$\Rightarrow \cos(P + Q + R) = \cos 180° = -1$

(iv) (d) : Given, $P = \cos^{-1}x$ ...(1)

Also, $\angle P=\tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan P=\frac{1}{3}$

$\therefore\ \cos P=\frac{3}{\sqrt{10}}$

$\Rightarrow P=\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$ ...(2)

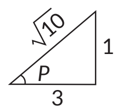
From (1) and (2), we get

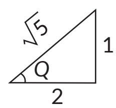
$x=\frac{3}{\sqrt{10}} \Rightarrow x^2=\frac{9}{10} \Rightarrow 10x^2=9$

(v) (c) : Given, $Q = \sin^{-1}x$ and $P = \cos^{-1}y$

$\Rightarrow x = \sin Q$ and $y = \cos P$ ...(1)

Also, $Q=\tan^{-1}\left(\frac{1}{2}\right)$ and $P=\tan^{-1}\left(\frac{1}{3}\right)$





$\Rightarrow \tan Q=\frac{1}{2}$ and $\tan P=\frac{1}{3}$

$\Rightarrow \sin Q=\frac{1}{\sqrt{5}}$ and $\cos P=\frac{3}{\sqrt{10}}$ ...(2)

From (1) and (2), we get

$x=\frac{1}{\sqrt{5}}$ and $y=\frac{3}{\sqrt{10}}$

$\therefore\ x^2+y^2=\left(\frac{1}{\sqrt{5}}\right)^2+\left(\frac{3}{\sqrt{10}}\right)^2=\frac{1}{5}+\frac{9}{10}=\frac{11}{10}$

2. (c) : $\tan^{-1}(\cot\theta) = 2\theta \Rightarrow \cot\theta = \tan 2\theta$

$\Rightarrow \cot\theta = \cot\left(\frac{\pi}{2}-2\theta\right) \Rightarrow \theta = \frac{\pi}{2} - 2\theta$

$\Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$

3. (a) : Let $\cot^{-1}\left(-\sqrt{3}\right) = \theta \Rightarrow \cot\theta = -\sqrt{3} = -\cot\frac{\pi}{6}$

$= \cot\left(\pi-\frac{\pi}{6}\right) = \cot\frac{5\pi}{6} \Rightarrow \theta=\frac{5\pi}{6}\in(0,\pi)$

$\therefore$ Principal value of $\cot^{-1}\left(-\sqrt{3}\right)$ is $\frac{5\pi}{6}$.

OR

(a) : Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$\Rightarrow \quad \theta = \frac{\pi}{4} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\therefore$ Principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

4. (b) : We have, $6\sin^{-1}(x^2 - 6x + 8.5) = \pi$

$\Rightarrow \quad \sin^{-1}(x^2 - 6x + 8.5) = \frac{\pi}{6}$

$\Rightarrow \quad x^2 - 6x + 8.5 = \sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow \quad (x - 4)(x - 2) = 0 \Rightarrow x = 4$ or $x = 2$

5. As we know, domain of $\sec^{-1}x$ is $R - (-1, 1)$

$\therefore \quad x \in (-\infty, -1) \cup (1, \infty)$

Therefore, there is no set of values exist for $\sec^{-1}\frac{1}{2}$.

Hence, ϕ is the answer.

OR

Given, $f(x) = \sin^{-1}\sqrt{x-1}$

Since, $x - 1 \geq 0$ and $-1 \leq \sqrt{x-1} \leq 1$

$\therefore \quad 0 \leq x - 1 \leq 1 \Rightarrow 1 \leq x \leq 2$

6. Since, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{3}\right)\right)$

$= \tan^{-1}\left(-\tan\frac{\pi}{3}\right)$ $\quad [\because \tan(\pi - \theta) = -\tan\theta]$

$= -\tan^{-1}\left(\tan\frac{\pi}{3}\right) = -\frac{\pi}{3}$ $\quad \left[\because \tan^{-1}(\tan x) = x, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$

OR

We have given, $f(x) = \cos^{-1}(2x - 1)$

Since, $-1 \leq 2x - 1 \leq 1$

$\Rightarrow \quad 0 \leq 2x \leq 2$

$\Rightarrow \quad 0 \leq x \leq 1$

7. Since, $\cos^{-1}\left(\cos\frac{14\pi}{3}\right) = \cos^{-1}\left(\cos\left(4\pi + \frac{2\pi}{3}\right)\right)$

$= \cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ $\quad \{\because \cos^{-1}(\cos x) = x, x \in [0, \pi]\}$

8. Given, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

$= \cos\left[\cos^{-1}\left(\cos\frac{5\pi}{6}\right) + \frac{\pi}{6}\right]$ $\quad \left[\because \cos\frac{5\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = \frac{-\sqrt{3}}{2}\right]$

$= \cos\left(\frac{5\pi}{6} + \frac{\pi}{6}\right)$ $\quad \{\because \cos^{-1}(\cos\theta) = \theta; \theta \in [0, \pi]\}$

$= \cos(\pi) = -1$

9. Given, $\sec^2(\tan^{-1}2) + \mathrm{cosec}^2(\cot^{-1}3)$

$= [\sec(\tan^{-1}2)]^2 + [\mathrm{cosec}(\cot^{-1}3)]^2$

$= [\sec(\sec^{-1}\sqrt{5})]^2 + [\mathrm{cosec}(\mathrm{cosec}^{-1}\sqrt{10})]^2$

$= (\sqrt{5})^2 + (\sqrt{10})^2 = 5 + 10 = 15$

OR

Given, $\sin^{-1}(x^2 - 7x + 12) = n\pi$

$\Rightarrow \quad x^2 - 7x + 12 = \sin n\pi$

$\Rightarrow \quad x^2 - 7x + 12 = 0$ $\quad (\because \sin n\pi = 0 \; \forall \; n \in I)$

$\Rightarrow \quad (x - 4)(x - 3) = 0 \Rightarrow x = 4, 3$

10. Given, $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$

Domain of $\sin^{-1}x = [-1, 1]$

Domain of $\tan^{-1}x = (-\infty, \infty)$

Domain of $\sec^{-1}x = (-\infty, \infty) - (-1, 1)$

Domain of $f(x) = [-1, 1] \cap (-\infty, \infty) \cap [(-\infty, \infty) - (-1, 1)]$

$= \{-1, 1\}$

Now, $f(-1) = \sin^{-1}(-1) + \tan^{-1}(-1) + \sec^{-1}(-1)$

$= -\frac{\pi}{2} - \frac{\pi}{4} + \pi = \frac{\pi}{4}$

and $f(1) = \sin^{-1}(1) + \tan^{-1}(1) + \sec^{-1}(1) = \frac{\pi}{2} + \frac{\pi}{4} + 0 = \frac{3\pi}{4}$

Range of $f(x) = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

11. We know that the minimum value of $\mathrm{cosec}^{-1}x$ is $-\frac{\pi}{2}$ which is attained at $x = -1$.

$\therefore \quad \mathrm{cosec}^{-1}x + \mathrm{cosec}^{-1}y + \mathrm{cosec}^{-1}z = -\frac{3\pi}{2}$

$\Rightarrow \quad \mathrm{cosec}^{-1}x + \mathrm{cosec}^{-1}y + \mathrm{cosec}^{-1}z = \left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right)$

$\Rightarrow \quad \mathrm{cosec}^{-1}x = -\frac{\pi}{2}, \mathrm{cosec}^{-1}y = -\frac{\pi}{2}, \mathrm{cosec}^{-1}z = -\frac{\pi}{2}$

$\Rightarrow \quad x = -1, y = -1, z = -1$

$\therefore \quad \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{(-1)}{(-1)} + \frac{(-1)}{(-1)} + \frac{(-1)}{(-1)} = 3$

12. Let $\cos^{-1}x = \theta$, then $\cos\theta = x$, where $\theta \in [0, \pi]$

$\therefore \quad \tan(\cos^{-1}x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}$

Hence, $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right) = \frac{\sqrt{1-(8/17)^2}}{8/17} = \frac{15}{8}$

OR

As we know, $\tan^{-1}(\tan x) = x; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\cos^{-1}(\cos x) = x; x \in [0, \pi]$

$\therefore \quad \tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

$= \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{6}\right)\right] + \cos^{-1}\left[\cos\left(\pi + \frac{7\pi}{6}\right)\right]$

$\left\{\because \frac{5\pi}{6} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \text{and} \frac{13\pi}{6} \notin [0, \pi]\right\}$

$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right) + \cos^{-1}\left(-\cos\frac{7\pi}{6}\right)$

$[\because \tan(\pi - \theta) = -\tan\theta$ and $\cos(\pi + \theta) = -\cos\theta]$

$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

$\{\because \tan^{-1}(-x) = -\tan^{-1}x; x \in R$ and $\cos^{-1}(-x) = \pi - \cos^{-1}x; x \in [-1, 1]\}$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\cos^{-1}\left[\cos\left(\pi+\frac{\pi}{6}\right)\right]$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\left[\cos^{-1}\left(-\cos\frac{\pi}{6}\right)\right]$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\pi+\cos^{-1}\left(\cos\frac{\pi}{6}\right)=-\frac{\pi}{6}+0+\frac{\pi}{6}=0$$

13. **(i)** Let $x = \sin^{-1}(1)$ and $y = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Then, $\sin x = 1$ and $\sin y = \frac{1}{\sqrt{2}}$

We know that the range of principal value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ and $\sin\left(\frac{\pi}{4}\right)=\frac{1}{2}$, $\sin\left(\frac{\pi}{2}\right)=1$

$\therefore$ Principal value of $\sin^{-1}(1)+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$

(ii) Let $p=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$, then $\sin p=\frac{\sqrt{3}}{2}$

We know that range of principal value branch of $\sin^{-1}$ is

$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ and $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$ $\Rightarrow p=\frac{\pi}{3}\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$\therefore$ Principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{3}$.

14. Since, $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

$$=\sin^{-1}\left(\sin\left(\pi+\frac{\pi}{6}\right)\right)=\sin^{-1}\left(-\sin\frac{\pi}{6}\right)=\sin^{-1}\left(\sin\left(\frac{-\pi}{6}\right)\right)=\frac{-\pi}{6}$$

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3},\ \tan^{-1}\left(\tan\frac{5\pi}{4}\right)=\tan^{-1}\left(\tan\left(\pi+\frac{\pi}{4}\right)\right)$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right]=\frac{\pi}{4}\text{ and }\cot^{-1}\left\{\cot\left(-\frac{\pi}{4}\right)\right\}=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

Hence, required value is

$$\frac{1}{\pi}\left\{216\times-\frac{\pi}{6}+27\times\frac{2\pi}{3}+28\times\frac{\pi}{4}+200\times\frac{3\pi}{4}\right\}$$

$= -36 + 18 + 7 + 150 = 139$

OR

We have, $x, y, z \in [-1, 1] \Rightarrow -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1$

$\Rightarrow$ $0 \le \cos^{-1}x \le \pi, 0 \le \cos^{-1}y \le \pi, 0 \le \cos^{-1}z \le \pi$

Given, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

$\Rightarrow$ $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi + \pi + \pi$

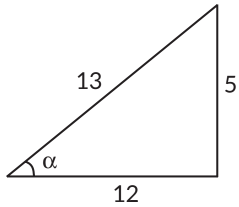
$\Rightarrow$ $\cos^{-1}x = \pi, \cos^{-1}y = \pi, \cos^{-1}z = \pi$

$\Rightarrow$ $x = -1, y = -1, z = -1$

$\therefore$ $xy + yz + zx = (-1)\times(-1) + (-1)\times(-1) + (-1)\times(-1)$

$= 1 + 1 + 1 = 3$

15. We have to prove, $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$

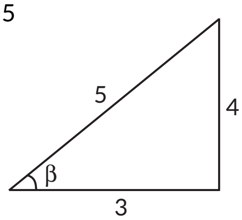


Consider, $\sin^{-1}\frac{5}{13}=\alpha$

$\Rightarrow \sin\alpha=\frac{5}{13}$ and $\cos\alpha=\frac{12}{13}$ $\therefore$ $\tan\alpha=\frac{5}{12}$

Again, consider $\cos^{-1}\frac{3}{5}=\beta \Rightarrow \cos\beta=\frac{3}{5}$

$\sin\beta=\frac{4}{5}$, $\tan\beta=\frac{4}{3}$



As we know,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$$\Rightarrow \tan(\alpha+\beta)=\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}} \Rightarrow \tan(\alpha+\beta)=\frac{63}{16}$$

$$\Rightarrow \alpha+\beta=\tan^{-1}\frac{63}{16} \Rightarrow \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$$

OR

We have to prove, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

Consider L.H.S. $=\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)\right]$

Let $\frac{1}{2}\sin^{-1}\frac{3}{4}=\alpha \Rightarrow \sin^{-1}\frac{3}{4}=2\alpha$

$$\Rightarrow \sin 2\alpha=\frac{3}{4} \Rightarrow \frac{2\tan\alpha}{1+\tan^2\alpha}=\frac{3}{4}$$

$\Rightarrow$ $3\tan^2\alpha - 8\tan\alpha + 3 = 0$

$$\Rightarrow \tan\alpha=\frac{8\pm\sqrt{64-4\times3\times3}}{2\times3}=\frac{8\pm\sqrt{28}}{6}$$

$$\Rightarrow \tan\alpha=\frac{4\pm\sqrt{7}}{3} \Rightarrow \alpha=\tan^{-1}\left[\frac{4\pm\sqrt{7}}{3}\right]$$

$$\therefore \text{L.H.S.}=\tan\left(\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right)\right)=\frac{4-\sqrt{7}}{3}=\text{R.H.S.}$$

Since, $-\frac{\pi}{2}\le\sin^{-1}\frac{3}{4}\le\frac{\pi}{2} \Rightarrow \frac{-\pi}{4}\le\frac{1}{2}\sin^{-1}\frac{3}{4}\le\frac{\pi}{4}$

$$\therefore \tan\left(\frac{-\pi}{4}\right)\le\tan\frac{1}{2}\left(\sin^{-1}\frac{3}{4}\right)\le\tan\frac{\pi}{4}$$

$$\Rightarrow -1\le\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)\le1 \quad \because \frac{4+\sqrt{7}}{3}>1$$

Hence, $\frac{4+\sqrt{7}}{3}$ is ignored.

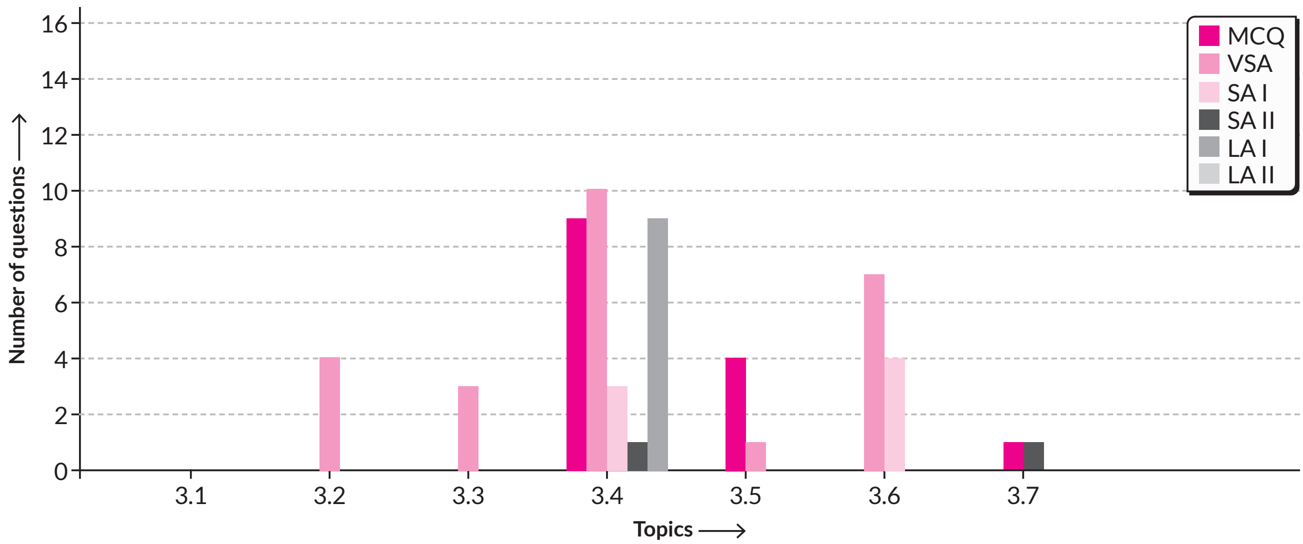
CHAPTER 3

Matrices

TOPICS

3.1 Introduction
3.2 Matrix
3.3 Types of Matrices
3.4 Operations on Matrices
3.5 Transpose of a Matrix
3.6 Symmetric and Skew Symmetric Matrices
3.7 Invertible Matrices

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 3.4 is highly scoring topic.
- Maximum weightage is of Topic 3.4 *Operations on Matrices.*
- Maximum MCQ, VSA and LA I type questions were asked from Topic 3.4 *Operations on Matrices.*

QUICK RECAP

Matrix

- A matrix is any rectangular array of numbers or functions in m rows and n columns within brackets.
  A matrix of m rows and n columns is usually written as

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The above matrix is also represented by

$A = [a_{ij}]_{m\times n}$ or, $A = [a_{ij}]$

Order of a Matrix

- A matrix having m rows and n columns has order $m \times n$.
- Types of Matrices
  - **Row Matrix :** A matrix having only one row.
  - **Column Matrix :** A matrix having only one column.
  - **Square Matrix :** A matrix in which number of rows is equal to the number of columns.
  - **Diagonal Matrix :** A square matrix whose all the non-diagonal elements are zero.

    For example : $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$ is a diagonal matrix and it can also be written as A = diag (1 2 3)
  - **Scalar Matrix :** A diagonal matrix in which all the diagonal elements are equal.
  - **Identity or Unit Matrix :** A square matrix is said to be identity matrix if all its diagonal entries are equal to 1 and rest are zero.
  - **Zero or Null Matrix :** A matrix whose all the elements are zero.

Equality of Matrices

- Two matrices are said to be equal, if their order is same and their corresponding elements are also equal.

Comparable Matrices

Two matrices A and B are said to be comparable if they have same order *i.e.*, number of rows and columns of A are equal to number of rows and columns of B respectively.

OPERATIONS ON MATRICES

Operations	Definition		Properties
Addition of two Matrices	Let A and B be two matrices each of order $m \times n$. Then, $A + B = [a_{ij} + b_{ij}]$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$	(i) (ii) (iii) (iv)	**Commutative Law :** For any two matrices A & B, $A + B = B + A$ **Associative Law :** For any three matrices A, B and C, $A + (B + C) = (A + B) + C$ **Existence of Additive Identity :** For any matrix A, if there exists a zero matrix O such that $A + O = A = O + A$. Then O is called additive identity. **Existence of Additive Inverse :** For any matrix A, if there exists a matrix $(-A)$ such that $A + (-A) = O = (-A) + A$. Then $(-A)$ is called additive inverse of A.
Multiplication of a Matrix by a Scalar	Let A be a matrix of order $m \times n$. Then, for any scalar k, $kA = [k \cdot a_{ij}]_{m\times n}$	(i) (ii)	Let A and B be two matrices each of order $m \times n$. Then, for any scalars k and l, we have $k(A + B) = kA + kB$ $(k + l) A = kA + lA$
Multiplication of two Matrices	Let A and B be any two matrices of orders $m \times n$ and $n \times p$ respectively. Then $AB = C = [c_{ik}]_{m\times p}$ where $c_{ik}=\sum_{r=1}^{n} a_{ir}b_{rk}$	(i) (ii) (iii) (iv)	Multiplication of two matrices is not commutative *i.e.*, $AB \neq BA$. **Associative Law :** For any three matrices A, B, and C, $(AB)C = A(BC)$ **Distributive Law :** For any three matrices A, B and C, • $A(B + C) = AB + AC$ • $(A + B)C = AC + BC$ **Existence of Multiplicative Identity :** For any square matrix, there exists a matrix I such that $AI = A = IA$, where I is called the identity matrix.

Transpose of a Matrix

- The matrix obtained by interchanging the rows and columns of matrix A is called the transpose of matrix A. It is represented by A' or A^T. In other words, if $A = [a_{ij}]_{m\times n}$, then $A' = [a_{ji}]_{n\times m}$.
- Properties of Transpose of a Matrix
  - For any comparable matrices A and B,

    (i) $(A + B)' = A' + B'$ (ii) $(A')' = A$

    (iii) $(kA)' = kA'$, where k is any constant.

    (iv) $(AB)' = B'A'$
  - **Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called a symmetric matrix, if $a_{ij} = a_{ji}$ for all i, j or we say, if $A = A'$.
  - **Skew-Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called a skew symmetric matrix, if $a_{ji} = -a_{ij}$ for all i, j or we say, if $A' = -A$.

Note :

(i) For any square matrix A with real entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

(ii) A matrix which is both symmetric and skew symmetric, is zero matrix.

(iii) If A is a square matrix such their $A = P + Q$, where P is symmetric matrix and Q is skew symmetric matrix,

then $P=\frac{1}{2}(A+A')$

and $Q=\frac{1}{2}(A-A')$, where A' is the transpose of A.

Invertible Matrices

- If A is a square matrix of order m and if there exists another square matrix B of the same order m, such that $AB = BA = I$, then B is called the inverse matrix of A and is denoted by A^{-1}.

Note : Inverse of a square matrix, if it exists, is unique.

BRAIN MAP

MATRIX

A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

	Operations	Properties
Addition and Subtraction	$A \pm B = C$ i.e., $[a_{ij}]_{m \times n} \pm [b_{ij}]_{m \times n} = [a_{ij} \pm b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$	• $A + B = B + A$ • $A + (B + C) = (A + B) + C$ • Additive identity $= O$ • Additive inverse $= -A$
Multiplication	$A_{m \times k} \times B_{k \times q} = C_{m \times q}$ i.e., $\left[\sum_{r=1}^{k} a_{ir} \cdot b_{rj}\right] = [c_{ij}]_{m \times q}$	• AB exists $\not\Rightarrow$ BA exists • AB may or may not be equal to BA • $(AB)\,C = A(BC)$ • $I_m \times A_{m \times m} = A_{m \times m} = A_{m \times m} \times I_m$ • $A\,(B + C) = AB + AC$, $(B + C)\,A = BA + CA$
Scalar Multiplication	$k\,A = B$ i.e., $k[a_{ij}]_{m \times n} = [k\,a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$	• $k\,(A + B) = kA + kB$ • $(k + m)A = kA + mA$

Trace of a Matrix

The sum of the main diagonal entries of a square matrix A is known as trace of matrix A.

Let square matrix $A = \begin{pmatrix} a & 1 & 2 \\ 1 & b & 2 \\ 2 & 1 & c \end{pmatrix}$, then trace of matrix A is given by $a + b + c$.

Equal Matrices

If matrices A and B are of same order, then $A = B$ iff $a_{ij} = b_{ij}\ \forall i, j$, where $A = [a_{ij}]$ and $B = [b_{ij}]$.

Comparable Matrices

Two matrices A and B are said to be comparable if they have same order *i.e.*, number of rows and columns of A are equal to number of rows and columns of B respectively.

Properties

- $(A')' = A$
- $(kA)' = kA'$
- $(AB)' = B'A'$
- $(A \pm B)' = A' \pm B'$
- $(ABC)' = C'\,B'\,A'$

TRANSPOSE OF A MATRIX

Transpose is obtained by interchanging rows and columns. If $A = [a_{ij}]_{m \times n}$, then A' or $A^T = [a_{ji}]_{n \times m}$

ORDER OF A MATRIX

A matrix having m rows and n columns is called a matrix of order $m \times n$.

Inverse of a Matrix

If A and B are two square matrices such that $AB = BA = I$, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B and is denoted by B^{-1}.

Properties

- Inverse of a square matrix, if it exists, is unique.
- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = A^{-1}/k$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

Types of Matrices

- Column Matrix : $A = [a_{ij}]_{m \times 1}$
- Row Matrix : $A = [a_{ij}]_{1 \times n}$
- Square Matrix : $A = [a_{ij}]_{m \times m}$
- Diagonal Matrix : $A = [a_{ij}]_{m \times m}$, where $a_{ij} = 0\ \forall\ i \neq j$
- Scalar Matrix : $A = [a_{ij}]_{n \times n}$ where $a_{ij} = \begin{cases} 0, \text{ if } i \neq j \\ k, \text{ if } i = j \end{cases}$ for some constant k
- Zero Matrix : $A = [a_{ij}]$, where $a_{ij} = \{0\ \forall\ i\ \&\ j\}$
- Identity Matrix : $A = [a_{ij}]_{n \times n}$, where $a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$
- Upper triangular matrix : A square matrix in which all the elements below the diagonal are zero.
- Lower triangular matrix : A square matrix in which all the elements above the diagonal are zero.
- Involutary Matrix : $A^2 = I$
- Orthogonal Matrix : $AA^T = A^T A = I$
- Idempotent Matrix : $A^2 = A$
- Unitary Matrix : $AA^\theta = A^\theta A = I$
- Symmetric Matrix : $A^T = A$
- Skew Symmetric Matrix : $A^T = -A$

Previous Years' CBSE Board Questions

3.2 Matrix

VSA (1 mark)

1. Construct a 2 × 2 matrix $A = [a_{ij}]$, whose elements are given by $a_{ij} = |(i)^2 - j|$. *(2020)* U

2. Write the number of all possible matrices of order 2 × 2 with each entry 1, 2 or 3. *(AI 2016)*

3. Write the element a_{23} of a 3 × 3 matrix $A = [a_{ij}]$ whose elements a_{ij} are given by $a_{ij} = \frac{|i-j|}{2}$. *(Delhi 2015)* U

4. The elements a_{ij} of a 3 × 3 matrix are given by $a_{ij} = \frac{1}{2}|-3i+j|$. Write the value of element a_{32}. *(AI 2014C)*

3.3 Types of Matrices

VSA (1 mark)

5. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $x + y$. *(AI 2014)* Ap

6. If $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{pmatrix}$, write the value of $a - 2b$. *(Foreign 2014)* Ap

7. If $\begin{bmatrix} x\cdot y & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$, write the value of $(x+y+z)$. *(Delhi 2014C)* Ap

3.4 Operations on Matrices

MCQ

8. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$ and $A = B^2$, then x equals
(a) ±1 (b) −1 (c) 1 (d) 2 *(2023)*

9. If $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$, then the value of $(2x + y - z)$ is
(a) 1 (b) 2 (c) 3 (d) 5 *(2023)*

10. If $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$, then
(a) $x = 1, y = 2$ (b) $x = 2, y = 1$
(c) $x = 1, y = -1$ (d) $x = 3, y = 2$ *(2023)*

11. If A is a square matrix and $A^2 = A$, then $(I + A)^2 - 3A$ is equal to
(a) I (b) A (c) $2A$ (d) $3I$ *(2023)*

12. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, then $(A - 2I)(A - 3I)$ is equal to
(a) A (b) I (c) $5I$ (d) O *(Term I, 2021-22)*

13. If order of matrix A is 2 × 3, of matrix B is 3 × 2, and of matrix C is 3 × 3, then which one of the following is not defined?
(a) $C(A + B')$ (b) $C(A + B')'$
(c) BAC (d) $CB + A'$ *(Term I, 2021-22)*

14. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, then $A^5 - A^4 - A^3 + A^2$ is equal to
(a) $2A$ (b) $3A$ (c) $4A$ (d) O *(Term I, 2021-22)*

15. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to
(a) I (b) O
(c) $I - A$ (d) $I + A$ *(2020)*

16. If $A = [2\ \ -3\ \ 4]$, $B = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$, $X = [1\ \ 2\ \ 3]$ and $Y = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, then $AB + XY$ equals
(a) [28] (b) [24] (c) 28 (d) 24 *(2020)*

VSA (1 mark)

17. If $A = [1\ \ 0\ \ 4]$ and $B = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, find AB. *(2021)* Ev

18. Find the order of the matrix A such that
$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}.$$
(2021) R

19. If $B = \begin{bmatrix} 1 & -5 \\ 0 & -3 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} 0 & 4 \\ -7 & 5 \end{bmatrix}$, find the matrix A. *(2021)*

20. If $A + B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A - 2B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, then A = ________. *(2020)*

21. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$. *(NCERT Exemplar, Delhi 2016)* An

22. If $[2\ \ 1\ \ 3]\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = A$, then write the order of matrix A. *(Foreign 2016)* Ap

23. Solve the following matrix equation for x :

$[x\ \ 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$ *(Delhi 2014)*

24. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find $(x - y)$. *(Delhi 2014)*

25. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix. *(AI 2014)* Ev

26. If $(2x\ \ 4)\begin{pmatrix} x \\ -8 \end{pmatrix} = O$, find the positive value of x. *(AI 2014C)*

SA I (2 mark)

27. If $A = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find scalar k so that $A^2 + I = kA$. *(2020)*

28. For what value of x is $[1\ \ 2\ \ 1]\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$? *(2020)* Ev

29. Find a matrix A such that $2A - 3B + 5C = O$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$. *(Delhi 2019)* Ev

SA II (3 mark)

30. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = O$. *(2023)*

LA I (4 marks)

31. Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}, C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$, find a matrix D such that $CD - AB = O$. *(Delhi 2017)* An

32. Find matrix A such that

$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$ *(AI 2017)*

33. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = O$ *(Delhi 2015)* An

34. Three schools A, B and C organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below.

Article/School	***A***	***B***	***C***
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also, find the total funds collected for the purpose.
Write one value generated by the above situation. *(Delhi 2015)*

35. To promote the making of toilets for women, an organisation tried to generate awareness through (i) house calls (ii) letters and (iii) announcements. The cost for each mode per attempt is given below :
(i) ₹ 50 (ii) ₹ 20 (iii) ₹ 40
The number of attempts made in three villages X, Y and Z are given below :

	(i)	(ii)	(iii)
X	400	300	100
Y	300	250	75
Z	500	400	150

Find the total cost incurred by the organisation for the three villages separately, using matrices. Write one value generated by the organisation in the society. *(AI 2015)* Ev

36. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, then find the values of a and b. *(Foreign 2015)*

37. In a parliament election, a political party hired a public relations firm to promote its candidates in three ways-telephone, house calls and letters. The cost per contact (in paise) is given in matrix A as

$A = \begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix} \begin{matrix} \text{Telephone} \\ \text{House call} \\ \text{Letters} \end{matrix}$

The number of contacts of each type made in two cities X and Y is given in matrix B as

$B = \begin{matrix} & \text{Telephone} & \text{House call} & \text{Letters} & \\ & \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} & & & \begin{matrix} \text{City } X \\ \text{City } Y \end{matrix} \end{matrix}$

Find the total amount spent by the party in the two cities. What should one consider before casting his/her vote-party's promotional activity or their social activities? *(Foreign 2015)*

38. If $[2x\ \ 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = O$, find x. *(Delhi 2015C)*

39. A trust fund, ₹ 35,000 is to be invested in two different types of bonds. The first bond pays 8% interest per annum which will be given to orphanage and second bond pays 10% interest per annum which will be given to an N.G.O. (Cancer Aid Society). Using matrix multiplication, determine how to divide ₹ 35,000 among two types of bonds if the trust fund obtains an annual total interest of ₹ 3,200. What are the values reflected in this question? *(AI 2015C)* Ev

3.5 Transpose of a Matrix

MCQ

40. If $A=\begin{bmatrix}5 & x\\ y & 0\end{bmatrix}$ and $A = A^T$, where A^T is the transpose of the matrix A, then
(a) $x = 0, y = 5$ (b) $x = y$
(c) $x + y = 5$ (d) $x = 5, y = 0$ *(2023)*

41. If a matrix $A = [1\ 2\ 3]$, then the matrix AA' (where A' is the transpose of A) is
(a) 14 (b) $\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$
(c) $\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{bmatrix}$ (d) [14] *(2023)*

42. If P is a 3 × 3 matrix such that $P' = 2P + I$, where P' is the transpose of P, then
(a) $P = I$ (b) $P = -I$ (c) $P = 2I$ (d) $P = -2I$ *(Term I, 2021-22)*

43. If $A = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$ and $A + A' = I$, then the value of α is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) π (d) $\frac{3\pi}{2}$
(Term I, 2021-22)

VSA (1 mark)

44. If A is a matrix of order 3 × 2, then the order of the matrix A' is ________. *(2020)* U

3.6 Symmetric and Skew Symmetric Matrices

VSA (1 mark)

45. A square matrix A is said to be symmetric, if ______. *(2020)* U

46. Given a skew-symmetric matrix $A = \begin{bmatrix}0 & a & 1\\ -1 & b & 1\\ -1 & c & 0\end{bmatrix}$, the value of $(a + b + c)^2$ is ________. *(2020)*

47. If the matrix $A=\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$ is skew symmetric, find the values of 'a' and 'b'. *(2018)*

48. Matrix $A=\begin{bmatrix}0 & 2b & -2\\ 3 & 1 & 3\\ 3a & 3 & -1\end{bmatrix}$ is given to be symmetric, find values of a and b. *(Delhi 2016)* Ap

49. If $A=\begin{bmatrix}3 & 5\\ 7 & 9\end{bmatrix}$ is written as $A = P + Q$, where P is a symmetric matrix and Q is a skew symmetric matrix, then write the matrix P. *(Foreign 2016)*

50. Express the matrix $A=\begin{bmatrix}2 & 4 & -6\\ 7 & 3 & 5\\ 1 & -2 & 4\end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.
(AI 2015C) Ev

51. Write a 2 × 2 matrix which is both symmetric and skew symmetric. *(Delhi 2014C)*

SA I (2 marks)

52. If the matrix $\begin{bmatrix}0 & 6-5x\\ x^2 & x+3\end{bmatrix}$ is symmetric, find the value of x. *(2021)* Ev

53. For the matrix $A=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}$, , verify that
(i) $(A + A')$ is a symmetric matrix.
(ii) $(A - A')$ is a skew-symmetric matrix. *(2020C)*

54. If A and B are symmetric matrices, such that AB and BA are both defined, then prove that $AB - BA$ is a skew symmetric matrix. *(AI 2019)* An

55. Show that all the diagonal elements of a skew symmetric matrix are zero. *(Delhi 2017)* An

3.7 Invertible Matrices

MCQ

56. If for a square matrix A, $A^2 - A + I = O$, then A^{-1} equals
(a) A (b) $A + I$ (c) $I - A$ (d) $A - I$
(2023)

SA II (3 marks)

57. If $A = \begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}$, then show that $A^3 - 4A^2 - 3A + 11I = O$. Hence find A^{-1}. *(2020)* U

CBSE Sample Questions

3.3 Types of Matrices

MCQ

1. If $A = [a_{ij}]$ is a skew-symmetric matrix of order n, then

(a) $a_{ij} = \frac{1}{a_{ji}} \forall i, j$ (b) $a_{ij} \neq 0 \ \forall\ i, j$

(c) $a_{ij} = 0$, where $i = j$ (d) $a_{ij} \neq 0$ where $i = j$ R (2022-23)

2. If $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$, then value of $a + b - c + 2d$ is

(a) 8 (b) 10 (c) 4 (d) −8

(Term I, 2021-22) Ap

3. Given that matrices A and B are of order $3 \times n$ and $m \times 5$ respectively, then the order of matrix $C = 5A + 3B$ is

(a) 3×5 and $m = n$ (b) 3×5

(c) 3×3 (d) 5×5 (Term I, 2021-22)

3.4 Operations on Matrices

MCQ

4. If $A = [a_{ij}]$ is a square matrix of order 2 such that $a_{ij} = \begin{cases} 1, \text{when } i \neq j \\ 0, \text{when } i = j \end{cases}$, then A^2 is

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(Term I, 2021-22) U

5. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$, then the values of k, a and b respectively are

(a) −6, −12, −18 (b) −6, −4, −9

(c) −6, 4, 9 (d) −6, 12, 18

(Term I, 2021-22) Ap

6. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(a) A (b) $I + A$

(c) $I - A$ (d) I (Term I, 2021-22)

7. Given that $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ and $A^2 = 3I$, then

(a) $1 + \alpha^2 + \beta\gamma = 0$ (b) $1 - \alpha^2 - \beta\gamma = 0$

(c) $3 - \alpha^2 - \beta\gamma = 0$ (d) $3 + \alpha^2 + \beta\gamma = 0$

(Term I, 2021-22) Ap

VSA (1 mark)

8. If A and B are matrices of order $3 \times n$ and $m \times 5$ respectively, then find the order of matrix $5A - 3B$, given that it is defined. (2020-21) Ap

9. Given that A is a square matrix of order 3×3 and $|A| = -4$. Find $|\text{adj } A|$. (2020-21) An

3.7 Invertible Matrices

MCQ

10. If A, B are non-singular square matrices of the same order, then $(AB^{-1})^{-1} =$

(a) $A^{-1}B$ (b) $A^{-1}B^{-1}$ (c) BA^{-1} (d) AB

(2022-23)

11. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, then

(a) $A^{-1} = B$ (b) $A^{-1} = 6B$

(c) $B^{-1} = B$ (d) $B^{-1} = \frac{1}{6}A$

(Term I, 2021-22) Ap

SA I (2 marks)

12. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}. (2020-21) Ap

Detailed SOLUTIONS

1. Here, $a_{11} = |(1)^2 - 1| = 0$, $a_{12} = |(1)^2 - 2| = 1$, $a_{21} = |(2)^2 - 1| = 3$ and $a_{22} = |(2)^2 - 2| = 2$

$\therefore$ Required matrix $= \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$.

Key Points

- A matrix is written as $A = [a_{ij}]_{m \times n}$, where a_{ij} is an element lying in the i^{th} row and j^{th} column.

2. As, matrix is of order 2×2, so there are 4 entries possible.

Each entry has 3 choices *i.e.*, 1, 2 or 3. So, the number of ways to make such matrices is $3 \times 3 \times 3 \times 3 = 81$.

3. Here, $a_{ij} = \frac{|i-j|}{2}$ $\therefore$ $a_{23} = \frac{|2-3|}{2} = \frac{1}{2}$ [For $i = 2, j = 3$]

4. Here, $a_{ij} = \frac{1}{2}|-3i + j|$

$\therefore \quad a_{32}=\frac{1}{2}|-3\cdot 3+2| \qquad$ [For $i=3, j=2$]

$=\frac{1}{2}|-9+2|=\frac{1}{2}|-7|=\frac{7}{2}$

5. Here, $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix}=\begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$

By equality of two matrices, we get

$x - y = -1, z = 4, 2x - y = 0, w = 5$

Solving these equations for x and y, we get

$x = 1, y = 2 \ \therefore \ x + y = 1 + 2 = 3.$

Answer Tips

➲ Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to equal if they are of a same order and $a_{ij} = b_{ij} \ \forall i, j$.

6. Given, $\begin{pmatrix} a+4 & 3b \\ 8 & -6 \end{pmatrix}=\begin{pmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{pmatrix}$

By equality of two matrices, we get

$a + 4 = 2a + 2, 3b = b + 2, -6 = a - 8b$

On solving these equations, we get $a = 2, b = 1$.

So, $a - 2b = 0$.

7. Here, $\begin{bmatrix} x\cdot y & 4 \\ z+6 & x+y \end{bmatrix}=\begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$

By equality of two matrices, we get

$x \cdot y = 8, w = 4, z + 6 = 0, x + y = 6$

$\Rightarrow \quad z = -6, x + y = 6 \Rightarrow x + y + z = 6 - 6 = 0.$

8. **(c):** We have, $A=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$

$\therefore \ B^2=\begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} x^2 & 0 \\ x+1 & 1 \end{bmatrix}$

Now, it is given that $A = B^2$

$\Rightarrow \ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} x^2 & 0 \\ x+1 & 1 \end{bmatrix}$

On comparing, we get

$x^2 = 1$ and $x + 1 = 2 \ \Rightarrow \ x = \pm 1$ and $x = 1$

$\therefore \quad x = 1$

9. **(d) :** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

$\therefore \quad x + y + z = 6$ ...(i)

$y + z = 3$ ...(ii)

$z = 2$ ...(iii)

$\Rightarrow \quad y + 2 = 3$ [Using (ii) and (iii)]

$\Rightarrow \quad y = 1$ ...(iv)

$\Rightarrow \quad x + 1 + 2 = 6$ [Using (i), (iii) and (iv)]

$\Rightarrow \quad x = 3$

So, $2x + y - z = (2 \times 3) + 1 - 2 = 6 + 1 - 2 = 5$

10. **(b):** We have, $x\begin{bmatrix} 1 \\ 2 \end{bmatrix}+y\begin{bmatrix} 2 \\ 5 \end{bmatrix}=\begin{bmatrix} 4 \\ 9 \end{bmatrix}$

$\begin{bmatrix} x+2y \\ 2x+5y \end{bmatrix}=\begin{bmatrix} 4 \\ 9 \end{bmatrix}$

$\Rightarrow \quad x + 2y = 4$ ...(i) and $2x + 5y = 9$ ...(ii)

Solving (i) and (ii), we get $x = 2, y = 1$

11. **(a):** Given that $A^2 = A$

Consider $(I + A)^2 - 3A$

$= I^2 + A^2 + 2AI - 3A$

$= I + A + 2A - 3A \qquad [\because I^2 = I, A^2 = A \text{ (given)}]$

$= I$

12. **(d)**

13. **(a) :** Consider $C(A+B')$ *i.e.*, $C_{3\times3}(A_{2\times3} + B'_{2\times3})$

$= C_{3\times3}(A + B')_{2\times3}$

Here, number of columns in the matrix C is 3 and number of rows in the matrix $(A + B')$ is 2. So, it is not defined .

14. **(d)**

15. **(a) :** We have, $A^2 = A$

Now, $(I - A)^3 + A = (I - A)(I - A)(I - A) + A$

$= (I\cdot I - I\cdot A - A\cdot I + A\cdot A)(I - A) + A$

$= (I - A - A + A)(I - A) + A \qquad [\because \ I\cdot A = A\cdot I = A \text{ and } A^2 = A]$

$= (I - A)(I - A) + A$

$= (I\cdot I - I\cdot A - A\cdot I + A\cdot A) + A$

$= (I - A - A + A) + A = (I - A) + A = I$

16. **(a) :** Consider, $AB=\begin{bmatrix} 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}=[6-6+8]=[8]$

and $XY=\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}=[2+6+12]=[20]$

$AB + XY = [8] + [20] = [28]$

17. Consider, $AB=\begin{bmatrix} 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}=[2+0+24]=[26]$

18. We have, $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}_{3\times2} A=\begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}_{3\times2}$

The order of matrix A should be 2×2.

19. Given, $B=\begin{bmatrix} 1 & -5 \\ 0 & -3 \end{bmatrix}$ and $A+2B=\begin{bmatrix} 0 & 4 \\ -7 & 5 \end{bmatrix}$

$\Rightarrow A+2\begin{bmatrix} 1 & -5 \\ 0 & -3 \end{bmatrix}=\begin{bmatrix} 0 & 4 \\ -7 & 5 \end{bmatrix}$

$\Rightarrow A+\begin{bmatrix} 2 & -10 \\ 0 & -6 \end{bmatrix}=\begin{bmatrix} 0 & 4 \\ -7 & 5 \end{bmatrix} \Rightarrow A=\begin{bmatrix} -2 & 14 \\ -7 & 11 \end{bmatrix}$

20. Given $A+B=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ ...(i)

and $A-2B=\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ ...(ii)

(i) – (ii), we get

$3B=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}-\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow \quad 3B=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow \quad B=\begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix}$

From (i),

$$A=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}-B=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}-\begin{bmatrix}2/3 & -1/3\\1/3 & 2/3\end{bmatrix}=\begin{bmatrix}1/3 & 1/3\\2/3 & 1/3\end{bmatrix}$$

21. Given, $A^2 = I$

$\therefore$ The simplified value of $(A - I)^3 + (A + I)^3 - 7A$

$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$

$= 2A^3 + 6AI^2 - 7A = 2AA^2 + 6AI - 7A$

$= 2AI + 6A - 7A = 2A - A = A$

Concept Applied

- $I \cdot A = A \cdot I = A$ and $A^2 = I$

22. Given, $A=[2 \quad 1 \quad 3]\begin{bmatrix}-1 & 0 & -1\\-1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$$=[-2-1 \quad 1+3 \quad -2+3]\begin{bmatrix}1\\0\\-1\end{bmatrix}=[-3 \quad 4 \quad 1]\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$= [-3 + 0 - 1] = [-4]$

$\therefore$ The order of matrix $A = 1 \times 1$

23. Given, $[x \quad 1]\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix}=O \Rightarrow [x-2 \quad 0]=[0 \quad 0]$

$\Rightarrow \quad x - 2 = 0 \Rightarrow x = 2$

Commonly Made Mistake

- Check the order of matrices before multiplying two matrices.

24. We have, $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}6 & 8\\10 & 2x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}7 & 8+y\\10 & 2x+1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

On comparing the corresponding elements, we get

$8 + y = 0$ and $2x + 1 = 5 \Rightarrow y = -8$ and $x = 2$

$\therefore \quad x - y = 2 + 8 = 10$

Key Points

- If $A = [a_{ij}]_{m\times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

25. Here, $A^2 = A$

Now, $7A - (I + A)^3 = 7A - (I + A)(I + A)(I + A)$

$= 7A - (I + A)(I{\cdot}I + I{\cdot}A + A{\cdot}I + A{\cdot}A)$

$= 7A - (I + A)(I + A + A + A)$ $\quad (\because I{\cdot}A = A{\cdot}I = A$ and $A^2 = A)$

$= 7A - (I + A)(I + 3A)$

$= 7A - (I{\cdot}I + I{\cdot}(3A) + A{\cdot}I + A{\cdot}(3A))$

$= 7A - (I + 3A + A + 3A) = 7A - I - 7A = -I.$

26. Here, $(2x \quad 4)\begin{pmatrix}x\\-8\end{pmatrix}=O$

$\Rightarrow \quad 2x \cdot x + 4 \cdot (-8) = 0 \Rightarrow 2x^2 - 32 = 0$

$\Rightarrow \quad x^2 = 16 = 4^2 \Rightarrow x = 4$

which is the required positive value of x.

27. We have, $A^2 + I = kA$

$$\Rightarrow \begin{bmatrix}-3 & 2\\1 & -1\end{bmatrix}\begin{bmatrix}-3 & 2\\1 & -1\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=k\begin{bmatrix}-3 & 2\\1 & -1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}11 & -8\\-4 & 3\end{bmatrix}+\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=k\begin{bmatrix}-3 & 2\\1 & -1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}12 & -8\\-4 & 4\end{bmatrix}=k\begin{bmatrix}-3 & 2\\1 & -1\end{bmatrix}$$

$$\Rightarrow -4\begin{bmatrix}-3 & 2\\1 & -1\end{bmatrix}=k\begin{bmatrix}-3 & 2\\1 & -1\end{bmatrix}$$

On comparing, we get $k = -4$

28. Given, $[1 \quad 2 \quad 1]\begin{bmatrix}1 & 2 & 0\\2 & 0 & 1\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\2\\x\end{bmatrix}=0$

$$\Rightarrow [6 \quad 2 \quad 4]\begin{bmatrix}0\\2\\x\end{bmatrix}=0 \Rightarrow 0+4+4x=0 \Rightarrow x=-1$$

29. Let $A=\begin{bmatrix}x & y & z\\p & q & r\end{bmatrix}$ [$\because B$ and C are matrices of order 2×3]

Given, $2A - 3B + 5C = O$

$\Rightarrow \quad 2A = 3B - 5C \Rightarrow A=\frac{1}{2}[3B-5C]$ ...(i)

Now, $3B-5C=3\begin{bmatrix}-2 & 2 & 0\\3 & 1 & 4\end{bmatrix}-5\begin{bmatrix}2 & 0 & -2\\7 & 1 & 6\end{bmatrix}$

$$=\begin{bmatrix}-6 & 6 & 0\\9 & 3 & 12\end{bmatrix}-\begin{bmatrix}10 & 0 & -10\\35 & 5 & 30\end{bmatrix}=\begin{bmatrix}-16 & 6 & 10\\-26 & -2 & -18\end{bmatrix}$$

From (i), we get $A=\begin{bmatrix}-8 & 3 & 5\\-13 & -1 & -9\end{bmatrix}$.

30. We have, $A=\begin{bmatrix}1 & 2 & 3\\3 & -2 & 1\\4 & 2 & 1\end{bmatrix}$

Now, $A^2=\begin{bmatrix}1 & 2 & 3\\3 & -2 & 1\\4 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\3 & -2 & 1\\4 & 2 & 1\end{bmatrix}$

$$=\begin{bmatrix}1+6+12 & 2-4+6 & 3+2+3\\3-6+4 & 6+4+2 & 9-2+1\\4+6+4 & 8-4+2 & 12+2+1\end{bmatrix}=\begin{bmatrix}19 & 4 & 8\\1 & 12 & 8\\14 & 6 & 15\end{bmatrix}$$

Now, $A^3=\begin{bmatrix}19 & 4 & 8\\1 & 12 & 8\\14 & 6 & 15\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\3 & -2 & 1\\4 & 2 & 1\end{bmatrix}$

$$=\begin{bmatrix}19+12+32 & 38-8+16 & 57+4+8\\1+36+32 & 2-24+16 & 3+12+8\\14+18+60 & 28-12+30 & 42+6+15\end{bmatrix}=\begin{bmatrix}63 & 46 & 69\\69 & -6 & 23\\92 & 46 & 63\end{bmatrix}$$

$$\text{Now, } A^3-23A-40I=\begin{bmatrix}63&46&69\\69&-6&23\\92&46&63\end{bmatrix}-\begin{bmatrix}23&46&69\\69&-46&23\\92&46&23\end{bmatrix}-\begin{bmatrix}40&0&0\\0&40&0\\0&0&40\end{bmatrix}$$

$$=\begin{bmatrix}63-23-40&46-46-0&69-69-0\\69-69-0&-6+46-40&23-23-0\\92-92-0&46-46-0&63-23-40\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$$

Hence proved.

31. We have, $A=\begin{bmatrix}2&-1\\3&4\end{bmatrix}, B=\begin{bmatrix}5&2\\7&4\end{bmatrix}, C=\begin{bmatrix}2&5\\3&8\end{bmatrix}$

Let $D=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

Now, $CD - AB = O$

$$\therefore \begin{bmatrix}2&5\\3&8\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}-\begin{bmatrix}2&-1\\3&4\end{bmatrix}\begin{bmatrix}5&2\\7&4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a+5c&2b+5d\\3a+8c&3b+8d\end{bmatrix}-\begin{bmatrix}10-7&4-4\\15+28&6+16\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a+5c-3&2b+5d\\3a+8c-43&3b+8d-22\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

On comparing the corresponding elements, we get

$2a + 5c - 3 = 0$ ...(i)

and $3a + 8c - 43 = 0$ ...(ii)

Also, $2b + 5d = 0$ ...(iii)

and $3b + 8d - 22 = 0$ ...(iv)

Solving (i) and (ii), we get

$a = -191,\ c = 77$

Solving (iii) and (iv), we get $b = -110$, $d = 44$

$$\therefore \quad D=\begin{bmatrix}-191&-110\\77&44\end{bmatrix}$$

32. Given that, $\begin{bmatrix}2&-1\\1&0\\-3&4\end{bmatrix}A=\begin{bmatrix}-1&-8\\1&-2\\9&22\end{bmatrix}$

Let $X=\begin{bmatrix}2&-1\\1&0\\-3&4\end{bmatrix}_{3\times2}$ and $Y=\begin{bmatrix}-1&-8\\1&-2\\9&22\end{bmatrix}_{3\times2}$

As order of X is 3×2, then A should be of order 2×2, so that we get Y matrix of order 3×2.

Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$\text{Now, } \begin{bmatrix}2&-1\\1&0\\-3&4\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}-1&-8\\1&-2\\9&22\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a-c&2b-d\\a+0&b+0\\-3a+4c&-3b+4d\end{bmatrix}=\begin{bmatrix}-1&-8\\1&-2\\9&22\end{bmatrix}$$

On comparing the corresponding elements, we get

$2a - c = -1$ ...(i)

$2b - d = -8$ ...(ii)

$a = 1$ ...(iii)

and $b = -2$ ...(iv)

Substituting $a = 1$ in (i), we get $c = 3$

and substituting $b = -2$ in (ii), we get $d = 4$

So, $A=\begin{bmatrix}1&-2\\3&4\end{bmatrix}$

33. Given, $A=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}$

Now, $A^2 - 5A + 4I$

$$=\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}-5\begin{bmatrix}2&0&1\\2&1&3\\1&-1&0\end{bmatrix}+4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}-\begin{bmatrix}10&0&5\\10&5&15\\5&-5&0\end{bmatrix}+\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$$=\begin{bmatrix}9&-1&2\\9&2&5\\0&-1&2\end{bmatrix}-\begin{bmatrix}10&0&5\\10&5&15\\5&-5&0\end{bmatrix}=\begin{bmatrix}-1&-1&-3\\-1&-3&-10\\-5&4&2\end{bmatrix}$$

Since, $A^2 - 5A + 4I + X = O \Rightarrow X = -(A^2 - 5A + 4I)$

$$\therefore \quad X=\begin{bmatrix}1&1&3\\1&3&10\\5&-4&-2\end{bmatrix}$$

Answer Tips

➲ O is the additive identity.

34. The number of articles sold by each school can be written in the matrix form as

$$X=\begin{bmatrix}40&25&35\\50&40&50\\20&30&40\end{bmatrix}$$

The cost of each article can be written in the matrix form as $Y = [25\ \ 100\ \ 50]$

The fund collected by each school is given by

$$YX=[25\ 100\ 50]\begin{bmatrix}40&25&35\\50&40&50\\20&30&40\end{bmatrix}=[7000\ \ 6125\ \ 7875]$$

Therefore, the funds collected by schools A, B and C are ₹ 7000, ₹ 6125 and ₹ 7875 respectively.

Thus, the total funds collected

= ₹ (7000 + 6125 + 7875) = ₹ 21000

The situation highlights the helping nature of the students.

35. Let ₹A, ₹B and ₹C be the cost incurred by the organisation for villages X, Y and Z respectively. Then, we get the matrix equation as

$$\begin{bmatrix} 400 & 300 & 100 \\ 300 & 250 & 75 \\ 500 & 400 & 150 \end{bmatrix}\begin{bmatrix} 50 \\ 20 \\ 40 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 400\times50+300\times20+100\times40 \\ 300\times50+250\times20+75\times40 \\ 500\times50+400\times20+150\times40 \end{bmatrix}$$

$$= \begin{bmatrix} 20{,}000+6{,}000+4{,}000 \\ 15{,}000+5{,}000+3{,}000 \\ 25{,}000+8{,}000+6{,}000 \end{bmatrix} = \begin{bmatrix} 30{,}000 \\ 23{,}000 \\ 39{,}000 \end{bmatrix}$$

$\therefore$ A = ₹ 30,000, B = ₹ 23,000 and C = ₹ 39,000

These are the costs incurred by the organisation for villages X, Y and Z respectively.

The value generated by the organisation in the society is cleanliness.

Key Points

- The product of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B.

36. We have, $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

Consider, $(A + B) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$

Now, $(A + B)^2 = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}\begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$

$= \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(1+a-2) & 4 \end{bmatrix} = \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix}$

Now, consider $A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1-2 & -1+1 \\ 2-2 & -2+1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

and $B^2 = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$

$\therefore \quad A^2 + B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$

It is given that, $(A + B)^2 = A^2 + B^2$

$\therefore \quad \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$

On comparing the corresponding elements, we get

$a - 1 = 0 \Rightarrow a = 1$ and $b = 4$

And $(1 + a)^2 = a^2 + b - 1$ and $(2 + b)(a - 1) = ab - b$ are also satisfied by $a = 1$ and $b = 4$

Therefore, $a = 1$ and $b = 4$.

37. The total amount spent by the party in two cities X and Y is represented in the matrix equation by matrix C as, $C = BA$

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix}\begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1000\times140+500\times200+5000\times150 \\ 3000\times140+1000\times200+10000\times150 \end{bmatrix}$$

$$= \begin{bmatrix} 990000 \\ 2120000 \end{bmatrix}$$

$\Rightarrow$ X = 990000 paise, Y = 2120000 paise

$\therefore$ X = ₹ 9900 and Y = ₹ 21200

i.e., Amount spent by the party in city X and Y are ₹ 9900 and ₹ 21200 respectively. One should consider about the social activities of a political party before casting his/her vote.

38. Here, $[2x \quad 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 3 \end{bmatrix} = O \Rightarrow [2x \quad 3]\begin{bmatrix} x+6 \\ -3x \end{bmatrix} = O$

$\Rightarrow \quad 2x(x + 6) + 3(-3x) = 0 \Rightarrow 2x^2 + 12x - 9x = 0$

$\Rightarrow \quad 2x^2 + 3x = 0 \Rightarrow x(2x + 3) = 0 \Rightarrow x = 0, \dfrac{-3}{2}$.

39. Trust fund = ₹ 35,000

Let ₹ x be invested in the first bond and then ₹ (35,000 – x) will be invested in the second bond.

Interest paid on the first bond = 8% = 0.08

Interest paid on the second bond = 10% = 0.10

Total annual interest = ₹ 3,200

$\therefore \quad$ In matrices, $[x \quad 35{,}000-x]\begin{bmatrix} 0.08 \\ 0.10 \end{bmatrix} = [3{,}200]$

$\Rightarrow \quad x \times 0.08 + (35{,}000 - x) \times 0.10 = 3{,}200$

$\Rightarrow \quad x \times \dfrac{8}{100} + (35{,}000 - x) \times \dfrac{10}{100} = 3{,}200$

$\Rightarrow \quad 8x + 3{,}50{,}000 - 10x = 3{,}20{,}000$

$\Rightarrow \quad 2x = 30{,}000 \Rightarrow x = 15{,}000$

$\therefore$ ₹ 15,000 should be invested in the first bond and ₹ 35,000 – ₹ 15,000 = ₹ 20,000 should be invested in the second bond.

The values reflected in this question are :

(i) Spirit of investment.

(ii) Giving charity to cancer patients.

(iii) Helping the orphans living in the society.

40. (b): We have, $A = A^T$

$\Rightarrow \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix} = \begin{bmatrix} 5 & y \\ x & 0 \end{bmatrix}$

On comparing, we get $x = y$.

41. (d): $A = [1 \;\; 2 \;\; 3]$

$A' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

So, $AA' = [1 \;\; 2 \;\; 3]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [1 + 4 + 9] = [14]$

42. (b): We have, $P' = 2P + I$ ...(i)

Now, $(P')' = (2P + I)' = 2P' + I$

$\Rightarrow \quad P = 2(2P + I) + I$ [Using (i)]

$\Rightarrow \quad P = 4P + 3I \Rightarrow P = -I$

43. **(b):** We have, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

and $A + A' = I$

$$\Rightarrow \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow 2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$$

44. If A is a matrix of order 3×2, then the order of the matrix A' is 2×3.

45. A square matrix A is said to be symmetric, if $A' = A$.

46. Given, $A = \begin{bmatrix} 0 & a & 1 \\ -1 & b & 1 \\ -1 & c & 0 \end{bmatrix}$

A is a skew-symmetric matrix.

$\therefore \quad A' = -A$

$$\Rightarrow \begin{bmatrix} 0 & -1 & -1 \\ a & b & c \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & -1 \\ 1 & -b & -1 \\ 1 & -c & 0 \end{bmatrix}$$

By comparing on both sides, we get $a = 1$,

$b = -b \Rightarrow 2b = 0 \Rightarrow b = 0; c = -1$

Now, $(a + b + c)^2 = (1 + 0 - 1)^2 = 0$

47. A square matrix A is said to be skew symmetric matrix if $A' = -A$ ...(i)

Now, $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix} \therefore A' = \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$

From (i), $A + A' = O$

$$\Rightarrow \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 0 & 2+a & b-3 \\ a+2 & 0 & 0 \\ b-3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad a + 2 = 0 \ \& \ b - 3 = 0 \ \therefore a = -2 \ \& \ b = 3$

Answer Tips

- If $A = [a_{ij}]$ be a m×n matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A.

48. Given, $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$

$\because \quad A$ is symmetric. $\therefore A' = A$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$$

On comparing the corresponding elements, we get

$$a = \frac{-2}{3} \text{ and } b = \frac{3}{2}.$$

49. Given, $A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix}$

$\because \quad P$ is symmetric matrix. So, $P = \frac{1}{2}(A + A')$

$$\therefore \quad P = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3+3 & 5+7 \\ 7+5 & 9+9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & 12 \\ 12 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix}$$

Hence, the matrix $P = \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix}$

Concept Applied

- A square matrix $A = [a_{ij}]$ is said to be symmetric if $A' = A$, that is $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

50. We know that a square matrix A can be written as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Out of which $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is skew symmetric matrix,

$\therefore \quad$ For the given matrix

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$$

$$\therefore \quad A + A^T = \begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix} \text{ and } A - A^T = \begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix}$$

Hence, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$= \begin{bmatrix} 2 & 11/2 & -5/2 \\ 11/2 & 3 & 3/2 \\ -5/2 & 3/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -3/2 & -7/2 \\ 3/2 & 0 & 7/2 \\ 7/2 & -7/2 & 0 \end{bmatrix}$$

In above case, first is symmetric and the second is skew symmetric matrix.

51. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 2×2 symmetric as well as skew symmetric matrix.

52. Let, $A = \begin{bmatrix} 0 & 6-5x \\ x^2 & x+3 \end{bmatrix}$

A is symmetric, then $A' = A$

$$\therefore \quad \begin{bmatrix} 0 & x^2 \\ 6-5x & x+3 \end{bmatrix} = \begin{bmatrix} 0 & 6-5x \\ x^2 & x+3 \end{bmatrix}$$

On comparing both sides, we get

$\Rightarrow \quad x^2 = 6 - 5x \Rightarrow x^2 + 5x - 6 = 0$

$\Rightarrow \quad (x + 6)(x - 1) = 0 \Rightarrow x = -6, 1$

53. Given, $A=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}$

(i) $A'=\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}$

$$A+A'=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}+\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}=\begin{bmatrix}2 & 11\\ 11 & 14\end{bmatrix}$$

$\therefore\ (A+A')'=\begin{bmatrix}2 & 11\\ 11 & 14\end{bmatrix}$

$\Rightarrow\ (A+A')'=A+A'$

$\therefore\ (A+A')$ is a symmetric matrix.

(ii) $A-A'=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}-\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

$$(A-A')'=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=-\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=-(A-A')$$

$\Rightarrow$ $(A - A')$ is a skew symmetric matrix.

Concept Applied

- A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $A' = -A$ *i.e.* $[a_{ij}] = -[a_{ji}]$ for all possible values of i and j

54. Given, A and B are symmetric matrices.

$\therefore\ A'=A$ and $B'=B$

Now, $(AB-BA)'=(AB)'-(BA)' \ =(B'A')-(A'B')$

$=(BA-AB) \qquad [\because A'=A \text{ and } B'=B]$

$=-(AB-BA)$

Thus, $(AB-BA)'=-(AB-BA)$

Hence, $(AB - BA)$ is a skew symmetric matrix.

55. Let $A = [a_{ij}]$ be a skew symmetric matrix.

Then, $a_{ji} = -a_{ij}\ \forall\ i, j$

$\Rightarrow\ a_{ii} = -a_{ii}\ \forall\ i \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0\ \forall\ i$

$\Rightarrow\ a_{11} = a_{22} = a_{33} = ... = a_{nn} = 0$

56. (c): We have, $A^2 - A + I = O$

Pre-multiplying with A^{-1} on both sides, we get

$(A^{-1}A)\cdot A - A^{-1}\cdot A + A^{-1}\cdot I = A^{-1}\cdot O$

$\Rightarrow\ I\cdot A - I + A^{-1} = O$

$\Rightarrow\ A^{-1} = -(A - I) = I - A$

57. $$A^2=\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}=\begin{bmatrix}9 & 7 & 5\\ 1 & 4 & 1\\ 8 & 9 & 9\end{bmatrix}$$

$$A^3=\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}9 & 7 & 5\\ 1 & 4 & 1\\ 8 & 9 & 9\end{bmatrix}=\begin{bmatrix}28 & 37 & 26\\ 10 & 5 & 1\\ 35 & 42 & 34\end{bmatrix}$$

Now, $$A^3-4A^2-3A+11I=\begin{bmatrix}28 & 37 & 26\\ 10 & 5 & 1\\ 35 & 42 & 34\end{bmatrix}-4\begin{bmatrix}9 & 7 & 5\\ 1 & 4 & 1\\ 8 & 9 & 9\end{bmatrix}-3\begin{bmatrix}1 & 3 & 2\\ 2 & 0 & -1\\ 1 & 2 & 3\end{bmatrix}+11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}28 & 37 & 26\\ 10 & 5 & 1\\ 35 & 42 & 34\end{bmatrix}-\begin{bmatrix}36 & 28 & 20\\ 4 & 16 & 4\\ 32 & 36 & 36\end{bmatrix}-\begin{bmatrix}3 & 9 & 6\\ 6 & 0 & -3\\ 3 & 6 & 9\end{bmatrix}+\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Hence, $A^3 - 4A^2 - 3A + 11I = O$

Now, $A^{-1}[A^3 - 4A^2 - 3A + 11I] = A^{-1}O$

$\Rightarrow\ A^2 - 4A - 3A^{-1}A + 11A^{-1}I = O \Rightarrow A^2 - 4A - 3I + 11A^{-1} = O$

$$\Rightarrow\ A^{-1}=\frac{-A^2+4A+3I}{11}$$

$$\Rightarrow A^{-1}=\frac{1}{11}\left(\begin{bmatrix}-9 & -7 & -5\\ -1 & -4 & -1\\ -8 & -9 & -9\end{bmatrix}+\begin{bmatrix}4 & 12 & 8\\ 8 & 0 & -4\\ 4 & 8 & 12\end{bmatrix}+\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}\right)$$

$$\Rightarrow A^{-1}=\frac{1}{11}\left(\begin{bmatrix}-2 & 5 & 3\\ 7 & -1 & -5\\ -4 & -1 & 6\end{bmatrix}\right)=\begin{bmatrix}-2/11 & 5/11 & 3/11\\ 7/11 & -1/11 & -5/11\\ -4/11 & -1/11 & 6/11\end{bmatrix}$$

CBSE Sample Questions

1. (c) : In a skew-symmetric matrix, the $(i, j)^{th}$ element is negative of the $(j, i)^{th}$ element. Hence, the $(i, i)^{th}$ element = 0. **(1)**

2. (a): From the definition of equality of two matrices, we have

$2a + b = 4$ ...(i) $\qquad a - 2b = -3$ ...(ii)

$5c - d = 11$ ...(iii) $\qquad 4c + 3d = 24$ ...(iv)

Solving (i) and (ii), we get

$5a = 5 \Rightarrow a = 1, b = 2$

Solving (iii) and (iv), we get

$19c = 57 \Rightarrow c = 3, d = 4$

$\therefore\ a + b - c + 2d = 1 + 2 - 3 + 8 = 8$ **(1)**

3. (b): We know that the sum of two matrices is defined only if both matrices have same order.

Here $5A + 3B$ is defined if A and B have same order.

$\Rightarrow\ 3 \times n = m \times 5 \Rightarrow n = 5, m = 3$

So, order of matrix C is 3×5 and $m \neq n$. **(1)**

4. (d): We have, $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

$\therefore\ A^2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0+1 & 0+0\\ 0+0 & 1+0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ **(1)**

5. (b): We have, $A=\begin{bmatrix}0 & 2\\ 3 & -4\end{bmatrix} \Rightarrow kA=\begin{bmatrix}0 & 2k\\ 3k & -4k\end{bmatrix}$

$\Rightarrow\ \begin{bmatrix}0 & 3a\\ 2b & 24\end{bmatrix}=\begin{bmatrix}0 & 2k\\ 3k & -4k\end{bmatrix}$ (Given)

$\Rightarrow\ -4k = 24,\ 3a = 2k,\ 2b = 3k$

$\Rightarrow\ k = -6,\ a = -4,\ b = -9$ **(1)**

6. **(d) :** We have, $(I + A)^3 - 7A$

$= I^3 + A^3 + 3I^2A + 3IA^2 - 7A = I + A \cdot A + 3A + 3A - 7A$ $(\because A^2 = A)$

$= I + A + 3A + 3A - 7A = I$ **(1)**

7. **(c) :** We have, $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} \alpha^2+\beta\gamma & 0 \\ 0 & \gamma\beta+\alpha^2 \end{bmatrix}$

But $A^2 = 3I$

$\Rightarrow \begin{bmatrix} \alpha^2+\beta\gamma & 0 \\ 0 & \alpha^2+\beta\gamma \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow \alpha^2 + \beta\gamma = 3$

$\Rightarrow 3 - \alpha^2 - \beta\gamma = 0$ **(1)**

8. For addition or subtraction of two matrices to be defined, the two matrices should be of same order.

$\therefore \quad 3 \times n = m \times 5 \Rightarrow m = 3$ and $n = 5$

So, order of matrix $(5A - 3B)$ is 3×5 and $m \neq n$. **(1)**

9. We know, $|\text{adj}A| = |A|^{n-1}$, where $n \times n$ is the order of non-singular matrix A.

$\therefore \quad |\text{adj } A| = (-4)^{3-1} = 16$ **(1)**

10. **(c) :** We know that if A and B are non-singular matries of same order, then

$(AB)^{-1} = B^{-1}A^{-1}$; $(AB^{-1})^{-1} = (B^{-1})^{-1}A^{-1} = BA^{-1}$ **(1)**

11. **(d) :** We have,

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I \Rightarrow B^{-1} = \frac{1}{6}A$$ **(1)**

12. We have, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

Also, $-5A = \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix}$ and $7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Now, $A^2 - 5A + 7I$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$ **(1)**

Now, $A^{-1}(A^2 - 5A + 7I) = A^{-1}O$

$\Rightarrow \quad A - 5I + 7A^{-1} = O$

$\Rightarrow \quad 7A^{-1} = 5I - A$

$\Rightarrow \quad A^{-1} = \frac{1}{7}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right) \Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ **(1)**

Self Assessment

Case Based Objective Questions (4 marks)

1. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices, then $A \pm B$ is of order $m \times n$ and is defined as

$(A \pm B)_{ij} = a_{ij} \pm b_{ij}$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$ are two matrices, then AB is of order $m \times p$ and is defined as

$$(AB)_{ik} = \sum_{r=1}^{n} a_{ir} b_{rk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}$$

Consider

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \text{ and } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Based on above information, attempt any 4 out of 5 subparts.

(i) Find the product AB.

(a) $\begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 \\ 22 & 43 \end{bmatrix}$

(c) $\begin{bmatrix} 43 & 22 \\ 0 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 22 & 43 \\ 3 & 0 \end{bmatrix}$

(ii) If A and B are any other two matrices such that AB exists, then

(a) BA does not exist

(b) BA will be equal to AB

(c) BA may or may not exist

(d) None of these

(iii) If $CD - AB = O$, then, find the value of $3c + a$.

(a) 30 (b) 40

(c) 50 (d) 60

(iv) If $CD - AB = O$, then find the value of $d - b$.

(a) 114 (b) 124

(c) 154 (d) 144

(v) Find $B + D$.

(a) $\begin{bmatrix} 80 & 200 \\ 115 & 105 \end{bmatrix}$ (b) $\begin{bmatrix} 84 & 48 \\ 180 & 181 \end{bmatrix}$

(c) $\begin{bmatrix} 186 & 108 \\ -84 & -48 \end{bmatrix}$ (d) $\begin{bmatrix} -186 & -108 \\ 84 & 48 \end{bmatrix}$

Multiple Choice Questions (1 mark)

2. If $AB = A$ and $BA = B$, then

(a) $B = I$ (b) $A = I$ (c) $A^2 = A$ (d) $B^2 = I$

3. A square matrix $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if $a_{ij} = 0$ for

(a) $i = j$ (b) $i < j$ (c) $i > j$ (d) $i \neq j$

OR

A square matrix $A = [a_{ij}]_{n \times n}$ is called a lower triangular matrix if $a_{ij} = 0$ for

(a) $i = j$ (b) $i < j$

(c) $i > j$ (d) None of these

4. If $A^2 - A + I = O$, then the inverse of A is

(a) $I - A$ (b) $A - I$ (c) A (d) $A + I$

5. If $A = \begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix}$ and $A = A^T$, then

(a) $x = 0, y = 5$ (b) $x + y = 5$

(c) $x = y$ (d) $x = 0, y = 9$

6. The order of the single matrix obtained from $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 3 \end{bmatrix} \left\{ \begin{bmatrix} -1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 23 \\ 1 & 0 & 21 \end{bmatrix} \right\}$ is

(a) 2×3 (b) 2×2 (c) 3×2 (d) 3×3

7. The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a

(a) unit matrix

(b) diagonal matrix

(c) symmetric matrix

(d) skew-symmetric matrix

VSA Type Questions (1 mark)

8. If A and B be two matrices of order 3×2 and 2×4 respectively, then write the order of matrix AB.

9. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, then find $A + A'$.

OR

If matrix $A = [1 \quad 2 \quad 3]$, then write AA'.

10. If $\begin{bmatrix} x & x-y \\ 2x+y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 7 \end{bmatrix}$, then find the value of y.

11. If $A^{-1} = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 6 & 9 \\ 0 & 0 & 5 \end{bmatrix}$, then find the value of $(A')^{-1}$.

12. A matrix $A = [a_{ij}]_{m \times n}$ is a null matrix iff $a_{ij} =$ ____ $\forall\, i, j$.

SA I Type Questions (2 marks)

13. Construct a 3×2 matrix $A = [a_{ij}]$, whose elements are given by $a_{ij} = e^{-ix} \cos\left(\frac{\pi}{2}i + jx\right)$.

14. Find the matrices A and B if

$$2A+3B=\begin{bmatrix}1 & -2 & 3\\ 2 & 0 & -1\end{bmatrix} \text{ and } A-2B=\begin{bmatrix}3 & 0 & 1\\ -1 & 6 & 2\end{bmatrix}.$$

OR

Find the set of all 2×2 matrices which is commutative with the matrix $\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$ with respect to matrix multiplication.

15. Show that $A'A$ and AA' are both symmetric matrices for any matrix A.

16. Show that the matrix $\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$ is the inverse of the matrix $\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$.

SA II Type Questions (3 marks)

17. If $\begin{bmatrix}x^2-4x & x^2\\ x^2 & x^3\end{bmatrix}=\begin{bmatrix}-3 & 1\\ -x+2 & 1\end{bmatrix}$, then find x.

18. Express $\begin{bmatrix}p & q\\ r & s\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

19. If A = diag[2 -1 3] and B = diag [3 0 -1], then find $4A + 2B$.

OR

Find A, if $\begin{bmatrix}4\\ 1\\ 3\end{bmatrix}A=\begin{bmatrix}-4 & 8 & 4\\ -1 & 2 & 1\\ -3 & 6 & 3\end{bmatrix}$.

20. If $A = [a_{ij}]$ is a matrix given by $\begin{bmatrix}4 & -2 & 1 & 3\\ 5 & 7 & 9 & 6\\ 21 & 15 & 18 & -25\end{bmatrix}$, then write the order of A. Also, show that $a_{32} = a_{23} + a_{24}$.

Case Based Questions (4 marks)

21. Consider 2 families A and B. Suppose there are 4 men, 4 women and 4 children in family A and 2 men, 2 women and 2 children in family B. The recommend daily amount of calories is 2400 for a man, 1900 for a woman, 1800 for a child and 45 grams of proteins for a man, 55 grams for a woman and 33 grams for a child.

Based on the given information, answer the following questions.

(i) Represent the requirement of calories and proteins for each person in matrix form.

(ii) What is the requirement of calories for family A?

LA Type Questions (4/6 marks)

22. Find x, y and z, if $A=\begin{bmatrix}0 & 2y & z\\ x & y & -z\\ x & -y & z\end{bmatrix}$ satisfies $A' = A^{-1}$.

23. If $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI_3 = O$, then find k.

OR

Three shopkeepers A, B and C went to a store to buy stationery. A purchased 12 dozen notebooks, 5 dozen pens and 6 dozen pencils. B purchased 10 dozen notebooks, 6 dozen pens and 7 dozens pencils. C purchased 11 dozen notebooks, 13 dozen pens and 8 dozen pencils. A notebook costs ₹ 40, a pen costs ₹ 8.50 and a pencil costs ₹ 3.50. Use matrix multiplication to calculate each individuals bill.

24. If $A=\begin{bmatrix}3 & \sqrt{3} & 2\\ 4 & 2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 2\\ 1 & 2 & 4\end{bmatrix}$, then verify that (i) $(A')' = A$ (ii) $(A + B)' = A' + B'$

(iii) $(kB)' = kB'$, where k is any constant.

25. If $\begin{bmatrix}x+3 & z+4 & 2y-7\\ 4x+6 & a-1 & 0\\ b-3 & 3b & z+2c\end{bmatrix}=\begin{bmatrix}0 & 6 & 3y-2\\ 2x & -3 & 2c+2\\ 2b+4 & -21 & 0\end{bmatrix}$, then find the values of a, b, c, x, y and z.

Detailed SOLUTIONS

1. **(i) (a) :** $AB=\begin{bmatrix}2 & -1\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 2\\7 & 4\end{bmatrix}$

$=\begin{bmatrix}10-7 & 4-4\\15+28 & 6+16\end{bmatrix}=\begin{bmatrix}3 & 0\\43 & 22\end{bmatrix}$

(ii) **(c)**

(iii) **(b) :** We have, $CD-AB=O$

$\Rightarrow \begin{bmatrix}2 & 5\\3 & 8\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix}-\begin{bmatrix}3 & 0\\43 & 22\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2a+5c & 2b+5d\\3a+8c & 3b+8d\end{bmatrix}-\begin{bmatrix}3 & 0\\43 & 22\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2a+5c-3 & 2b+5d\\3a+8c-43 & 3b+8d-22\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

By equality of matrices, we get $2a+5c-3=0$ ...(I)

$3a+8c-43=0$ ...(II)

$2b+5d=0$ ...(III)

$3b+8d-22=0$ ...(IV)

Solving (I) and (II), we get $a=-191$, $c=77$

$\therefore\ 3c+a=3\times 77-191=231-191=40$

(iv) **(c) :** Solving (III) and (IV), we get $b=-110$, $d=44$

$\therefore\ d-b=44-(-110)=44+110=154$

(v) **(d) :** We have, $B+D=\begin{bmatrix}5 & 2\\7 & 4\end{bmatrix}+\begin{bmatrix}-191 & -110\\77 & 44\end{bmatrix}$

$=\begin{bmatrix}-186 & -108\\84 & 48\end{bmatrix}$

2. **(c) :** $A=AB=A(BA)=(AB)A=A\cdot A=A^2$

[$\because$ Given, $AB=A$ and $BA=B$]

$B=BA=B(AB)=(BA)B=B\cdot B=B^2$

3. **(d) :** $A=[a_{ij}]_{n\times n}$ is a diagonal matrix iff all non-diagonal entries are 0 *i.e.*, $a_{ij}=0$ for $i\neq j$.

OR

(b) : $A=[a_{ij}]_{n\times n}$ is a lower triangular matrix iff all entries above the diagonal non-singular vanish, *i.e*, $a_{ij}=0$ for $i<j$

4. **(a) :** If A is any non-singular square matrix, then $AA^{-1}=I$ and $A^{-1}I=A^{-1}$

Since, $A^2-A+I=O$

$\Rightarrow A^{-1}A^2-A^{-1}A+A^{-1}I=O\Rightarrow(A^{-1}A)A-(A^{-1}A)+A^{-1}=O$

$\Rightarrow IA-I+A^{-1}=O \Rightarrow A-I+A^{-1}=O\Rightarrow A^{-1}=I-A$

5. **(c) :** We have, $A=\begin{bmatrix}5 & x\\y & 0\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}5 & y\\x & 0\end{bmatrix}$

Now, $A=A^T\Rightarrow \begin{bmatrix}5 & x\\y & 0\end{bmatrix}=\begin{bmatrix}5 & y\\x & 0\end{bmatrix} \Rightarrow x=y$

6. **(d) :** When a 3×2 matrix is post multiplied by a 2×3 matrix, the product is a 3×3 matrix.

7. **(c) :** $A'=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=A$

Hence, A is a symmetric matrix.

8. Order of A is 3×2 and order of B is 2×4.

$\therefore$ Order of AB is 3×4.

9. We have, $A=\begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}\Rightarrow A'=\begin{bmatrix}1 & 3\\2 & 5\end{bmatrix}$

$\therefore\ A+A'=\begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}+\begin{bmatrix}1 & 3\\2 & 5\end{bmatrix}=\begin{bmatrix}2 & 5\\5 & 10\end{bmatrix}$

OR

We have, $A=[1\quad 2\quad 3]$

$AA'=[1\quad 2\quad 3][1\quad 2\quad 3]'=[1\quad 2\quad 3]\begin{bmatrix}1\\2\\3\end{bmatrix}=[1+4+9]=[14]$

10. By definition of equal matrices, we have $x=3$

$\therefore\ x-y=1\Rightarrow 3-y=1\Rightarrow y=2$

11. We have, $A^{-1}=\begin{bmatrix}5 & 4 & 3\\2 & 6 & 9\\0 & 0 & 5\end{bmatrix}$

Now, $(A')^{-1}=(A^{-1})'$

$=\begin{bmatrix}5 & 4 & 3\\2 & 6 & 9\\0 & 0 & 5\end{bmatrix}'=\begin{bmatrix}5 & 2 & 0\\4 & 6 & 0\\3 & 9 & 5\end{bmatrix}.$

12. A matrix $A=[a_{ij}]_{m\times n}$ is a null matrix iff each of its element is zero *i.e.*, $a_{ij}=0\ \forall\ i,j$

13. We have, $A=[a_{ij}]_{3\times 2}$, where $a_{ij}=e^{-ix}\cos\left(\frac{\pi}{2}i+jx\right)$

$\therefore\ a_{11}=e^{-x}\cos\left(\frac{\pi}{2}+x\right)=-e^{-x}\sin x,$

$a_{12}=e^{-x}\cos\left(\frac{\pi}{2}+2x\right)=-e^{-x}\sin 2x,$

$a_{21}=e^{-2x}\cos(\pi+x)=-e^{-2x}\cos x,$

$a_{22}=e^{-2x}\cos(\pi+2x)=-e^{-2x}\cos 2x,$

$a_{31}=e^{-3x}\cos\left(\frac{3\pi}{2}+x\right)=e^{-3x}\sin x,$

$a_{32}=e^{-3x}\cos\left(\frac{3\pi}{2}+2x\right)=e^{-3x}\sin 2x$

$\therefore\ A=\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\\a_{31} & a_{32}\end{bmatrix}=\begin{bmatrix}-e^{-x}\sin x & -e^{-x}\sin 2x\\-e^{-2x}\cos x & -e^{-2x}\cos 2x\\e^{-3x}\sin x & e^{-3x}\sin 2x\end{bmatrix}$

14. Given, $2A+3B=\begin{bmatrix}1 & -2 & 3\\2 & 0 & -1\end{bmatrix}$ ... (i)

and $A-2B=\begin{bmatrix}3 & 0 & 1\\-1 & 6 & 2\end{bmatrix}$ ... (ii)

Multiplying (ii) by 2 and subtracting it from (i), we get

$$2A + 3B - 2(A - 2B) = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix} - 2\begin{bmatrix} 3 & 0 & 1 \\ -1 & 6 & 2 \end{bmatrix}$$

$$\Rightarrow 7B = \begin{bmatrix} 1-6 & -2-0 & 3-2 \\ 2+2 & 0-12 & -1-4 \end{bmatrix} \Rightarrow B = \frac{1}{7}\begin{bmatrix} -5 & -2 & 1 \\ 4 & -12 & -5 \end{bmatrix}$$

Multiplying (i) by 2, (ii) by 3 and adding, we get

$$2(2A+3B)+3(A-2B) = 2\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 3 & 0 & 1 \\ -1 & 6 & 2 \end{bmatrix}$$

$$\Rightarrow 7A = \begin{bmatrix} 2+9 & -4+0 & 6+3 \\ 4-3 & 0+18 & -2+6 \end{bmatrix} \Rightarrow A = \frac{1}{7}\begin{bmatrix} 11 & -4 & 9 \\ 1 & 18 & 4 \end{bmatrix}$$

OR

Let $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ be a matrix, which is commutative with matrix $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Then, $AB = BA$

$$\Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+y & x \\ z+t & z \end{bmatrix} = \begin{bmatrix} x+z & y+t \\ x & y \end{bmatrix}$$

Here, both matrices are equal. So, we equate the corresponding elements.

$\therefore \quad x + y = x + z \Rightarrow y = z$

$z + t = x \Rightarrow t = x - z = x - y$

$\therefore$ Required matrix is $\begin{bmatrix} x & y \\ y & x-y \end{bmatrix}$.

15. Consider, $(A'A)' = A'(A')'$ $\quad [\because (AB)' = B'A']$

$= A'A$ $\quad [\because (A')' = A]$

Hence, $A'A$ is symmetric matrix for any matrix A.

Now, $(AA')' = (A')'A' = AA'$ $\quad [\because (AB)' = B'A']$

Hence, AA' is also symmetric matrix for any matrix A.

16. Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$B = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $AB = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & 0 \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

17. We have, $\begin{bmatrix} x^2 - 4x & x^2 \\ x^2 & x^3 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -x+2 & 1 \end{bmatrix}$

Equating the corresponding elements of two matrices, we get

$x^2 - 4x = -3 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 1)(x - 3) = 0$

$\Rightarrow \quad x = 1, 3$

$x^2 = 1 \Rightarrow x = \pm 1$

$x^2 = -x + 2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x - 1)(x + 2) = 0 \Rightarrow x = 1, -2$

$x^3 = 1 \Rightarrow x^3 - 1 = 0 \Rightarrow (x - 1)(x^2 + x + 1) = 0 \Rightarrow x = 1, \omega, \omega^2$,

where, $\omega = \frac{-1+\sqrt{3}i}{2}, \omega^2 = \frac{-1-\sqrt{3}i}{2}$

Since, common value of x is 1.

$\therefore \quad x = 1$

18. We have, $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow A' = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$

$$\therefore \quad A + A' = \begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} p & r \\ q & s \end{bmatrix} = \begin{bmatrix} 2p & q+r \\ q+r & 2s \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 2p & q+r \\ q+r & 2s \end{bmatrix} = \begin{bmatrix} p & \frac{q+r}{2} \\ \frac{q+r}{2} & s \end{bmatrix}$$

$$\text{Also, } A - A' = \begin{bmatrix} p & q \\ r & s \end{bmatrix} - \begin{bmatrix} p & r \\ q & s \end{bmatrix} = \begin{bmatrix} 0 & q-r \\ r-q & 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & q-r \\ r-q & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{q-r}{2} \\ \frac{r-q}{2} & 0 \end{bmatrix}$$

$$\therefore \quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$= \begin{bmatrix} p & \frac{q+r}{2} \\ \frac{q+r}{2} & s \end{bmatrix} + \begin{bmatrix} 0 & \frac{q-r}{2} \\ \frac{r-q}{2} & 0 \end{bmatrix}$$

19. We have, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\text{Now, } 4A + 2B = 4\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 12 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$\therefore \quad 4A + 2B = \text{diag}\,[14 \ \ -4 \ \ 10]$

OR

Let $A = [x \ \ y \ \ z]$

Given, $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}_{3\times1} A = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}_{3\times3}$

$$\therefore \begin{bmatrix}4\\1\\3\end{bmatrix}_{3\times1}\begin{bmatrix}x & y & z\end{bmatrix}_{1\times3}=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}_{3\times3}$$

$$\Rightarrow \begin{bmatrix}4x & 4y & 4z\\x & y & z\\3x & 3y & 3z\end{bmatrix}=\begin{bmatrix}-4 & 8 & 4\\-1 & 2 & 1\\-3 & 6 & 3\end{bmatrix}$$

On comparing the corresponding elements of two matrices, we get

$4x = -4 \quad \Rightarrow x = -1;\ 4y = 8 \quad \Rightarrow y = 2$

$4z = 4 \quad \Rightarrow z = 1$

Hence, $A = [-1\ \ 2\ \ 1]$

20. We observe that there are 3 rows and 4 columns in matrix A.

$\therefore$ It is of order 3×4.

Here, $a_{32} = 15, a_{23} = 9, a_{24} = 6$

$\therefore \quad a_{23} + a_{24} = 9 + 6 = 15 = a_{32}$, which is true.

21. (i) Let F be the matrix representing the number of family members. Let R be the matrix representing the requirement of calories and proteins for each person. Then

	Men	Women	Children
$F =$ Family A	4	4	4
Family B	2	2	2

	Calories	Proteins
$R =$ Man	2400	45
Woman	1900	55
Child	1800	33

(ii) The requirement of calories and proteins for each of the two families is given by the product matrix FR.

$$FR=\begin{bmatrix}4 & 4 & 4\\2 & 2 & 2\end{bmatrix}\begin{bmatrix}2400 & 45\\1900 & 55\\1800 & 33\end{bmatrix}$$

$$=\begin{bmatrix}4(2400+1900+1800) & 4(45+55+33)\\2(2400+1900+1800) & 2(45+55+33)\end{bmatrix}$$

	Calories	Proteins	
$FR=$	24400	532	Family A
	12200	266	Family B

Hence, the calories required for family A is 24400.

22. We have given,

$$A=\begin{bmatrix}0 & 2y & z\\x & y & -z\\x & -y & z\end{bmatrix} \text{and } A'=\begin{bmatrix}0 & x & x\\2y & y & -y\\z & -z & z\end{bmatrix}$$

Since, $A' = A^{-1}$

$\therefore \quad AA' = AA^{-1}$ [Multiplying by A on both sides]

$\Rightarrow \quad AA' = I$

$$\Rightarrow \begin{bmatrix}0 & 2y & z\\x & y & -z\\x & -y & z\end{bmatrix}\begin{bmatrix}0 & x & x\\2y & y & -y\\z & -z & z\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}4y^2+z^2 & 2y^2-z^2 & -2y^2+z^2\\2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2\\-2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

By equality of two matrices, we get

$\Rightarrow \quad 2y^2 - z^2 = 0 \Rightarrow 2y^2 = z^2$ ...(i)

and $4y^2 + z^2 = 1$

$\Rightarrow \quad 2 \cdot z^2 + z^2 = 1$ [Using (i)]

$$\Rightarrow \quad z=\pm\frac{1}{\sqrt{3}} \quad \because\ y^2=\frac{z^2}{2}\Rightarrow y=\pm\frac{1}{\sqrt{6}}$$

Also, $x^2 + y^2 + z^2 = 1$

$$\Rightarrow x^2=1-y^2-z^2=1-\frac{1}{6}-\frac{1}{3}=\frac{1}{2}$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}} \quad \therefore x=\pm\frac{1}{\sqrt{2}}, y=\pm\frac{1}{\sqrt{6}} \text{ and } z=\pm\frac{1}{\sqrt{3}}$$

23. Given, $A=\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix}$

$$A^2=\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+0+4 & 0+0+0 & 2+0+6\\0+0+2 & 0+4+0 & 0+2+3\\2+0+6 & 0+0+0 & 4+0+9\end{bmatrix}=\begin{bmatrix}5 & 0 & 8\\2 & 4 & 5\\8 & 0 & 13\end{bmatrix}$$

$$A^3=A\cdot A^2=\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix}\begin{bmatrix}5 & 0 & 8\\2 & 4 & 5\\8 & 0 & 13\end{bmatrix}$$

$$=\begin{bmatrix}5+0+16 & 0+0+0 & 8+0+26\\0+4+8 & 0+8+0 & 0+10+13\\10+0+24 & 0+0+0 & 16+0+39\end{bmatrix}=\begin{bmatrix}21 & 0 & 34\\12 & 8 & 23\\34 & 0 & 55\end{bmatrix}$$

Since, $A^3 - 6A^2 + 7A + kI_3 = O$

$$\Rightarrow \begin{bmatrix}21 & 0 & 34\\12 & 8 & 23\\34 & 0 & 55\end{bmatrix}-6\begin{bmatrix}5 & 0 & 8\\2 & 4 & 5\\8 & 0 & 13\end{bmatrix}+7\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 0 & 3\end{bmatrix}+k\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}21 & 0 & 34\\12 & 8 & 23\\34 & 0 & 55\end{bmatrix}-\begin{bmatrix}30 & 0 & 48\\12 & 24 & 30\\48 & 0 & 78\end{bmatrix}+\begin{bmatrix}7 & 0 & 14\\0 & 14 & 7\\14 & 0 & 21\end{bmatrix}+\begin{bmatrix}k & 0 & 0\\0 & k & 0\\0 & 0 & k\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-2+k & 0 & 0\\0 & -2+k & 0\\0 & 0 & -2+k\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

On equating the corresponding elements, we get

$-2 + k = 0 \Rightarrow k = 2$

OR

A purchased 144 notebooks, 60 pens and 72 pencils, *B* purchased 120 notebooks, 72 pens and 84 pencils, *C* purchased 132 notebooks, 156 pens and 96 pencils.

Let $D=\begin{bmatrix}144 & 60 & 72\\120 & 72 & 84\\132 & 156 & 96\end{bmatrix}$ be the matrix of purchased stationary.

Let *E* be the price matrix.

$$\therefore \quad E=\begin{bmatrix}40\\8.50\\3.50\end{bmatrix}$$

$$\therefore\ DE=\begin{bmatrix}144 & 60 & 72\\120 & 72 & 84\\132 & 156 & 96\end{bmatrix}\begin{bmatrix}40\\8.50\\3.50\end{bmatrix}$$

$$=\begin{bmatrix}5760+510+252\\4800+612+294\\5280+1326+336\end{bmatrix}=\begin{bmatrix}6522\\5706\\6942\end{bmatrix}$$

∴ Bill of *A*, *B* and *C* are ₹ 6522, ₹ 5706 and ₹ 6942 respectively.

24. We have, $A=\begin{bmatrix}3 & \sqrt{3} & 2\\4 & 2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 2\\1 & 2 & 4\end{bmatrix}$

$$\therefore \quad A'=\begin{bmatrix}3 & 4\\\sqrt{3} & 2\\2 & 0\end{bmatrix} \text{ and } B'=\begin{bmatrix}2 & 1\\-1 & 2\\2 & 4\end{bmatrix}$$

(i) $$(A')'=\begin{bmatrix}3 & 4\\\sqrt{3} & 2\\2 & 0\end{bmatrix}'=\begin{bmatrix}3 & \sqrt{3} & 2\\4 & 2 & 0\end{bmatrix}=A$$

(ii) $$A+B=\begin{bmatrix}3 & \sqrt{3} & 2\\4 & 2 & 0\end{bmatrix}+\begin{bmatrix}2 & -1 & 2\\1 & 2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}3+2 & \sqrt{3}-1 & 2+2\\4+1 & 2+2 & 0+4\end{bmatrix}=\begin{bmatrix}5 & \sqrt{3}-1 & 4\\5 & 4 & 4\end{bmatrix}$$

$$\therefore \quad (A+B)'=\begin{bmatrix}5 & \sqrt{3}-1 & 4\\5 & 4 & 4\end{bmatrix}'=\begin{bmatrix}5 & 5\\\sqrt{3}-1 & 4\\4 & 4\end{bmatrix}$$

$$\text{Also, } A'+B'=\begin{bmatrix}3 & 4\\\sqrt{3} & 2\\2 & 0\end{bmatrix}+\begin{bmatrix}2 & 1\\-1 & 2\\2 & 4\end{bmatrix}=\begin{bmatrix}5 & 5\\\sqrt{3}-1 & 4\\4 & 4\end{bmatrix}$$

Thus, $(A + B)' = A' + B'$

(iii) $$kB=k\begin{bmatrix}2 & -1 & 2\\1 & 2 & 4\end{bmatrix}=\begin{bmatrix}2k & -k & 2k\\k & 2k & 4k\end{bmatrix}$$

$$\text{and } (kB)'=\begin{bmatrix}2k & -k & 2k\\k & 2k & 4k\end{bmatrix}'=\begin{bmatrix}2k & k\\-k & 2k\\2k & 4k\end{bmatrix}=k\begin{bmatrix}2 & 1\\-1 & 2\\2 & 4\end{bmatrix}=kB'$$

Thus $(kB)' = kB'$.

25. We have,

$$\begin{bmatrix}x+3 & z+4 & 2y-7\\4x+6 & a-1 & 0\\b-3 & 3b & z+2c\end{bmatrix}=\begin{bmatrix}0 & 6 & 3y-2\\2x & -3 & 2c+2\\2b+4 & -21 & 0\end{bmatrix}$$

Equating the corresponding elements of two matrices, we get

$x + 3 = 0 \quad \Rightarrow x = -3;\ z + 4 = 6 \quad \Rightarrow z = 2$

$2y - 7 = 3y - 2 \quad \Rightarrow y = -5;\ a - 1 = -3 \quad \Rightarrow a = -2$

$b - 3 = 2b + 4 \quad \Rightarrow b = -7$

$2c + 2 = 0 \quad \Rightarrow c = -1$

Thus, $a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$.

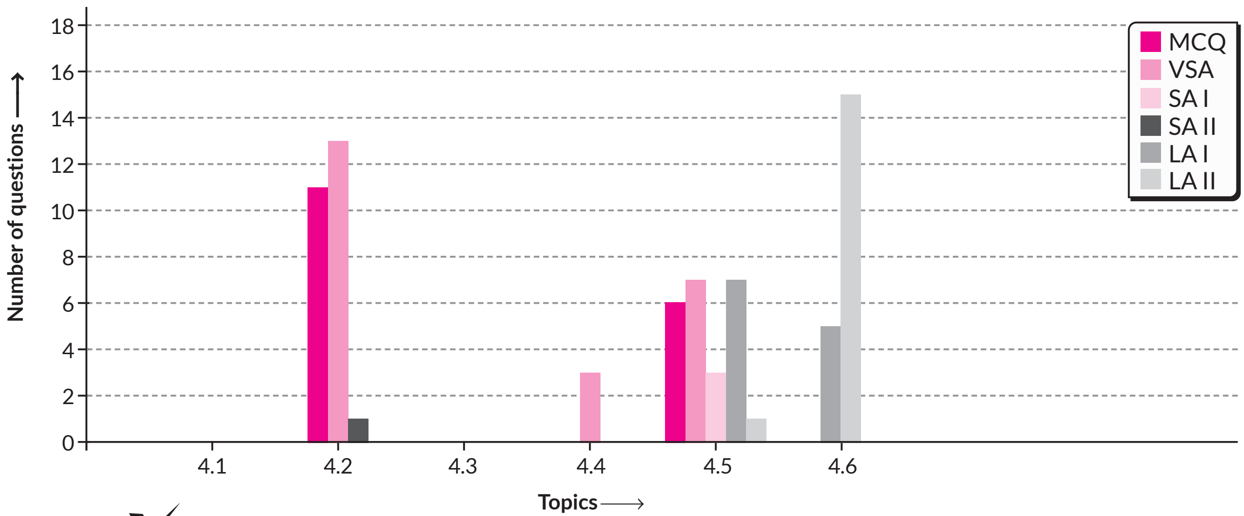
CHAPTER 4

Determinants

TOPICS

4.1 Introduction
4.2 Determinant
4.3 Area of a Triangle
4.4 Minors and Cofactors
4.5 Adjoint and Inverse of a Matrix
4.6 Applications of Determinants and Matrices

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 4.5 is highly scoring topic.
- Maximum weightage is of Topic 4.5 *Adjoint and Inverse of a Matrix.*
- Maximum MCQ and VSA type questions were asked from Topic 4.2 *Determinant.*
- Maximum LA II type questions were asked from Topic 4.6 *Applications of Determinants and Matrices.*

QUICK RECAP

Determinant

- If $A = [a_{ij}]$ is a square matrix of order n, then a number (real or complex) associated to matrix A is called determinant of A. It is denoted by det A or $|A|$ or Δ.
  - If $A = [a]$ be a matrix of order 1, then det $(A) = a$.
  - If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2, then $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$
  - If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a matrix of order 3,

then $|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$+(-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+(-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

Note : $|KA| = K^n|A|$, where A is of order n.

Area of a Triangle

- The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$$

Note :

(i) Since area is always a positive quantity, therefore we always take the absolute value of the determinant for the area.

(ii) Area of a triangle formed by three collinear points is always zero.

Minor of an element

- Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column. Minor of a_{ij} is denoted by M_{ij}.
  - Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$.

Cofactor of an element

- Cofactor of an element a_{ij} of determinant of matrix A is, $A_{ij} = (-1)^{i+j} M_{ij}$.
  - The determinant of a matrix A can also be obtained by using cofactors *i.e.*, sum of product of elements of a row (or column) with their corresponding cofactors.

    $\therefore \quad |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
  - If the elements of one row (or column) are multiplied with the cofactors of elements of any other row (or column), then their sum is zero *i.e.*, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$

Adjoint of a matrix

- The adjoint of a square matrix is the transpose of the matrix of cofactors.
  - Adjoint of A is denoted by adj$A$.

  Remark : For a matrix A of order n,
  $A(\text{adj}A) = (\text{adj}A)A = |A|I_n$

Singular and Non-Singular matrix

- Let A be a square matrix, then A is called
  - Singular matrix, iff $|A| = 0$
  - Non-singular matrix, iff $|A| \neq 0$

  Note : If A and B are non-singular matrices of same order, then AB and BA are also non-singular matrices of same order.

Inverse of a matrix

- A square matrix A is invertible iff A is non-singular matrix and

$$A^{-1}=\frac{1}{|A|}(\text{adj}A)$$

Solution of a System of Linear Equations

- For a square matrix A, a system of equations $AX = B$ is said to be

  (i) Consistent, if it has one or more solutions.

  (ii) Inconsistent, if its solution doesn't exist.

Conditions for Consistency

- For a system of equations, $AX = B$.

  (i) If $|A| \neq 0$, then the given system of equations is consistent and has a unique solution.

  (ii) If $|A| = 0$ and $(\text{adj}A)B \neq O$, then the solution does not exist and the given system is inconsistent.

  (iii) If $|A| = 0$ and $(\text{adj}A)B = O$, then the given system may be either consistent or inconsistent, according as the system have either infinitely many solutions or no solution.

BRAIN MAP

DETERMINANTS

Corresponding to every sqa re matrix A, there exists a number called the determinant of A and denoted by $|A|$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then,

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Minors and Cofactors

- For any matrix $A = [a_{ij}]_{n \times n}$, if we leave the row and the column of the element a_{ij}, then the value of determinant thus obtained is called the minor of a_{ij} and it is denoted by M_{ij}.
- The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij} and denoted by A_{ij}.

$\therefore \quad A_{ij} = (-1)^{i+j} M_{ij}$

Solution of System of Linear Equations

Let $AX = B$ be the given system of eqa tions :

- If $|A| \neq 0$, the system is consistent and has a uniqe solution.
- If $|A| = 0$ and $(\text{adj } A)B \neq O$, then the system is inconsistent and hence it has no solution.
- If $|A| = 0$ and $(\text{adj } A)B = O$, then the system may be either consistent or inconsistent according as the system has either infinitely many solutions or no solution.

Inv rse of a Matrix

For any sqa re matrix A, inverse of A is defined as

$$A^{-1} = \frac{1}{|A|}(\text{adj } A), |A| \neq 0$$

Properties

- $(A^{-1})^{-1} = A$
- $(A')^{-1} = (A^{-1})'$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Singular and Non-Singular Matrices

A square matrix A of order n is said to be

- Singular if $|A| = 0$
- Non-singular if $|A| \neq 0$

Note : If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

Area of a Triangle

Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then area of ΔABC is $\Delta = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$.

Adjoint of a Matrix

Let $B = [A_{ij}]$ be the matrix of cofactors of matrix $A = [a_{ij}]$. Then the transpose of B is called the adjoint of matrix A.

Points to be Remember on Area of Triangle

- Area is a positive qa ntity, we always tak the absolute value of the determinant.
- If area is given, use both positive and negative values of the determinant for calculation.
- The area of the triangle formed by three collinear points is zero.

Properties

If A is non-singular matrix of order n, then

- $A(\text{adj } A) = (\text{adj } A)\, A = |A|\, I_n$
- $|A \text{ adj } A| = |A|^n$
- $\text{adj } (AB) = (\text{adj } B) \,.\, (\text{adj } A)$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj } (\text{adj } A) = |A|^{n-2} A$
- $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$

Previous Years' CBSE Board Questions

4.2 Determinant

MCQ

1. The value of the determinant $\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$ is
(a) 47 (b) –79 (c) 49 (d) –51 *(2023)*

2. If $\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0$, then the value of α is
(a) 1 (b) 2 (c) 3 (d) 4 *(2023)*

3. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 27$, then the value of α is
(a) ± 1 (b) ± 2 (c) $\pm\sqrt{5}$ (d) $\pm\sqrt{7}$ *(Term I, 2021-22)* U

4. If $\begin{vmatrix} 5 & 3 & -1 \\ -7 & x & -3 \\ 9 & 6 & -2 \end{vmatrix} = 0$, then the value of x is
(a) 3 (b) 5 (c) 7 (d) 9 *(Term I, 2021-22)* R

5. The determinant $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$ is equal to
(a) $k(3y + k^2)$ (b) $3y + k^3$
(c) $3y + k^2$ (d) $k^2(3y + k)$ *(Term I, 2021-22)*

6. The value of $\begin{vmatrix} \lfloor 1 & \lfloor 2 & \lfloor 3 \\ 2\lfloor 2 & 3\lfloor 3 & 4\lfloor 4 \\ \lfloor 3 & \lfloor 4 & \lfloor 5 \end{vmatrix}$ is
(a) 12 (b) –12 (c) 24 (d) –24 *(Term I, 2021-22)*

7. If A is a non-singular square matrix of order 3 such that $A^2 = 3A$, then value of $|A|$ is
(a) – 3 (b) 3
(b) 9 (d) 27 *(2020)*

8. The roots of the equation $\begin{vmatrix} x & 0 & 8 \\ 4 & 1 & 3 \\ 2 & 0 & x \end{vmatrix} = 0$ are
(a) – 4, 4 (b) 2, – 4 (c) 2, 4 (d) 2, 8 *(2020 C)*

9. If A is a square matrix of order 3 and $|A| = 5$, then the value of $|2A'|$ is
(a) –10 (b) 10
(c) –40 (d) 40 *(2020)* U

10. If A is a skew-symmetric matrix of order 3, then the value of $|A|$ is
(a) 3 (b) 0
(c) 9 (d) 27 *(2020)*

11. If A is a 3×3 matrix such that $|A| = 8$, then $|3A|$ equals
(a) 8 (b) 24
(c) 72 (d) 216 *(2020)* U

VSA (1 mark)

12. If $A = \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 17 \\ 0 & -10 \end{bmatrix}$, then $|AB|$ = ______ . *(2020 C)*

13. If A and B are square matrices each of order 3 and $|A| = 5$, $|B| = 3$, then the value of $|3AB|$ is ______. *(2020)*

14. If A and B are square matrices of the same order 3, such that $|A| = 2$ and $AB = 2I$, write the value of $|B|$. *(Delhi 2019)* An

15. Find the maximum value of
$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$. *(Delhi 2016)*

16. If $x \in N$ and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x. *(AI 2016)*

17. If $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$, write the value of x. *(Foreign 2016)* An

18. Write the value of $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$. *(AI 2015)*

19. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$, write the value of $|AB|$. *(Delhi 2015C)*

20. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x. *(Delhi 2014)* U

21. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x. *(AI 2014)*

22. If A is a 3×3 matrix, $|A| \neq 0$ and $|3A| = k|A|$, then write the value of k. *(Foreign 2014)*

23. Write the value of the determinant $\begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix}$. *(Delhi 2014C)*

24. Write the value of $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$. *(AI 2014C)* Ap

SA II (3 marks)

25. Show that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ. *(2023)*

4.4 Minors and Cofactors

VSA (1 mark)

26. Find the cofactors of all the elements of $\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$. *(2020)* Ap

27. Find the cofactor of the element a_{23} of the determinant $\begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. *(2019C)*

28. If $A = \begin{bmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{bmatrix}$, then write the cofactor of the element a_{21} of its 2^{nd} row. *(Foreign 2015)*

4.5 Adjoint and Inverse of a Matrix

MCQ

29. The inverse of $\begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$ is

(a) $\begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & 7 \\ 3 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & -3 \\ -7 & -4 \end{bmatrix}$

(Term I, 2021-22) Ap

30. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 59 & 69 & -1 \end{bmatrix}$, then A^{-1}

(a) is A (b) is $(-A)$

(c) is A^2 (d) does not exist

(Term I, 2021-22)

31. If $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}$ is the adjoint of a square matrix B, then B^{-1} is equal to

(a) $\pm A$ (b) $\pm\sqrt{2}A$ (c) $\pm\frac{1}{\sqrt{2}}B$ (d) $\pm\frac{1}{\sqrt{2}}A$

(Term I, 2021-22) Cr

32. If $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then

(a) $a = 1 = b$ (b) $a = \cos 2\theta, b = \sin 2\theta$

(c) $a = \sin 2\theta, b = \cos\theta$ (d) $a = \cos\theta, b = \sin\theta$

(Term I, 2021-22)

33. If $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then the value of $|\text{adj } A|$ is

(a) 64 (b) 16 (c) 0 (d) −8

(2020)

34. If A is a square matrix of order 3, such that $A(\text{adj } A) = 10I$, then $|\text{adj } A|$ is equal to

(a) 1 (b) 10 (c) 100 (d) 101

(2020)

VSA (1 mark)

35. If A is a square matrix of order 3 such that $A(\text{adj } A) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then find $|A|$. *(2021 C)*

36. If $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$, then find $A(\text{adj } A)$. *(2020)* Ap

37. If A is a square matrix of order 3 with $|A| = 9$, then write the value of $|2 \cdot \text{adj } A|$. *(AI 2019)*

38. If A is a 3×3 invertible matrix, then what will be the value of k if $\det(A^{-1}) = (\det A)^k$? *(Delhi 2017)* R

39. If for any 2×2 square matrix A, $A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of $|A|$. *(AI 2017)*

40. In the interval $\pi/2 < x < \pi$, find the value of x for which the matrix $\begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$ is singular. *(AI 2015C)*

41. Find $(\text{adj } A)$, if $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$. *(Delhi 2014C)* Ev

SA I (2 marks)

42. For the matrix $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$, verify the following :

$A(\text{adj } A) = (\text{adj } A)A = |A|I$ *(2020C)*

43. Find $(AB)^{-1}$ if $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$. *(2020)* Ap

44. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute I^{-1} and show that $2A^{-1} = 9I - A$. *(2018)* Cr

LA I (4 marks)

45. Gautam buys 5 pens, 3 bags and 1 instrument box and pays a sum of ₹160. From the same shop Vikram buys 2 pens, 1 bag and 3 instrument boxes and pays a sum of ₹190. Also Ankur buys 1 pen, 2 bags and 4 instrument boxes and pays a sum of ₹250.

(i) Convert the given above situation into a matrix equation of the form $AX = B$.
(ii) Find $|A|$.
(iii) Find A^{-1}.

OR

Determine $P = A^2 - 5A$. *(2023)*

46. If $A^{-1}=\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$, find $(AB)^{-1}$. *(2021 C)*

47. If $A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$, find $(A')^{-1}$. *(Delhi 2015)*

48. Find the adjoint of the matrix $A = \begin{bmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$ and hence show that $A\cdot(\text{adj } A) = |A|I_3$. *(AI 2015)* Cr

49. If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ and I is the identity matrix of order 2, then show that $A^2 = 4A - 3I$. Hence find A^{-1}. *(Foreign 2015)*

50. Find the inverse of the matrix $A = \begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$. *(Delhi 2015C)* Ap

51. If $A=\begin{bmatrix}2 & 3\\ 1 & -4\end{bmatrix}$ and $B=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$. *(AI 2015C)*

LA II (5/6 marks)

52. If $A=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$, find adj A and verify that $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_3$. *(Foreign 2016)* Cr

4.6 Applications of Determinants and Matrices

LA I (4 marks)

53. If $\begin{bmatrix}3 & 4 & 2\\ 0 & 2 & -3\\ 1 & -2 & 6\end{bmatrix}$, find A^{-1}.
Hence, solve the following system of equations :
$3x + 4y + 2z = 8$
$2y - 3z = 3$
$x - 2y + 6z = -2$ *(2021 C)*

54. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value? *(Delhi 2016)* Cr

55. A trust invested some money in two type of bonds. The first bond pays 10% interest and second bond pays 12% interest. The trust received ₹ 2,800 as interest. However, if trust had interchanged money in bonds they would have got ₹ 100 less as interest. Using matrix method, find the amount invested by the trust. Interest received on this amount will be given to Helpage India as donation. Which value is reflected in this question? *(AI 2016)*

56. A coaching institute of English (Subject) conducts classes in two batches I and II and fees for rich and poor children are different. In batch I, it has 20 poor and 5 rich children and total monthly collection is ₹ 9,000, whereas in batch II, it has 5 poor and 25 rich children and total monthly collection is ₹ 26,000. Using matrix method, find monthly fees paid by each child of two types. What values the coaching institute is inculcating in the society? *(Foreign 2016)* Cr

57. Two schools A and B decided to award prizes to their students for three values, team spirit, truthfulness and tolerance at the rate of ₹ x, ₹ y and ₹ z per student respectively. School A, decided to award a total of ₹ 1,100 for the three values to 3, 1 and 2 students respectively while school B decided to award ₹ 1,400 for the three values to 1, 2 and 3 students respectively. If one prize for all the three values together amount to ₹ 600 then
(i) Represent the above situation by a matrix equation after forming linear equations.
(ii) Is it possible to solve the system of equations so obtained using matrices?
(iii) Which value you prefer to be rewarded most and why? *(Delhi 2015C)* Cr

LA II (5/6 marks)

58. Find the inverse of the matrix $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$.
Using the inverse, A^{-1}, solve the system of linear equations
$x - y + 2z = 1;\ 2y - 3z = 1;\ 3x - 2y + 4z = 3.$ *(2023)*

59. If $A = \begin{bmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{bmatrix}$, then find A^{-1} and use it to solve the following system of the equations :
$x + 2y - 3z = 6,\ 3x + 2y - 2z = 3,$
$2x - y + z = 2$ *(2020)*

60. Solve the following system of equations by matrix method :
$x - y + 2z = 7,$
$2x - y + 3z = 12,$
$3x + 2y - z = 5$ *(2020)*

61. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}.

Using A^{-1}, solve the following system of equations :
$2x - 3y + 5z = 11, 3x + 2y - 4z = -5$
$x + y - 2z = -3$ *(NCERT, 2020, 2018)* Ev

62. If $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of equations
$y + 2z = 5$
$x + 2y + 3z = 10$
$3x + y + z = 9$ *(2019C)*

63. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1}. Hence, solve the system of equations $x + y + z = 6, x + 2z = 7, 3x + y + z = 12$. *(Delhi 2019)*

64. Using matrices, solve the following system of linear equations:
$x + 2y - 3z = -4, 2x + 3y + 2z = 2$
$3x - 3y - 4z = 11$ *(AI 2019)*

65. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations $x + 3z = 9, -x + 2y - 2z = 4,$
$2x - 3y + 4z = -3$ *(Delhi 2017)*

66. Determine the product
$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x - y + z = 4,$
$x - 2y - 2z = 9, 2x + y + 3z = 1.$ *(AI 2017)* Ev

67. A shopkeeper has 3 varieties of pens '*A*', '*B*' and '*C*'. Meenu purchased 1 pen of each variety for a total of ₹ 21. Jeevan purchased 4 pens of '*A*' variety, 3 pens of '*B*' variety and 2 pens of '*C*' variety for ₹ 60. While Shikha purchased 6 pens of '*A*' variety, 2 pens of '*B*' variety and 3 pens of '*C*' variety for ₹ 70. Using matrix method, find cost of each variety of pen. *(AI 2016)* Cr

68. Two schools *P* and *Q* want to award their selected students on the values of discipline, politeness and punctuality. The school *P* wants to award ₹ *x* each, ₹ *y* each and ₹ *z* each for the three respective values to its 3, 2 and 1 students with a total award money of ₹ 1,000. School *Q* wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ₹ 600, using matrices, find the award money for each value.
Apart from the above three values, suggest one more value for awards. *(Delhi 2014)* Cr

69. Two schools *A* and *B* want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school *A* wants to award ₹ *x* each, ₹ *y* each and ₹ *z* each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School *B* wants to spend ₹ 2,300 to award its, 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award. *(AI 2014)* Cr

70. Two schools *P* and *Q* want to award their selected students on the values of tolerance, kindness and leadership. The school *P* wants to award ₹ *x* each, ₹ *y* each and ₹ *z* each for the three respective values to its 3, 2 and 1 students respectively with a total award money of ₹ 2200. School *Q* wants to spend ₹ 3100 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as school *P*). If the total amount of award for one prize on each value is ₹ 1200, using matrices, find the award money for each value.
Apart from the above these three values, suggest one more value which should be considered for award. *(Foreign 2014)*

71. A total amount of ₹ 7,000 is deposited in three different savings bank accounts with annual interest rates of 5%, 8% and $8\frac{1}{2}\%$ respectively. The total annual interest from these three accounts is ₹ 550. Equal amounts have been deposited in the 5% and 8% savings accounts. Find the amount deposited in each of the three accounts, with the help of matrices. *(Delhi 2014C)* Cr

72. Two schools, *P* and *Q*, want to award their selected students for the values of sincerity, truthfulness and hard work at the rate of ₹ *x*, ₹ *y* and ₹ *z* for each respective value per student. School *P* awards its 2, 3 and 4 students on the above respective values with a total prize money of ₹ 4,600. School *Q* wants to award its 3, 2 and 3 students on the respective values with a total award money of ₹ 4,100. If the total amount of award money for one prize on each value is ₹ 1,500, using matrices find the award money for each value. Suggest one other value which the school can consider for awarding the students. *(AI 2014C)*

CBSE Sample Questions

4.2 Determinant

MCQ

1. If $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$, then the possible value(s) of 'x' is/are
(a) 3 (b) $\sqrt{3}$
(c) $-\sqrt{3}$ (d) $\sqrt{3}, -\sqrt{3}$ *(2022-23)*

2. If A is a square matrix of order 3, $|A'| = -3$, then $|AA'| =$
(a) 9 (b) −9 (c) 3 (d) −3 *(2022-23)*

3. Value of k, for which $A = \begin{bmatrix} k & 8 \\ 4 & 2k \end{bmatrix}$ is a singular matrix is
(a) 4 (b) −4 (c) ±4 (d) 0
(Term I, 2021-22) Cr

4. Given that A is a non-singular matrix of order 3 such that $A^2 = 2A$, then value of $|2A|$ is
(a) 4 (b) 8 (c) 64 (d) 16
(Term I, 2021-22)

SA I (2 marks)

5. If A is a square matrix of order 3 such that $A^2 = 2A$, then find the value of $|A|$. *(2020-21)* Cr

4.4 Minors and Cofactors

MCQ

6. Given that $A = [a_{ij}]$ is a square matrix of order 3 × 3 and $|A| = -7$, then the value of $\sum_{i=1}^{3} a_{i2}A_{i2}$, where A_{ij} denotes the cofactor of element a_{ij} is
(a) 7 (b) −7
(c) 0 (d) 49 *(Term I, 2021-22)*

VSA (1 mark)

7. Let $A = [a_{ij}]$ be a square matrix of order 3 × 3 and $|A| = -7$. Find the value of $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$ where A_{ij} is the cofactor of element a_{ij}.
(2020-21) Cr

4.5 Adjoint and Inverse of a Matrix

MCQ

8. If A is a square matrix of order 3 and $|A| = 5$, then $|\text{adj } A| =$
(a) 5 (b) 25 (c) 125 (d) $\frac{1}{5}$ *(2022-23)*

9. Given that A is a square matrix of order 3 and $|A| = -4$, then $|\text{adj } A|$ is equal to
(a) −4 (b) 4
(c) −16 (d) 16
(Term I, 2021-22)

10. For matrix $A = \begin{bmatrix} 2 & 5 \\ -11 & 7 \end{bmatrix}$, $(\text{adj } A)'$ is equal to
(a) $\begin{bmatrix} -2 & -5 \\ 11 & -7 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 \\ 11 & 2 \end{bmatrix}$
(c) $\begin{bmatrix} 7 & 11 \\ -5 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & -5 \\ 11 & 2 \end{bmatrix}$
(Term I, 2021-22) Ap

11. For $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $14A^{-1}$ is given by
(a) $14\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -2 \\ 2 & 6 \end{bmatrix}$
(c) $2\begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$ (d) $2\begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix}$
(Term I, 2021-22) Ap

VSA (1 mark)

12. Given that A is a square matrix of order 3 × 3 and $|A| = -4$. Find $|\text{adj } A|$. *(2020-21)*

4.6 Applications of Determinants and Matrices

LA II (5/6 marks)

13. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}. Use A^{-1} to solve the following system of equations. *(2022-23)*
$2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$

14. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Hence solve the system of equations;
$x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$ *(2020-21)* Ev

15. Evaluate the product AB, where
$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$
Hence solve the system of linear equations
$x - y = 3$, $2x + 3y + 4z = 17$, $y + 2z = 7$ *(2020-21)* Cr

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. **(a):** Let $|A| = \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$

Expanding along R_1, we get

$|A| = 2(1 - 8) - 7(1 - 10) + 1(8 - 10)$
$= 2(-7) - 7(-9) + 1(-2)$
$= -14 + 63 - 2 = 47$

2. **(d):** $\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0 \Rightarrow \alpha(2-4) - 3(1-1) + 4(4-2) = 0$

$\Rightarrow -2\alpha + 8 = 0 \Rightarrow 2\alpha = 8 \Rightarrow \alpha = 4$

3. **(d):** Given, $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ ...(i)

$|A^3| = 27$

$\Rightarrow |A|^3 = 27$ $[\because |A^n| = |A|^n]$ $\Rightarrow |A| = 3$ ...(ii)

From (i) and (ii), we get

$\Rightarrow \alpha^2 - 4 = 3 \Rightarrow \alpha^2 = 7 \Rightarrow \alpha = \pm\sqrt{7}$

4. **(d):** We have, $\begin{vmatrix} 5 & 3 & -1 \\ -7 & x & -3 \\ 9 & 6 & -2 \end{vmatrix} = 0$

$\Rightarrow 5(-2x + 18) - 3(14 + 27) - 1(-42 - 9x) = 0$
$\Rightarrow -x + 9 = 0$
$\Rightarrow x = 9$

Concept Applied

➲ $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

5. **(d):** Let $D = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

$= (y+k)((y+k)^2 - y^2) - y(y^2 + ky - y^2) + y(y^2 - y^2 - yk)$
$= k^3 + 3k^2y = k^2(k + 3y)$

6. **(d):** Given, $\begin{vmatrix} \lfloor 1 & \lfloor 2 & \lfloor 3 \\ 2\lfloor 2 & 3\lfloor 3 & 4\lfloor 4 \\ \lfloor 3 & \lfloor 4 & \lfloor 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 4 & 18 & 96 \\ 6 & 24 & 120 \end{vmatrix}$

$= 1(2160 - 2304) - 2(480 - 576) + 6(96 - 108)$
$= -144 - 2(-96) + 6(-12) = -144 + 192 - 72 = -24$

7. **(d):** Given $A^2 = 3A$

$\Rightarrow |A^2| = |3A| \Rightarrow |A|^2 = 3^3|A| \Rightarrow |A|^2 - 27|A| = 0$
$\Rightarrow |A|\,[|A| - 27] = 0$

As, A is a non-singular matrix

$\therefore |A| \neq 0 \Rightarrow |A| - 27 = 0 \Rightarrow |A| = 27$

8. **(a):** Given, $\begin{vmatrix} x & 0 & 8 \\ 4 & 1 & 3 \\ 2 & 0 & x \end{vmatrix} = 0$

Expanding along R_1, we get

$x(x - 0) - 0 + 8(0 - 2) = 0$
$\Rightarrow x^2 - 16 = 0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

9. **(d):** Given, A is a 3×3 matrix and $|A| = 5$

Now, $|2A'| = 2^3|A'| = 2^3|A| = 8 \times 5 = 40$

Answer Tips

➲ In a square matrix, $|A'| = |A|$

10. **(b) :** We have, $A^T = -A$ [$\because$ A is skew-symmetric matrix]

$\therefore |A^T| = |-A| \Rightarrow |A| = (-1)^3|A|$ [$\because$ A is of order 3]
$\Rightarrow |A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0$

11. **(d) :** We have, $|3A| = 3^3|A| = 3^3 \cdot 8$ [Given $|A| = 8$]
$= 27 \cdot 8 = 216$

12. If $A = \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 17 \\ 0 & -10 \end{bmatrix}$, then

$AB = \begin{bmatrix} 3 & 101 \\ 2 & 34 \end{bmatrix}$

Now, $|AB| = \begin{vmatrix} 3 & 101 \\ 2 & 34 \end{vmatrix} = 102 - 202 = -100$

13. We have, $|A| = 5$, $|B| = 3$

Now, $|3AB| = 3^3|AB|$ ($\because$ Order of AB is 3)
$= 3^3|A||B| = 3^3 \times 5 \times 3 = 405$

14. We have, $AB = 2I \Rightarrow |AB| = |2I| \Rightarrow |A|\,|B| = 2^3|I|$
[$\because$ A and B are order 3]

$\Rightarrow 2|B| = 8$ [$\because |A| = 2$]
$\Rightarrow |B| = 4$

15. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$

$\Rightarrow \Delta = 1[(1 + \sin\theta)(1 + \cos\theta) - 1] - 1(1 + \cos\theta - 1) + 1(1 - 1 - \sin\theta)$
$= 1 + \cos\theta + \sin\theta + \sin\theta\cos\theta - 1 - \cos\theta - \sin\theta = \sin\theta\cos\theta$

$\therefore$ Maximum value of Δ is $\frac{1}{2}$.

16. Given, $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$

$\Rightarrow (x + 3)(2x) - (-2)(-3x) = 8$
$\Rightarrow 2x^2 + 6x - 6x = 8 \Rightarrow 2x^2 = 8$
$\Rightarrow x^2 = 4 \Rightarrow x = 2$ [$x \neq -2$, $\because x \in N$]

17. Given, $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$

$\Rightarrow x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta) = 8$

$\Rightarrow$ $-x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta = 8$

$\Rightarrow$ $-x^3 - x + x(\sin^2\theta + \cos^2\theta) = 8$

$\Rightarrow$ $-x^3 - x + x = 8 \Rightarrow x^3 + 8 = 0$

$\Rightarrow$ $(x+2)(x^2 - 2x + 4) = 0 \Rightarrow x + 2 = 0$ $[\because x^2 - 2x + 4 > 0\ \forall x]$

$\Rightarrow$ $x = -2$

18. $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

$= (x+y)(-3x+3y) - (y+z)(-3z+3y) + (z+x)(-3z+3x)$

$= 3[(y+x)(y-x) - (y+z)(y-z) + (x+z)(x-z)]$

$= 3[y^2 - x^2 - y^2 + z^2 + x^2 - z^2] = 0$

19. Given that $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

$\therefore\ AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 4 & 8 \end{bmatrix}$

$\therefore\ |AB| = \begin{vmatrix} -1 & 5 \\ 4 & 8 \end{vmatrix} = (-1)\cdot 8 - 4\cdot 5 = -28.$

20. Given, $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow$ $2x^2 - 40 = 18 + 14$

$\Rightarrow$ $2x^2 = 72 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

21. Given, $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$\Rightarrow$ $12x + 14 = 32 - 42$

$\Rightarrow$ $12x = -10 - 14 = -24 \Rightarrow x = -2.$

22. We have, $|3A| = k|A|$

A is 3×3 matrix, so we have

$3^3|A| = k|A|$ [Using $|mA| = m^n|A|$, where n is order of A]

$\Rightarrow$ $k = 27.$

23. $\begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix} = p^2 - (p-1)(p+1) = p^2 - (p^2 - 1) = 1.$

24. Let $\Delta = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

$\Rightarrow$ $\Delta = 2[8(86) - 9(75)] - 7[3(86) - 5(75)] + 65[3(9) - 5(8)]$

$= 2(688 - 675) - 7(258 - 375) + 65(27 - 40)$

$= 2(13) - 7(-117) + 65(-13) = 26 + 819 - 845 = 0$

25. We have, $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$

$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x$ $[\because \sin^2\theta + \cos^2\theta = 1]$

$= -x^3$, which is independent of θ.

26. Let $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

Cofactor of 1 = 3, Cofactor of −2 = −4.

Cofactor of 4 = 2, Cofactor of 3 = 1.

Concept Applied

- Cofactor, $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}.

27. Given, $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Cofactor of $a_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = (-1)(10-3) = -7$

28. We have, $A = \begin{bmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{bmatrix}$

$\therefore$ Cofactor of $a_{21} = (-1)^{2+1}\begin{vmatrix} 6 & -3 \\ -7 & 3 \end{vmatrix} = -1(18 - 21) = 3$

29. **(b):** Given, $A = \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$

$\therefore$ $|A| = 20 - 21 = -1$

And adj $A = \begin{bmatrix} -5 & -7 \\ -3 & -4 \end{bmatrix}^T = \begin{bmatrix} -5 & -3 \\ -7 & -4 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{|A|}(\text{adj} A) = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$

30. **(a):** Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 59 & 69 & -1 \end{bmatrix}$

Here, $|A| = -1$

And adj$A = \begin{bmatrix} -1 & 0 & -59 \\ 0 & -1 & -69 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -59 & -69 & 1 \end{bmatrix}$

$\therefore\ A^{-1} = \dfrac{1}{|A|}(\text{adj}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 59 & 69 & -1 \end{bmatrix} = A$

Key Points

- The adjoint of a square matrix is the transpose of the matrix of cofactors.

31. **(d):** $|A| = \begin{vmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \\ 4 & 2 & 0 \end{vmatrix}$

$= 1(0 - 6) + 2(0 - 12) + 4(4 + 4) = 2$ ...(i)

Given, $A = \text{adj}B$

$\Rightarrow$ $|A| = |\text{adj}B| \Rightarrow |\text{adj}B| = 2$ (Using (i))

$\Rightarrow$ $|B|^2 = 2$ $[\because |\text{adj}B| = |B|^{3-1}$, where B is 3×3 matrix$]$

$\Rightarrow$ $|B| = \pm\sqrt{2}$

$\therefore\ B^{-1} = \pm\dfrac{1}{\sqrt{2}}A$ $\left[\because B^{-1} = \dfrac{1}{|B|}(\text{adj}B)\right]$

Concept Applied

- $|\text{adj } A| = |A|^{n-1}$, where n is the order of matrix A.

32. (b): We have,

$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Now, $\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \dfrac{1}{\sec^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta & -\tan\theta\cos^2\theta \\ \cos^2\theta\tan\theta & \cos^2\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} \cos^2\theta & -\tan\theta\cos^2\theta \\ \cos^2\theta\tan\theta & \cos^2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos^2\theta - \cos^2\theta\tan^2\theta & -2\tan\theta\cos^2\theta \\ 2\tan\theta\cos^2\theta & \cos^2\theta - \cos^2\theta\tan^2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$\therefore$ $a = \cos^2\theta - \cos^2\theta\tan^2\theta$ and $b = 2\tan\theta\cos^2\theta$

$\Rightarrow$ $a = \cos^2\theta\left(1 - \dfrac{\sin^2\theta}{\cos^2\theta}\right)$ and $b = \dfrac{2\sin\theta}{\cos\theta}\cdot\cos^2\theta$

$\Rightarrow$ $a = \cos^2\theta - \sin^2\theta = \cos 2\theta$ and $b = 2\sin\theta\cos\theta = \sin 2\theta$

33. (a) : We have, $|A| = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$

$|A| = -2(4 - 0) - 0 + 0 = -8$

$\therefore$ $|\text{adj } A| = (-8)^2 = 64$

Answer Tips

➲ We know that $|A \text{ adj } A| = |A|^n$, where n is the order of A.

34. (c): Given, $A(\text{adj } A) = 10\, I$

$\Rightarrow$ $|A(\text{adj}) A)| = |10\, I|$

$\Rightarrow$ $|A|\,|\text{adj } A| = 10^3\,|I|$ ...(i)

$\Rightarrow$ $|A||A|^2 = 10^3$

[$\because$ A is matrix of order n, then $|\text{adj } A| = |A|^{n-1}$ and $n = 3$]

$\Rightarrow$ $|A|^3 = 10^3$

$\Rightarrow$ $|A| = 10$

Now, from (i), we get

$|(\text{adj}) A)| = |A|^2 = 10^2 = 100$

35. Given $A(\text{adj } A) = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$

$$|A(\text{adj } A)| = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = (-2)^3 = -8$$

$\Rightarrow$ $|A|^3 = -8$ [$\because$ $|A \text{ adj } A| = |A|^n$]

$\Rightarrow$ $|A| = -2$

36. Given, $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$

Now, $A(\text{adj } A) = |A| = \begin{vmatrix} 2 & 0 & 0 \\ -1 & 2 & 3 \\ 3 & 3 & 5 \end{vmatrix}$

$= 2(10 - 9) - 0 + 0 = 2(1) = 2$

37. Given, $|A| = 9$

$\therefore$ $|2\cdot\text{adj } A| = 2^3|A|^2 = 2^3(9)^2 = 8\times 81 = 648$

Concept Applied

➲ We know that, $|k \text{ adj } A| = k^n|A|^{n-1}$, where n is the order of the matrix A.

38. Given that, $\det(A^{-1}) = (\det A)^k$

i.e., $|A^{-1}| = |A|^k$

We know that $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$

$\therefore$ $k = -1$

39. We have, $A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

$|A|\cdot I = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ [$\because$ $A(\text{adj } A) = |A|\times I$]

$\Rightarrow$ $|A| = 8$

40. For the matrix $\begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$ to be singular, its determinant = 0

$\therefore$ $\begin{vmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{vmatrix} = 0$

$\Rightarrow$ $4\sin^2 x - 3 = 0 \Rightarrow \sin^2 x = \dfrac{3}{4} \Rightarrow \sin x = \pm\dfrac{\sqrt{3}}{2}$

$\therefore$ $x = \dfrac{2\pi}{3}\left(\because \dfrac{\pi}{2} < x < \pi\right)$

41. Here, $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

Cofactor of matrix $A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$\therefore$ $\text{adj } A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

42. Given, $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \Rightarrow |A| = -12 + 12 = 0$

$\therefore$ A is a singular matrix

$\text{adj } A = \begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

Now, $A(\text{adj}A) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ...(i)

and $(\text{adj}A)A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ ...(ii)

Similarly, $|A|I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ...(iii) ($\because$ $|A| = 0$)

From equation (i), (ii) and (iii);

we have $A(\text{adj } A) = (\text{adj } A)\, A = |A|I$

43. Given, $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 0 \\ -4 & 2 \end{vmatrix} = 2$ and adj $A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}' = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

Now, $(AB)^{-1} = B^{-1}A^{-1}$

$= \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 & 1 \\ 18 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1/2 \\ 9 & 1 \end{bmatrix}$

44. We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$

So, A is a non-singular matrix and therefore it is invertible.

$\therefore \quad \text{adj } A = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$\Rightarrow \quad 2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ ...(i)

Now, $9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ [From (ii)]

Hence, $2A^{-1} = 9I - A$.

45. (i) Let the cost of each pen, each bag and each instrument box be x, y and z respectively.

According to question, we have

$5x + 3y + z = 160$ ....(i)

$2x + y + 3z = 190$ ...(ii)

$x + 2y + 4z = 250$ ...(iii)

$\Rightarrow \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 160 \\ 190 \\ 250 \end{bmatrix}$

i.e., $AX = B$

(ii) Now, $|A| = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$

$= 5(4 - 6) - 3(8 - 3) + 1(4 - 1) = -10 - 15 + 3 = -22$

(iii) $\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{-1}{22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1/11 & 5/11 & -4/11 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix}$

OR

$A^2 = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 32 & 20 & 18 \\ 15 & 13 & 17 \\ 13 & 13 & 23 \end{bmatrix}$

$P = A^2 - 5A$

$= \begin{bmatrix} 32 & 20 & 18 \\ 15 & 13 & 17 \\ 13 & 13 & 23 \end{bmatrix} - \begin{bmatrix} 25 & 15 & 5 \\ 10 & 5 & 15 \\ 5 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 13 \\ 5 & 8 & 2 \\ 8 & 3 & 3 \end{bmatrix}$

46. Given $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Here, $|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 1(3 - 0) - 2(-1 - 0) - 2(2 - 0)$

$= 3 + 2 - 4 = 1$

$\Rightarrow \quad \text{adj} B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

$\therefore \quad B^{-1} = \frac{1}{|B|}(\text{adj} B) = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

As we know that $(AB)^{-1} = B^{-1}A^{-1}$

$\therefore \quad B^{-1}A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

$|A'| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = 1(-1 - 8) - 2(-8 + 3)$

$= -9 + 10 = 1 \neq 0.$

So, $(A')^{-1}$ exists.

Let the cofactors of a_{ij}'s are A_{ij} in A'

Now, $A_{11} = -9, A_{12} = 8, A_{13} = -5,$

$A_{21} = -8, A_{22} = 7, A_{23} = -4,$

$A_{31} = -2, A_{32} = 2, A_{33} = -1$

$\therefore \quad \text{adj}(A') = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

$\therefore (A')^{-1} = \frac{\text{adj}(A')}{|A'|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

48. Here, $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$\Rightarrow \quad |A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$

$= -1(1-4) - (-2)(2+4) - 2(-4-2)$
$= 3 + 12 + 12 = 27$
Now, $A_{11} = -3, A_{12} = -6, A_{13} = -6,$
$A_{21} = 6, A_{22} = 3, A_{23} = -6,$
$A_{31} = 6, A_{32} = -6, A_{33} = 3$

$$\therefore \text{ adj } A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\therefore \quad A \cdot (\text{adj } A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} = 27 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3.$$

49. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad \text{...(i)}$$

Now, $4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad 4A - 3I = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad \text{...(ii)}$$

From (i) and (ii), we get
$A^2 = 4A - 3I$
Pre-multiplying by A^{-1} on both sides, we get
$A^{-1}(A^2) = 4A^{-1}A - 3A^{-1}I$
$\Rightarrow \quad A = 4I - 3A^{-1}$ $\quad [\because AA^{-1} = I]$
$\Rightarrow \quad 3A^{-1} = 4I - A$

$$\Rightarrow \quad A^{-1} = \frac{4}{3}I - \frac{1}{3}A$$

$$= \frac{4}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

50. Given $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Now, $A_{11} = 7, A_{12} = -1, A_{13} = -1,$
$A_{21} = -3, A_{22} = 1, A_{23} = 0,$
$A_{31} = -3, A_{32} = 0, A_{33} = 1$

$$\therefore \text{ adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

and $|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$

$= 1(16-9) - 3(4-3) + 3(3-4) = 7 - 3 - 3 = 1 \neq 0$
So, A^{-1} exists and it is given by

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

51. Here, $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\Rightarrow \quad |A| = -11, |B| = 1$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj } A = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$\therefore \quad$ R.H.S. $= B^{-1}A^{-1}$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \cdot \left(-\frac{1}{11}\right)\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = \left(-\frac{1}{11}\right)\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} \quad \text{...(i)}$$

Now, $A \cdot B = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$\Rightarrow \quad |AB| = 14 - 25 = -11$

$$\therefore \quad \text{L.H.S.} = (AB)^{-1} = \left(-\frac{1}{11}\right)\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} \quad \text{...(ii)}$$

From (i) and (ii), $(AB)^{-1} = B^{-1}A^{-1}$.

52. Given, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $A_{11} = \cos\alpha, A_{12} = -\sin\alpha, A_{13} = 0,$
$A_{21} = \sin\alpha, A_{22} = \cos\alpha, A_{23} = 0,$
$A_{31} = 0, A_{32} = 0, A_{33} = 1$

$$\therefore \text{ adj}(A) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A.\text{adj}(A) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{...(i)}$$

$$\text{adj}(A) \cdot A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{...(ii)}$$

$$|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \cos\alpha(\cos\alpha - 0) + \sin\alpha(\sin\alpha - 0) + 0 = 1 \quad$ ...(iii)
From (i), (ii) and (iii), we get
$A(\text{adj } A) = (\text{adj } A)A = |A| I_3.$

53. Given, $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & -3 \\ 1 & -2 & 6 \end{bmatrix}$

$|A| = 3(12-6) - 4(0+3) + 2(0-2)$
$\quad = 18 - 12 - 4 = 2 \neq 0$

So, A^{-1} exists and $A^{-1} = \frac{1}{|A|}$ adj(A)

Now, $\text{adj}\, A=\begin{bmatrix}6 & -3 & -2\\ -28 & 16 & 10\\ -16 & 9 & 6\end{bmatrix}'=\begin{bmatrix}6 & -28 & -16\\ -3 & 16 & 9\\ -2 & 10 & 6\end{bmatrix}$

$\Rightarrow A^{-1}=\frac{1}{2}\begin{bmatrix}6 & -28 & -16\\ -3 & 16 & 9\\ -2 & 10 & 6\end{bmatrix}$ ...(i)

Given,

$3x + 4y + 2z = 8$

$2y - 3z = 3$

$x - 2y + 6z = -2$

Given system of equations can be written as $AX = B$,

where $A=\begin{bmatrix}3 & 4 & 2\\ 0 & 2 & -3\\ 1 & -2 & 6\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}8\\ 3\\ -2\end{bmatrix}$

$X = A^{-1} B$

$$=\frac{1}{2}\begin{bmatrix}6 & -28 & -16\\ -3 & 16 & 9\\ -2 & 10 & 6\end{bmatrix}\begin{bmatrix}8\\ 3\\ -2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}48-84+32\\ -24+48-18\\ -16+30-12\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-4\\ 6\\ 2\end{bmatrix}=\begin{bmatrix}-2\\ 3\\ 1\end{bmatrix}$$

$\therefore$ $x = -2, y = 3, z = 1$

54. Let the monthly income of Aryan be ₹ $3x$ and that of Babban be ₹ $4x$.

Also, let monthly expenditure of Aryan be ₹ $5y$ and that of Babban be ₹ $7y$.

According to question,

$3x - 5y = 15000$

$4x - 7y = 15000$

These equations can be written as $AX = B$

where, $A=\begin{bmatrix}3 & -5\\ 4 & -7\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}15000\\ 15000\end{bmatrix}$

$|A|=\begin{vmatrix}3 & -5\\ 4 & -7\end{vmatrix}=(-21+20)=-1\neq 0$

Thus, A^{-1} exists. So, system of equations has a unique solution and given by $X= A^{-1} B$

Now, $\text{adj}(A)=\begin{bmatrix}-7 & 5\\ -4 & 3\end{bmatrix}$

$\therefore$ $A^{-1}=\frac{\text{adj}(A)}{|A|}=\begin{bmatrix}7 & -5\\ 4 & -3\end{bmatrix}$

Now, $X=A^{-1}B=\begin{bmatrix}7 & -5\\ 4 & -3\end{bmatrix}\begin{bmatrix}15000\\ 15000\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}30000\\ 15000\end{bmatrix} \Rightarrow x = 30000$ and $y = 15000$

So, monthly income of Aryan = 3 × 30000
= ₹90000

Monthly income of Babban = 4 × 30000 = ₹120000

From this question we are encouraged to save a part of money every month.

Key Points

- When $|A| \neq 0$, then A^{-1} exists, so the system of equations has a unique solution given by $X = A^{-1} B$.

55. Let ₹ x be invested in the first bond and ₹ y be invested in the second bond.

According to question,

$\frac{10x}{100}+\frac{12y}{100}=2800 \Rightarrow 10x + 12y = 280000$ ...(i)

If the rate of interest had been interchanged, then the total interest earned is ₹ 100 less than the previous interest *i.e.*, ₹ 2700.

$\therefore \frac{12x}{100}+\frac{10y}{100}=2700 \Rightarrow 12x + 10y = 270000$ ...(ii)

The system of equations (i) and (ii) can be represented as

$AX=B$, where $A=\begin{bmatrix}10 & 12\\ 12 & 10\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, $B=\begin{bmatrix}280000\\ 270000\end{bmatrix}$

$|A|=\begin{vmatrix}10 & 12\\ 12 & 10\end{vmatrix}=100-144=-44\neq 0$

Thus A^{-1} exists. So, system of equations has a unique solution and given by $X = A^{-1} B$

$\text{adj}\, A=\begin{bmatrix}10 & -12\\ -12 & 10\end{bmatrix}$

Now, $X = A^{-1}B \Rightarrow X=\frac{\text{adj}\, A}{|A|}B$

$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{(-44)}\begin{bmatrix}10 & -12\\ -12 & 10\end{bmatrix}\begin{bmatrix}280000\\ 270000\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{(-44)}\begin{bmatrix}-440000\\ -660000\end{bmatrix}=\begin{bmatrix}10000\\ 15000\end{bmatrix}$

$\Rightarrow$ $x = 10000$ and $y = 15000$

Therefore, ₹ 10,000 be invested in the first bond and ₹ 15,000 be invested in the second bond. Thus, the total amount invested by the trust = 10,000 + 15,000 = ₹ 25,000. The interest received will be given to Helpage India as donation reflects the helping and caring nature of the trust.

56. Let the monthly fees paid by poor and rich children be ₹ x and ₹ y respectively.

For batch I : $20x + 5y = 9000$ ...(i)

For batch II : $5x + 25y = 26000$ ...(ii)

The system of equations (i) and (ii) can be written as $AX = B$

where, $A=\begin{bmatrix}20 & 5\\ 5 & 25\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}9000\\ 26000\end{bmatrix}$

$|A|=\begin{vmatrix}20 & 5\\ 5 & 25\end{vmatrix}=500-25=475\neq 0$

Thus, A^{-1} exists. So, the given system has a unique solution and it is given by $X = A^{-1}B$.

$\text{adj}\, A=\begin{bmatrix}25 & -5\\ -5 & 20\end{bmatrix}$

$A^{-1}=\frac{\text{adj}\, A}{|A|}=\frac{1}{475}\begin{bmatrix}25 & -5\\ -5 & 20\end{bmatrix}$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{475}\begin{bmatrix} 25 & -5 \\ -5 & 20 \end{bmatrix}\begin{bmatrix} 9000 \\ 26000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{475}\begin{bmatrix} 95000 \\ 475000 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \end{bmatrix}$$

$\Rightarrow \quad x = 200, y = 1000$

Hence, the monthly fees paid by each poor child is ₹ 200 and the monthly fees paid by each rich child is ₹ 1000.

By offering discount to the poor children, the coaching institute offers an unbiased chance for the development and enhancement of the weaker section of our society.

57. (i) Given, value of prize for team spirit = ₹ x

Value of prize for truthfulness = ₹ y

Value of prize for tolerance = ₹ z

Linear equation for School A is $3x + y + 2z = 1100$

Linear equation for School B is $x + 2y + 3z = 1400$

Linear equation for Prize is $x + y + z = 600$

The corresponding matrix equation is $PX = Q$

where, $P = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $Q = \begin{bmatrix} 1100 \\ 1400 \\ 600 \end{bmatrix}$

(ii) Now $|P| = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$

$= 3\cdot(2 - 3) - 1\cdot(1 - 3) + 2\cdot(1 - 2)$

$= -3 + 2 - 2 = -3 \neq 0$

Thus, P^{-1} exists. So, system of equations has unique solution and it is given by $X = P^{-1}Q$

Now, cofactors of elements of P are

$A_{11} = -1, A_{12} = 2, A_{13} = -1,$

$A_{21} = 1, A_{22} = 1, A_{23} = -2,$

$A_{31} = -1, A_{32} = -7, A_{33} = 5$

$$\therefore \quad \text{adj}\, P = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}\cdot \text{adj}\, P = -\frac{1}{3}\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}$$

Now, $X = P^{-1}Q$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -7 \\ -1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1100 \\ 1400 \\ 600 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -300 \\ -600 \\ -900 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} \Rightarrow x = 100, y = 200, z = 300.$$

Thus, the above system of equations is solvable.

(iii) The value truthfulness should be rewarded the most because a student who is truthful will be also tolerant and will work with a team spirit in the school.

58. Given $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

$|A| = 1(8 - 6) + 1(0 + 9) + 2(0 - 6) = 2 + 9 - 12 = -1$

$\therefore \quad |A| \neq 0$, so A^{-1} exists

Now, Co-factors are

$A_{11} = 2, A_{12} = -9, A_{13} = -6, \quad A_{21} = 0, \quad A_{22} = -2$

$A_{23} = -1, A_{31} = -1, A_{32} = 3, \quad A_{33} = 2$

$$\text{Now, } adj(A) = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

We know that $A^{-1} = \frac{1}{|A|} adj(A)$

$$= -1\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

We know that $AX = B \Rightarrow X = A^{-1}B$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

and $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+0+3 \\ 9+2-9 \\ 6+1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Hence $x = 1, y = 2$ and $z = 1$

59. Given, $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{vmatrix} = 1(2 - 2) - 2(3 + 4) - 3(-3 - 4)$

$= -14 + 21 = 7 \neq 0$

$\therefore \quad A^{-1}$ exists

Now, $A_{11} = 0, A_{12} = -7, A_{13} = -7, A_{21} = 1, A_{22} = 7,$

$A_{23} = 5, A_{31} = 2, A_{32} = -7, A_{33} = -4$

$$\therefore \quad \text{adj}\, A = \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj}\, A = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$$

The given system of equations is

$x + 2y - 3z = 6$

$3x + 2y - 2z = 3$

$2x - y + z = 2$

The system of equations can be written as $AX = B$

where $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

$\because$ A^{-1} exists, so system of equations has a unique solution given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -35 \\ -35 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = -5, z = -5$

60. We have, $x - y + 2z = 7$
$2x - y + 3z = 12$
$3x + 2y - z = 5$

The given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 12 \\ 5 \end{bmatrix}$$

Here, $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix}$

$= 1(1 - 6) + 1(-2 - 9) + 2(4 + 3)$
$= -5 - 11 + 14 = -2 \neq 0$

$\therefore$ A^{-1} exists. So system of equations has a unique solution given by $X = A^{-1}B$

$$\therefore \text{adj } A = \begin{bmatrix} -5 & 11 & 7 \\ 3 & -7 & -5 \\ -1 & 1 & 1 \end{bmatrix}' = \begin{bmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{-1}{2}\begin{bmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{bmatrix}$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 12 \\ 5 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x = 2, y = 1, z = 3$

61. We have, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$\Rightarrow$ $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$= 2(-4 + 4) + 3(-6 + 4) + 5(3 - 2)$
$= -6 + 5 = -1 \neq 0$

$\therefore$ A^{-1} exists.

Now, $A_{11} = 0$, $A_{12} = 2$, $A_{13} = 1$, $A_{21} = -1$, $A_{22} = -9$, $A_{23} = -5$, $A_{31} = 2$, $A_{32} = 23$, $A_{33} = 13$

$$\therefore \text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|}\text{adj } A = (-1)\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

The given system of equations is

$2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$

The system of equations can be written as $AX = B$,

where $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Since A^{-1} exists, therefore, system of equations has a unique solution given by

$$X = A^{-1}B = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = 2$ and $z = 3$.

62. Given, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$\Rightarrow$ $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 0(2 - 3) - 1(1 - 9) + 2(1 - 6)$

$= 8 - 10 = -2$

As $|A| \neq 0$. $\therefore$ A^{-1} exists and A^{-1} is given by $\frac{1}{|A|}(\text{adj } A)$

$$\text{adj } A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} \quad \text{...(i)}$$

Given,
$y + 2z = 5$
$x + 2y + 3z = 10$
$3x + y + z = 9$

The given system of equations can be written as $AX = B$

Where, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 10 \\ 9 \end{bmatrix}$

$X = A^{-1}B$

by using equation (i), we get

$$X = \frac{-1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 9 \end{bmatrix} \Rightarrow \frac{-1}{2}\begin{bmatrix} -5+10-9 \\ 40-60+18 \\ -25+30-9 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 2, y = 1, z = 2$

63. We have $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

Now, $|A| = 1(0 - 2) - 1(1 - 6) + 1(1)$
$= -2 + 5 + 1 = 4 \neq 0$

$\therefore$ A^{-1} exists.

Now, $A_{11} = -2, A_{12} = 5, A_{13} = 1, A_{21} = 0, A_{22} = -2, A_{23} = 2, A_{31} = 2, A_{32} = -1, A_{33} = -1$

$$\therefore \text{ adj } A = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

Now the given equations are

$x + y + z = 6$

$x + 0y + 2z = 7$

$3x + y + z = 12$

The given system of equations can be written as $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$\because$ A^{-1} exists, so system has a unique solution given by $X = A^{-1}B$.

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -12+0+24 \\ 30-14-12 \\ 6+14-12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ $x = 3, y = 1, z = 2$

64. The given system of equations is

$x + 2y - 3z = -4$

$2x + 3y + 2z = 2$

$3x - 3y - 4z = 11$

The system of equations can be written as $AX = B$

$$\text{where, } A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$\text{Now, } A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix}$$

$= 1(-12 + 6) - 2(-8 - 6) - 3(-6 - 9)$

$= -6 + 28 + 45 = 67 \neq 0$

$\therefore$ A^{-1} exists.

Now, $A_{11} = -6, A_{12} = 14, A_{13} = -15, A_{21} = 17, A_{22} = 5, A_{23} = 9, A_{31} = 13, A_{32} = -8, A_{33} = -1$

$$\therefore \text{ adj } A = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\cdot\text{adj } A = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

As A^{-1} exists, so system of equations has a unique solution given by $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$= \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 3, y = -2, z = 1.$

Concept Applied

- For a system of equations, $AX = B$, if $|A| \neq 0$, then the given system of equations is consistent and has a unique solution.

65. We have,
$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

Now the given system of equations is

$x + 3z = 9$

$-x + 2y - 2z = 4$

$2x - 3y + 4z = -3$

The system of equations can be written as $AX = B$

$$\text{where, } A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

Since, A^{-1} exists, so system of equations has a unique solution, given by

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -18+36-18 \\ 8-3 \\ 9-12+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x = 0, y = 5, z = 3$

66. We have,
$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I$$

$$\Rightarrow \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

The given system of equations is

$x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$ and it can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

Here, $|A| = 1(-6+2) + 1(3+4) + 1(1+4)$

$= -4 + 7 + 5 = 8 \neq 0$

So, the given system of equations has a unique solution given by $X = A^{-1}B$.

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \Rightarrow x = 3, y = -2, z = -1$$

67. Let one pen of variety 'A' costs ₹ x, one pen of variety 'B' costs ₹ y and one pen of variety 'C' costs ₹ z.

According to question,

$x + y + z = 21$ (For Meenu)

$4x + 3y + 2z = 60$ (For Jeevan)

$6x + 2y + 3z = 70$ (For Shikha)

The given system of equations can be written as $AX = B$

$$\text{where, } A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{vmatrix} = 1(9-4) - 1(12-12) + 1(8-18) = -5 \neq 0$$

$\therefore$ A^{-1} exists and system of equations has a unique solution given by $X = A^{-1}B$.

Now, $A_{11} = 5, A_{12} = 0, A_{13} = -10$,

$A_{21} = -1, A_{22} = -3, A_{23} = 4$,

$A_{31} = -1, A_{32} = 2, A_{33} = -1$

$$\therefore \text{ adj } A = \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{(-5)}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{(-5)}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{(-5)}\begin{bmatrix} -25 \\ -40 \\ -40 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix} \Rightarrow x = 5, y = 8, z = 8$$

$\therefore$ Cost of 1 pen of variety 'A' = ₹ 5

Cost of 1 pen of variety 'B' = ₹ 8

Cost of 1 pen of variety 'C' = ₹ 8

68. According to question, we have,

$3x + 2y + z = 1000$ ...(i)

$4x + y + 3z = 1500$ ...(ii)

$x + y + z = 600$ ...(iii)

The given system of equations can be written as $AX = B$

$$\text{where, } A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

$\therefore$ A is invertible and system of equations has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = -2, A_{12} = -1, A_{13} = 3$,

$A_{21} = -1, A_{22} = 2, A_{23} = -1$,

$A_{31} = 5, A_{32} = -5, A_{33} = -5$

$$\therefore \text{ adj } A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -500 \\ -1000 \\ -1500 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} \Rightarrow x = 100, y = 200, z = 300$$

Hence the money awarded for discipline, politeness and punctuality are ₹ 100, ₹ 200 and ₹ 300 respectively.

Apart from the above three values schools can award children for sincerity.

69. According to question, we have

$3x + 2y + z = 1600$ ...(i)

$4x + y + 3z = 2300$ ...(ii)

$x + y + z = 900$ ...(iii)

The given system of equations can be written as $AX = B$

$$\text{where } A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

$\therefore$ A is invertible and system of equations has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = -2, A_{12} = -1, A_{13} = 3,$
$A_{21} = -1, A_{22} = 2, A_{23} = -1,$
$A_{31} = 5, A_{32} = -5, A_{33} = -5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -1000 \\ -1500 \\ -2000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

$\Rightarrow \quad x = 200, y = 300, z = 400$

Hence, the money awarded for sincerity, truthfulness and helpfulness are ₹ 200, ₹ 300 and ₹ 400 respectively.

Apart from the above three values schools can award children for discipline.

70. According to question, we have

$3x + 2y + z = 2200$ ...(i)

$4x + y + 3z = 3100$ ...(ii)

$x + y + z = 1200$ ...(iii)

The given system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$$

$\therefore$ A is invertible and system of equations has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = -2, A_{12} = -1, A_{13} = 3,$
$A_{21} = -1, A_{22} = 2, A_{23} = -1,$
$A_{31} = 5, A_{32} = -5, A_{33} = -5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -1500 \\ -2000 \\ -2500 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix}$$

$\Rightarrow \quad x = 300, y = 400, z = 500$

Hence, the money awarded for tolerance, kindness and leadership are ₹ 300, ₹ 400 and ₹ 500 respectively.

Apart from the above three values schools can award children for sincerity.

71. Let ₹ x, ₹ y and ₹ z be deposited at the rates of interest 5%, 8% and $8\frac{1}{2}$% respectively.

According to question,

$x + y + z = 7000$

$x - y = 0$

$$x \cdot \frac{5}{100} + y \cdot \frac{8}{100} + z \cdot \frac{17}{2} \times \frac{1}{100} = 550$$

$\Rightarrow \quad 10x + 16y + 17z = 110000$

The system of equations can be written as $AX = B$

$$\text{where, } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 10 & 16 & 17 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7000 \\ 0 \\ 110000 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 10 & 16 & 17 \end{vmatrix} = 1(-17) - (17) + 1(16 + 10) = -8 \neq 0$$

$\therefore$ A^{-1} exists. So, system of equations has a unique solution and it is given by $X = A^{-1}B$

Now, $A_{11} = -17, A_{12} = -17, A_{13} = 26,$
$A_{21} = -1, A_{22} = 7, A_{23} = -6,$
$A_{31} = 1, A_{32} = 1, A_{33} = -2$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -17 & -1 & 1 \\ -17 & 7 & 1 \\ 26 & -6 & -2 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-8}\begin{bmatrix} -17 & -1 & 1 \\ -17 & 7 & 1 \\ 26 & -6 & -2 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -17 & -1 & 1 \\ -17 & 7 & 1 \\ 26 & -6 & -2 \end{bmatrix} \cdot \begin{bmatrix} 7000 \\ 0 \\ 110000 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 4 & 0 & 0 \\ -4 & 4 & 0 \\ 0 & -4 & 4 \end{bmatrix}$$

$\Rightarrow \quad x = 1125 = y, z = 4750$

Answer Tips

- First, make the linear equations from given word problem and then solve the equations.

72. According to question, we have

$x + y + z = 1500$
$2x + 3y + 4z = 4600$
$3x + 2y + 3z = 4100$

The system of equations can be written as $AX = B$

where, $A=\begin{bmatrix}1&1&1\\2&3&4\\3&2&3\end{bmatrix}; X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}1500\\4600\\4100\end{bmatrix}$

$|A|=\begin{vmatrix}1&1&1\\2&3&4\\3&2&3\end{vmatrix}$

$= 1(9 - 8) - 1(6 - 12) + 1(4 - 9)$

$= 1 + 6 - 5 = 2 \neq 0$

$\therefore$ A^{-1} exists and so, system of equations has a unique solution given by $X = A^{-1}B$

Now, $A_{11} = 1, A_{12} = 6, A_{13} = -5,$
$A_{21} = -1, A_{22} = 0, A_{23} = 1,$
$A_{31} = 1, A_{32} = -2, A_{33} = 1$

$\therefore \quad \text{adj}\,A=\begin{bmatrix}1&-1&1\\6&0&-2\\-5&1&1\end{bmatrix}$

$A^{-1}=\frac{1}{|A|}\cdot\text{adj}\,A=\frac{1}{2}\begin{bmatrix}1&-1&1\\6&0&-2\\-5&1&1\end{bmatrix}$

Now, $X = A^{-1}B$

$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&-1&1\\6&0&-2\\-5&1&1\end{bmatrix}\begin{bmatrix}1500\\4600\\4100\end{bmatrix}$

$=\frac{1}{2}\begin{bmatrix}1000\\800\\1200\end{bmatrix}=\begin{bmatrix}500\\400\\600\end{bmatrix} \Rightarrow x = 500; y = 400; z = 600.$

Apart from sincerity, truthfulness and hard work, the schools can include an award for regularity.

CBSE Sample Questions

1. **(d) :** We have, $\begin{vmatrix}2&4\\5&1\end{vmatrix}=\begin{vmatrix}2x&4\\6&x\end{vmatrix}$

$\Rightarrow \quad 2 - 20 = 2x^2 - 24 \Rightarrow 2x^2 = 6 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$ **(1)**

2. **(a) :** $|AA'| = |A|\,|A'| = |A'|^2 = (-3)^2 = 9$ **(1)**

3. **(c) :** $\because$ A is singular matrix.

$\therefore \quad |A| = 0$

$\Rightarrow \begin{vmatrix}k&8\\4&2k\end{vmatrix}=0 \Rightarrow 2k^2-32=0 \Rightarrow k^2=16 \Rightarrow k=\pm 4$ **(1)**

4. **(c) :** We have, $A^2 = 2A$

$\Rightarrow \quad |A^2| = |2A|$

$\Rightarrow \quad |A|^2 = 2^3|A|$ [As $|kA| = k^n|A|$ for a matrix of order n]

$\Rightarrow \quad |A|\,[|A| - 8] = 0$

$\Rightarrow$ either $|A| = 0$ or $|A| = 8$

But A is non-singular matrix $\Rightarrow |A| \neq 0$

$\therefore \quad |2A| = 2^3\cdot|A| = 64$ **(1)**

5. We have, $A^2 = 2A$

$\Rightarrow \quad |AA| = |2A|$

$\Rightarrow \quad |A||A| = 8|A|$ **(1/2)**

($\because$ $|AB| = |A||B|$ and $|kA| = k^n|A|$, where n is order of square matrix A)

$\Rightarrow \quad |A|\,(|A| - 8) = 0$ **(1)**

$\Rightarrow \quad |A| = 0$ or 8 **(1/2)**

6. **(b):** We have, $|A| = -7$

$\therefore \sum_{i=1}^{3} a_{i2}A_{i2} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = |A| = -7$ **(1)**

7. We know that, if elements of a row are multiplied with cofactors of any other row, then their sum is 0.

$\therefore \quad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0.$ **(1)**

8. **(b) :** Given, $|A| = 5$, order of matrix, $n = 3$.

$|\text{adj}\,A| = |A|^{n-1} \Rightarrow |\text{adj}\,A| = 25$ **(1)**

9. **(d):** We know that, $|\text{adj}\,A| = |A|^{n-1}$, where n is the order of A.

Here, $|\text{adj}\,A| = |A|^2 = (-4)^2 = 16$ **(1)**

10. **(c) :** We know that, $(\text{adj}\,A)'$ = cofactor matrix of A

Here, cofactor matrix of $A=\begin{bmatrix}7&11\\-5&2\end{bmatrix}=(\text{adj}\,A)'$ **(1)**

11. **(b):** We have, $|A| = 6 + 1 = 7$

Also, $\text{adj}\,A=\begin{bmatrix}2&-1\\1&3\end{bmatrix}$

Now, $A^{-1}=\frac{1}{|A|}\text{adj}(A)=\frac{1}{7}\begin{bmatrix}2&-1\\1&3\end{bmatrix}$

$\therefore \; 14A^{-1}=\begin{bmatrix}4&-2\\2&6\end{bmatrix}$ **(1)**

12. We know, $|\text{adj}A| = |A|^{n-1}$, where $n \times n$ is the order of non-singular matrix A.

$\therefore \quad |\text{adj}\,A| = (-4)^{3-1} = 16$ **(1)**

13. Given, $A=\begin{bmatrix}2&-3&5\\3&2&-4\\1&1&-2\end{bmatrix}$

$\therefore \quad |A| = 2(0) + 3(-2) + 5(1) = -1 \neq 0$ **(1/2)**

As, $|A| \neq 0$, so A^{-1} exists and given by

$A^{-1}=\frac{\text{adj}\,A}{|A|}$

Now, $\text{adj}\,A=\begin{bmatrix}0&-1&2\\2&-9&23\\1&-5&13\end{bmatrix}, A^{-1}=\frac{1}{(-1)}\begin{bmatrix}0&-1&2\\2&-9&23\\1&-5&13\end{bmatrix}$ **(1½)**

The given system of equations are :

$2x - 3y + 5z = 11$
$3x + 2y - 4z = -5$
$x + y - 2z = -3$

The given system of equations can be written as $AX = B$

where, $A=\begin{bmatrix}2&-3&5\\3&2&-4\\1&1&-2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}11\\-5\\-3\end{bmatrix}$

$$\Rightarrow \quad X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{(-1)} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$ **(1½)**

$$= \frac{1}{(-1)} \begin{bmatrix} 0+5-6 \\ 22+45-69 \\ 11+25-39 \end{bmatrix}$$ **(1/2)**

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{(-1)} \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \Rightarrow x = 1, y = 2 \text{ and } z = 3.$$ **(1)**

14. We have, $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore$ $|A| = 1(-1-2) - 2(-2-0) + 0 = -3 + 4 = 1 \neq 0$

$\therefore$ A^{-1} exists. **(1)**

Now, adj $A = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

$$\therefore \; A^{-1} = \frac{1}{|A|}(\text{adj } A) = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$ **(1½)**

We have, $x - 2y = 10$

$2x - y - z = 8$

$-2y + z = 7$

The given equations can be written as

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix},$$ **(1)**

which is of the form $AX = B$

where, $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}$

$\Rightarrow$ $X = A^{-1}B$ **(1)**

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

$\Rightarrow$ $x = 0, y = -5, z = -3$ **(1½)**

15. We have,

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$ **(1½)**

$\Rightarrow$ $AB = 6I \;\Rightarrow\; A^{-1} = \frac{1}{6}(B)$ **(1)**

The given equations can be written as

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

which is of the form $AX = C$

$\Rightarrow$ $X = A^{-1}C$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix}$$ **(1)**

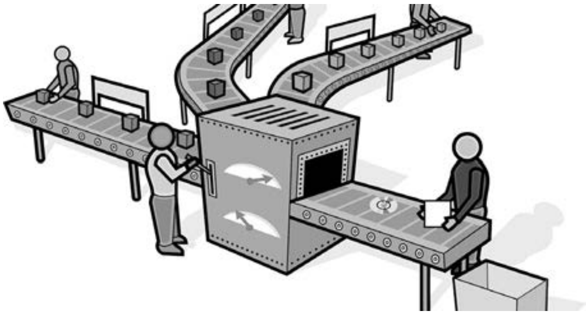
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$\Rightarrow$ $x = 2, y = -1, z = 4$ **(1½)**

Self Assessment

Case Based Objective Questions (4 marks)

1. A company produces three products every day. Their production on certain day is 45 tons. It is found that the production of third product exceeds the production of first product by 8 tons while the total production of first and third product is twice the production of second product.



Based on the above information, attempt any 4 out of 5 subparts.

(i) If x, y and z respectively denotes the quantity (in tons) of first, second and third product produced, then which of the following is true?

(a) $x+y+z=45$ (b) $x+8=z$
(c) $x-2y+z=0$ (d) all of these

(ii) The matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -2\\ 1 & -1 & 1\end{bmatrix}$ is a/an

(a) a symmetric matrix
(b) an identity matrix
(c) a non-singular matrix
(d) null matrix

(iii) If $\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & -2\\ 1 & -1 & 1\end{pmatrix}^{-1}=\frac{1}{6}\begin{pmatrix}2 & 2 & 2\\ 3 & 0 & -3\\ 1 & -2 & 1\end{pmatrix}$, then the inverse of $\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 1 & -2 & 1\end{pmatrix}$ is

(a) $\begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & 0 & \frac{-1}{2}\\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6}\end{pmatrix}$ (b) $\begin{pmatrix}\frac{1}{2} & 0 & -\frac{1}{2}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6}\end{pmatrix}$

(c) $\begin{pmatrix}\frac{1}{3} & \frac{1}{2} & \frac{1}{6}\\ \frac{1}{3} & 0 & \frac{-1}{3}\\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6}\end{pmatrix}$ (d) None of these

(iv) $x:y:z$ is equal to

(a) 12 : 13 : 20 (b) 11 : 15 : 19
(c) 15 : 19 : 11 (d) 13 : 12 : 20

(v) Which of the following options is not correct in a given determinant of A, where $A=[a_{ij}]_{3\times3}$.

(a) order of minor is less than order of det (A).
(b) minor of an element is always equal to cofactor of the same element.
(c) cofactor of an element a_{ij} is obtained by multiplying the minor by $(-1)^{i+j}$.
(d) value of a determinant is obtained by multiplying elements of a row or column of corresponding co-factors.

Multiple Choice Questions (1 mark)

2. The area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units. The value of k will be

(a) 9 (b) 3 (c) –9 (d) 6

3. For what value of x is the matrix $\begin{bmatrix}2x+4 & 4\\ x+5 & 3\end{bmatrix}$ a singular matrix?

(a) 2 (b) 3 (c) 4 (d) 5

4. If $A=\begin{bmatrix}2 & 4\\ 4 & 3\end{bmatrix}, X=\begin{bmatrix}n\\ 1\end{bmatrix}$ and $B=\begin{bmatrix}8\\ 11\end{bmatrix}$ and $AX=B$, then value of n is

(a) 0 (b) 2 (c) 4 (d) 6

5. The area of triangle with vertices (k, 0), (5, 1), (1, 3) is 4 square units, then value of k is

(a) 9 (b) 3 or –9
(c) 3 (d) 3 or 11

OR

The system of linear equations
$x+y+z=7$, $2x+y-z=3$, $3x+2y+kz=4$,
has a unique solution, if

(a) $k\neq0$ (b) $-1<k<1$
(c) $-2<k<2$ (d) $k=0$

6. Find the value of x, for which det A vanishes, where $A=\begin{bmatrix}x+1 & -1 & 0\\ 2 & x+4 & 0\\ 0 & 0 & x\end{bmatrix}$.

(a) 0, –2, –3 (b) 0, –2, 3
(c) 0, 2, –3 (d) 0, 2, 3

VSA Type Questions (1 mark)

7. If A_{ij} is the co-factor of element a_{ij} of the determinant $\begin{vmatrix}2 & -3 & 5\\ 6 & 0 & 4\\ 1 & 5 & -7\end{vmatrix}$, then write the value of $a_{32}\cdot A_{32}$.

8. Let $\begin{vmatrix} 3 & y \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$. Find the possible value of x and y if x, y are natural numbers.

9. Evaluate : $\begin{vmatrix} a+ib & -c+id \\ c+id & a-ib \end{vmatrix}$

10. If A is a matrix of order 3×3, then find the value $(A^2)^{-1}$.

OR

There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then what is the sum of these numbers?

11. If A is a matrix of order 3×3, then the number of minors in determinant of A are _______ .

SA I Type Questions (2 marks)

12. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then find the value of adj (adj A).

13. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1}, if exists.

14. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of an equilateral triangle whose each side is equal to a, then the value of $\begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix}^2$.

OR

Find the minor and co-factor of each element of third row of the determinant $\Delta = \begin{vmatrix} 6 & -7 & 8 \\ 1 & -3 & 1 \\ 2 & 1 & -4 \end{vmatrix}$ and hence evaluate det Δ.

15. Find the inverse of matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

SA II Type Questions (3 marks)

16. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then find $(AB)^{-1}$.

17. If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then show that $A^{-1} = A^2$.

18. Solve the following system of linear equations using matrix method.
$x - 2y = 4$ and $-3x + 5y = -7$

19. Using matrices, solve the following system of equations : $x + y - z = 3$; $2x + 3y + z = 10$; $3x - y - 7z = 1$.

OR

If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y such that $A^2 + xI = yA$.
Hence, find A^{-1}.

Case Based Questions (4 marks)

20. Each triangular face of the Pyramid of Peace in Kazakhstan is made up of 25 smaller equilateral triangles as shown in the figure.

Using the above information and concept of determinants, answer the following questions.

(i) If the vertices of one of the smaller equilateral triangle are (0, 0), $(3, \sqrt{3})$ and $(3, -\sqrt{3})$, then find the area of such triangle.

(ii) $A(2, 4)$, $B(2, 6)$ are two vertices of triangle, if $C(k, 0)$ is a point such that area of triangle ABC is $3\sqrt{3}$ sq. units.

LA Type Questions (4/6 marks)

21. Find $(AB)^{-1}$, where

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

22. If A, B, C are angles of a triangle, then find out the value of $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$.

23. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$ and show that $aA^{-1} = (a^2 + bc + 1)I - aA$.

OR

Evaluate the determinant $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$.
Also prove that $2 \le \Delta \le 4$.

24. Find the equation of line joining $P(11, 7)$ and $Q(5, 5)$ using determinant. Also, find the value of k, if $R(-1, k)$ is the point such that area of ΔPQR is 9 sq. units.

Detailed SOLUTIONS

1. (i) (d) : According to given condition, we have the following system of linear equations.

$x + y + z = 45$

$x + 8 = z$ or $x + 0 \cdot y - z = -8$

and $x + z = 2y$ or $x - 2y + z = 0$

(ii) (c) : We have $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

So, $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$ As, $A \neq A^T$

So A is not symmetric matrix.

Now, $|A| = 1(0 - 2) - 1(1 + 2) + 1(-1 - 0)$

$= -2 - 3 - 1 = -6$

$\therefore \quad |A| \neq 0$

Hence, it is a non-singular matrix.

(iii) (c) : Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$, then we have

$A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

Now, $(A')^{-1} = (A^{-1})'$

$$(A^{-1})' = \frac{1}{6}\begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & -2 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} \end{pmatrix}$$

(iv) (b) : The above system of equations can be written in matrix form as

$AX = B$, where,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 45 \\ -8 \\ 0 \end{bmatrix}$$

$$\Rightarrow X = (A)^{-1}B = \frac{1}{6}\begin{bmatrix} 2 & 3 & 1 \\ 2 & 0 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 45 \\ -8 \\ 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 90-24 \\ 90 \\ 90+24 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 66 \\ 90 \\ 114 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 19 \end{bmatrix}$$

Thus, $x : y : z = 11 : 15 : 19$

(v) (b): Minor of an element is always equal to cofactor of the same element.

2. (b) : Since, area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \therefore \quad \Delta = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

Expanding along C_2,

$$9 = \frac{1}{2}[-k(-3-3)] \Rightarrow 9 = \frac{1}{2} \times 6k \Rightarrow k = \frac{9}{3} = 3$$

3. (c) : Let $A = \begin{bmatrix} 2x+4 & 4 \\ x+5 & 3 \end{bmatrix}$

For a matrix to be singular, $|A| = 0$

$\Rightarrow \quad (2x + 4)3 - 4(x + 5) = 0 \Rightarrow 6x + 12 - 4x - 20 = 0$

$\Rightarrow \quad 2x = 8 \Rightarrow x = 4$

4. (b) : Since, $AX = B$

$$\therefore \quad \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} 2n+4 \\ 4n+3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$\Rightarrow \quad 2n + 4 = 8$ or $4n + 3 = 11$

$\Rightarrow \quad 2n = 4$ or $4n = 8$

$\Rightarrow \quad n = 2$ or $n = 2$

5. (d) : Given, area of triangle with vertices $(k, 0)$, $(5, 1)$, $(1, 3)$ is 4 sq. units *i.e.*,

$$\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 5 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \pm 4 \Rightarrow \frac{-2k+14}{2} = \pm 4$$

$\Rightarrow \quad k = 3$ or 11

OR

(a) : The given system of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix} \text{ i.e., } AX = B$$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix}$

$= 1(k + 2) - 1(2k + 3) + 1(4 - 3)$

$= k + 2 - 2k - 3 + 1 = -k$

To have unique solution, $|A| \neq 0$

$\Rightarrow \quad k \neq 0$

6. (a) : $|A| = \begin{vmatrix} x+1 & -1 & 0 \\ 2 & x+4 & 0 \\ 0 & 0 & x \end{vmatrix}$

Expanding along C_1, we get

$$(x+1)\begin{vmatrix} x+4 & 0 \\ 0 & x \end{vmatrix} - (-1)\begin{vmatrix} 2 & 0 \\ 0 & x \end{vmatrix} + 0$$

$= (x + 1)[(x + 4)x - 0] + (2x - 0)$

$= x(x + 1)(x + 4) + 2x = x[(x + 1)(x + 4) + 2]$

$= x(x^2 + 5x + 6)$

Since, $\det A = 0$

$\Rightarrow \quad x(x^2 + 5x + 6) = 0$

$\Rightarrow \quad x(x + 2)(x + 3) = 0$

$\Rightarrow \quad x = 0, -2, -3$

7. Let $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Here, $a_{32} = 5$

Also, cofactor of $a_{32} = A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = -(8 - 30) = 22$

Hence, $a_{32} \cdot A_{32} = 5 \times 22 = 110$

8. Given, $\begin{vmatrix} 3 & y \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

$\Rightarrow \quad 3 \times 1 - x \times y = 3 \times 1 - 4 \times 2$
$\Rightarrow \quad 3 - xy = 3 - 8$
$\Rightarrow \quad xy = 8$
If $x = 1, y = 8; x = 2, y = 4; x = 4, y = 2; x = 8, y = 1$.

9. Since, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$\therefore \quad \begin{vmatrix} a+ib & -c+id \\ c+id & a-ib \end{vmatrix} = (a + ib)(a - ib) - (-c + id)(c + id)$

$= a^2 - i^2b^2 - (-c^2 + i^2d^2)$

$= a^2 + b^2 + c^2 + d^2$ $\quad [\because i^2 = -1]$

10. If A is a matrix of order 3×3, then $(A^2)^{-1} = (A^{-1})^2$.

OR

We have, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

$\Rightarrow 1(2a^2 + 4) - 2(-4a - 20) + 0 = 86$

[Expanding along first column]

$\Rightarrow \quad 2a^2 + 4 + 8a + 40 = 86$
$\Rightarrow \quad 2a^2 + 8a - 42 = 0$
$\Rightarrow \quad a^2 + 4a - 21 = 0$
$\Rightarrow \quad (a + 7)(a - 3) = 0$
$\Rightarrow \quad a = -7$ or 3
$\therefore \quad$ Required sum $= -7 + 3 = -4$

11. $\because$ In a 3×3 matrix, there are 9 elements, therefore, if A is a matrix of order 3×3, then the number of minors in determinant of A are 9.

12. We have, $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

$\therefore \quad \text{adj}(A) = \begin{bmatrix} 4 & 0 & 0 \\ -4 & 4 & 0 \\ 0 & -4 & 4 \end{bmatrix}$

$\therefore \quad \text{adj}(\text{adj} A) = \begin{bmatrix} 16 & 0 & 0 \\ 16 & 16 & 0 \\ 16 & 16 & 16 \end{bmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

13. We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$

So, A is a non-singular matrix and therefore it is invertible.

$\therefore \quad \text{adj } A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

14. Let Δ be the area of triangle ABC. Then,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow 2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 8\Delta = 4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix}$$

$$\Rightarrow \quad 64\Delta^2 = \begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix}^2 \quad \text{...(i)}$$

But, the area of an equilateral triangle with each side a is $\frac{\sqrt{3}}{4}a^2$.

$\therefore \quad \Delta = \frac{\sqrt{3}}{4}a^2 \Rightarrow 16\Delta^2 = 3a^4$ ...(ii)

From (i) and (ii), we get

$$\begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix}^2 = 12a^4$$

OR

$M_{31} = \begin{vmatrix} -7 & 8 \\ -3 & 1 \end{vmatrix} = -7 + 24 = 17$

$M_{32} = \begin{vmatrix} 6 & 8 \\ 1 & 1 \end{vmatrix} = 6 - 8 = -2$ and

$M_{33} = \begin{vmatrix} 6 & -7 \\ 1 & -3 \end{vmatrix} = -18 + 7 = -11$

Also, $A_{31} = (-1)^{3+1} M_{31} = (-1)^4(17) = 17$

$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-2) = 2$

$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 (-11) = -11$

Now, det $\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$

$= 2 \times 17 + 1 \times 2 + (-4) \times (-11) = 34 + 2 + 44 = 80$

15. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1, \forall\, \theta$

Since, $|A| \neq 0 \quad \therefore A^{-1}$ exists

Now, $A_{11} = \cos\theta; A_{12} = -(-\sin\theta) = \sin\theta,$

$A_{21} = -\sin\theta; A_{22} = \cos\theta$

adj $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^{-1}=\frac{1}{|A|}\text{adj}\,A=\frac{1}{1}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

16. Given, $A=\begin{bmatrix}2 & 3\\ 1 & -4\end{bmatrix}$ and $B=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$

Now, $AB=\begin{bmatrix}2 & 3\\ 1 & -4\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 5 & -14\end{bmatrix}$

$\therefore\quad |AB|=\begin{vmatrix}-1 & 5\\ 5 & -14\end{vmatrix}=14-25=-11\neq 0$

$\therefore\quad (AB)^{-1}$ exists.

$\therefore\quad (AB)^{-1}=\frac{1}{|AB|}[\text{adj}\,(AB)]$

$$=\frac{-1}{11}\begin{bmatrix}-14 & -5\\ -5 & -1\end{bmatrix}=\frac{1}{11}\begin{bmatrix}14 & 5\\ 5 & 1\end{bmatrix}$$

17. By definition, a matrix B is an inverse of A if $AB = I = BA$

Here, A^2 is inverse of A.

$\therefore$ It is sufficient to prove that

$A^2A = I = A\cdot A^2$ *i.e.*, $A^3 = I$.

Now, $A^2=A\cdot A=\begin{bmatrix}-1 & 2 & 0\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 2 & 0\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$

$$=\begin{bmatrix}1-2+0 & -2+2+0 & 0+2+0\\ 1-1+0 & -2+1+1 & 0+1+0\\ 0-1+0 & 0+1+0 & 0+1+0\end{bmatrix}=\begin{bmatrix}-1 & 0 & 2\\ 0 & 0 & 1\\ -1 & 1 & 1\end{bmatrix}$$

$\therefore\quad A^3=A^2\cdot A=\begin{bmatrix}-1 & 0 & 2\\ 0 & 0 & 1\\ -1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 2 & 0\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$

$$=\begin{bmatrix}1+0+0 & -2+0+2 & 0+0+0\\ 0+0+0 & 0+0+1 & 0+0+0\\ 1-1+0 & -2+1+1 & 0+1+0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

Hence, $A^{-1} = A^2$.

18. The given system of equations can be written as

$\begin{bmatrix}1 & -2\\ -3 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ -7\end{bmatrix}$ *i.e.*, $AX = B$, where

$A=\begin{bmatrix}1 & -2\\ -3 & 5\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}4\\ -7\end{bmatrix}$

Now, $|A|=\begin{vmatrix}1 & -2\\ -3 & 5\end{vmatrix}=5-6=-1\neq 0$

$\therefore$ The given system has unique solution given by $X = A^{-1}B$.

$A_{11} = (-1)^{1+1}5 = 5$; $A_{12} = (-1)^{1+2}(-3) = 3$

$A_{21} = (-1)^{2+1}(-2) = 2$; $A_{22} = (-1)^{2+2}(1) = 1$

$$\text{adj}\,A=\begin{bmatrix}5 & 3\\ 2 & 1\end{bmatrix}'=\begin{bmatrix}5 & 2\\ 3 & 1\end{bmatrix}$$

$\therefore\quad A^{-1}=\frac{\text{adj}\,A}{|A|}=\frac{1}{-1}\begin{bmatrix}5 & 2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}-5 & -2\\ -3 & -1\end{bmatrix}$

$\therefore$ Solution is given by, $X = A^{-1}B$

$$=\begin{bmatrix}-5 & -2\\ -3 & -1\end{bmatrix}\begin{bmatrix}4\\ -7\end{bmatrix}=\begin{bmatrix}-20+14\\ -12+7\end{bmatrix}=\begin{bmatrix}-6\\ -5\end{bmatrix}$$

$\Rightarrow\ X=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-6\\ -5\end{bmatrix}$

$\therefore\quad x = -6, y = -5$ are required solutions.

19. The given system of equations can be written as

$\begin{bmatrix}1 & 1 & -1\\ 2 & 3 & 1\\ 3 & -1 & -7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 10\\ 1\end{bmatrix}$ *i.e.*, $AX=B$,

where $A=\begin{bmatrix}1 & 1 & -1\\ 2 & 3 & 1\\ 3 & -1 & -7\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 10\\ 1\end{bmatrix}$

Now, $|A|=\begin{vmatrix}1 & 1 & -1\\ 2 & 3 & 1\\ 3 & -1 & -7\end{vmatrix}$

$= 1(-21 + 1) - 1(-14 - 3) - 1(-2 - 9) = -20 + 17 + 11$

$= 8 \neq 0$

$\therefore\quad A^{-1}$ exists.

So, the system of equations has a unique solution which given by $X = A^{-1}B$.

To find A^{-1}: $A_{11} = -20, A_{12} = 17, A_{13} = -11, A_{21} = 8, A_{22} = -4, A_{23} = 4, A_{31} = 4, A_{32} = -3, A_{33} = 1$

Now, $\text{adj}\,A=\begin{bmatrix}-20 & 17 & -11\\ 8 & -4 & 4\\ 4 & -3 & 1\end{bmatrix}'=\begin{bmatrix}-20 & 8 & 4\\ 17 & -4 & -3\\ -11 & 4 & 1\end{bmatrix}$

$\therefore\quad A^{-1}=\frac{1}{|A|}\text{adj}\,A=\frac{1}{8}\begin{bmatrix}-20 & 8 & 4\\ 17 & -4 & -3\\ -11 & 4 & 1\end{bmatrix}$

Now, $X=A^{-1}B=\frac{1}{8}\begin{bmatrix}-20 & 8 & 4\\ 17 & -4 & -3\\ -11 & 4 & 1\end{bmatrix}\begin{bmatrix}3\\ 10\\ 1\end{bmatrix}$

$\Rightarrow\quad \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}24\\ 8\\ 8\end{bmatrix}=\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}\ \Rightarrow\ x = 3, y = 1, z = 1$

OR

We have, $A=\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$

$\therefore\quad A^2=\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}=\begin{bmatrix}9+7 & 3+5\\ 21+35 & 7+25\end{bmatrix}=\begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix}$

Now, $A^2 + xI = yA$

$\Rightarrow\quad \begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix}+x\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=y\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$

$\Rightarrow\quad \begin{bmatrix}16+x & 8\\ 56 & 32+x\end{bmatrix}=\begin{bmatrix}3y & y\\ 7y & 5y\end{bmatrix}$

On equating like terms, we get

$16 + x = 3y$ ...(i)

$y = 8$ ...(ii)

From (i) and (ii), $x = 24 - 16 = 8$

Also, $|A| = \begin{vmatrix} 3 & 1 \\ 7 & 5 \end{vmatrix} = 15 - 7 = 8 \neq 0 \Rightarrow A^{-1}$ exists

Now, $A^2 + xI = yA$

$\Rightarrow A^2 + 8I = 8A, 8I = 8A - A^2$

Pre-multiplying both sides by A^{-1}, we get

$8A^{-1}I = 8A^{-1}A - A^{-1}A^2$

$\Rightarrow 8A^{-1} = 8I - A \Rightarrow A^{-1} = \frac{1}{8}(8I - A)$

$\therefore A^{-1} = \frac{1}{8}\left(8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right) = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix}$

$= \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$

20. (i) Area of triangle $= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$.

$\therefore$ Required area $= \left|\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 3 & \sqrt{3} & 1 \\ 3 & -\sqrt{3} & 1 \end{vmatrix}\right|$

$= \left|\frac{1}{2}[1(-3\sqrt{3} - 3\sqrt{3})]\right|$ [Expanding along R_1]

$= 3\sqrt{3}$ sq. units

(ii) Since, area of triangle ABC is $3\sqrt{3}$ sq. units, we have

$\frac{1}{2}\begin{vmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3\sqrt{3}$

$\Rightarrow \frac{1}{2}[2(6-0) - 4(2-k-x) + 1(2(0)y - 6k] = \pm 3\sqrt{3}$

$\Rightarrow 12 - 8 + 4k - 6k = \pm 6\sqrt{3}$

$\Rightarrow 4 - 2k = \pm 6\sqrt{3}$

$\Rightarrow 2 - k = \pm 3\sqrt{3} \Rightarrow k = 2 \pm 3\sqrt{3}$

21. We know that $(AB)^{-1} = B^{-1}A^{-1}$ ...(i)

Now, $|A| = \begin{vmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 5(3-4) - 0(2-2) + 4(4-3) = -1 \neq 0$

Hence A^{-1} exists.

Let A_{ij} be the cofactor of the element in the ith row and jth column of A, then

$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1, A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1, A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} = 8$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = 1, A_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 0 \\ 1 & 2 \end{vmatrix} = -10$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 4 \\ 3 & 2 \end{vmatrix} = -12, A_{32} = (-1)^{3+2}\begin{vmatrix} 5 & 4 \\ 2 & 2 \end{vmatrix} = -2$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 5 & 0 \\ 2 & 3 \end{vmatrix} = 15$

Now, adj A = transpose of $\begin{bmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & -2 & 15 \end{bmatrix}$

$= \begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{(-1)}\text{adj}A = -\text{adj}A = \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$

From (i), $(AB)^{-1} = B^{-1}A^{-1}$

$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$

$= \begin{bmatrix} 1+0-3 & -8-3+30 & 12+6-45 \\ 1+0-3 & -8-4+30 & 12+8-45 \\ 1+0-4 & -8-3+40 & 12+6-60 \end{bmatrix}$

$= \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$

22. Let $\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

Applying $C_2 \to C_2 + (\cos C)C_1$ and $C_3 \to C_3 + (\cos B)C_1$, we get

$\Delta = \begin{vmatrix} -1 & 0 & 0 \\ \cos C & -1+\cos^2 C & \cos A + \cos C\cos B \\ \cos B & \cos A + \cos B\cos C & -1+\cos^2 B \end{vmatrix}$

$= \begin{vmatrix} -1 & 0 & 0 \\ \cos C & -\sin^2 C & \cos A + \cos C\cos B \\ \cos B & \cos A + \cos B\cos C & -\sin^2 B \end{vmatrix}$

Expanding along R_1, we get

$\Delta = -1[\sin^2 C\sin^2 B - (\cos A + \cos C\cos B)(\cos A + \cos B\cos C)]$

$= -[\sin^2 C\sin^2 B - \cos^2 A - \cos A\cos B\cos C - \cos A\cos B\cos C - \cos^2 B\cos^2 C]$

$= -[(1-\cos^2 C)(1-\cos^2 B) - \cos^2 A - 2\cos A\cos B\cos C - \cos^2 B\cos^2 C]$

$= -[1 - \cos^2 C - \cos^2 B + \cos^2 B\cos^2 C - \cos^2 A - 2\cos A\cos B\cos C - \cos^2 B\cos^2 C]$

$= -[1 - \cos^2 C - \cos^2 B - \cos^2 A - 2\cos A\cos B\cos C]$

$= -1 + \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C$

Now, in a triangle

$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$

$\Rightarrow \quad \Delta = -1 + 1 = 0$

23. We have, $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$

$\therefore \quad |A| = a\left(\frac{1+bc}{a}\right) - bc = 1 + bc - bc = 1 \neq 0$

$\therefore \quad A^{-1}$ exists

Also, $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

Now, R.H.S. = $(a^2 + bc + 1) I - aA$

$= (a^2 + bc + 1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - a\begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$

$= \begin{bmatrix} a^2+bc+1 & 0 \\ 0 & a^2+bc+1 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ac & 1+bc \end{bmatrix}$

$= \begin{bmatrix} bc+1 & -ab \\ -ac & a^2 \end{bmatrix} = a\begin{bmatrix} \frac{bc+1}{a} & -b \\ -c & a \end{bmatrix} = a A^{-1} = \text{L.H.S.}$

OR

We have, $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$= 1\begin{vmatrix} 1 & \sin\theta \\ -\sin\theta & 1 \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & \sin\theta \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} -\sin\theta & 1 \\ -1 & -\sin\theta \end{vmatrix}$

[Expanding along R_1]

$= 1(1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$= 1 + \sin^2\theta - 0 + \sin^2\theta + 1$

$= 2 + 2\sin^2\theta = 2(1 + \sin^2\theta)$

We know that, $-1 \leq \sin\theta \leq 1 \ \forall\ \theta$

$\Rightarrow \quad 0 \leq \sin^2\theta \leq 1 \ \forall\ \theta$

$\Rightarrow \quad 1 + 0 \leq 1 + \sin^2\theta \leq 1 + 1 \ \forall\ \theta \Rightarrow 1 \leq 1 + \sin^2\theta \leq 2 \ \forall\ \theta$

$\Rightarrow \quad 2 \leq 2(1 + \sin^2\theta) \leq 4 \ \forall\ \theta$

$\Rightarrow \quad 2 \leq \Delta \leq 4$

Hence proved.

24. Let $A(x, y)$ be any point on line PQ. Then, the points P, Q and A are collinear.

So, $\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$\Rightarrow \quad 11(5 - y) - 7(5 - x) + 1(5y - 5x) = 0$

$\Rightarrow \quad 55 - 11y - 35 + 7x + 5y - 5x = 0$

$\Rightarrow \quad x - 3y + 10 = 0,$

which is the required equation of line joining points P and Q.

Now, according to the question,

area of ΔPQR = 9 sq. units.

$\therefore \quad \frac{1}{2}\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ -1 & k & 1 \end{vmatrix} = \pm 9$

$\Rightarrow \quad 11(5 - k) - 7(6) + 1(5k + 5) = \pm 18$

$\Rightarrow \quad -6k + 18 = \pm 18 \Rightarrow k = \frac{-18 \pm 18}{-6}$

For positive sign, $k = \frac{-18+18}{-6} = 0$

For negative sign, $k = \frac{-18-18}{-6} = 6$

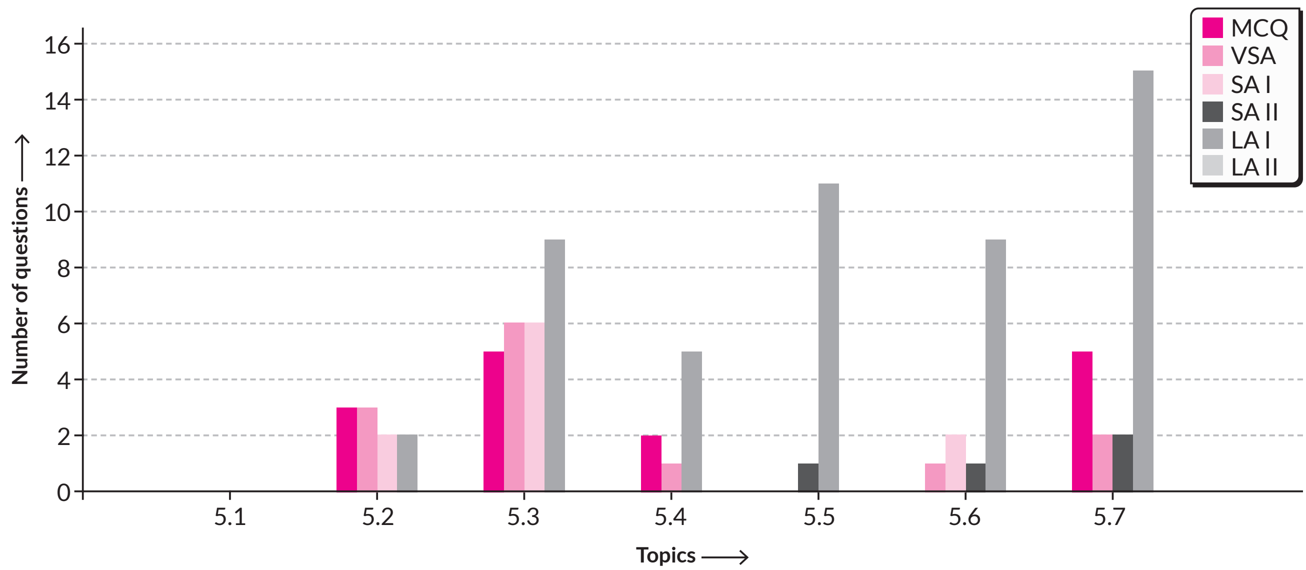
$\therefore$ Required values of k are 0 and 6.

CHAPTER 5

Continuity and Differentiability

TOPICS

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 5.3 is highly scoring topic.
- Maximum weightage is of Topic 5.3 *Differentiability.*
- Maximum LA I type questions were asked from Topic 5.7 *Second Order Derivative.*

QUICK RECAP

Continuity

- A real valued function f is said to be continuous at a point $x = c$, if the function is defined at $x = c$ and $\lim_{x \to c} f(x) = f(c)$ or we say f is continuous at $x = c$ iff $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$
- **Discontinuity of a Function :** A real function f is said to be discontinuous at $x = c$, if it is not continuous at $x = c$. *i.e.*, f is discontinuous if any of the following reasons arise :

  (i) $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ or both does not exist.

  (ii) $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

  (iii) $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \neq f(c)$

- A function f is said to be continuous in an interval (a, b) iff f is continuous at every point in the interval (a, b) ; and f is said to be continuous in the interval $[a, b]$ iff f is continuous in the interval (a, b) and it is continuous at a from the right and at b from the left.
- A function f is said to be discontinuous in the interval (a, b) if it is not continuous at atleast one point in the given interval.
- **Algebra of Continuous Functions :** If f and g be two real valued functions, continuous at $x = c$, then
  (i) $f + g$ is continuous at $x = c$.
  (ii) $f - g$ is continuous at $x = c$.
  (iii) $f \cdot g$ is continuous at $x = c$.
  (iv) $\left(\frac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).
- Composition of two continuous functions is continuous *i.e.*, if f and g are two real valued functions and g is continuous at c and f is continuous at $g(c)$, then fog is continuous at c.
- The following functions are continuous everywhere.
  (i) Constant function
  (ii) Identity function
  (iii) Polynomial function
  (iv) Modulus function
  (v) Sine and cosine functions
  (vi) Exponential function

Differentiability

- Let $f(x)$ be a real function and a be any real number. Then, we define
  (i) **Right-hand derivative :**
  $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$ and is denoted by $Rf'(a)$.
  (ii) **Left-hand derivative :**
  $\lim_{h\to 0^-} \frac{f(a-h)-f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$ and is denoted by $Lf'(a)$.
  A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.
  The common value of $Rf'(a)$ and $Lf'(a)$is denoted by $f'(a)$ and is known as the derivative of $f(x)$ at $x = a$. If, however, $Rf'(a) \neq Lf'(a)$ we say that $f(x)$ is not differentiable at $x = a$.
- A function is said to be differentiable in (a, b), if it is differentiable at every point of (a, b).
- Every differentiable function is continuous but the converse is not necessarily true.
- **Some General Derivatives**

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}; x\in(-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}; x\in(-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2}; x\in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2}; x\in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}; x\in R-[-1, 1]$	$\operatorname{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}; x\in R-[-1,1]$
$\log_e x$	$\frac{1}{x}; x>0$	a^x	$a^x \log_e a; a>0$	$\log_a x$	$\frac{1}{x\log_e a}; x>0$ and $a>0$

Exponential Function

- If a is any positive real number, then the function f defined by $f(x) = a^x$ is called the exponential function.

Logarithmic Function

- Let $a > 1$ be a real number. The logarithmic function of x to the base a is the function $y = f(x) = \log_a x$ *i.e.*, $\log_a x = b$, if $x = a^b$.
- The logarithm function, with base $a = 10$, is called common logarithm and with base $a = e$, is called natural logarithm.
- The function $\log_a x$ ($a > 0, \neq 1$) has the following properties :
  (i) $\log_a(mn) = \log_a m + \log_a n\,; m, n > 0$
  (ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n; m, n > 0$
  (iii) $\log_a m^n = n\log_a m\,; m > 0$
  (iv) $\log_a x = \frac{\log x}{\log a}; x > 0$
  (v) $\log_a a = 1, \log_a 1 = 0$

Some Properties of Derivatives

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.
6.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v \left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
7.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, f'(t) \neq 0$
8.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$

BRAIN MAP

CONTINUITY AND DIFFERENTIABILITY

Continuity

A real valued function $f(x)$ is continuous at $x = a$ if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$

In an open interval (a, b), $f(x)$ is continuous if it is continuous at every point between (a, b).

In a closed interval $[a, b]$, $f(x)$ is continuous if

(i) $f(x)$ is continuous in (a, b)

(ii) $\lim\limits_{x \to a^+} f(x) = f(a)$, $\lim\limits_{x \to b^-} f(x) = f(b)$

Algebra of Continuous Functions

If f and g are two real functions continuous at a, then $f \pm g$, $f.g$, f/g (where $g \neq 0$), kf (where $k \in R$), fg, gf are continuous at $x = a$.

Note : Constant, Polynomial, Modulus, Logarithmic and Exponential functions are everywhere continuous.

Discontinuity at a Point

(i) Removable discontinuity : Discontinuity at $x = a$ $\lim\limits_{x \to a} f(x) \neq f(a)$

(ii) Discontinuity of 1st kind : $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

(iii) Discontinuity of 2nd kind : Neither L.H.L. nor R.H.L. exists

Differentiability

A real valued function $f(x)$ is differentiable at $x = c$ if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely or $Lf'(a) = Rf'(a)$

Note : If a function is differentiable at a point then it is continuous at that point. But the converse is not always true.

Logarithmic Differentiation

If $y = u^v$, where u, v are functions of x, then

$$\log y = v \log u \Rightarrow \frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$$

Algebra of Derivatives

- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + uv'$
- $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}, v \neq 0$

Derivative of Functions

- Composite Function (Chain Rule)
  - Let $y = f(t)$ and $t = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$.
  - Let $y = f(t)$, $t = g(u)$ and $u = m(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{du} \times \dfrac{du}{dx}$.
- Implicit Function
  - Here, we differentiate the function of type $f(x, y) = 0$
- Parametric Function
  - If $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}$, $f'(t) \neq 0$

Second Order Derivative

Let $y = f(x)$, then $\dfrac{dy}{dx} = f'(x)$

If $f'(x)$ is differentiable, then

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x) \text{ or } \frac{d^2y}{dx^2} = f''(x)$$

Previous Years' CBSE Board Questions

5.2 Continuity

MCQ

1. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is continuous at
(a) $x = 1$ (b) $x = 1.5$ (c) $x = -2$ (d) $x = 4$
(2023)

2. If the function $f(x) = \begin{cases} 3x-8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$ is continuous, then the value of k is
(a) 2/7 (b) 7/2 (c) 3/7 (d) 4/7
(Term I, 2021-22) Ap

3. The function $f(x) = [x]$, where $[x]$ is the greatest integer function that is less than or equal to x, is continuous at
(a) 4 (b) −2 (c) 1.5 (d) 1
(Term I, 2021-22) U

VSA (1 mark)

4. The value of λ so that the function f defined by $f(x) = \begin{cases} \lambda x, \text{if } x \le \pi \\ \cos x, \text{if } x > \pi \end{cases}$
is continuous at $x = \pi$ is ________. *(2020)* Ap

5. Determine the value of the constant 'k' so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases}$ is continuous at $x = 0$. *(Delhi 2017)* Ap

6. Determine the value of 'k' for which the following function is continuous at $x = 3$.
$$f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3}, & x \ne 3 \\ k, & x = 3 \end{cases}$$
(AI 2017) Ap

SA I (2 marks)

7. Find the value(s) of 'λ', if the function
$f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, \text{if } x \ne 0 \\ 1, \text{if } x = 0 \end{cases}$ is continuous at $x = 0$. *(2023)*

8. Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax+1 & \text{if } x \le 3 \\ bx+3 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. *(2021C)* Ap

LA I (4 marks)

9. Find the values of p and q, for which
$$f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \pi/2 \\ p, & \text{if } x = \pi/2 \\ \frac{q(1-\sin x)}{(\pi - 2x)^2}, & \text{if } x > \pi/2 \end{cases}$$
is continuous at $x = \pi/2$. *(Delhi 2016)* Ap

10. Find the value of the constant k so that the function f, defined below, is continuous at $x = 0$, where
$$f(x) = \begin{cases} \left(\frac{1-\cos 4x}{8x^2}\right), & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$$
(AI 2014C) Ap

5.3 Differentiability

MCQ

11. The function $f(x) = |x|$ is
(a) continuous and differentiable everywhere.
(b) continuous and differentiable nowhere.
(c) continuous everywhere, but differentiable everywhere except at $x = 0$.
(d) continuous everywhere, but differentiable nowhere.
(2023)

12. The derivative of x^{2x} w.r.t. x is
(a) x^{2x-1} (b) $2x^{2x}\log x$
(c) $2x^{2x}(1 + \log x)$ (d) $2x^{2x}(1 - \log x)$ *(2023)*

13. If $y^2(2 - x) = x^3$, then $\left(\frac{dy}{dx}\right)_{(1,1)}$ is equal to
(a) 2 (b) −2 (c) 3 (d) −3/2
(Term I, 2021-22) Ap

14. The function $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 2-x & \text{for } x \ge 1 \end{cases}$ is
(a) not differentiable at $x = 1$
(b) differentiable at $x = 1$
(c) not continuous at $x = 1$
(d) neither continuous nor differentiable at $x = 1$
(Term I, 2021-22) Ev

15. If $\sec^{-1}\left(\frac{1+x}{1-y}\right) = a$, then $\frac{dy}{dx}$ is equal to
(a) $\frac{x-1}{y-1}$ (b) $\frac{x-1}{y+1}$
(c) $\frac{y-1}{x+1}$ (d) $\frac{y+1}{x-1}$ *(2020C)* Ap

VSA (1 mark)

16. If $y = \tan^{-1} x + \cot^{-1} x$, $x \in R$, then $\frac{dy}{dx}$ is equal to ________. *(2020)* Ev

17. If $\cos(xy) = k$, where k is a constant and $xy \ne n\pi$, $n \in Z$, then $\frac{dy}{dx}$ is equal to ________. *(2020)* Ev

18. Differentiate $\sin^2(\sqrt{x})$ with respect to x. *(2020)* Ap

19. Let $f(x) = x|x|$, for all $x \in R$ check its differentiability at $x = 0$. *(2020)* Ev

20. If $y = f(x^2)$ and $f'(x) = e^{\sqrt{x}}$, then find $\frac{dy}{dx}$. *(2020)* Ev

21. If $f(x)=x+1$, find $\frac{d}{dx}(fof)(x)$. *(Delhi 2019)* U

SA I (2 marks)

22. If $(x^2+y^2)^2 = xy$, then find $\frac{dy}{dx}$. *(2023)*

23. If $f(x)=\begin{cases} x^2, \text{if } x\geq 1 \\ x, \text{if } x<1 \end{cases}$, then show that f is not differentiable at $x = 1$. *(2023)*

24. Check the differentiability of $f(x) = |x - 3|$ at $x = 3$. *(2021C)* Ev

25. If $y=\sqrt{a+\sqrt{a+x}}$, then find $\frac{dy}{dx}$. *(2020C)* Ap

26. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x. *(2018)* Ev

27. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$. *(Delhi 2017)* Ev

LA I (4 marks)

Here, question 28(i) to (iii) is a case study based question of 4 marks.

28. Let $f(x)$ be a real valued function. Then its
   - Left Hand Derivative (L.H.D.) :
   $Lf'(a)=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$
   - Right Hand Derivative (R.H.D.) :
   $Rf'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

   Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$ exist and both are equal.

   For the function $f(x)=\begin{cases} |x-3|, x\geq 1 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, x<1 \end{cases}$

   answer the following questions :

   (i) What is R.H.D. of $f(x)$ at $x = 1$?

   (ii) What is L.H.D. of $f(x)$ at $x = 1$?

   (iii) Check if the function $f(x)$ is differentiable at $x = 1$.

   OR

   (iii) Find $f'(2)$ and $f'(-1)$. *(2023)*

29. Find the values of a and b, if the function f defined by
$$f(x)=\begin{cases} x^2+3x+a, & x\leq 1 \\ bx+2, & x>1 \end{cases}$$
is differentiable at $x = 1$. *(Foreign 2016)* Ap

30. If $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\leq 1$ then find $\frac{dy}{dx}$. *(NCERT Exemplar, Delhi 2015)* Ap

31. If $f(x)=\sqrt{x^2+1}$; $g(x)=\frac{x+1}{x^2+1}$ and $h(x) = 2x-3$, then find $f'[h'(g'(x))]$. *(AI 2015)* Ap

32. Show that the function $f(x) = |x - 1| + |x + 1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$. *(AI 2015)* Ev

33. Find whether the following function is differentiable at $x = 1$ and $x = 2$ or not.
$$f(x)=\begin{cases} x, & x<1 \\ 2-x, & 1\leq x\leq 2 \\ -2+3x-x^2, & x>2 \end{cases}$$
(Foreign 2015) Ap

34. For what value of λ the function defined by $f(x)=\begin{cases} \lambda(x^2+2), & \text{if } x\leq 0 \\ 4x+6, & \text{if } x>0 \end{cases}$ is continuous at $x = 0$? Hence check the differentiability of $f(x)$ at $x = 0$. *(AI 2015C)* Ap

35. If $\cos y = x\cos(a + y)$, where $\cos a \neq \pm 1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$. *(Foreign 2014)* Ev

36. If $y=\sin^{-1}\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\}$ and $0 < x < 1$, then find $\frac{dy}{dx}$. *(AI 2014C)* Ap

5.4 Exponential and Logarithmic Functions

MCQ

37. If $y = \log(\sin e^x)$, then $\frac{dy}{dx}$ is

(a) $\cot e^x$ (b) $\operatorname{cosec} e^x$
(c) $e^x \cot e^x$ (d) $e^x \operatorname{cosec} e^x$ *(2023)*

38. If $y = \tan^{-1}(e^{2x})$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{2e^{2x}}{1+e^{4x}}$ (b) $\frac{1}{1+e^{4x}}$
(c) $\frac{2}{e^{2x}+e^{-2x}}$ (d) $\frac{1}{e^{2x}-e^{-2x}}$
(Term I, 2021-22) Ev

VSA (1 mark)

39. If $y = \log(\cos e^x)$, then find $\frac{dy}{dx}$. *(NCERT, AI 2019)* Ap

LA I (4 marks)

40. If $y = e^{x^2\cos x} + (\cos x)^x$, then find $\frac{dy}{dx}$. *(2020)* Ev

41. If $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx}=\frac{x+y}{x-y}$. *(Delhi 2019)* Ev

42. If $y=\frac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log\sqrt{1-x^2}$, then prove that $\frac{dy}{dx}=\frac{\cos^{-1}x}{(1-x^2)^{3/2}}$. *(Delhi 2015C)* Ap

43. If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx}+e^{y-x}=0$ *(Foreign 2014)* Ev

44. If $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx}=\frac{2a^3}{x^4-a^4}$. *(AI 2014C)* Ap

5.5 Logarithmic Differentiation

SA II (3 marks)

45. If $e^{y-x}=y^x$, prove that $\frac{dy}{dx}=\frac{y(1+\log y)}{x\log y}$. *(2021C)* Ev

LA I (4 marks)

46. If $y=x^3(\cos x)^x+\sin^{-1}\sqrt{x}$, find $\frac{dy}{dx}$. *(2020)* Ev

47. If $y=(\log x)^x+x^{\log x}$, then find $\frac{dy}{dx}$. *(NCERT, 2020)* Ap

48. Find $\frac{dy}{dx}$, if $x^y\cdot y^x=x^x$. *(2019C)* Ap

49. If $x^y-y^x=a^b$, find $\frac{dy}{dx}$. *(Delhi 2019)* Ev

50. If $y=(x)^{\cos x}+(\cos x)^{\sin x}$, then find $\frac{dy}{dx}$. *(AI 2019)*

51. Differentiate the function $(\sin x)^x+\sin^{-1}\sqrt{x}$ with respect to x. *(Delhi 2017)* Ap

OR

If $y=(\sin x)^x+\sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$. *(Delhi 2015C)* Ev

52. If $x^y+y^x=a^b$, then find $\frac{dy}{dx}$. *(AI 2017)* Ev

53. Differentiate $x^{\sin x}+(\sin x)^{\cos x}$ with respect to x. *(AI 2016)* Ap

54. If $x^my^n=(x+y)^{m+n}$, prove that $\frac{dy}{dx}=\frac{y}{x}$. *(Foreign 2014)* Ev

55. If $(x-y)\cdot e^{\frac{x}{x-y}}=a$, prove that $y\frac{dy}{dx}+x=2y$. *(Delhi 2014C)* Ev

56. If $(\tan^{-1}x)^y+y^{\cot x}=1$, then find $\frac{dy}{dx}$. *(AI 2014C)* Ev

5.6 Derivatives of Functions in Parametric Forms

VSA (1 mark)

57. If $x=e^t\sin t$, $y=e^t\cos t$, then the value of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ is ______. *(2020C)* Ev

SA I (2 marks)

58. If $x=a\sec\theta$, $y=b\tan\theta$, then find $\frac{dy}{dx}$ at $\theta=\frac{\pi}{3}$. *(2020)* Ev

59. Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$. *(NCERT, 2020)* Ap

SA II (3 marks)

60. Differentiate $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ w.r.t. $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$. *(2023)*

LA I (4 marks)

61. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ with respect to $\cos^{-1}x^2$. *(AI 2019)* Ev

62. If $x=a(2\theta-\sin 2\theta)$ and $y=a(1-\cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$. *(2018)* Ev

63. If $x=a\sin 2t(1+\cos 2t)$ and $y=b\cos 2t(1-\cos 2t)$, find the values of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ and $t=\frac{\pi}{3}$. *(Delhi 2016, AI 2016)* Ap

OR

If $x=a\sin 2t\,(1+\cos 2t)$ and $y=b\cos 2t(1-\cos 2t)$, show that at $t=\frac{\pi}{4}$, $\left(\frac{dy}{dx}\right)=\frac{b}{a}$. *(NCERT Exemplar, AI 2014)* Ev

64. Differentiate

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) \text{ w.r.t. } \sin^{-1}\frac{2x}{1+x^2}, \text{ if } x\in(-1,1)$$

(Foreign 2016, Delhi 2014) Ev

65. If $x=ae^t(\sin t+\cos t)$ and $y=ae^t(\sin t-\cos t)$, prove that $\frac{dy}{dx}=\frac{x+y}{x-y}$. *(AI 2015C)* Ap

66. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, when $x\neq 0$. *(Delhi 2014)* Ev

67. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. *(Delhi 2014)* Ev

68. Find the value of $\frac{dy}{dx}$ at $\theta=\frac{\pi}{4}$, if $x=ae^\theta(\sin\theta-\cos\theta)$ and $y=ae^\theta(\sin\theta+\cos\theta)$. *(AI 2014)* Ap

69. If $x=\cos t(3-2\cos^2 t)$ and $y=\sin t\,(3-2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$. *(AI 2014)* Ev

5.7 Second Order Derivative

MCQ

70. If $x=A\cos 4t+B\sin 4t$, then $\frac{d^2x}{dt^2}$ is equal to

(a) x (b) $-x$ (c) $16x$ (d) $-16x$

(2023)

71. If $y = e^{-x}$, then $\frac{d^2y}{dx^2}$ is equal to
(a) $-y$ (b) y (c) x (d) $-x$
(Term I, 2021-22) U

72. If $x = t^2 + 1$, $y = 2at$, then $\frac{d^2y}{dx^2}$ at $t = a$ is
(a) $-\frac{1}{a}$ (b) $-\frac{1}{2a^2}$ (c) $\frac{1}{2a^2}$ (d) 0
(Term I, 2021-22) Ev

73. If $y = \sin(2\sin^{-1}x)$, then $(1 - x^2)\, y_2$ is equal to
(a) $-xy_1 + 4y$ (b) $-xy_1 - 4y$
(c) $xy_1 - 4y$ (d) $xy_1 + 4y$
(Term I, 2021-22) Ap

74. If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals
(a) $-\frac{1}{x}$ (b) $-\frac{1}{x^2}$ (c) $\frac{2}{x^2}$ (d) $-\frac{2}{x^2}$
(2020) Ev

SA I (2 marks)

75. If $x = at^2$, $y = 2at$, then find $\frac{d^2y}{dx^2}$. *(2020)* Ev

76. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$. *(2020)* Ap

SA II (3 marks)

77. If $y = \tan x + \sec x$, then prove that
$$\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}.$$ *(2023)*

78. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, then show that $\frac{dy}{dx} = -\frac{x}{y}$ and hence show that
$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$ *(2021C)* Ap

LA I (4 marks)

79. If $x = a\sec^3\theta$, $y = a\tan^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$.
(2020, Delhi 2015C) Ev

80. If $y = a\cos(\log x) + b\sin(\log x)$, show that
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$ *(2019C)* Ap

81. If $y = (\sin^{-1}x)^2$, prove that
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$ *(Delhi 2019)* Ap

82. If $x = \sin t$, $y = \sin pt$, prove that
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0.$$ *(AI 2019, Foreign 2016)* Ap

83. If $y = \sin(\sin x)$, prove that
$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2x = 0.$$ *(2018)* Ap

84. If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.
(Delhi 2017) Ev

85. If $e^y(x + 1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.
(AI 2017) Ap

86. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.
(Delhi 2016, 2014) Ap

87. If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$ *(AI 2016)* Ap

88. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that
$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$ *(Delhi 2015, Foreign 2014)* Ap

89. If $y = e^{m\sin^{-1}x}, -1 \le x \le 1$, then show that
$$(1-x^2)\frac{d^2y}{dx^2} - \frac{x\,dy}{dx} - m^2y = 0.$$ *(AI 2015)* Ap

90. If $y = (x + \sqrt{1+x^2})^n$, then show that
$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y.$$ *(Foreign 2015)* Ap

91. If $y = Ae^{mx} + Be^{nx}$, show that
$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$$ *(AI 2015C, 2014)* Ap

92. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. *(Delhi 2014C)* Ev

93. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t$, evaluate
$\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. *(Delhi 2014C)* Ev

CBSE Sample Questions

5.2 Continuity

MCQ

1. The value of 'k' for which the function
$$f(x) = \begin{cases} \frac{1-\cos 4x}{8x^2}, & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$$ is continuous at $x = 0$ is
(a) 0 (b) -1 (c) 1 (d) 2
(2022-23) Ap

2. The value of $k (k < 0)$ for which the function f defined as $f(x)=\begin{cases}\frac{1-\cos kx}{x\sin x}, & x\neq 0\\ \frac{1}{2}, & x=0\end{cases}$ is continuous at $x = 0$ is

(a) ± 1 (b) -1 (c) $\pm\frac{1}{2}$ (d) $\frac{1}{2}$

(Term I, 2021-22) Ap

3. The point(s), at which the function f given by $f(x)=\begin{cases}\frac{x}{|x|}, x<0\\ -1, x\geq 0\end{cases}$ is continuous, is/are

(a) $x \in R$ (b) $x = 0$
(c) $x \in R - \{0\}$ (d) $x = -1$ and 1

(Term I, 2021-22) Ap

SA I (2 marks)

4. Find the value(s) of k so that the following function is continuous at $x = 0$.

$f(x)=\begin{cases}\frac{1-\cos kx}{x\sin x} & \text{if } x\neq 0\\ \frac{1}{2} & \text{if } x=0\end{cases}$ *(2020-21)* Ap

5.3 Differentiability

SA I (2 marks)

5. If $y\sqrt{1-x^2}+x\sqrt{1-y^2}=1$, then prove that $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$. *(2022-23)* Ev

SA II (3 marks)

6. Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 2$ is not differentiable at $x = 1$. *(2020-21)* Ap

5.4 Exponential and Logarithmic Functions

MCQ

7. If $e^x + e^y = e^{x+y}$, then $\frac{dy}{dx}$ is

(a) e^{y-x} (b) e^{y+x} (c) $-e^{y-x}$ (d) $2e^{x-y}$

(Term I, 2021-22) Ev

8. If $y = \log(\cos e^x)$, then $\frac{dy}{dx}$ is

(a) $\cos e^{x-1}$ (b) $e^{-x}\cos e^x$
(c) $e^x \sin e^x$ (d) $-e^x \tan e^x$

(Term I, 2021-22) Ev

5.5 Logarithmic Differentiation

SA I (2 marks)

9. If $y=e^{x\sin^2 x}+(\sin x)^x$, find $\frac{dy}{dx}$. *(2020-21)* Ap

5.6 Derivatives of Functions in Parametric Forms

MCQ

10. The derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ w.r.t. $\sin^{-1}x, \frac{1}{\sqrt{2}}<x<1$, is

(a) 2 (b) $\frac{\pi}{2}-2$ (c) $\frac{\pi}{2}$ (d) -2

(Term I, 2021-22) Ev

5.7 Second Order Derivative

MCQ

11. If $y = 5\cos x - 3\sin x$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $-y$ (b) y (c) $25y$ (d) $9y$

(Term I, 2021-22) Ap

12. If $x = a\sec\theta, y = b\tan\theta$, then $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$ is

(a) $\frac{-3\sqrt{3}b}{a^2}$ (b) $\frac{-2\sqrt{3}b}{a}$ (c) $\frac{-3\sqrt{3}b}{a}$ (d) $\frac{-b}{3\sqrt{3}a^2}$

(Term I, 2021-22) Ap

SA II (3 marks)

13. If $x = a\sec\theta, y = b\tan\theta$, find $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$. *(2020-21)* Ap

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. (b): Let $x = 1.5$

$\therefore$ L.H.L. $=\underset{x\to 1.5^-}{\text{Lt}} f(x)=\underset{h\to 0}{\text{Lt}}[1.5-h]=1$

and R.H.L. $=\underset{x\to 1.5^+}{\text{Lt}} f(x)=\underset{h\to 0}{\text{Lt}}[1.5+h]=1$

$\because$ L. H. L. = R. H. L.

$\therefore$ $f(x)$ is continuous at $x = 1.5$

Also, greatest integer function is discontinuous at all integral values of x.

2. (b): Since $f(x)$ is continuous at $x = 5$,

$\Rightarrow \lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$

$\Rightarrow 3(5) - 8 = 2k \Rightarrow 7 = 2k \Rightarrow k=\frac{7}{2}$

3. (c): Since, greatest integer function *i.e.*, $[x]$ is continuous at all points except at integers.

$\therefore$ $f(x)$ is continuous at 1.5.

4. $\because$ $f(x)$ is continuous at $x = \pi$

$\therefore$ $f(\pi) = \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x)$ ...(i)

Here, $f(\pi) = \lambda\pi$ ...(ii)

And $\lim_{x\to\pi^+} f(x) = \lim_{h\to 0} f(\pi+h)$

$= \lim_{h\to 0} \cos(\pi+h) = -1$ ...(iii)

From (i), (ii) and (iii), we get

$\lambda\pi = -1 \Rightarrow \lambda = -\frac{1}{\pi}$

Concept Applied

➲ A real function f is said to be continuous if it is continuous at every point in the domain of f.

5. We have, $f(x) = \begin{cases} \frac{kx}{|x|}, & x<0 \\ 3\ , & x \ge 0 \end{cases}$

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{kx}{-x} = -k$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3 = 3$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ L.H.L. = R.H.L. = $f(0)$

$\Rightarrow$ $-k = 3 \Rightarrow k = -3$

6. Given, $f(x)$ is continuous at $x = 3$.

So, $\lim_{x\to 3} f(x) = f(3) \Rightarrow \lim_{x\to 3} \frac{(x+3)^2-36}{x-3} = k$

$\Rightarrow \lim_{x\to 3} \frac{(x+3)^2-6^2}{x-3} = k \Rightarrow \lim_{x\to 3} \frac{(x+3+6)(x+3-6)}{x-3} = k$

$\Rightarrow$ $3 + 3 + 6 = k \Rightarrow k = 12$

7. We have, $f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, & x \ne 0 \\ 1\ , & x = 0 \end{cases}$ is continuous at $x = 0$

$\therefore$ L.H.L. = R.H.L.

$\Rightarrow \underset{x\to 0}{\text{Lt}} \frac{\sin^2 \lambda x}{x^2} = 1 \Rightarrow \underset{x\to 0}{\text{Lt}} \frac{\sin^2 \lambda x}{\lambda^2 x^2} \cdot \lambda^2 = 1$

$\Rightarrow \lambda^2 \underset{\lambda x\to 0}{\text{Lt}} \left(\frac{\sin \lambda x}{\lambda x}\right)^2 = 1 \Rightarrow \lambda^2 \cdot 1 = 1 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

8. We have, $f(x) = \begin{cases} ax+1 & \text{if } x \le 3 \\ bx+3 & \text{if } x > 3 \end{cases}$

Since, $f(x)$ is continuos at $x = 3$

$\therefore$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \Rightarrow \lim_{x\to 3} ax+1 = \lim_{x\to 3} bx+3$

$\Rightarrow$ $3a + 1 = 3b + 3 \Rightarrow 3a - 3b = 2 \Rightarrow a - b = 2/3$

9. $\because$ $f(x)$ is continuous at $\pi/2$.

$\therefore$ $\lim_{x\to \pi/2^-} f(x) = \lim_{x\to \pi/2^+} f(x) = f(\pi/2)$ ...(1)

Now, $\lim_{x\to \pi/2^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$= \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h}$

$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$

$= \lim_{h\to 0} \frac{(1+\cos^2 h+\cos h)}{3(1+\cos h)} = \frac{1+1+1}{3(1+1)} = \frac{1}{2}$

and $\lim_{x\to \pi/2^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$

$= \lim_{h\to 0} \frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2} = \lim_{h\to 0} \frac{q(1-\cos h)}{4h^2}$

$= \frac{q}{4} \times \lim_{h\to 0} \frac{2\sin^2 \frac{h}{2}}{\frac{h^2}{4}\times 4} = \frac{q}{4}\times\frac{2}{4} = \frac{q}{8}$ and $f(\pi/2) = p$

$\therefore$ $\frac{1}{2} = \frac{q}{8} = p$ [From (1)]

$\Rightarrow$ $p = \frac{1}{2}$ and $q = 4$

Commonly Made Mistake

➲ Remember the difference between left hand limit and right hand limit.

10. $\because$ $f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = k$

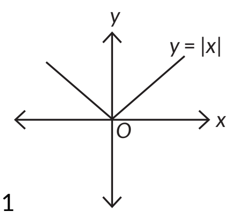
and $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos 4x}{8x^2}$

$= \lim_{x\to 0} \frac{2\cdot \sin^2 2x}{8x^2} = \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)^2 = 1$

$\because$ f is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x\to 0} f(x) \Rightarrow k = 1$

11. **(c):** $f(x) = |x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$



The function $f(x)$ is continuous everywhere but not differentiable at $x = 0$ as at $x = 0$

$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{-x-0}{x} = -1$

$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = 1$

$\therefore$ $Lf'(0) \ne Rf'(0)$, so $f(x)$ is not differentiable at $x = 0$.

12. **(c):** Let $y = x^{2x}$

Taking log on both sides, we get

$\log y = 2x \log x$

Differentiating both sides w.r.t. x, we get

$\frac{1}{y}\frac{dy}{dx} = 2\left\{x \cdot \frac{1}{x} + \log x \cdot 1\right\}$

$\Rightarrow \frac{dy}{dx} = 2y\{1+\log x\} = 2x^{2x}(1+\log x)$

13. **(a):** Given, $y^2(2 - x) = x^3$

$\Rightarrow y^2 = \frac{x^3}{2-x} \Rightarrow 2y \cdot \frac{dy}{dx} = \frac{(2-x)\times 3x^2 - x^3(-1)}{(2-x)^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{6x^2-2x^3}{2y(2-x)^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)}=\frac{6-2}{2\times1}=2$$

14. (a): At $x = 1$ $\lim_{x\to1^-} f(x)=\lim_{x\to1} x^2=1$

And $\lim_{x\to1^+} f(x)=\lim_{x\to1} 2-x=1$

Also, $f(1) = 2 - 1 = 1$ $\because \lim_{x\to1^-} f(x)=\lim_{x\to1^+} f(x)=f(x)$

$\therefore$ $f(x)$ is continuous at $x = 1$

Now, $\text{L.H.D.}=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{x^2-1}{x-1}=\lim_{x\to1}(x+1)=2$

$\text{R.H.D.}=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{(2-x)-1}{x-1}=-1$

$\because$ L.H.D. $\neq$ R.H.D. $\therefore$ $f(x)$ is not differentiable at $x = 1$.

Commonly Made Mistake

- Every continuous function is not differentiable.

15. (c): Given, $\sec^{-1}\left(\frac{1+x}{1-y}\right)=a \Rightarrow \sec a=\frac{1+x}{1-y}$

On differentiating, we get

$$\frac{(1-y)+(1+x)\frac{dy}{dx}}{(1-y)^2}=0 \Rightarrow (1+x)\frac{dy}{dx}=y-1 \Rightarrow \frac{dy}{dx}=\frac{y-1}{1+x}$$

16. We have, $y = \tan^{-1}x + \cot^{-1}x$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x^2+1}-\frac{1}{x^2+1} \Rightarrow \frac{dy}{dx}=0$$

17. We have, $\cos(xy) = k$

$$\Rightarrow -\sin(xy)\left(y+x\frac{dy}{dx}\right)=0 \Rightarrow y+x\frac{dy}{dx}=0 \qquad [\because xy\neq n\pi]$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y}{x}$$

18. Let $y=\sin^2(\sqrt{x})$

$$\therefore \frac{dy}{dx}=2\sin(\sqrt{x})\cdot\cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}}\sin(2\sqrt{x})$$

Key Points

- $\sin 2x = 2\sin x\cos x$

19. To check the differentiability of $f(x) = x|x|$ at $x = 0$.

Consider, $\lim_{h\to0}\frac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to0}\frac{f(h)-f(0)}{h}=\lim_{h\to0}\frac{h|h|-0}{h}=\lim_{h\to0}|h|=0$$

Hence, $f'(0)$ exists, so $f(x) = x|x|$ is differentiable at $x = 0$.

Answer Tips

- $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

20. We have, $y = f(x^2) \Rightarrow \frac{dy}{dx}=f'(x^2)\cdot 2x$

$=e^x\cdot 2x \qquad [\because f'(x)=e^{\sqrt{x}}]$

$= 2x\,e^x$

21. Given, $f(x) = x + 1$

Now, $(f\circ f)(x) = f(f(x)) = f(x + 1) = (x + 1) + 1 = x + 2$

$$\therefore \frac{d}{dx}(f\circ f)(x)=\frac{d}{dx}(x+2)=1$$

22. We have, $(x^2 + y^2)^2 = xy$

$\Rightarrow$ $x^4 + y^4 + 2x^2y^2 = xy$

On differentiating both sides w.r.t. x, we get

$$4x^3+4y^3\frac{dy}{dx}+4xy^2+4x^2y\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow \frac{dy}{dx}(4y^3+4x^2y-x)=y-4x^3-4xy^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4y^3+4x^2y-x}$$

23. We have, $f(x)=\begin{cases}x^2, & \text{if } x\ge1\\ x, & \text{if } x<1\end{cases}$

$$\text{R.H.D.}=\underset{x\to1^+}{\text{Lt}}\frac{f(x)-f(1)}{x-1}=\underset{x\to1}{\text{Lt}}\frac{x^2-1}{x-1}=\underset{x\to1}{\text{Lt}}\frac{(x-1)(x+1)}{x-1}$$

$$=\underset{x\to1}{\text{Lt}}(x+1)=2$$

$$\text{L.H.D.}=\underset{x\to1^-}{\text{Lt}}\frac{f(x)-f(1)}{x-1}=\underset{x\to1}{\text{Lt}}\frac{x-1}{x-1}=1$$

$\because$ L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x)$ is not differentiable at $x = 1$

24. We have, $f(x) = |x - 3|$

$$f(x)=\begin{cases}x-3, & x\ge3\\ -x+3, & x<3\end{cases}$$

At $x = 3$

$$f'(3^+)=\lim_{h\to0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0}\frac{3+h-3-(3-3)}{h}=\lim_{h\to0}1=1$$

$$f'(3^-)=\lim_{h\to0}\frac{f(3-h)-f(3)}{-h}=\lim_{h\to0}\frac{-3+h+3-(0)}{-h}=\lim_{h\to0}-1=-1$$

$\because$ $f'(3^+)\neq f'(3^-)$ $\therefore$ $f(x)$ is not differentiable at $x = 3$.

25. We have, $y=\sqrt{a+\sqrt{a+x}}$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{1}{\sqrt{a+\sqrt{a+x}}}\times\frac{1}{2}\frac{1}{\sqrt{a+x}}=\frac{1}{4}\cdot\frac{1}{\sqrt{a\sqrt{(a+x)}+a+x}}$$

26. Let, $y=\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left[\tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) \Rightarrow \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right] \Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right)=-\frac{1}{2}$$

27. We have, $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x on both sides, we get

$$2\sin y\cos y\frac{dy}{dx}+(-\sin xy)\left(x\frac{dy}{dx}+y\right)=0$$

($\because$ Product rule : $(uv)' = u'v + uv'$)

$$\Rightarrow \frac{dy}{dx}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{\left(1,\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2}-\sin\frac{\pi}{4}} = \frac{\pi}{4(\sqrt{2}-1)}$$

28. Given $f(x)=\begin{cases} |x-3|, & x\ge 1 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1 \end{cases}$

$$f(x)=\begin{cases} -(x-3), & 1\le x<3 \\ x-3, & x\ge 3 \\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}, & x<1 \end{cases}$$

(i) R.H.D. at $x = 1$

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{-(1+h-3)-[-(1-3)]}{h}$$

$$= \lim_{h\to 0}\frac{(-h+2-2)}{h} = \lim_{h\to 0}\frac{-h}{h}=-1$$

(ii) L.H.D. at $x = 1$

$$f'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$=\lim_{h\to 0}\left[\frac{\left[\frac{(1-h)^2}{4}-\frac{3}{2}(1-h)+\frac{13}{4}\right]-\left[\frac{1}{4}-\frac{3}{2}+\frac{13}{4}\right]}{-h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\frac{1+h^2-2h}{4}-\frac{3}{2}+\frac{3h}{2}+\frac{13}{4}-\frac{1}{4}+\frac{3}{2}-\frac{13}{4}}{-h}\right]$$

$$=\lim_{h\to 0}\left[\frac{\frac{h^2}{4}-\frac{h}{2}+\frac{3h}{2}}{-h}\right]=\lim_{h\to 0}\left[\frac{h^2}{-4h}+\frac{h}{-h}\right]=0-1=-1$$

(iii) (a) We know, that function $f(x)$ is differentiable at $x = 1$ if L.H.D. = R.H.D. = $f'(1)$

$\Rightarrow$ $-1 = -1 = -1$

Hence, the given function $f(x)$ is differentiable at $x = 1$.

OR

(b) Differentiate $f(x)$ w.r.t. x, we get

$$f'(x)=\begin{cases} -1, & 1\le x<3 \\ 1, & x\ge 3 \\ \frac{x}{2}-\frac{3}{2}, & x<1 \end{cases}$$

$$f'(2)=-1 \text{ and } f'(-1)=\frac{-1}{2}-\frac{3}{2}=\frac{-4}{2}=-2$$

29. Given that $f(x)$ is differentiable at $x = 1$. Therefore, $f(x)$ is continuous at $x = 1$

$$\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$$

$$\Rightarrow \lim_{x\to 1}(x^2+3x+a)=\lim_{x\to 1}(bx+2)$$

$\Rightarrow$ $1 + 3 + a = b + 2 \Rightarrow a - b + 2 = 0$ ...(1)

Again, $f(x)$ is differentiable at $x = 1$. So,

(L.H.D. at $x = 1$) = (R.H.D. at $x = 1$)

$$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x^2+3x+a)-(4+a)}{x-1}=\lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{bx-2-a}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x+4)(x-1)}{x-1}=\lim_{x\to 1}\frac{bx-b}{x-1} \quad \text{[From (1)]}$$

$$\Rightarrow \lim_{x\to 1}(x+4)=\lim_{x\to 1}\frac{b(x-1)}{x-1} \Rightarrow 5=b$$

Putting $b = 5$ in (1), we get $a = 3$. Hence, $a = 3$ and $b = 5$

Key Points

- If a function f is differentiable at a point c, then it is also continuous at that point.

30. We have,

$$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$$

Putting $x^2 = \cos\theta \Rightarrow \theta = \cos^{-1}(x^2)$ we get

$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)=\tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)=\frac{\pi}{4}+\frac{\theta}{2} \Rightarrow y=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Differentiating w.r.t. x on both sides, we get

$$\frac{dy}{dx}=-\frac{1\times 2x}{2\sqrt{1-x^4}}=\frac{-x}{\sqrt{1-x^4}}$$

31. Here $f(x)=\sqrt{x^2+1}=(x^2+1)^{\frac{1}{2}}$

$$\Rightarrow f'(x)=\frac{1}{2}\cdot(x^2+1)^{-\frac{1}{2}}\cdot 2x=\frac{x}{\sqrt{x^2+1}}, \quad \text{...(1)}$$

and $g(x)=\dfrac{x+1}{x^2+1}$

$$\Rightarrow g'(x)=\frac{(x^2+1)\cdot 1-(x+1)\cdot 2x}{(x^2+1)^2}=\frac{-x^2-2x+1}{(x^2+1)^2} \quad \text{...(2)}$$

and $h(x) = 2x - 3 \Rightarrow h'(x) = 2$ ...(3)

$$\therefore f'[h'(g'(x))]=f'\left[h'\left(\frac{-x^2-2x+1}{(x^2+1)^2}\right)\right] \quad \text{[Using (2)]}$$

$$=f'(2) \quad \text{[Using (3)]}$$

$$=\frac{2}{\sqrt{2^2+1}}=\frac{2}{\sqrt{5}} \quad \text{[Using (1)]}$$

Concept Applied

- Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{du}\times\dfrac{du}{dx}$

32. The given function is $f(x) = |x - 1| + |x + 1|$

$$=\begin{cases} -(x-1)-(x+1), & x<-1 \\ -(x-1)+x+1, & -1\le x\le 1 \\ x-1+x+1, & x>1 \end{cases}=\begin{cases} -2x, & x<-1 \\ 2, & -1\le x\le 1 \\ 2x, & x>1 \end{cases}$$

At $x = 1$,

$$f'(1^-)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{2-2}{-h}=0$$

$$f'(1^+)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{2(1+h)-2}{h}=\lim_{h\to 0}\frac{2h}{h}=2$$

$$\therefore f'(1^-)\ne f'(1^+)$$

$\Rightarrow$ f is not differentiable at $x = 1$.

At $x = -1$,

$$f'(-1^-) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h}$$

$$= \lim_{h\to 0}\frac{-2(-1-h)-(2)}{-h} = \lim_{h\to 0}\frac{2h}{-h} = -2$$

$$f'(-1^+) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{2-2}{h} = 0$$

$\therefore$ $f'(-1^-) \neq f'(-1^+)$

$\Rightarrow$ f is not differentiable at $x = -1$.

33. At $x = 1$

$$f'(1^-) = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x-1}{x-1} = 1$$

$$f'(1^+) = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{2-x-1}{x-1} = \lim_{x\to 1}\frac{1-x}{x-1} = -1$$

Since, $f'(1^-) \neq f'(1^+)$

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

At $x = 2$

$$f'(2^-) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{2-x-0}{x-2} = -1$$

$$f'(2^+) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2}\frac{-2+3x-x^2-0}{x-2} = \lim_{x\to 2}\frac{(1-x)(x-2)}{x-2} = -1$$

Since, $f'(2^-) = f'(2^+)$

$\therefore$ $f(x)$ is differentiable at $x = 2$.

34. Here $f(x) = \begin{cases}\lambda(x^2+2), & \text{if } x \le 0\\ 4x+6, & \text{if } x > 0\end{cases}$

At $x = 0$, $f(0) = \lambda(0^2 + 2) = 2\lambda$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\lambda[(0-h)^2+2] = 2\lambda$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}[4(0+h)+6] = 6$$

$\therefore$ For f to be continuous at $x = 0$

$2\lambda = 6 \Rightarrow \lambda = 3$.

Hence the function becomes

$$f(x) = \begin{cases}3(x^2+2), & \text{if } x \le 0\\ 4x+6 & \text{if } x > 0\end{cases}$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h} = \lim_{h\to 0}\frac{3(h^2+2)-6}{-h} = \lim_{h\to 0}(-3h) = 0$$

$$\text{and } f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h} = \lim_{h\to 0}\frac{4h+6-6}{h} = 4$$

$\Rightarrow$ $f'(0^-) \neq f'(0^+)$ $\therefore$ f is not differentiable at $x = 0$.

35. We have $\cos y = x\cos(a + y)$

$$\Rightarrow \quad x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y on both sides, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\left(\frac{d}{dy}\cos y\right) - \cos y\left(\frac{d}{dy}\cos(a+y)\right)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y)+\cos y\sin(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos y\sin(a+y)-\cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin[(a+y)-y]}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)} \quad \therefore \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Concept Applied

- Quotient rule : $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$

36. We have, $y = \sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$

$$\Rightarrow \quad y = \sin^{-1}\left(x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right)$$

$$\Rightarrow y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{d}{dx}(\sqrt{x}) \quad \left(\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{x-x^2}}$$

37. **(c)** : $y = \log(\sin e^x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin e^x}\cdot\frac{d}{dx}(\sin e^x) = \frac{1}{\sin e^x}\cos e^x\cdot\frac{d}{dx}e^x = \frac{1}{\sin e^x}\cos e^x\cdot e^x$$

$= e^x \cot e^x$

38. **(a)**: Given, $y = \tan^{-1}(e^{2x})$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+e^{4x}}\times 2e^{2x} = \frac{2e^{2x}}{1+e^{4x}}$$

Answer Tips

- $\frac{d}{dx}(e^x) = e^x$

39. Given $y = \log(\cos e^x)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\cos e^x}(-\sin e^x\cdot e^x) = -e^x\tan e^x$$

40. We have, $y = e^{x^2\cos x} + (\cos x)^x$

$= e^{x^2\cos x} + e^{x(\ln\cos x)}$

$$\therefore \quad \frac{dy}{dx} = e^{x^2\cos x}\frac{d}{dx}(x^2\cos x) + e^{x\ln\cos x}\frac{d}{dx}(x\ln\cos x)$$

$$= e^{x^2\cos x}(2x\cos x - x^2\sin x) + e^{x\ln\cos x}\left(\ln\cos x - \frac{x}{\cos x}\sin x\right)$$

$$= e^{x^2\cos x}(2x\cos x - x^2\sin x) + (\cos x)^x(\ln\cos x - x\tan x)$$

41. Given, $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

On differentiating w.r.t. x on both sides, we get

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right) = 2\times\frac{1}{1+\frac{y^2}{x^2}}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\frac{dy}{dx} = \frac{2x^2}{x^2+y^2}\left(\frac{1}{x}\frac{dy}{dx} + y\left(\frac{-1}{x^2}\right)\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{2y}{x^2+y^2} - \frac{2x}{x^2+y^2}\right] = \frac{2x^2}{x^2+y^2}\left[\frac{-y}{x^2} - \frac{1}{x}\right]$$

$$\Rightarrow \frac{2(y-x)}{x^2+y^2}\frac{dy}{dx}=\frac{-2x^2}{x^2+y^2}\left(\frac{y+x}{x^2}\right) \Rightarrow \frac{dy}{dx}=\frac{x+y}{x-y}$$

Key Points

- To differentiate an implicit function, consider y as a function of x then apply derivative rules.

42. Here $y=\frac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log\sqrt{1-x^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{\sqrt{1-x^2}\cdot\frac{d}{dx}(x\cos^{-1}x)-x\cos^{-1}x\cdot\frac{d}{dx}\sqrt{1-x^2}}{1-x^2}-\frac{1}{\sqrt{1-x^2}}\cdot\frac{d}{dx}(\sqrt{1-x^2})$$

$$=\frac{\sqrt{1-x^2}\cdot\left(1\cdot\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}\right)-x\cos^{-1}x\left(\frac{-x}{\sqrt{1-x^2}}\right)}{1-x^2}-\frac{1}{\sqrt{1-x^2}}\left(\frac{-x}{\sqrt{1-x^2}}\right)$$

$$=\frac{\sqrt{1-x^2}\cos^{-1}x-x+\frac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{1-x^2}+\frac{x}{1-x^2}$$

$$=\frac{(1-x^2)\cos^{-1}x+x^2\cos^{-1}x}{(1-x^2)\sqrt{1-x^2}}=\frac{\cos^{-1}x}{(1-x^2)^{3/2}}$$

Concept Applied

- Product rule of derivative : $(uv)'=u'v+uv'$

43. Given $e^x+e^y=e^{x+y} \Rightarrow 1+e^{y-x}=e^y$ ...(1)

Differentiating (1) w.r.t. x, we get

$$e^{y-x}\cdot\frac{d}{dx}(y-x)=e^y\frac{dy}{dx}$$

$$\Rightarrow e^{y-x}\left(\frac{dy}{dx}-1\right)=e^y\frac{dy}{dx} \Rightarrow \frac{dy}{dx}(e^{y-x}-e^y)=e^{y-x}$$

$$\Rightarrow \frac{dy}{dx}(-1)=e^{y-x} \Rightarrow \frac{dy}{dx}+e^{y-x}=0 \quad \text{[Using (1)]}$$

44. Here, $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}\log\left(\frac{x-a}{x+a}\right)$$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}[\log(x-a)-\log(x+a)]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{1+\frac{a^2}{x^2}}\cdot\frac{d}{dx}\left(\frac{a}{x}\right)+\frac{1}{2}\left[\frac{1}{x-a}-\frac{1}{x+a}\right] \quad \left(\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right)$$

$$=\frac{x^2}{x^2+a^2}\cdot a\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{2}\left[\frac{(x+a)-(x-a)}{x^2-a^2}\right]$$

$$=\frac{-a}{x^2+a^2}+\frac{a}{x^2-a^2}=\frac{-a(x^2-a^2)+a(x^2+a^2)}{x^4-a^4}=\frac{2a^3}{x^4-a^4}$$

Concept Applied

- $\log\left(\frac{a}{b}\right)=\log a-\log b$

45. We have, $e^{y-x}=y^x$

Taking log on both sides

$(y-x)\log e=x\log y \Rightarrow y-x=x\log y$ ...(i)

On differentiating, we get

$$\frac{dy}{dx}-1=x\cdot\frac{1}{y}\frac{dy}{dx}+\log y \Rightarrow \frac{dy}{dx}-\frac{x}{y}\frac{dy}{dx}=1+\log y$$

$$\Rightarrow \frac{dy}{dx}\left(1-\frac{x}{y}\right)=1+\log y \Rightarrow \frac{dy}{dx}=\frac{y(1+\log y)}{y-x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y(1+\log y)}{x\log y} \quad \text{[From (i)]}$$

46. We have $y=x^3(\cos x)^x+\sin^{-1}\sqrt{x}$ ...(i)

Let $z=(\cos x)^x=e^{x\log\cos x}$

$$\therefore \frac{dz}{dx}=e^{x\log\cos x}\left[\frac{x(-\sin x)}{\cos x}+\log\cos x\right]$$

$=(\cos x)^x\times[-x\tan x+\log\cos x]$ ...(ii)

Now, differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=3x^2(\cos x)^x+x^3(\cos x)^x[-x\tan x+\log\cos x]+\frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right) \quad \text{[Using (ii)]}$$

$$=x^2(\cos x)^x[3-x^2\tan x+x\log\cos x]+\frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)$$

47. Let $y=(\log x)^x+x^{\log x}$ $\therefore$ $y=e^{x\log(\log x)}+e^{(\log x)^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=(\log x)^x\frac{d}{dx}\{x\log(\log x)\}+x^{\log x}\frac{d}{dx}\{(\log x)^2\}$$

$$=(\log x)^x\left\{x\left(\frac{1}{\log x}\right)\frac{1}{x}+\log(\log x)\right\}+x^{\log x}\left(2(\log x)\frac{1}{x}\right)$$

$$=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}+2\left(\frac{\log x}{x}\right)x^{\log x}$$

48. Given, $x^y\cdot y^x=x^x$

Taking log on both sides, we get

$y\log x+x\log y=x\log x$

$$\Rightarrow y\cdot\frac{1}{x}+\log x\frac{dy}{dx}+\frac{x}{y}\frac{dy}{dx}+\log y=x\cdot\frac{1}{x}+\log x$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x}{y}+\log x\right)=1+\log x-\frac{y}{x}-\log y=1-\frac{y}{x}+\log\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-\frac{y}{x}+\log\frac{x}{y}}{\frac{x}{y}+\log x}$$

49. We have, $x^y-y^x=a^b$

Taking log on both sides, we get

$y\log x-x\log y=b\log a$ ...(i)

Now, differentiating (i) w.r.t. x on both sides, we get

$$\frac{y}{x}+(\log x)\frac{dy}{dx}-\log y-\frac{x\cdot 1}{y}\frac{dy}{dx}=0$$

$$\Rightarrow \left(\frac{y}{x}-\log y\right)=\frac{dy}{dx}\left(\frac{x}{y}-\log x\right)$$

$$\Rightarrow \frac{1}{x}(y-x\log y)=\frac{dy}{dx}\left(\frac{x-y\log x}{y}\right) \Rightarrow \frac{dy}{dx}=\frac{y(y-x\log y)}{x(x-y\log x)}$$

50. Given $y = (x)^{\cos x} + (\cos x)^{\sin x}$

Let $u = (x)^{\cos x}$, $v = (\cos x)^{\sin x}$ $\therefore\ y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = x^{\cos x}$

$\Rightarrow$ $\log u = \cos x \log x$

Differentiating with respect to x, we get

$$\frac{1}{u}\frac{du}{dx} = \cos x \cdot \frac{1}{x} + \log x(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \log x\right] \quad \text{...(ii)}$$

Now, $v = (\cos x)^{\sin x}$

$\Rightarrow$ $\log v = \sin x \log \cos x$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\cos x}(-\sin x) + \log \cos x \cdot \cos x$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x}\left[\frac{-\sin^2 x + \cos^2 x \log \cos x}{\cos x}\right] \quad \text{... (iii)}$$

Putting value of (ii) and (iii) into (i), we get

$$\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \log x\right] + (\cos x)^{\sin x}\left[\frac{-\sin^2 x + \cos^2 x \log \cos x}{\cos x}\right]$$

51. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$\Rightarrow$ $y = u + v$ [where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$]

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = (\sin x)^x$

Taking logarithm on both sides, we get $\log u = x \log \sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}(\cos x) + \log \sin x$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\ (x \cot x + \log \sin x) \quad \text{...(ii)}$$

and $v = \sin^{-1}\sqrt{x}$

$$\Rightarrow \frac{dv}{dx} = \left(\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2\sqrt{x}} \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^x\,(x \cot x + \log \sin x) + \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)$$

Answer Tips

➲ $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

52. We have, $x^y + y^x = a^b$

Taking log on both sides, we get

$y \log x + x \log y = b \log a$ ...(i)

Now, differentiating (i) w.r.t. x on both sides, we get

$$\frac{y}{x} + \log x\left(\frac{dy}{dx}\right) + \log y + x\cdot\frac{1}{y}\cdot\frac{dy}{dx} = 0 \qquad \left(\because\ \frac{d}{dx}(\log x) = \frac{1}{x}\right)$$

$$\Rightarrow \frac{y}{x} + \log y = -\left(\frac{x}{y} + \log x\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-y}{x}\left(\frac{y + x\log y}{x + y\log x}\right)$$

53. We have, $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$, $v = (\sin x)^{\cos x}$

$\therefore$ $y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = x^{\sin x}$ $\Rightarrow$ $\log u = \sin x \cdot (\log x)$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \cos x \cdot (\log x) + \frac{1}{x}\cdot \sin x$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\cos x \cdot (\log x) + \frac{1}{x}\sin x\right] \quad \text{...(ii)}$$

Also, $v = (\sin x)^{\cos x} \Rightarrow \log v = \cos x\,(\log \sin x)$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = -\sin x(\log \sin x) + \cos x \cdot \frac{1}{\sin x}\cdot \cos x$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x - \sin^2 x(\log \sin x)}{\sin x}\right] \quad \text{...(iii)}$$

From (i), (ii) & (iii), we get

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{x\cos x \cdot (\log x) + \sin x}{x}\right] + (\sin x)^{\cos x}\left[\frac{\cos^2 x - \sin^2 x(\log \sin x)}{\sin x}\right]$$

Key Points

For function $y(f(x))^{g(x)}$, we must take 'log' on both sides of the function to remove $g(x)$ as the power of $f(x)$.

54. Given $x^m y^n = (x + y)^{m+n}$

Taking log on both the sides, we get

$\log x^m + \log y^n = (m + n)\log(x + y)$

$\Rightarrow$ $m \log x + n \log y = (m + n) \log(x + y)$

Differentiating w.r.t. x, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\cdot\frac{dy}{dx} = (m+n)\cdot\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx + ny - my - ny}{y(x+y)}\right) = \frac{mx + nx - mx - my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx - my}{y(x+y)}\right) = \frac{nx - my}{x(x+y)} \quad \therefore \quad \frac{dy}{dx} = \frac{y}{x}$$

55. Here, $(x-y)\cdot e^{\frac{x}{x-y}} = a$

Taking log on both sides, we get

$$\log\left\{(x-y)\cdot e^{\frac{x}{x-y}}\right\} = \log a \ \Rightarrow\ \log(x-y) + \frac{x}{x-y} = \log a$$

Differentiating w.r.t. x, we get

$$\frac{1}{x-y}\cdot\left(1 - \frac{dy}{dx}\right) + \frac{(x-y)\cdot 1 - x\left(1 - \frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$\Rightarrow (x-y)\left(1 - \frac{dy}{dx}\right) + x - y - x\left(1 - \frac{dy}{dx}\right) = 0$$

$$\Rightarrow -y\left(1 - \frac{dy}{dx}\right) + x - y = 0 \Rightarrow y\frac{dy}{dx} + x = 2y$$

56. Here, $(\tan^{-1}x)^y + y^{\cot x} = 1$

$\Rightarrow$ $u + v = 1$ where $u = (\tan^{-1}x)^y$ and $v = y^{\cot x}$

Differentiating w.r.t. x, we get

$$\frac{du}{dx}+\frac{dv}{dx}=0 \quad ...(1)$$

Now, $u = (\tan^{-1}x)^y$

$\Rightarrow$ $\log u = y \log(\tan^{-1}x)$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{dy}{dx}\log(\tan^{-1}x)+y\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \frac{du}{dx}=(\tan^{-1}x)^y\times\left[\frac{dy}{dx}\log(\tan^{-1}x)+\frac{y}{(1+x^2)\tan^{-1}x}\right] \quad ...(2)$$

And $v = y^{\cot x}$

$\Rightarrow$ $\log v = \cot x \cdot \log y$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx}=\cot x\cdot\frac{1}{y}\frac{dy}{dx}-\operatorname{cosec}^2 x\cdot\log y$$

$$\Rightarrow \frac{dv}{dx}=y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx}-\operatorname{cosec}^2 x\cdot\log y\right] \quad ...(3)$$

From (1), (2) and (3), we get

$$(\tan^{-1}x)^y\left[\frac{dy}{dx}\log(\tan^{-1}x)+\frac{y}{(1+x^2)\tan^{-1}x}\right]+y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx}-\operatorname{cosec}^2x\cdot\log y\right]=0$$

$$\Rightarrow \frac{dy}{dx}[(\tan^{-1}x)^y\cdot\log(\tan^{-1}x)+y^{\cot x-1}\cdot\cot x]$$

$$=y^{\cot x}\cdot\operatorname{cosec}^2x\cdot\log y-(\tan^{-1}x)^{y-1}\cdot\frac{y}{(1+x^2)}$$

$$\therefore \frac{dy}{dx}=\frac{y^{\cot x}\cdot\operatorname{cosec}^2x\cdot\log y-(\tan^{-1}x)^{y-1}\cdot\frac{y}{(1+x^2)}}{(\tan^{-1}x)^y\log(\tan^{-1}x)+y^{\cot x-1}\cot x}$$

57. We have, $x = e^t \sin t$

$$\Rightarrow \frac{dx}{dt}=e^t\sin t+e^t\cos t$$

and $y = e^t\cos t \Rightarrow \dfrac{dy}{dt}=e^t\cos t-e^t\sin t$

$$\frac{dy}{dx}=\frac{e^t\cos t-e^t\sin t}{e^t(\cos t+\sin t)}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=\frac{\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)}=\frac{\frac{1}{\sqrt2}-\frac{1}{\sqrt2}}{\frac{1}{\sqrt2}+\frac{1}{\sqrt2}}=0$$

58. We have, $x = a\sec\theta$, $y = b\tan\theta$

$$\therefore \frac{dx}{d\theta}=a\sec\theta\tan\theta \text{ and } \frac{dy}{d\theta}=b\sec^2\theta$$

Now, $$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b}{a}\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}=\frac{b}{a}\cdot\frac{1}{\sin\theta}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}=\frac{b}{a}\cdot\frac{1}{\sin\frac{\pi}{3}}=\frac{b}{a}\cdot\frac{1}{\frac{\sqrt3}{2}}=\frac{2b}{a\sqrt3}$$

59. Let $u = \sin^2x$, $v = e^{\cos x}$

$$\therefore \frac{du}{dx}=2\sin x\cos x \text{ and } \frac{dv}{dx}=e^{\cos x}(-\sin x)$$

Now, $$\frac{du}{dv}=\frac{du}{dx}\cdot\frac{dx}{dv}=\frac{2\sin x\cos x}{e^{\cos x}(-\sin x)}=\frac{-2\cos x}{e^{\cos x}}$$

60. Let $u=\sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$ ...(i)

and $v=\sin^{-1}(2x\sqrt{1-x^2})$ ...(ii)

Put $x = \sin t$ in (i) and (ii), we get

$$u=\sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2t}}\right)=\sec^{-1}\left(\frac{1}{\cos t}\right)=\sec^{-1}(\sec t)=t$$

and $v=\sin^{-1}(2\sin t\sqrt{1-\sin^2t})=\sin^{-1}(\sin 2t)=2t$

$$\therefore \frac{du}{dt}=1 \text{ and } \frac{dv}{dt}=2. \text{ So, } \frac{du}{dv}=\frac{du}{dt}\times\frac{dt}{dv}=\frac{1}{2}$$

61. Let $u=\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ ...(i)

Putting $x^2 = \cos 2\theta$ in (i), we get

$$u=\tan^{-1}\left[\frac{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]=\tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right]=\frac{\pi}{4}-\theta$$

$$=\frac{\pi}{4}-\frac{\cos^{-1}(x^2)}{2} \quad \left(\because x^2=\cos2\theta\Rightarrow\theta=\frac{\cos^{-1}x^2}{2}\right)$$

$$\Rightarrow \frac{du}{dx}=\frac{1}{2}\frac{1}{\sqrt{1-x^4}}2x=\frac{x}{\sqrt{1-x^4}} \quad ...(ii)$$

Let $v=\cos^{-1}(x^2) \Rightarrow \dfrac{dv}{dx}=\dfrac{-2x}{\sqrt{1-x^4}}$ ...(iii)

Dividing (ii) by (iii), we get $\dfrac{du}{dv}=-\dfrac{1}{2}$.

Key Points

- $1 + \cos2\theta = 2\cos^2\theta$, $1 - \cos2\theta = 2\sin^2\theta$.

62. We have, $x = a(2\theta - \sin2\theta)$ ...(i)

and $y = a(1 - \cos2\theta)$ ...(ii)

Differentiating (i) w.r.t. θ, we get

$$\frac{dx}{d\theta}=a(2-2\cos2\theta) \quad ...(iii)$$

Differentiating (ii) w.r.t. θ, we get

$$\frac{dy}{d\theta}=2a\sin2\theta \quad ...(iv)$$

Now, $$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2a\sin2\theta}{a(2-2\cos2\theta)}=\frac{\sin2\theta}{1-\cos2\theta}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin 2\left(\frac{\pi}{3}\right)}{1-\cos\frac{2\pi}{3}} = \frac{\sin\left(\pi-\frac{\pi}{3}\right)}{1-\cos\left(\pi-\frac{\pi}{3}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{3}\right)}{1+\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

Concept Applied

➲ If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, f'(t) \neq 0$

63. $x = a\sin 2t\,(1+\cos 2t)$, $y = b\cos 2t\,(1-\cos 2t)$

Now, $\frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) + a\sin 2t(-2\sin 2t)$

$= 2a\cos 2t + 2a[\cos^2 2t - \sin^2 2t]$

$= 2a\cos 2t + 2a\cos 4t$ $\quad (\because \cos^2(a) - \sin^2(a) = \cos(2a))$

Also, $\frac{dy}{dt} = -2b\sin 2t(1-\cos 2t) + b\cos 2t(2\sin 2t)$

$= -2b\sin 2t + 4b(\sin 2t\cos 2t)$

$= -2b\sin 2t + 2b\sin 4t$ $\quad (\because 2\sin(a)\cos(a) = \sin(2a))$

So, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(\sin 4t - \sin 2t)}{2a(\cos 4t + \cos 2t)}$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=\pi/4} = \frac{b}{a}\left[\frac{\sin\pi - \sin(\pi/2)}{\cos\pi + \cos(\pi/2)}\right] = \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}$$

$$\left(\frac{dy}{dx}\right)_{\text{at } t=\pi/3} = \frac{b}{a}\left[\frac{\sin(4\pi/3) - \sin(2\pi/3)}{\cos(4\pi/3) + \cos(2\pi/3)}\right]$$

$$= \frac{b}{a}\left[\frac{\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\frac{-1}{2} - \frac{1}{2}}\right] = \frac{\sqrt{3}b}{a}$$

64. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\therefore u = \frac{\theta}{2} \Rightarrow u = \frac{1}{2}\tan^{-1}x$$

$$\left(\because 2\sin^2\frac{x}{2} = 1-\cos x, 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x\right)$$

Differentiating w.r.t. x, we get $\frac{du}{dx} = \frac{1}{2(1+x^2)}$

Also, let $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow v = 2\tan^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{1+x^2} \quad \therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} \Rightarrow \frac{du}{dv} = \frac{1}{4}$$

65. We have $x = ae^t(\sin t + \cos t)$

$\Rightarrow \frac{dx}{dt} = ae^t(\sin t + \cos t) + ae^t(\cos t - \sin t) = 2ae^t\cos t$

and $y = ae^t(\sin t - \cos t)$

$\Rightarrow \frac{dy}{dt} = ae^t(\sin t - \cos t) + ae^t(\cos t + \sin t) = 2ae^t\sin t$

$$\therefore \text{L.H.S.} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2ae^t\sin t}{2ae^t\cos t} = \tan t$$

$$\text{Also, R.H.S.} = \frac{x+y}{x-y} = \frac{ae^t(\sin t+\cos t) + ae^t(\sin t - \cos t)}{ae^t(\sin t + \cos t) - ae^t(\sin t - \cos t)}$$

$$= \frac{2ae^t\sin t}{2ae^t\cos t} = \tan t = \text{L.H.S.}$$

66. Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Put $x = \cos\theta$

$$\therefore u = \tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right] = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$= \tan^{-1}(\tan\theta) = \theta \Rightarrow \frac{du}{d\theta} = 1$

Also let,

$v = \cos^{-1}(2x\sqrt{1-x^2}) \Rightarrow v = \cos^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$

$= \cos^{-1}(2\cos\theta\sin\theta) = \cos^{-1}(\sin 2\theta)$

$$= \cos^{-1}\left(\cos\left(\frac{\pi}{2}-2\theta\right)\right) = \frac{\pi}{2} - 2\theta \Rightarrow \frac{dv}{d\theta} = -2$$

Now $\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{-1}{2}$

67. Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$\therefore u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}(\tan\theta)$$

$\Rightarrow u = \theta \Rightarrow u = \sin^{-1}x$

Differentiating w.r.t. x, we get $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ ...(i)

Again, let $v = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta$

$\therefore v = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow v = 2\sin^{-1}x$

Differentiating w.r.t. x, we get $\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$ ...(ii)

From (i) & (ii), we get $\frac{du}{dv} = \frac{1}{2}$

Concept Applied

➲ Power rule : $\frac{d}{dx}(x^n) = nx^{n-1}, n \neq 0$

68. We have, $x = ae^{\theta}(\sin\theta - \cos\theta)$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = ae^{\theta}(\sin\theta - \cos\theta) + ae^{\theta}(\cos\theta + \sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = ae^{\theta}(2\sin\theta) \quad ...(i)$$

Also $y = ae^{\theta}(\sin\theta + \cos\theta)$

Differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = ae^{\theta}(\sin\theta + \cos\theta) + ae^{\theta}(\cos\theta - \sin\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = ae^{\theta}(2\cos\theta) \quad ...(ii)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta \quad \text{[From (i) \& (ii)]}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

69. Here, $x = \cos t(3 - 2\cos^2 t)$, $y = \sin t(3 - 2\sin^2 t)$

$$\Rightarrow \frac{dx}{dt} = -\sin t(3 - 2\cos^2 t) + \cos t[2 \cdot 2\cos t \sin t]$$

$$= -3\sin t + 6\cos^2 t \sin t$$

and $\dfrac{dy}{dt} = \cos t(3 - 2\sin^2 t) + \sin t(-2 \cdot 2\sin t \cos t)$

$$= 3\cos t - 6\sin^2 t \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t - 6\sin^2 t\cos t}{-3\sin t + 6\cos^2 t\sin t} = \frac{3\cos t \cdot \cos 2t}{3\sin t \cdot \cos 2t} = \cot t$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t = \frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

70. **(d):** We have, $x = A\cos 4t + B\sin 4t$

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = A \cdot (-\sin 4t) \cdot 4 + B\cos 4t \cdot 4 \quad ...(i)$$

Again differentiating both sides of (i) w.r.t. t, we get

$$\frac{d^2x}{dt^2} = -4A(\cos 4t) \cdot 4 + 4B(-\sin 4t) \cdot 4$$

$= -16A\cos 4t - 16B\sin 4t = -16(A\cos 4t + B\sin 4t) = -16x$

71. **(b):** Given, $y = e^{-x}$

$$\Rightarrow \frac{dy}{dx} = -e^{-x} \Rightarrow \frac{d^2y}{dx^2} = e^{-x} = y$$

72. **(b):** Given, $x = t^2 + 1$ and $y = 2at$

$$\Rightarrow \frac{dx}{dt} = 2t \Rightarrow \frac{dy}{dt} = 2a \quad \therefore \frac{dy}{dx} = \frac{a}{t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a}{t^2} \cdot \frac{dt}{dx} = \frac{-a}{2t^3} \quad \therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at } t = a} = \frac{-a}{2a^3} = \frac{-1}{2a^2}$$

73. **(c):** We have, $y = \sin(2\sin^{-1}x)$

$$\Rightarrow y = \sin\left[\sin^{-1}\left(2x\sqrt{1-x^2}\right)\right]$$

$$\left[\because 2\sin^{-1}x = \sin^{-1}2x\sqrt{1-x^2}\right]$$

$$\Rightarrow y = 2x\sqrt{1-x^2} \quad ...(i)$$

$$\Rightarrow y_1 = 2x \times \frac{-2x}{2\sqrt{1-x^2}} + 2\sqrt{1-x^2} = \frac{-4x^2 + 2}{\sqrt{1-x^2}} \quad ...(ii)$$

$$\therefore y_2 = \frac{\sqrt{1-x^2}(-8x) - (-4x^2 + 2) \times \frac{-2x}{2\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{4x^3 - 6x}{(1-x^2)\sqrt{1-x^2}} \Rightarrow (1-x^2)y_2 = \frac{4x^3 - 6x}{\sqrt{1-x^2}}$$

Now, consider $xy_1 - 4y$

$$= \frac{-4x^3 + 2x}{\sqrt{1-x^2}} - 8x\sqrt{1-x^2} \quad \text{[Using (i) and (ii)]}$$

$$= \frac{4x^3 - 6x}{\sqrt{1-x^2}}$$

Thus, $(1 - x^2)y_2 = xy_1 - 4y$

74. **(d):** We have, $y = \log_e\left(\dfrac{x^2}{e^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{e^2}{x^2} \cdot \frac{1}{e^2} \cdot 2x = \frac{2}{x} \Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{x^2}$$

75. Given, $x = at^2$, $y = 2at$

$$\therefore \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a \quad \therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

So, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{t^2} \cdot \dfrac{dt}{dx} = -\dfrac{1}{t^2} \cdot \dfrac{1}{2at} = -\dfrac{1}{2at^3}$

76. We have, $x = a\cos\theta$, $y = b\sin\theta$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$$

So, $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx} = \dfrac{b\cos\theta}{-a\sin\theta} = -\dfrac{b}{a}\cot\theta$

Now, $\dfrac{d^2y}{dx^2} = -\dfrac{b}{a}(-\text{cosec}^2\theta)\dfrac{d\theta}{dx}$

$$= \frac{b}{a}\text{cosec}^2\theta\left(-\frac{1}{a}\text{cosec}\,\theta\right) = -\frac{b}{a^2}\text{cosec}^3\theta$$

77. We have, $y = \tan x + \sec x$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \sec^2 x + \sec x \tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x} + \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos^2 x}$$

Now, again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos^2 x(\cos x) - (1 + \sin x)(-2\cos x \sin x)}{(\cos^2 x)^2}$$

$$= \frac{\cos^3 x + (1 + \sin x) \cdot 2\sin x \cdot \cos x}{(1 - \sin^2 x)^2}$$

$$= \frac{\cos^3 x + 2\sin x\cos x + 2\sin^2 x\cos x}{(1 - \sin x)^2(1 + \sin x)^2}$$

$$= \frac{\cos x(\cos^2 x+\sin^2 x)+\sin^2 x\cos x+2\sin x\cos x}{(1-\sin x)^2(1+\sin x)^2}$$

$$= \frac{\cos x\{1+\sin^2 x+2\sin x\}}{(1-\sin x)^2(1+\sin x)^2} = \frac{\cos x\cdot(1+\sin x)^2}{(1-\sin x)^2(1+\sin x)^2} = \frac{\cos x}{(1-\sin x)^2}$$

Hence proved.

78. We have, $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$

$$\therefore \quad \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta, \quad \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a\cos\theta+b\sin\theta}{-a\sin\theta+b\cos\theta} = \frac{x}{-y} \quad \frac{d^2y}{dx^2} = \frac{-y\cdot 1 - x\left(-\frac{dy}{dx}\right)}{(-y)^2}$$

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx} \Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

79. Here $x = a\sec^3\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = a\cdot 3\sec^2\theta\cdot\sec\theta\tan\theta = 3a\sec^3\theta\tan\theta \text{ and } y = a\tan^3\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = a\cdot 3\tan^2\theta\cdot\sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

On differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cos\theta\cdot\frac{d\theta}{dx} = \frac{\cos\theta}{3a\sec^3\theta\tan\theta} = \frac{1}{3a}\cos^4\theta\cdot\cot\theta$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}} = \frac{1}{3a}\cos^4\frac{\pi}{4}\cdot\cot\frac{\pi}{4} = \frac{1}{3a}\cdot\left(\frac{1}{\sqrt{2}}\right)^4\cdot 1 = \frac{1}{3a}\cdot\frac{1}{4} = \frac{1}{12a}$$

80. We have, $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$= \frac{-a\sin(\log x)+b\cos(\log x)}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x\left[-a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}\right] - (-a\sin(\log x)+b\cos(\log x))}{x^2}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = -y - \frac{xdy}{dx} \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

81. We have, $y = (\sin^{-1}x)^2$ ...(i)

$$\Rightarrow \quad \frac{dy}{dx} = 2(\sin^{-1}x)\times\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{4(\sin^{-1}x)^2}{(1-x^2)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{4y}{1-x^2}$$ [From (i)]

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(1-x^2) = 4y$$ ...(ii)

Again, differentiating (ii) w.r.t. x on both sides, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}(1-x^2) + \left(\frac{dy}{dx}\right)^2(-2x) = 4\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

Concept Applied

➲ Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$. If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$

82. We have, $x = \sin t$ and $y = \sin pt$

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = p\cos pt \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^2 t}\times\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{p^2\sin pt\cos t}{\cos^3 t} + \frac{p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-p^2y}{\cos^2 t} + \frac{x\frac{dy}{dx}}{\cos^2 t} \Rightarrow \cos^2 t\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$

$$\Rightarrow \quad (1-\sin^2 t)\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

83. Here, $y = \sin(\sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\sin x)\cdot\cos x$$

Again, differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cdot\cos x\cdot\cos x + (-\sin x)\cos(\sin x)$$

$$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$$

Now, L.H.S. $= \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x$

$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$
$+ \tan x(\cos x\cdot\cos(\sin x)) + \cos^2 x\cdot\sin(\sin x)$

$= -\sin x\cdot\cos(\sin x) + \sin x\cdot\cos(\sin x)$

$= 0 =$ R.H.S.

84. Given $x^m y^n = (x+y)^{m+n}$

Taking log on both the sides, we get

$\log x^m + \log y^n = (m+n)\log(x+y)$

$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y)$

Differentiating w.r.t. x, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\cdot\frac{dy}{dx} = (m+n)\cdot\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \frac{mx+nx-mx-my}{x(x+y)}$$

$\Rightarrow \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right)=\frac{nx-my}{x(x+y)} \quad \therefore \quad \frac{dy}{dx}=\frac{y}{x}.$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{x\frac{dy}{dx}-y}{x^2}=\frac{x\left(\frac{y}{x}\right)-y}{x^2}=0 \therefore \frac{d^2y}{dx^2}=0$$

85. Given that $e^y\cdot(x+1)=1$ ...(i)

Differentiating (i) w.r.t. x, we get

$$e^y\frac{d}{dx}(x+1)+(x+1)\frac{d}{dx}e^y=\frac{d}{dx}(1) \Rightarrow e^y+(x+1)e^y\frac{dy}{dx}=0$$

$$\Rightarrow e^y\left[1+(x+1)\frac{dy}{dx}\right]=0 \Rightarrow (x+1)\frac{dy}{dx}=-1$$

and $\frac{dy}{dx}=\frac{-1}{x+1}$ ...(ii)

or $\left(\frac{dy}{dx}\right)^2=\frac{1}{(x+1)^2}$ ...(iii)

Again, differentiating (ii) w.r.t. x, we get

$\frac{d^2y}{dx^2}=\frac{1}{(x+1)^2} \Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$ [From (iii)]

86. We have, $y=x^x \Rightarrow y=e^{x\log x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{x\log x}\left(x\times\frac{1}{x}+\log x\right)$$

$\Rightarrow \frac{dy}{dx}=x^x(1+\log x) \Rightarrow \frac{dy}{dx}=y(1+\log x)$ ...(1)

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=(1+\log x)\cdot\frac{dy}{dx}+y\times\frac{1}{x}$$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{y}\left(\frac{dy}{dx}\right)^2+\frac{y}{x}$ [From (1)]

$$\Rightarrow \frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$$

87. We have, $y=2\cos(\log x)+3\sin(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=-2\sin(\log x)\times\frac{1}{x}+3\cos(\log x)\times\frac{1}{x}$$

$\Rightarrow x\frac{dy}{dx}=-2\sin(\log x)+3\cos(\log x)$ ...(1)

Again, differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-2\cos(\log x)\times\frac{1}{x}-3\sin(\log x)\times\frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-[2\cos(\log x)+3\sin(\log x)]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y \Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

Answer Tips

- $\frac{d}{dx}(\log(x))=\frac{1}{x}$

88. Given, $x=a\cos\theta+b\sin\theta$, $y=a\sin\theta-b\cos\theta$

$\Rightarrow x^2=a^2\cos^2\theta+b^2\sin^2\theta+2ab\cos\theta\sin\theta$ ...(1)

and $y^2=a^2\sin^2\theta+b^2\cos^2\theta-2ab\sin\theta\cos\theta$ ...(2)

Adding (1) and (2), we get $x^2+y^2=a^2+b^2$

Differentiating w.r.t. x, we get

$2x+2y\frac{dy}{dx}=0 \Rightarrow x+y\frac{dy}{dx}=0$ ...(3)

Again, differentiating w.r.t. x, we get

$$1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

Multiplying by y on both sides, we get

$$y^2\frac{d^2y}{dx^2}+\left(y\cdot\frac{dy}{dx}\right)\frac{dy}{dx}+y=0$$

$\Rightarrow y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$ [From (3)]

89. We have, $y=e^{m\sin^{-1}x}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=e^{m\sin^{-1}x}\left(\frac{m}{\sqrt{1-x^2}}\right)=\frac{my}{\sqrt{1-x^2}}$ ...(1)

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=m\left[\frac{\sqrt{1-x^2}\frac{dy}{dx}-\frac{y}{2\sqrt{1-x^2}}\cdot(-2x)}{(1-x^2)}\right]$$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m\left[my+\frac{xy}{\sqrt{1-x^2}}\right]$ [From (1)]

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m\left[my+x\cdot\left(\frac{1}{m}\cdot\frac{dy}{dx}\right)\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m^2y+x\frac{dy}{dx} \Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$$

90. We have, $y=(x+\sqrt{1+x^2})^n$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$\Rightarrow \frac{dy}{dx}=\frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}=\frac{ny}{\sqrt{1+x^2}}$ ...(1)

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=n\left[\frac{\sqrt{1+x^2}\cdot\frac{dy}{dx}-\frac{2(xy)}{2\sqrt{1+x^2}}}{1+x^2}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n\left[\sqrt{1+x^2}\times\frac{ny}{\sqrt{1+x^2}}-\frac{xy}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-\frac{nxy}{\sqrt{1+x^2}}$$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-x\frac{dy}{dx}$ [From (1)]

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

91. Given $y = Ae^{mx} + Be^{nx}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{mx}\cdot m + Be^{nx}\cdot n \Rightarrow \frac{d^2y}{dx^2} = m^2Ae^{mx} + n^2Be^{nx}$$

Now, L.H.S. $= \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= m^2Ae^{mx} + n^2Be^{nx} - (m+n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$
$= Ae^{mx}[m^2 - (m+n)m + mn] + Be^{nx}[n^2 - (m+n)n + mn]$
$= Ae^{mx}\times 0 + Be^{nx}\times 0 = 0 =$ R.H.S.

92. Here, $x = a(\cos t + t\sin t)$

$\Rightarrow \frac{dx}{dt} = a[-\sin t + 1\cdot\sin t + t\cos t] = at\cos t$ ...(1)

and $y = a(\sin t - t\cos t)$

$\Rightarrow \frac{dy}{dt} = a[\cos t - (1\cdot\cos t - t\sin t)] = at\sin t$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \frac{\sec^2 t}{at\cos t}$ [Using (1)]

$= \frac{1}{a}\cdot\frac{1}{t\cos^3 t}$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = \frac{1}{a}\cdot\frac{1}{\frac{\pi}{4}\cos^3\frac{\pi}{4}} = \frac{4}{\pi a}\cdot(\sqrt{2})^3 = \frac{8\sqrt{2}}{\pi a}$$

93. Here, $x = a(\cos t + \log\tan\frac{t}{2})$

$$\Rightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\cdot\frac{1}{2\cos^2\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\frac{(-\sin^2 t + 1)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

Also, $y = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a\cos t\cdot\frac{\sin t}{a\cos^2 t} = \tan t$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \frac{\sec^2 t}{a\cos^2 t/\sin t} = \frac{\sin t}{a\cos^4 t}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{a\cos^4\frac{\pi}{3}} = \frac{\sqrt{3}/2}{a(1/2)^4} = \frac{8\sqrt{3}}{a}$$

CBSE Sample Questions

1. **(c):** Given, the function f is continuous at $x = 0$.

$\therefore \lim_{x\to 0} f(x) = f(0)$.

Now, $\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2} = \lim_{x\to 0}\frac{\sin^2 2x}{4x^2}$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$$ **(1)**

Also, $f(0) = k$
Hence, $k = 1$

2. **(b):** We have, $f(x) = \begin{cases}\frac{1-\cos kx}{x\sin x}, & x\neq 0\\ \frac{1}{2}, & x = 0\end{cases}$

$\because$ $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x\to 0}\frac{1-\cos kx}{x\sin x} = \frac{1}{2} \Rightarrow \lim_{x\to 0}\frac{2\sin^2\frac{kx}{2}}{x^2\frac{\sin x}{x}} = \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} 2\cdot\frac{k^2}{4}\left\{\frac{\sin\left(\frac{kx}{2}\right)}{\frac{kx}{2}}\right\}^2\frac{1}{\frac{(\sin x)}{x}} = \frac{1}{2}$$

$\Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1$

But $k < 0$ $\therefore$ $k = -1$ **(1)**

3. **(a):** We have, $f(x) = \begin{cases}\frac{x}{|x|}, & x < 0\\ -1, & x\geq 0\end{cases}$

$$\Rightarrow f(x) = \begin{cases}\frac{x}{-x} = -1, & x < 0\\ -1, & x\geq 0\end{cases}$$

$\Rightarrow$ $f(x) = -1\ \forall\ x\in R$

$\Rightarrow$ $f(x)$ is continuous $\forall\ x\in R$ as it is a constant function. **(1)**

4. We have, $\lim_{x\to 0}\frac{1-\cos kx}{x\sin x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{kx}{2}\right)}{x\sin x}$

$$= \frac{\lim_{x\to 0}\frac{2\sin^2\left(\frac{kx}{2}\right)}{\left(\frac{kx}{2}\right)^2}\times\left(\frac{k}{2}\right)^2}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{2\times 1\times\frac{k^2}{4}}{1} = \frac{k^2}{2}$$ **(1½)**

$\because$ $f(x)$ is continuous at $x = 0$.

$\therefore \lim_{x\to 0} f(x) = f(0)$

$\Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$ **(1/2)**

5. We have, $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$ ...(i)

Let $\sin^{-1}x = A$ and $\sin^{-1}y = B$.
Then, $x = \sin A$ and $y = \sin B$
From (i) $\sin B\cos A + \sin A\cos B = 1$

$\Rightarrow$ $\sin(A + B) = 1$ **(1)**

$\Rightarrow \quad A + B = \sin^{-1}1 = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$

Differentiating w.r.t. x, we obtain $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$ **(1)**

6. We have, $f(x) = [x], 0 < x < 2$.

R.H.D.(at $x = 1$) $= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{[1+h]-[1]}{h}$

$= \lim_{h\to 0}\frac{(1-1)}{h} = 0$ **(1)**

L.H.D.(at $x = 1$) $= \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[1-h]-[1]}{-h}$

$= \lim_{h\to 0}\frac{0-1}{-h} = \lim_{h\to 0}\frac{1}{h} = \infty$ **(1)**

Since, R.H.D. $\neq$ L.H.D.

Therefore $f(x)$ is not differentiable at $x = 1$. **(1)**

7. (c): We have, $e^x + e^y = e^{x+y}$

$\Rightarrow \quad e^{-y} + e^{-x} = 1$

Differentiating w.r.t. x, we get

$-e^{-y}\frac{dy}{dx} - e^{-x} = 0 \Rightarrow \frac{dy}{dx} = -e^{y-x}$ **(1)**

8. (d): We have, $y = \log(\cos e^x)$

Differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{\cos e^x}\cdot(-\sin e^x)\cdot e^x$

$\Rightarrow \frac{dy}{dx} = -e^x \tan e^x$ **(1)**

9. We have, $y = e^{x\sin^2 x} + (\sin x)^x \Rightarrow y = u + v$,

where $u = e^{x\sin^2 x}$ and $v = (\sin x)^x$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (i)

Now, consider $u = e^{x\sin^2 x}$

Differentiating both sides with respect to x, we get

$\frac{du}{dx} = e^{x\sin^2 x}[x\cdot(2\sin x\cos x) + \sin^2 x]$ **(1)**

$= e^{x\sin^2 x}[x(\sin 2x) + \sin^2 x]$ ... (ii)

Also, $v = (\sin x)^x$

$\Rightarrow \quad \log v = x\log(\sin x)$

Differentiating both sides with respect to x, we get

$\frac{1}{v}\frac{dv}{dx} = x\cdot\frac{1}{\sin x}\cdot\cos x + \log(\sin x)$

$\Rightarrow \frac{dv}{dx} = (\sin x)^x[x\cot x + \log(\sin x)]$ ... (iii) **(1/2)**

From (i), (ii) and (iii), we get

$\frac{dy}{dx} = e^{x\sin^2 x}[x\sin 2x + \sin^2 x] + (\sin x)^x[x\cot x + \log(\sin x)]$ **(1/2)**

10. (a): Let $u = \sin^{-1}(2x\sqrt{1-x^2})$ ...(i)

and $v = \sin^{-1}x, \frac{1}{\sqrt{2}} < x < 1$

$\Rightarrow \quad \sin v = x$ ...(ii)

From (i) and (ii), we get

$u = \sin^{-1}(2\sin v\cos v) = \sin^{-1}(\sin 2v)$

$\Rightarrow \quad u = 2v$

Differentiating with respect to v both sides, we get

$\frac{du}{dv} = 2$ **(1)**

11. (a): We have, $y = 5\cos x - 3\sin x$

$\Rightarrow \frac{dy}{dx} = -5\sin x - 3\cos x$

$\Rightarrow \frac{d^2y}{dx^2} = -5\cos x + 3\sin x = -y$ **(1)**

12. (a): We have, $x = a\sec\theta$

$\Rightarrow \frac{dx}{d\theta} = a\tan\theta\sec\theta$ and $y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{b\sec^2\theta}{a\tan\theta\sec\theta} = \frac{b}{a}\operatorname{cosec}\theta$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-b}{a}\operatorname{cosec}\theta\cot\theta\frac{d\theta}{dx}$

$= \frac{-b}{a}\operatorname{cosec}\theta\cot\theta\frac{1}{a\tan\theta\sec\theta} = \frac{-b}{a^2}\cot^3\theta$

$\therefore \left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{6}} = \frac{-3\sqrt{3}b}{a^2}$ **(1)**

13. We have, $y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$ ... (i)

and $x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$ ... (ii)

$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\operatorname{cosec}\theta$ **(1½)**

Differentiating both sides with respect to x, we get

$\frac{d^2y}{dx^2} = \frac{-b}{a}\operatorname{cosec}\theta\cot\theta\times\frac{d\theta}{dx}$

$= -\frac{b}{a}\operatorname{cosec}\theta\cot\theta\times\frac{1}{a\sec\theta\tan\theta}$ [using (ii)] **(1)**

$= -\frac{b}{a^2}\cot^3\theta$

$\therefore \left[\frac{d^2y}{dx^2}\right]_{\theta=\pi/6} = \frac{-b}{a^2}\left[\cot\frac{\pi}{6}\right]^3 = \frac{-b}{a^2}(\sqrt{3})^3 = -\frac{3\sqrt{3}b}{a^2}$ **(1/2)**

Self Assessment

Case Based Objective Questions (4 marks)

1. The function $f(x)$ will be discontinuous at $x = a$ if $f(x)$ has

Discontinuity of first kind : $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ both exist but are not equal. It is also known as irremovable discontinuity.

Discontinuity of second kind : If none of the limits $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ exist.

Removable discontinuity : $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ both exist and equal but not equal to $f(a)$.

Based on the above information, attempt any 4 out of 5 subparts.

(i) If $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & \text{for } x\neq 3\\ 4, & \text{for } x=3\end{cases}$, then at $x=3$

(a) f has removable discontinuity
(b) f is continuous
(c) f has irremovable discontinuity
(d) none of these

(ii) Let $f(x)=\begin{cases}x+2, & \text{if } x\leq 4\\ x+4, & \text{if } x>4\end{cases}$ then at $x = 4$

(a) f is continuous
(b) f has removable discontinuity
(c) f has irremovable discontinuity
(d) none of these

(iii) Consider the function $f(x)$ defined as

$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{for } x\neq 2\\ 5, & \text{for } x=2\end{cases}$, then at $x = 2$

(a) f has removable discontinuity
(b) f has irremovable discontinuity
(c) f is continuous
(d) f is continuous if $f(2) = 3$

(iv) If $f(x)=\begin{cases}\dfrac{x-|x|}{x}, & x\neq 0\\ 2, & x=0\end{cases}$, then at $x = 0$

(a) f is continuous
(b) f has removable discontinuity
(c) f has irremovable discontinuity
(d) none of these

(v) If $f(x)=\begin{cases}\dfrac{e^x-1}{\log(1+2x)}, & \text{if } x\neq 0\\ 7, & \text{if } x=0\end{cases}$, then at $x = 0$

(a) f is continuous if $f(0) = 2$
(b) f is continuous
(c) f has irremovable discontinuity
(d) f has removable discontinuity

Multiple Choice Questions (1 mark)

2. Let f be defined on $[-5, 5]$ as $f(x)=\begin{cases}x, & \text{if } x \text{ is rational}\\ -x, & \text{if } x \text{ is irrational}\end{cases}$

Then $f(x)$ is
(a) continuous at every x except $x = 0$
(b) discontinuous at every x except $x = 0$
(c) continuous everywhere
(d) discontinuous everywhere

3. If $u = x^2 + y^2$ and $x = s + 3t$, $y = 2s - t$, then $\dfrac{d^2u}{ds^2}$ is equal to
(a) 12 (b) 32 (c) 36 (d) 10

4. Let $f(x)=\begin{cases}\sin x, & \text{for } x\geq 0\\ 1-\cos x, & \text{for } x\leq 0\end{cases}$ and $g(x) = e^x$.

Then, the value of $(gof)'(0)$ is
(a) 1 (b) -1
(c) 0 (d) None of these

OR

If the function $f(x)=\begin{cases}\dfrac{\sin x^2}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, is differentiable at $x = 0$, then right hand derivative of $f(x)$ at $x = 0$ is
(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$
(c) 1 (d) -1

5. If $x^y \cdot y^x = 16$, then the value of $\dfrac{dy}{dx}$ at $(2, 2)$ is
(a) -1 (b) 0
(c) 1 (d) None of these

VSA Type Questions (1 mark)

6. Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

7. If $f(x) = |\cos x|$, then find $f'\left(\dfrac{\pi}{4}\right)$

OR

If $f(x) = |\cos x - \sin x|$, then find $f'\left(\dfrac{\pi}{3}\right)$.

8. Find the derivative of $2^{\cos^2 x}$.

9. Differentiate a^{7x+4} w.r.t. x.

10. For the curve $\sqrt{x}+\sqrt{y}=1$, find $\dfrac{dy}{dx}$ at $\left(\dfrac{1}{4},\dfrac{1}{4}\right)$

11. If $f(x) = \sqrt{x^2+9}$, write the value of $\lim_{x\to 4}\dfrac{f(x)-f(4)}{x-4}$.

SA I Type Questions (2 marks)

12. Prove that the function $f(x)=\frac{x+3}{(x-2)(x-5)}$ is continuous in its domain.

13. Differentiate $\frac{x^2}{1+x^2}$ w.r.t. x^2.

OR

Find the differential coefficient of cos x w.r.t. x^3.

14. If $y = |x - x^2|$, then find $\frac{d^2y}{dx^2}$.

15. If $f(x)=\begin{cases} mx+1, \text{ if } x\le\frac{\pi}{2} \\ \sin x+n, \text{if } x>\frac{\pi}{2}\end{cases}$ is continuous at $x=\frac{\pi}{2}$, prove that $n=\frac{m\pi}{2}$.

SA II Type Questions (3 marks)

16. Differentiate $\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$ w.r.t. x.

17. Find the differential coefficient of $\frac{e^x}{\log x}$ w.r.t. x.

18. If $u = \sin(m\cos^{-1}x)$, $v = \cos(m\sin^{-1}x)$, prove that $\frac{du}{dv}=\frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}$.

19. Find the value of k so that the function $f(x)=\begin{cases}\frac{2^{x+2}-16}{4^x-16}, \text{if } x\ne 2 \\ k \quad, \text{if } x=2\end{cases}$ is continuous at $x = 2$.

OR

If $y=e^{x+e^{x+e^{x+\ldots \text{to} \infty}}}$, prove that $\frac{dy}{dx}=\frac{y}{1-y}$.

Case Based Questions (4 marks)

20. Derivative of $y = f(x)$ w.r.t. x (if exists) is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is called the first order derivative of y. If we take derivative of $\frac{dy}{dx}$ again, then we get $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$ or $f''(x)$ and is called the second order derivative of y. Similarly, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is denoted and defined as $\frac{d^3y}{dx^3}$ or $f'''(x)$ and is known as third order derivative of y and so on.
Based on the given information, answer the following questions.

(i) If $y=\tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$, then find $\frac{d^2y}{dx^2}$.

(ii) If $y^2 = ax^2 + bx + c$, then evaluate $\frac{d}{dx}(y^3y_2)$.

LA Type Questions (4/6 marks)

21. If $y=Ae^{-kt}\cos(pt+c)$, prove that $\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+n^2y=0$, where $n^2 = p^2 + k^2$.

22. Find the value of $f(0)$, so that the function $f(x)=\frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}$, $x \neq 0$ is continuous everywhere.

OR

Show that the function $f(x)$ defined as follows is continuous at $x = 2$, but not differentiable there :

$$f(x)=\begin{cases}3x-2 \ , & 0<x\le 1 \\ 2x^2-x \ , & 1<x\le 2 \\ 5x-4 \ , & x>2\end{cases}$$

23. Find $\frac{dy}{dx}$, when $\sin x=\frac{2t}{1+t^2}$, $\tan y=\frac{2t}{1-t^2}$.

24. Discuss the differentiability of $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\ne 0 \\ 0 \ , & x=0\end{cases}$ at $x=0$.

25. Find whether the function $f(x)=\begin{cases}\frac{2x^2-3x-2}{x-2}, \text{if } x\ne 2 \\ 5 \quad, \text{if } x=2\end{cases}$ at $x = 2$ is continuous or discontinuous.

Detailed SOLUTIONS

1. (i) (a) : $f(3) = 4$

$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x+3)(x-3)}{(x-3)}$

$=\lim_{x\to 3}(x+3)=6 \because \lim_{x\to 3} f(x)\ne f(3)$

$\therefore$ $f(x)$ has removable discontinuity at $x = 3$.

(ii) (c) : $\lim_{x\to 4^-} f(x)=\lim_{x\to 4}(x+2)=4+2=6$

$\lim_{x\to 4^+} f(x)=\lim_{x\to 4}(x+4)=4+4=8$

$\therefore \lim_{x\to 4^-} f(x)\ne \lim_{x\to 4^+} f(x)$

$\therefore$ $f(x)$ has an irremovable discontinuity at $x = 4$.

(iii) (a): $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x^2-4)}{(x-2)} = \lim_{x\to 2}(x+2) = 4$

and $f(2) = 5$ (given) $\therefore \lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ $f(x)$ has removable discontinuity at $x = 2$.

(iv) (c): $f(0) = 2$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{x+x}{x} = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{x-x}{x} = 0$$

$\because \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore$ $f(x)$ has an irremovable discontinuity at $x = 0$.

(v) (d): $f(0) = 7$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x-1}{\log(1+2x)} = \lim_{x\to 0} \frac{\left(\frac{e^x-1}{x}\right)}{\frac{\log(1+2x)}{2x}\cdot 2} = \frac{1}{2}$$

$\because \lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ $f(x)$ has removable discontinuity at $x = 0$.

2. (b): As $x \to 0$ both x and $-x$ tend to zero, $f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

For $x \neq 0, x \neq -x$, $f(x)$ is discontinuous.

3. (d): Given, $u = x^2 + y^2$, $x = s + 3t$, $y = 2s - t$

$\Rightarrow \frac{dx}{ds} = 1, \frac{dy}{ds} = 2$

Now, $u = x^2 + y^2 \Rightarrow \frac{du}{ds} = 2x\frac{dx}{ds} + 2y\frac{dy}{ds} = 2x + 4y$

$$\Rightarrow \frac{d^2u}{ds^2} = 2\left(\frac{dx}{ds}\right) + 4\left(\frac{dy}{ds}\right) \Rightarrow \frac{d^2u}{ds^2} = 2(1) + 4(2) = 10$$

4. (c): Given, $f(x) = \begin{cases} \sin x, & \text{for } x \geq 0 \\ 1-\cos x, & \text{for } x \leq 0 \end{cases}$ and $g(x) = e^x$

$$\therefore \quad gof(x) = \begin{cases} e^{\sin x}, & x \geq 0 \\ e^{1-\cos x}, & x \leq 0 \end{cases}$$

$\therefore$ L.H.D. $= (gof)'(0-h) = \lim_{h\to 0} \frac{gof(0-h) - gof(h)}{-h}$

$$= \lim_{h\to 0} \frac{e^{1-\cos(0-h)} - e^{1-\cos h}}{-h} = 0$$

R.H.D. $= (gof)'(0+h)$

$$= \lim_{h\to 0} \frac{gof(0+h) - gof(h)}{h} = \lim_{h\to 0} \frac{e^{\sin h} - e^{\sin h}}{h} = 0$$

$\therefore$ R.H.D. = L.H.D. = 0 $\Rightarrow (gof)'(0) = 0$

OR

(c) : At $x = 0$, right hand derivative

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{\sin h^2}{h} - 0}{h} = \lim_{h\to 0} \frac{\sin h^2}{h^2} = 1$$

5. (a): Given, $x^y \cdot y^x = 16$

Taking log on both sides, we get

$\log x^y + \log y^x = \log 2^4 \Rightarrow y\log x + x\log y = 4\log 2$

Differentiating both sides w.r.t. x, we get

$$\frac{y}{x} + \log x\frac{dy}{dx} + \frac{x}{y}\frac{dy}{dx} + \log y \cdot 1 = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\left(\log y + \frac{y}{x}\right)}{\left(\log x + \frac{x}{y}\right)} \quad \therefore \left[\frac{dy}{dx}\right]_{(2,2)} = -\frac{(\log 2+1)}{(\log 2+1)} = -1$$

6. We have given, $f(x) = x^3 + 2x^2 - 1$.

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0}(1+h)^3 + 2(1+h)^2 - 1 = 2$$

and $\lim_{x\to 1^-} f(x) = \lim_{h\to 0}(1-h)^3 + 2(1-h)^2 - 1 = 2$

and $f(1) = 1 + 2 - 1 = 2 \therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

Therefore, $f(x)$ is continuous at $x = 1$.

7. Given, $f(x) = |\cos x|$

We have to find out $f'\left(\frac{\pi}{4}\right)$

Since, $\cos x > 0$ when $x \in \left[0, \frac{\pi}{2}\right]$

$\therefore$ $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

OR

Given, $f(x) = |\cos x - \sin x|$,

As we know, $\frac{\pi}{4} < x < \frac{\pi}{2}, \sin x > \cos x$

$\Rightarrow$ $\cos x - \sin x \leq 0$

$\therefore$ $f(x) = -(\cos x - \sin x)$

$\Rightarrow$ $f'(x) = -[-\sin x - \cos x]$

$$\therefore f'\left(\frac{\pi}{3}\right) = -\left(\frac{-\sqrt{3}}{2} - \frac{1}{2}\right) = \left(\frac{\sqrt{3}+1}{2}\right)$$

8. Let $y = 2^{\cos^2 x}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(2^{\cos^2 x}\right) = 2^{\cos^2 x}\log 2\frac{d}{dx}(\cos^2 x)$$

$$= 2^{\cos^2 x}\log 2[2\cos x(-\sin x)]$$

$$= -2^{\cos 2x}\log 2 \cdot \sin 2x$$

9. Let $y = a^{7x+4}$

$$\therefore \frac{dy}{dx} = a^{7x+4}\log_e a \cdot \frac{d}{dx}(7x+4) = 7a^{7x+4}\log_e a.$$

10. We have given, $\sqrt{x} + \sqrt{y} = 1$

Taking derivative w.r.t. 'x' on both sides, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \therefore \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$$

11. $\lim_{x\to4}\frac{f(x)-f(4)}{x-4}=\lim_{x\to4}\frac{\sqrt{x^2+9}-5}{x-4}\left(\frac{0}{0}\text{ form}\right)$

By L'Hospital rule, we get $\lim_{x\to4}\frac{\frac{1}{2}\times\frac{1}{\sqrt{x^2+9}}\times2x}{1}=\frac{4}{5}$

12. Let $f(x)=\frac{x+3}{(x-2)(x-5)}, D_f=R-\{2,5\}$

Since a rational function is continuous at every point of its domain, therefore f is continuous at every point of its domain. Hence, f is a continuous function.

13. Let $y=\frac{x^2}{1+x^2}$ and $z=x^2$

Now, $\frac{dy}{dx}=\frac{(1+x^2)(2x)-x^2(2x)}{(1+x^2)^2}=\frac{2x}{(1+x^2)^2}$ and $\frac{dz}{dx}=2x$

$\therefore \quad \frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{2x/(1+x^2)^2}{2x}=\frac{1}{(1+x^2)^2}$

OR

Here, $u=\cos x$ and $v=x^3$

So, $\frac{du}{dx}=-\sin x$ and $\frac{dv}{dx}=3x^2$

$\therefore \quad \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{-\sin x}{3x^2}$

14. Here, $y=|x-x^2|\Rightarrow y=\begin{cases}x-x^2, & \text{if } 0\le x\le1\\ x^2-x, & \text{if } x<0, x>1\end{cases}$

$\Rightarrow \frac{dy}{dx}=\begin{cases}1-2x, & \text{if } 0<x<1\\ 2x-1, & \text{if } x<0, x>1\end{cases}$

$\Rightarrow \frac{d^2y}{dx^2}=\begin{cases}-2, & \text{if } 0<x<1\\ 2, & \text{if } x<0, x>1\end{cases}$

15. We have given, $f(x)=\begin{cases}mx+1, & \text{if } x\le\frac{\pi}{2}\\ (\sin x+n), & \text{if } x>\frac{\pi}{2}\end{cases}$

Since $f(x)$ is continuous at $x=\frac{\pi}{2}$

$\therefore$ L.H.L. $=\lim_{x\to\left(\frac{\pi}{2}\right)^-}(mx+1)=\lim_{h\to0}\left[m\left(\frac{\pi}{2}-h\right)+1\right]=\frac{m\pi}{2}+1$

and R.H.L. $=\lim_{x\to\left(\frac{\pi}{2}\right)^+}(\sin x+n)=\lim_{h\to0}\left[\sin\left(\frac{\pi}{2}+h\right)+n\right]$

$=\lim_{h\to0}\cos h+n=1+n$

$\because$ L.H.L. = R.H.L.

$\Rightarrow m\cdot\frac{\pi}{2}+1=n+1 \Rightarrow n=m\cdot\frac{\pi}{2}.$

16. Let $y=\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$

Then, $y=\tan^{-1}\left(\frac{\sqrt{x}-x}{1+\sqrt{x}\cdot x}\right)=\tan^{-1}(\sqrt{x})-\tan^{-1}(x)$

$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}\sqrt{x})-\frac{d}{dx}(\tan^{-1}x)$

$=\frac{d(\tan^{-1}\sqrt{x})}{d\sqrt{x}}\cdot\frac{d\sqrt{x}}{dx}-\frac{1}{1+x^2}=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}-\frac{1}{1+x^2}$

$=\frac{1}{2\sqrt{x}(1+x)}-\frac{1}{1+x^2}$

17. Let $y=\frac{e^x}{\log x}$

Now, $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^x}{\log x}\right)=\frac{\log x\cdot\frac{d}{dx}(e^x)-e^x\cdot\frac{d}{dx}(\log x)}{(\log x)^2}$

$=\frac{\log x\cdot e^x-e^x\cdot\frac{1}{x}}{(\log x)^2}=\frac{xe^x\log x-e^x}{x(\log x)^2}=\frac{e^x(x\log x-1)}{x(\log x)^2}$

18. We have $u=\sin(m\cos^{-1}x), v=\cos(m\sin^{-1}x)$

$\Rightarrow \sin^{-1}u=m\cos^{-1}x, \cos^{-1}v=m\sin^{-1}x$

Now, $\sin^{-1}u+\cos^{-1}v=m(\cos^{-1}x+\sin^{-1}x)$

$=m\frac{\pi}{2} \quad \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

Differentiating both sides w.r.t. v, we get

$$\frac{1}{\sqrt{1-u^2}}\frac{du}{dv}-\frac{1}{\sqrt{1-v^2}}=0 \Rightarrow \frac{du}{dv}=\frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}$$

19. We have, $f(x)=\begin{cases}\frac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne2\\ k, & \text{if } x=2\end{cases}$

At $x=2$, $\lim_{x\to2}\frac{2^{x+2}-16}{4^x-16}=\lim_{x\to2}\frac{2^x\cdot2^2-2^4}{4^x-4^2}=\lim_{x\to2}\frac{4\cdot(2^x-4)}{(2^x)^2-(4)^2}$

$=\lim_{x\to2}\frac{4\cdot(2^x-4)}{(2^x-4)(2^x+4)} \quad [\because a^2-b^2=(a+b)(a-b)]$

$=\lim_{x\to2}\frac{4}{2^x+4}=\frac{4}{8}=\frac{1}{2}$

Since, $f(x)$ is continuous at $x=2$.

$\therefore$ L.H.L. = R.H.L. = $f(2)$

But given $f(2)=k$

$\therefore \quad k=\frac{1}{2}.$

OR

Given, $y=e^{x+e^{x+e^{x+\ldots\text{ to }\infty}}}$

$\therefore \quad y=e^{x+y}$ ...(i)

Taking logarithm, we get $\log y=(x+y)\log_e e$

or $\log y=x+y \quad [\because \log_e e=1]$

Differentiating w.r.t. x, we get

$\frac{1}{y}\cdot\frac{dy}{dx}=1+\frac{dy}{dx} \Rightarrow \left(\frac{1}{y}-1\right)\frac{dy}{dx}=1 \Rightarrow \left(\frac{1-y}{y}\right)\frac{dy}{dx}=1$

$\therefore \quad \frac{dy}{dx}=\frac{y}{1-y}$

20. (i) Given, $y=\tan^{-1}\left(\frac{\log\left(\frac{e}{x^2}\right)}{\log ex^2}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

$$=\tan^{-1}\left(\frac{1-\log x^2}{1+\log x^2}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$$

$= \tan^{-1}(1) - \tan^{-1}(2\log x) + \tan^{-1}(3) + \tan^{-1}(2\log x)$

$\Rightarrow\ y = \tan^{-1}(1) + \tan^{-1}(3) \Rightarrow \frac{dy}{dx}=0 \Rightarrow \frac{d^2y}{dx^2}=0$

(ii) Given $y^2 = ax^2 + bx + c$

$\Rightarrow\ 2yy_1 = 2ax + b$ ...(i)

$\Rightarrow\ 2yy_2 + y_1(2y_1) = 2a$

$\Rightarrow\ yy_2 = a - y_1^2 \Rightarrow yy_2 = a-\left(\frac{2ax+b}{2y}\right)^2$ (Using (i))

$$=\frac{4y^2a-(4a^2x^2+b^2+4abx)}{4y^2}$$

$\Rightarrow\ y^3y_2=\frac{4a(ax^2+bx+c)-(4a^2x^2+b^2+4abx)}{4}=\frac{4ac-b^2}{4}$

$\Rightarrow\ \frac{d}{dx}(y^3y_2)=0$

21. Here, $y = Ae^{-kt}\cos(pt + c)$

Differentiating w.r.t. t, we get

$$\frac{dy}{dt}=-k\,Ae^{-kt}\cos(pt+c)-pAe^{-kt}\sin(pt+c)$$

or $\frac{dy}{dt}=-ky-pAe^{-kt}\sin(pt+c)$ ...(i)

Differentiating (i) again w.r.t. 't', we get

$$\frac{d^2y}{dt^2}=-k\frac{dy}{dt}+pkAe^{-kt}\sin(pt+c)-p^2Ae^{-kt}\cos(pt+c)$$

$$=-\frac{k\,dy}{dt}+pk\,Ae^{-kt}\sin(pt+c)-p^2y$$

$\Rightarrow\ \frac{d^2y}{dt^2}=-\frac{kdy}{dt}+k\left(-ky-\frac{dy}{dt}\right)-p^2y$ [From (i)]

$\Rightarrow\ \frac{d^2y}{dt^2}=-2k\frac{dy}{dt}-k^2y-p^2y$

$\Rightarrow\ \frac{d^2y}{dt^2}+2k\frac{dy}{dt}+y(p^2+k^2)=0$

$\Rightarrow\ \frac{d^2y}{dt^2}+2k\frac{dy}{dt}+n^2y=0$, where $n^2=p^2+k^2$

22. We have $f(x)=\frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}, x\neq 0$

For $f(x)$ to be continuous at $x = 0$, we have

$f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow\ f(0)=\lim_{x\to 0}\left[\frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}\right]$

$$=\lim_{x\to 0}\left[\frac{(256)^{\frac{1}{8}}-(256-7x)^{\frac{1}{8}}}{(5x+32)^{\frac{1}{5}}-(32)^{\frac{1}{5}}}\right]=-\lim_{x\to 0}\left[\frac{(256-7x)^{\frac{1}{8}}-256^{\frac{1}{8}}}{(5x+32)^{\frac{1}{5}}-(32)^{\frac{1}{5}}}\right]$$

$$=-\lim_{x\to 0}\left(\frac{7}{5}\right)\left[\frac{\frac{(256-7x)^{\frac{1}{8}}-256^{\frac{1}{8}}}{7x}}{\frac{(5x+32)^{\frac{1}{5}}-(32)^{\frac{1}{5}}}{5x}}\right]$$

$$=\frac{7}{5}\lim_{x\to 0}\left[\frac{\frac{(256-7x)^{\frac{1}{8}}-(256)^{\frac{1}{8}}}{(256-7x)-256}}{\frac{(5x+32)^{\frac{1}{5}}-(32)^{\frac{1}{5}}}{(5x+32)-32}}\right]$$

$$=\frac{7}{5}\times\frac{\frac{1}{8}(256)^{\frac{1}{8}-1}}{\frac{1}{5}(32)^{\frac{1}{5}-1}}=\frac{7}{5}\times\frac{\frac{1}{8}(256)^{-\frac{7}{8}}}{\frac{1}{5}(32)^{-\frac{4}{5}}}=\frac{7}{8}\times\frac{2^4}{2^7}$$

$$=\frac{7}{8}\times\frac{1}{8}=\frac{7}{64}.$$

OR

Continuity at $x = 2$:

L.H.L. $= \lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x^2-x)$

$=\lim_{h\to 0}[2(2-h)^2-(2-h)] = 2(2)^2 - 2 = 8 - 2 = 6$

R.H.L. $= \lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(5x-4)$

$=\lim_{h\to 0}[5(2+h)-4]=5(2)-4=6$

Also, $f(2) = 2(2)^2 - 2 = 8-2 = 6$

$\therefore$ L.H.L = R.H.L = $f(2)$

Hence, $f(x)$ is continuous at $x = 2$.

Differentiability at $x = 2$:

L.H.D. $= \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{(2x^2-x)-6}{x-2}$

$=\lim_{h\to 0}\frac{\left[2(2-h)^2-(2-h)\right]-6}{-h}$ [Putting $x = 2 - h$]

$=\lim_{h\to 0}\frac{\left[2(4+h^2-4h)-2+h\right]-6}{-h}$

$=\lim_{h\to 0}\frac{8-2-6+2h^2-7h}{-h}$

$=\lim_{h\to 0}\frac{h(2h-7)}{-h}=7$

R.H.D. $= \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to 2^+}\frac{[5x-4]-6}{x-2}=\lim_{h\to 0}\frac{[5(2+h)-4]-6}{h}$ [Putting $x = 2 + h$]

$=\lim_{h\to 0}\frac{10+5h-4-6}{h}=\lim_{h\to 0}\frac{5h}{h}=5$

$\therefore$ L.H.D. $\neq$ R.H.D.

Hence, $f(x)$ is not differentiable at $x = 2$.

23. We have, $\sin x = \dfrac{2t}{1+t^2}$ ...(i)

and $\tan y = \dfrac{2t}{1-t^2}$ ...(ii)

Taking derivative of (i) and (ii) w.r.t. 't', we get

$$\frac{d}{dt}(\sin x) = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$\Rightarrow \cos x \frac{dx}{dt} = \frac{(1+t^2)\cdot\frac{d}{dt}(2t) - (2t)\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

[Apply quotient rule]

$$= \frac{2(1+t^2) - 2t\cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\cos x}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\sin^2 x}} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{(1+t^2)}{(1-t^2)} = \frac{2}{1+t^2} \quad \text{...(iii)}$$

Also, $\dfrac{d}{dt}\tan y = \dfrac{d}{dt}\left(\dfrac{2t}{1-t^2}\right)$

$$\Rightarrow \sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\frac{d}{dt}\cdot(2t) - 2t\cdot\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}\cdot\frac{1}{\sec^2 y}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{(1+\tan^2 y)} = \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{1+\frac{4t^2}{(1-t^2)^2}} \quad \text{(Using (ii))}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{(1-t^2)^2}{(1+t^2)^2} = \frac{2}{1+t^2} \quad \text{...(iv)}$$

Dividing (iv) by (iii), we get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 1$$

24. We observe that

$$f(x) = \begin{cases} x e^{-\left(\frac{1}{x}+\frac{1}{x}\right)} = x e^{-2/x} & , \quad x > 0 \\ 0 & , \quad x = 0 \\ x e^{-\left(\frac{-1}{x}+\frac{1}{x}\right)} = x & , \quad x < 0 \end{cases}$$

Now, (L.H.D. at $x = 0$) $= \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0}$

$= \lim\limits_{x\to 0} \dfrac{x-0}{x-0} = 1$ [$\because f(x) = x$ for $x < 0$ and $f(0) = 0$]

and, (R.H.D. at $x = 0$) $= \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0}$

$= \lim\limits_{x\to 0} \dfrac{x e^{-2/x} - 0}{x}$ [$\because f(x) = x e^{-2/x}$ for $x > 0$ and $f(0) = 0$]

$= \lim\limits_{x\to 0} e^{-2/x} = 0.$

$\therefore$ (L. H. D. at $x = 0$) $\neq$ (R. H. D. at $x = 0$)

So, $f(x)$ is not differentiable at $x = 0$.

25. We have given,

$$f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$$

At $x = 2$, L.H.L. $= \lim\limits_{x\to 2^-} \dfrac{2x^2-3x-2}{x-2}$

$$= \lim_{h\to 0} \frac{2(2-h)^2 - 3(2-h) - 2}{(2-h)-2}$$

$$= \lim_{h\to 0} \frac{8+2h^2-8h-6+3h-2}{-h}$$

$= \lim\limits_{h\to 0} -(2h-5) = 5$ and R.H.L. $= \lim\limits_{x\to 2^+} \dfrac{2x^2-3x-2}{x-2}$

$$= \lim_{h\to 0} \frac{2(2+h)^2 - 3(2+h) - 2}{(2+h)-2}$$

$$= \lim_{h\to 0} \frac{8+2h^2+8h-6-3h-2}{h} = \lim_{h\to 0}(2h+5) = 5$$

also, $f(2) = 5$ (given)

Hence, L.H.L. = R.H.L. = $f(2)$

Therefore, $f(x)$ is continuous at $x = 2$.

CHAPTER

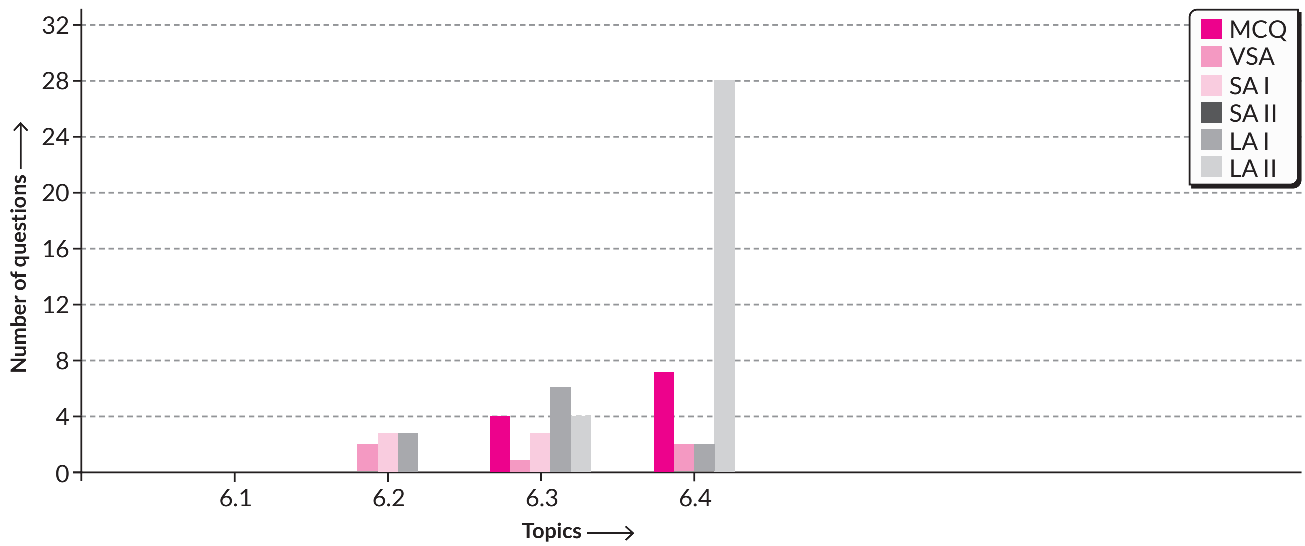
6

Application of Derivatives

TOPICS

6.1 Introduction
6.2 Rate of Change of Quantities
6.3 Increasing and Decreasing Functions
6.4 Maxima and Minima

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 6.4 is highly scoring topic.
- Maximum weightage is of Topic 6.4 *Maxima and Minima.*
- Maximum LA I type questions were asked from Topic 6.3 *Increasing and Decreasing Functions.*
- Maximum LA II type questions were asked from Topic 6.4 *Maxima and Minima.*

QUICK RECAP

Rate of Change of Quantities

- Let $y = f(x)$ be a function. If the change in one quantity y varies with another quantity x, then $\frac{dy}{dx}$ or $f'(x)$ denotes the rate of change of y (or $f(x)$) with respect to x.

Increasing and Decreasing Functions

- A function f is said to be
  - an increasing function on an interval (a, b) if, for all $x_1, x_2 \in (a, b)$, $x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$ and strictly increasing if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$.
  - a decreasing function on an interval (a, b) if, for all $x_1, x_2 \in (a, b)$, $x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$ and strictly decreasing if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

Note : If f is a continuous function on $[a, b]$ and differentiable on (a, b), then for each $x \in (a, b)$
(i) $f'(x) > 0 \Rightarrow f$ is strictly increasing on (a, b)
(ii) $f'(x) < 0 \Rightarrow f$ is strictly decreasing on (a, b)
(iii) $f'(x) = 0 \Rightarrow f$ is a constant function on (a, b).

Maxima and Minima

- Let f be a real valued function defined on I (subset of R), then
  - f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$ for all $x \in I$. The number $f(c)$ is called the (absolute) maximum value of f in I and the point c is called point of maxima of f in I.
  - f is said to have a minimum value in I, if there exists a point d in I such that $f(d) < f(x)$ for all $x \in I$. The number $f(d)$ is called the (absolute) minimum value of f in I and the point d is called point of minima of f in I.

Note : If $f : I \to R$ be a differentiable function and c be any interior point of I, then $f'(c) = 0$ if f attains its absolute maximum (minimum) value at c.

  - A point c is local maximum of a function $f(x)$ if there is an open interval I containing c such that $f(x) < f(c)$ for all $x \in I$.
  - A point c is local minimum of a function $f(x)$ if there is an open interval I containing c such that $f(c) < f(x)$ for all $x \in I$.

Note : If a function f is either increasing or decreasing in an interval I, then f is said to be a monotonic function.

  - **Critical Point** : If $f : I \to R$, then a point $c \in I$ is called the critical point of f, if either $f'(c) = 0$ or f is not differentiable at c.
  - **Methods of Finding Local Maxima and Local Minima**
    - **First Derivative Test :** Let $f(x)$ be a function defined on an open interval I and $f(x)$ be continuous at a critical point c in I. Then,
      (i) If $f'(x)$ changes sign from positive to negative as x increases through c, then $x = c$ is a point of local maxima.
      (ii) If $f'(x)$ changes sign from negative to positive as x increases through c, then $x = c$ is a point of local minima.
      (iii) If $f'(x)$ does not change sign as x increases through c, we say that c is neither a point of local maxima nor a point of local minima. In this case, $x = c$ is called a point of inflection.
    - **Second Derivative Test :** Let $f(x)$ be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,
      (i) If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a point of local maxima.
      (ii) If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.
      (iii) If $f'(c) = 0$ and $f''(c) = 0$, then use the first-derivative test.
  - **Working Rule for Finding Absolute Maxima and Minima**
    (i) Find all critical points of f in the interval $I = [a, b]$ *i.e.*, find all points x where either $f'(x) = 0$ or f is not differentiable.
    (ii) Find the end points of the interval *i.e.*, a and b.
    (iii) Find the values of f at all its critical points in interval I and at the end points of the interval I.
    (iv) Absolute maximum value of f in I = greatest of values of f at end points and at all critical points.
    (v) Absolute minimum value of f in I = least of values of f at end points and at all critical points.

BRAIN MAP

APPLICATION OF DERIVATIVES

Rate of Change of Quantities

Let $y = f(x)$ then $\frac{dy}{dx}$ or $f'(x)$ denotes the rate of change of y w.r.t. x and its value at $x = a$ is denoted as $\left[\frac{dy}{dx}\right]_{x=a}$.

Marginal Cost and Marginal Revenue

- Let C be the total cost of producing and marketing x units of a product, then marginal cost (MC), is $MC = \frac{dC}{dx}$.
- The rate of change of total revenue with respect to the quantity sold is the marginal revenue, $MR = \frac{dR}{dx}$.

Increasing and Decreasing Functions

Decreasing Function	**without derivative test**	If $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)\ \forall\ x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) \leq 0$ for each $x \in (a, b)$
Strictly Decreasing Function	**without derivative test**	If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) < 0$ for each $x \in (a, b)$
Increasing Function	**without derivative test**	If $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)\ \forall\ x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) \geq 0$ for each $x \in (a, b)$
Strictly increasing Function	**without derivative test**	If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in (a, b)$
	with derivative test	If $f'(x) > 0$ for each $x \in (a, b)$

Mak ma and Minima

Maxima and Minima
They exist at critical points only.

Local Maxima : At $x = a$
$f(a - h) < f(a) > f(a + h)\ (h \to 0)$

Local Minima : At $x = a$
$f(a - h) > f(a) < f(a + h)\ (h \to 0)$

Critical points
(i) $f(x)$ doesn't exist (or)
(ii) $f'(x)$ doesn't exist
(iii) $f'(x) = 0$

Global Max : Max value of $f(x)$ in domain
Global Min : Least value of $f(x)$ in domain

First derivative test
(i) $x = a$ is local max if $f'(a) = 0$ and $f'(x)$ changes sign from +ve to –ve
(ii) $x = a$ is local min if $f'(a) = 0$ and $f'(x)$ changes sign from –ve to +ve

Second derivative test
(i) Find the roots of $f'(x) = 0$
(ii) Suppose $x = a$ is one of the roots.
(iii) At those points $f''(x) < 0 \Rightarrow f(x)$ is maximum at $x = a$
$f''(x) > 0 \Rightarrow f(x)$ is minimum at $x = a$.

Previous Years' CBSE Board Questions

6.2 Rate of Change of Quantities

VSA (1 mark)

1. The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of ________ cm^2/s. *(2020)* Ap

2. The rate of change of the area of a circle with respect to its radius r, when r = 3 cm, is _______. *(2020)*

SA I (2 marks)

3. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. *(2018)* Ap

4. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. *(Delhi 2017)*

5. The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm? *(NCERT, AI 2017)* Ev

LA I (4 marks)

6. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? *(AI 2019)*

7. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm? *(Delhi 2015)* Ev

8. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area increases, when the side is 10 cm. *(AI 2014C)*

6.3 Increasing and Decreasing Functions

MCQ

9. The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is
(a) $(-1, \infty)$ (b) $(-2, -1)$ (c) $(-\infty, -2)$ (d) $[-1, 1]$ *(2023)*

10. The function $f(x) = x^3 + 3x$ is increasing in interval
(a) $(-\infty, 0)$ (b) $(0, \infty)$ (c) R (d) $(0, 1)$ *(2023)*

11. The interval, in which function $y = x^3 + 6x^2 + 6$ is increasing, is
(a) $(-\infty,-4)\cup(0,\infty)$ (b) $(-\infty,-4)$
(c) $(-4, 0)$ (d) $(-\infty, 0)\cup(4,\infty)$ *(Term I, 2021-22)*

12. The function $(x - \sin x)$ decreases for
(a) all x (b) $x<\frac{\pi}{2}$
(c) $0<x<\frac{\pi}{4}$ (d) no value of x *(Term I, 2021-22)*

VSA (1 mark)

13. Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing. *(2020)*

SA I (2 marks)

14. Find the interval in which the function $f(x) = 2x^3 - 3x$ is strictly increasing. *(2023)*

15. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R. *(Delhi 2017)*

16. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R. *(AI 2017)* Ap

LA I (4 marks)

17. Find whether the function $f(x) = \cos\left(2x+\frac{\pi}{4}\right)$; is increasing or decreasing in the interval $\frac{3\pi}{8}<x<\frac{5\pi}{8}$. *(2019)* U

18. Find the intervals in which the function
$f(x)=\frac{x^4}{4}-x^3-5x^2+24x+12$ is
(a) strictly increasing
(b) strictly decreasing *(2018)* Ap

19. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is
(a) strictly increasing
(b) strictly decreasing *(Delhi 2014)*

20. Find the value(s) of x for which $y = [x(x-2)]^2$ is an increasing function. *(AI 2014)*

21. Find the intervals in which the function $f(x)=\frac{3}{2}x^4-4x^3-45x^2+51$ is
(i) strictly increasing
(ii) strictly decreasing *(Foreign 2014)* Ap

22. Find the intervals in which the function $f(x)=\frac{3}{10}x^4-\frac{4}{5}x^3-3x^2+\frac{36}{5}x+11$ is
(a) strictly increasing
(b) strictly decreasing. *(NCERT, AI 2014C)* Ev

LA II **(5/6 marks)**

23. Find the intervals on which the function $f(x) = (x - 1)^3 (x - 2)^2$ is (a) strictly increasing (b) strictly decreasing. *(2020)*

24. Find the intervals in which the function f defined as $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is strictly increasing or decreasing. *(2020)* U

25. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing. *(Delhi 2016)* An

26. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is (i) strictly increasing (ii) strictly decreasing. *(Delhi 2014C)*

6.4 Maxima and Minima

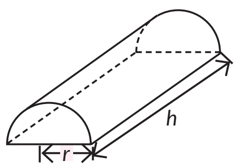
MCQ

27. The value of x for which $(x - x^2)$ is maximum, is
(a) 3/4 (b) 1/2 (c) 1/3 (d) 1/4
(Term I, 2021-22) U

28. A wire of length 20 cm is bent in the form of a sector of a circle. The maximum area that can be enclosed by the wire is
(a) 20 sq.cm (b) 25 sq.cm
(c) 10 sq.cm (d) 30 sq.cm
(Term I, 2021-22)

Case study-Some young entrepreneur started a industry "young achievers" for casting metal into various shapes. They put up an advertisement online stating the same and expecting order to cast metal for toys, sculptures, decorative pieces and more.

A group of friends wanted to make innovative toys and hence contacted the "young achievers" to order them to cast metal into solid half cylinders with a rectangular base and semi-circular ends.



Based on the above information, answer the following questions (29 to 33) :

29. The volume (V) of the casted half cylinder will be
(a) $\pi r^2 h$ (b) $\frac{1}{3}\pi r^2 h$
(c) $\frac{1}{2}\pi r^2 h$ (d) $\pi r^2(r + h)$
(Term I, 2021-22)

30. The total surface area (S) of the casted half cylinder will be
(a) $\pi rh + 2\pi r^2 + rh$ (b) $\pi rh + \pi r^2 + 2rh$
(c) $2\pi rh + \pi r^2 + 2rh$ (d) $\pi rh + \pi r^2 + rh$
(Term I, 2021-22) Ev

31. The total surface area S can be expressed in terms of V and r as
(a) $2\pi r + \frac{2V(\pi+2)}{\pi r}$ (b) $\pi r + \frac{2V}{\pi r}$
(c) $\pi r^2 + \frac{2V(\pi+2)}{\pi r}$ (d) $2\pi r^2 + \frac{2V(\pi+2)}{\pi r}$
(Term I, 2021-22) Ap

32. For the given half-cylinder of volume V, the total surface area S is minimum, when
(a) $(\pi + 2)\,V = \pi^2 r^3$ (b) $(\pi + 2)\,V = \pi^2 r^2$
(c) $2(\pi + 2)\,V = \pi^2 r^3$ (d) $(\pi + 2)\,V = \pi^2 r$
(Term I, 2021-22)

33. The ratio $h : 2r$ for which S to be minimum will be equal to
(a) $2\pi : \pi + 2$ (b) $2\pi : \pi + 1$
(c) $\pi : \pi + 1$ (d) $\pi : \pi + 2$
(Term I, 2021-22) Cr

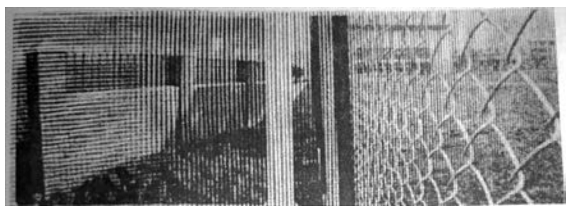
VSA **(1 mark)**

34. The absolute minimum value of $f(x) = 2 \sin x$ in $\left[0, \frac{3\pi}{2}\right]$ is ________. *(2020)*

35. The least value of the function $f(x) = ax + \frac{b}{x}$ $(a > 0, b > 0, x > 0)$ is ________. *(2020)*

LA I **(4 marks)**

36. **Case-study :** Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.



Based on the above information, answer the following questions :
(i) Let 'x' metres denote the length of the side of the garden perpendicular to the brick wall and 'y' metres denote the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write $A(x)$, the area of the garden.
(ii) Determine the maximum value of $A(x)$. *(2023)*

37. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question? *(2018)* Ap

LA II **(5 / 6 marks)**

38. The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing. *(2023)*

39. Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers. *(2023)*

40. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right

circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone. *(2020)*

41. Find the minimum value of $(ax + by)$, where $xy = c^2$. *(2020, Foreign 2015)*

42. Amongst all open (from the top) right circular cylindrical boxes of volume 125π cm^3, find the dimensions of the box which has the least surface area. *(2020)*

43. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its side. Also, find the maximum volume. *(2020)* Ap

44. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. *(2020C)*

45. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. *(2019)*

46. Show that the height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base. *(2019)*

47. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank? *(NCERT, Delhi 2019)* Cr

48. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ *(AI 2019)*

49. Show that the right circular cylinder of given surface area and maximum volume is such that its height is equal to the diameter of the base. *(2019C)*

50. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$. *(NCERT Exemplar, Delhi 2017, AI 2016, 2014)*

51. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. *(AI 2017)* Ev

52. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find maximum volume in terms of volume of the sphere. *(Delhi 2016, AI 2014)* Ap

53. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. *(AI 2016)*

54. The sum of the surface areas of a cuboid with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes. *(Foreign 2016)* Ap

OR

The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}, x$ and $2x$, is constant. Show that the sum of their volumes is minimum if x is equal to three times the radius of sphere. *(AI 2015C)*

55. Find the local maxima and local minima of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum and local minimum values. *(Delhi 2015)*

56. Find the coordinates of a point of the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$. *(Foreign 2015)*

57. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 m^3. If building of tank costs ₹ 100 per square metre for the base and ₹ 50 per square metre for the sides, find the cost of least expensive tank. *(Delhi 2015C)* Ap

58. A point on the hypotenuse of a right triangle is at distance 'a' and 'b' from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$. *(NCERT, Delhi 2015C)*

59. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area. *(Delhi 2014)*

60. Show that the semi vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. *(Delhi 2014)* Ev

61. Prove that the semi vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. *(Delhi 2014)*

62. The sum of the perimeters of a circle and a square is k, where k is some constant. Prove that the sum of their areas is least when the side of the square is equal to the diameter of the circle. *(Foreign 2014, Delhi 2014C)*

63. Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base. *(Foreign 2014)* Ap

64. A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10 m. Find the dimension of the window to admit maximum light through the whole opening. *(Foreign 2014)*

65. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles. *(AI 2014C)*

66. Find the point P on the curve $y^2 = 4ax$ which is nearest to the point $(11a, 0)$. *(AI 2014C)* Ev

67. If the length of three sides of a trapezium other than base is 10 cm each, then find the area of the trapezium when it is maximum. *(NCERT, AI 2014C)* Ap

CBSE Sample Questions

6.2 Rate of Change of Quantities

SA I **(2 marks)**

1. A man 1.6 m tall walks at the rate of 0.3 m/sec away from a street light that is 4 m above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow lengthening? *(2022-23)*

6.3 Increasing and Decreasing Functions

MCQ

2. Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is strictly increasing.
   (a) $(-\infty, 2) \cup (2, \infty)$ (b) $(2, \infty)$
   (c) $(-\infty, 2)$ (d) $(-\infty, 2] \cup (2, \infty)$
   (Term I, 2021-22) Ev

3. The real function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is
   (a) Strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$
   (b) Strictly decreasing in $(-2, 3)$
   (c) Strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$
   (d) Strictly decreasing in $(-\infty, -2) \cup (3, \infty)$
   (Term I, 2021-22)

4. The value of b for which the function $f(x) = x + \cos x + b$ is strictly decreasing over R is
   (a) $b < 1$ (b) No value of b exists
   (c) $b \leq 1$ (d) $b \geq 1$
   (Term I, 2021-22) Ap

SA II **(3 marks)**

5. Find the intervals in which the function f given by $f(x) = \tan x - 4x,\ x \in \left(0, \frac{\pi}{2}\right)$ is
   (a) strictly increasing (b) strictly decreasing
   (2020-21)

6.4 Maxima and Minima

MCQ

6. The least value of the function $f(x) = 2\cos x + x$ in the closed interval $\left[0, \frac{\pi}{2}\right]$ is
   (a) 2 (b) $\frac{\pi}{6} + \sqrt{3}$
   (c) $\frac{\pi}{2}$
   (d) The least value does not exist.
   (Term I, 2021-22)

7. The area of a trapezium is defined by function f and given by $f(x) = (10 + x)\sqrt{100 - x^2}$, then the area when it is maximised is
   (a) 75 cm^2 (b) $7\sqrt{3} \text{ cm}^2$
   (c) $75\sqrt{3} \text{ cm}^2$ (d) 5 cm^2
   (Term I, 2021-22) Cr

8. The maximum value of $[x(x - 1) + 1]^{1/3}, 0 \leq x \leq 1$ is
   (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt[3]{\frac{1}{3}}$
   (Term I, 2021-22)

Case Study : The fuel cost per hour for running a train is proportional to the square of the speed it generates in km per hour. If the fuel costs ₹ 48 per hour at speed 16 km per hour and the fixed charges to run the train amount is ₹ 1200 per hour.

Assume the speed of the train as v km/h.
Based on the given information, answer the following questions (9 to 13).

9. Given that the fuel cost per hour is k times the square of the speed the train generates in km/h, the value of k is
   (a) $\frac{16}{3}$ (b) $\frac{1}{3}$ (c) 3 (d) $\frac{3}{16}$

10. If the train has travelled a distance of 500 km, then the total cost of running the train is given by function
    (a) $\frac{15}{16}v + \frac{600000}{v}$ (b) $\frac{375}{4}v + \frac{600000}{v}$
    (c) $\frac{5}{16}v^2 + \frac{150000}{v}$ (d) $\frac{3}{16}v + \frac{6000}{v}$

11. The most economical speed to run the train is
    (a) 18 km/h (b) 5 km/h
    (c) 80 km/h (d) 40 km/h

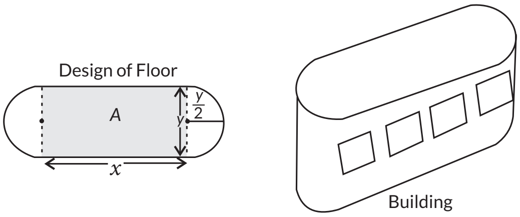
12. The fuel cost for the train to travel 500 km at the most economical speed is

(a) ₹ 3750 (b) ₹ 750
(c) ₹ 7500 (d) ₹ 75000

13. The total cost of the train to travel 500 km at the most economical speed is
(a) ₹ 3750 (b) ₹ 75000
(c) ₹ 7500 (d) ₹ 15000
(Term I, 2021-22)

Case study based questions are compulsory. Attempt any 4 sub parts from question. Each sub-part carrie 1 mark.

14. An architect designs a building for a multi-national company. The floor consists of a rectangular region with semicircular ends having a perimeter of 200 m as shown below:



Based on the above information answer the following:

(i) If x and y represents the length and breadth of the rectangular region, then the relation between the variables is
(a) $x + \pi y = 100$ (b) $2x + \pi y = 200$
(c) $\pi x + y = 50$ (d) $x + y = 100$

(ii) The area of the rectangular region A expressed as a function of x is
(a) $\frac{2}{\pi}(100x - x^2)$ (b) $\frac{1}{\pi}(100x - x^2)$
(c) $\frac{x}{\pi}(100 - x)$ (d) $\pi y^2 + \frac{2}{\pi}(100x - x^2)$

(iii) The maximum value of area A is
(a) $\frac{\pi}{3200}\text{m}^2$ (b) $\frac{3200}{\pi}\text{m}^2$
(c) $\frac{5000}{\pi}\text{m}^2$ (d) $\frac{1000}{\pi}\text{m}^2$

(iv) The CEO of the multi-national company is interested in maximizing the area of the whole floor including the semi-circular ends. For this to happen the value of x should be
(a) 0 m (b) 30 m
(c) 50 m (d) 80 m

(v) The extra area generated if the area of the whole floor is maximized is
(a) $\frac{3000}{\pi}\text{m}^2$ (b) $\frac{5000}{\pi}\text{m}^2$
(c) $\frac{7000}{\pi}\text{m}^2$
(d) No change, Both areas are equal *(2020-21)* Cr

LA I (4 marks)

15. **Case-Study :** Read the following passage and answer the questions given below.

In an elliptical sport field the authority wants to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) If the length and the breadth of the rectangular field be $2x$ and $2y$ respectively, then find the area function in terms of x.
(ii) Find the critical point of the function.
(iii) Use First Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximize its area.

OR

Use Second Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximize its area. *(2022-23)*

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. Let r be the radius and A be the area of circle.

Given that $\frac{dr}{dt} = 3\text{ cm/sec}$ ...(i)

We know that, area of circle $A = \pi r^2$

$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r \cdot 3$ [Using (i)]
$= 6\pi r$

$\therefore \left(\frac{dA}{dt}\right)_{r = 2\text{ cm}} = 12\pi\text{ cm}^2\text{/s}$

2. Let r be the radius and A be the area of circle.

We know that, area of circle, $A = \pi r^2$

$\therefore \frac{dA}{dr} = 2\pi r \Rightarrow \left(\frac{dA}{dr}\right)_{r=3} = 6\pi\text{ cm}$

3. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

$\Rightarrow \frac{dC}{dx} = 0.015x^2 - 0.04x + 30$

Now, $\left(\frac{dC}{dx}\right)_{x=3} = 0.015 \times 3^2 - 0.04 \times 3 + 30 = 30.015$

4. Let *r*, *S* and *V* respectively be the radius, surface area and volume of sphere at any time *t*.

Given, $\frac{dV}{dt} = 3\,\text{cm}^3/\text{sec}$

We know that, volume of sphere $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$ cm/sec

We know that, surface area of sphere $S = 4\pi r^2$

$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r\left(\frac{3}{4\pi r^2}\right) \Rightarrow \frac{dS}{dt} = \frac{6}{r}$

$\therefore \left(\frac{dS}{dt}\right)_{r=2\text{cm}} = \frac{6}{2} = 3\,\text{cm}^2/\text{sec}$

5. Let *l* be the length of an edge and *V* be the volume of a cube respectively.

Given, $\frac{dV}{dt} = 9\text{cm}^3/\text{s}$ and $l = 10\,\text{cm}$

We know that, volume of cube $(V) = l^3$

$\therefore \frac{dV}{dt} = \frac{d}{dt}(l)^3 \Rightarrow 9 = 3l^2 \frac{dl}{dt}$

$\Rightarrow \frac{dl}{dt} = \frac{3}{l^2}$ ...(i)

And, surface area of cube $(A) = 6l^2$

$\therefore \frac{dA}{dt} = \frac{d}{dt}(6l^2) = 12l\frac{dl}{dt} = 12l \times \frac{3}{l^2}$ (From (i))

$= \frac{36}{l}$

$\therefore \left(\frac{dA}{dt}\right)_{l=10} = \frac{36}{10} = 3.6\text{cm}^2/\text{s}$

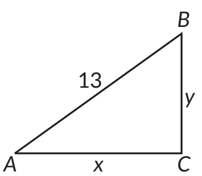
6. Let foot of the ladder is at a distance *x* m from the wall and height on the wall is *y* m.

Here, $x^2 + y^2 = (13)^2$ [Using Pythagoras theorem]

Differentiating with respect to *t*, we get

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$



When $x = 5$ m, $y^2 = (13)^2 - (5)^2 = 169 - 25 = 144$

$\therefore y = 12$ m

Also, $\frac{dx}{dt} = 2$ cm/sec [Given]

$\therefore \frac{dy}{dt} = \frac{-5}{12} \times 2 = \frac{-5}{6}$ cm/sec

7. Let '*a*' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2\text{cm/sec}$

Let '*A*' be the area of an equilateral triangle, then

$A = \frac{\sqrt{3}}{4}a^2 \Rightarrow \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4}a\frac{da}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}$

$\therefore \left(\frac{dA}{dt}\right)_{a=20} = \frac{\sqrt{3}}{2} \times 20 \times 2 = 20\sqrt{3}\,\text{cm}^2/\text{sec}$

8. Let '*a*' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2\text{cm/sec}$

Let '*A*' be the area of an equilateral triangle, then

$A = \frac{\sqrt{3}}{4}a^2 \Rightarrow \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4}a\frac{da}{dt} = \frac{\sqrt{3}}{2}a\frac{da}{dt}$

$\therefore \left(\frac{dA}{dt}\right)_{a=10} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3}\,\text{cm}^2/\text{sec}$

Concept Applied

- Power rule of derivative : $\frac{d}{dx}(x^n) = nx^{n-1}$

9. **(b):** We have, $f(x) = 2x^3 + 9x^2 + 12x - 1$

$\Rightarrow f'(x) = 6x^2 + 18x + 12$

For decreasing, $f'(x) < 0$

$\therefore 6x^2 + 18x + 12 < 0$

$\Rightarrow x^2 + 3x + 2 < 0 \Rightarrow (x + 1)(x + 2) < 0 \Rightarrow -2 < x < -1$

So, $f(x)$ is decreasing, if $x \in (-2, -1)$.

10. **(c):** $f(x) = x^3 + 3x$

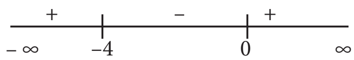
For increasing, we must have $f'(x) > 0$

$\therefore f'(x) = 3x^2 + 3 > 0 \Rightarrow 3(x^2 + 1) > 0$

$\Rightarrow x^2 + 1 > 0$, which is true $\forall\, x \in R$.

11. **(a):** Given, $y = x^3 + 6x^2 + 6 \Rightarrow \frac{dy}{dx} = 3x^2 + 12x$

For increasing, $\frac{dy}{dx} > 0 \Rightarrow 3x^2 + 12x > 0 \Rightarrow 3x(x + 4) > 0$



So, *y* is strictly increasing in $(-\infty, -4) \cup (0, \infty)$.

12. **(d):** Let $f(x) = x - \sin x$

Differentiating w.r.t. *x*, we get $f'(x) = 1 - \cos x$

For function to be decreasing, $f'(x) < 0$

$\Rightarrow 1 - \cos x < 0 \Rightarrow \cos x > 1$,

which is not possible, because maximum value of cos*x* is 1.

$\therefore f(x) = (x - \sin x)$ doesn't decrease at any value of *x*.

13. Let $y = f(x) = 7 - 4x - x^2$

$\therefore \frac{dy}{dx} = -4 - 2x$

For strictly increasing, $\frac{dy}{dx} > 0$

$\Rightarrow -4 - 2x > 0 \Rightarrow x < -2$

$\therefore$ Required interval is $(-\infty, -2)$.

14. Let $y = f(x) = 2x^3 - 3x \quad \therefore \frac{dy}{dx} = 6x^2 - 3$

For strictly increasing, $\frac{dy}{dx} > 0$

$\Rightarrow 6x^2 - 3 > 0 \Rightarrow x^2 > \frac{1}{2}$

So, $f(x)$ is strictly increasing in $x \in \left(-\infty, \frac{-1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$.

15. We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$

$\Rightarrow f'(x) = 12x^2 - 36x + 27$

$= 12\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) + 27$

$= 12\left(x - \frac{3}{2}\right)^2 - 27 + 27 = 12\left(x - \frac{3}{2}\right)^2 \geq 0 \; \forall x \in R$

Hence, $f(x)$ is always increasing on R.

16. We have, $f(x) = x^3 - 3x^2 + 6x - 100$ ...(i)

Differentiating (i) w.r.t. x, we get

$f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 1) + 3 = 3(x - 1)^2 + 3 > 0$

($\because$ For all values of x, $(x - 1)^2$ is always positive)

$\therefore \quad f'(x) > 0$

So, $f(x)$ is an increasing function on R.

Answer Tips

- If $f'(x) > 0 \Rightarrow f$ is strictly increasing function.

17. We have, $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$

$\therefore \quad f'(x) = -2\sin\left(2x + \frac{\pi}{4}\right)$

Given, $\frac{3\pi}{8} < x < \frac{5\pi}{8}$

$\Rightarrow \frac{3\pi}{4} < 2x < \frac{5\pi}{4} \Rightarrow \frac{\pi}{4} + \frac{3\pi}{4} < 2x + \frac{\pi}{4} < \frac{5\pi}{4} + \frac{\pi}{4}$

$\Rightarrow \pi < 2x + \frac{\pi}{4} < \frac{3\pi}{2} \Rightarrow \sin\left(2x + \frac{\pi}{4}\right) < 0$

[$\because$ sin function is negative in III^rd and IV^th quadrant]

$\Rightarrow -2\sin\left(2x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad f'(x) > 0$

Hence, $f(x)$ is increasing in $\left(\frac{3\pi}{8}, \frac{5\pi}{8}\right)$.

18. We have, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ ...(i)

$f(x)$ being polynomial function is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$

$= x^3 - 3x^2 - 10x + 24$

$= (x - 2)(x^2 - x - 12) = (x - 2)(x - 4)(x + 3)$

(a) For strictly increasing, $f'(x) > 0$

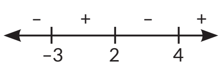
$\Rightarrow \quad (x - 2)(x - 4)(x + 3) > 0$

$\Rightarrow \quad x \in (-3, 2) \cup (4, \infty)$

(b) For strictly decreasing, $f'(x) < 0$

$\Rightarrow \quad (x - 2)(x - 4)(x + 3) < 0$

$\Rightarrow \quad x \in (-\infty, -3) \cup (2, 4)$



19. We have, $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$

$\Rightarrow \quad f'(x) = 12x(x + 1)(x - 2)$

Now, $f'(x) = 0$

$\Rightarrow \quad 12x(x + 1)(x - 2) = 0$

$\Rightarrow \quad x = -1, x = 0$ or $x = 2$

Hence, these points divide the whole real line into four disjoint open intervals namely $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -1)$	$(-)(-)(-) < 0$	Strictly decreasing
$(-1, 0)$	$(-)(+)(-) > 0$	Strictly increasing
$(0, 2)$	$(+)(+)(-) < 0$	Strictly decreasing
$(2, \infty)$	$(+)(+)(+) > 0$	Strictly increasing

(a) $f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$.

(b) $f(x)$ is strictly decreasing in $(-\infty, -1) \cup (0, 2)$.

20. Here, $y = [x(x - 2)]^2 = x^2(x - 2)^2$

$\Rightarrow \quad \frac{dy}{dx} = 2x(x - 2)^2 + 2x^2(x - 2)$

$= 2x(x - 2)(x - 2 + x) = 4x(x - 1)(x - 2)$

For y to be an increasing function, $\frac{dy}{dx} \geq 0$

$\Rightarrow \quad x(x - 1)(x - 2) \geq 0$

Case 1: When $-\infty < x \leq 0$

$\frac{dy}{dx} \leq 0 \Rightarrow y$ is a decreasing function.

Case 2: When $0 \leq x \leq 1$

$\frac{dy}{dx} \geq 0 \Rightarrow y$ is an increasing function.

Case 3: When $1 \leq x \leq 2$

$\frac{dy}{dx} \leq 0 \Rightarrow y$ is a decreasing function.

Case 4: When $2 \leq x \leq \infty$

$\frac{dy}{dx} \geq 0 \Rightarrow y$ is an increasing function.

$\therefore \quad y$ is an increasing function in $[0, 1] \cup [2, \infty)$

21. We have, $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ ...(i)

$f(x)$ being a polynomial function is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x) = \frac{3}{2} \times 4x^3 - 12x^2 - 90x$

$\Rightarrow \quad f'(x) = 6x^3 - 12x^2 - 90x = 6x(x^2 - 2x - 15)$

$= 6x(x - 5)(x + 3)$

(i) For strictly increasing, $f'(x) > 0$

$\Rightarrow \quad 6x(x - 5)(x + 3) > 0$

$\Rightarrow \quad x \in (-3, 0) \cup (5, \infty)$

(ii) For strictly decreasing, $f'(x) < 0$

$\Rightarrow \quad 6x(x - 5)(x + 3) < 0$

$\Rightarrow \quad x \in (-\infty, -3) \cup (0, 5)$

Answer Tips

- If $f'(x) < 0 \Rightarrow f$ is strictly decreasing function.
- If $f'(x) > 0 \Rightarrow f$ is strictly increasing function.

22. Here, $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$

Differentiating w.r.t. x, we get

$f'(x)=\frac{3}{10}\cdot 4x^3-\frac{4}{5}\cdot 3x^2-3\cdot 2x+\frac{36}{5}\cdot 1$

$=\frac{6}{5}(x^3-2x^2-5x+6)\ =\frac{6}{5}(x-1)(x^2-x-6)$

$=\frac{6}{5}(x-1)(x+2)(x-3)$

$\therefore\ \ f'(x) = 0 \Rightarrow x = -2, 1, 3.$

Hence, the points divide the real line into four disjoint intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -2)$	$(-)\,(-)\,(-) < 0$	Strictly decreasing
$(-2, 1)$	$(-)\,(+)\,(-) > 0$	Strictly increasing
$(1, 3)$	$(+)\,(+)\,(-) < 0$	Strictly decreasing
$(3, \infty)$	$(+)\,(+)\,(+) > 0$	Strictly increasing

(a) $f(x)$ is strictly increasing in $(-2, 1) \cup (3, \infty)$.
(b) $f(x)$ is strictly decreasing in $(-\infty, -2) \cup (1, 3)$.

23. We have, $f(x) = (x-1)^3 (x-2)^2$ ...(1)

Differentiating equation (1) w.r.t. x, we get

$f'(x)=(x-1)^3\frac{d}{dx}(x-2)^2+(x-2)^2\frac{d}{dx}(x-1)^3$

$= (x-1)^2 (x-2)[2(x-1) + 3(x-2)]$

$= (x-1)^2 (x-2)(2x-2+3x-6)$

$\Rightarrow\ \ f'(x) = (x-1)^2 (x-2)(5x-8)$

Now put $f'(x) = 0$

$\Rightarrow\ \ (x-1)^2 (x-2)(5x-8) = 0 \Rightarrow x = 1, 2, 8/5$

The points divide the real line into four disjoint intervals

$(-\infty, 1)$, $(1, 8/5)$, $(8/5, 2)$ and $(2, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, 1)$	$(+)(-)(-) > 0$	Strictly increasing
$(1, 8/5)$	$(+)(-)(-) > 0$	Strictly increasing
$(8/5, 2)$	$(+)(-)(+) < 0$	Strictly decreasing
$(2, \infty)$	$(+)(+)(+) > 0$	Strictly increasing

(a) $f(x)$ is strictly increasing in $(-\infty, 1) \cup (1, 8/5) \cup (2, \infty)$.

(b) $f(x)$ is strictly decreasing in $\left(\frac{8}{5}, 2\right)$.

24. The given function is

$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$

$\Rightarrow\ \ f'(x) = \cos x - \sin x$

Now $f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$\Rightarrow \tan x = 1$

$\Rightarrow x=\frac{\pi}{4},\frac{5\pi}{4}$

The points $x=\frac{\pi}{4}$ and $x=\frac{5x}{4}$ divide the interval $[0, 2\pi]$ into three disjoint intervals, $[0, \pi/4)$, $(\pi/4, 5\pi/4)$, $\left(\frac{5\pi}{4}, 2\pi\right]$

Now $f'(x) > 0$ in $\left[0, \frac{\pi}{4}\right)$

$\therefore\ \ f$ is strictly increasing in $\left[0\ \frac{\pi}{4}\right)$.

$f'(x) < 0$ in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

$\therefore\ \ f$ is strictly decreasing in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

and $f'(x) > 0$ in $\left(\frac{5\pi}{4}, 2\pi\right]$

$\therefore\ \ f$ is strictly increasing in $\left(\frac{5\pi}{4}, 2\pi\right]$.

Thus, the function f is strictly increasing in $\left[0,\frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$.

25. $f(x) = \sin 3x - \cos 3x$

$\Rightarrow\ \ f'(x) = 3\cos 3x + 3\sin 3x$

$f'(x) = 0 \Rightarrow\ 3\cos 3x = -3\sin 3x$

$\Rightarrow\ \ \cos 3x = -\sin 3x \Rightarrow \tan 3x = -1$

which gives $3x=\frac{3\pi}{4}$ or $\frac{7\pi}{4}$ or $\frac{11\pi}{4}$

$\Rightarrow\ \ x=\frac{\pi}{4}$ or $\frac{7\pi}{12}$ or $\frac{11\pi}{12}$ $\quad [\because\ 0<x<\pi]$

The points $x=\frac{\pi}{4}, x=\frac{7\pi}{12}$ and $x=\frac{11\pi}{12}$ divide the interval $(0, \pi)$ into four disjoint intervals,

$\left(0,\frac{\pi}{4}\right),\left(\frac{\pi}{4},\frac{7\pi}{12}\right),\left(\frac{7\pi}{12},\frac{11\pi}{12}\right),\left(\frac{11\pi}{12},\pi\right)$

Now, $f'(x) > 0$ in $\left(0,\frac{\pi}{4}\right)$

$\Rightarrow\ \ f$ is strictly increasing in $\left(0,\frac{\pi}{4}\right)$

$f'(x) < 0$ in $\left(\frac{\pi}{4},\frac{7\pi}{12}\right)$

$\Rightarrow\ \ f$ is strictly decreasing in $\left(\frac{\pi}{4},\frac{7\pi}{12}\right)$

$f'(x) > 0$ in $\left(\frac{7\pi}{12},\frac{11\pi}{12}\right)$

$\Rightarrow\ \ f$ is strictly increasing in $\left(\frac{7\pi}{12},\frac{11\pi}{12}\right)$

$f'(x) < 0$ in $\left(\frac{11\pi}{12},\pi\right)$

$\Rightarrow\ \ f$ is strictly decreasing in $\left(\frac{11\pi}{12},\pi\right)$

Hence, f is strictly increasing in the intervals

$\left(0,\frac{\pi}{4}\right)\cup\left(\frac{7\pi}{12},\frac{11\pi}{12}\right)$

and f is strictly decreasing in the intervals

$\left(\frac{\pi}{4},\frac{7\pi}{12}\right)\cup\left(\frac{11\pi}{12},\pi\right)$

26. Here, $f(x) = x^2 - x + 1;\ x \in (-1, 1)$

$\Rightarrow\ \ f'(x) = 2x - 1$

$f'(x)=0 \Rightarrow x=\frac{1}{2}$

Now $f'(x)=2\left(x-\frac{1}{2}\right)>0$ for $\frac{1}{2}<x<1$

$\Rightarrow\ \ f$ is strictly increasing in $\left(\frac{1}{2}, 1\right)$

Also $f'(x)=2\left(x-\frac{1}{2}\right)<0$ for $-1<x<\frac{1}{2}$

$\Rightarrow$ f is strictly decreasing in $\left(-1, \frac{1}{2}\right)$.

Thus f is neither increasing nor decreasing in $(-1, 1)$.

27. (b): Let $f(x) = x - x^2$

$\therefore$ $f'(x) = 1 - 2x$

For critical point, $f'(x) = 0$

$\Rightarrow$ $1 - 2x = 0 \Rightarrow x = 1/2$

Now, at $x = 1/2$, $f''(x) = -2 < 0$

So, $f(x)$ has maximum value at $x = 1/2$.

Concept Applied

➲ $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value of $f(c)$ is local maximum value of f.

28. (b): Let r be the radius, θ be the central angle and l be the length of the circular sector.

Given, $l + 2r = 20$

$\Rightarrow$ $r\theta + 2r = 20$ $(\because l = r\theta)$ $\Rightarrow$ $\theta=\frac{20-2r}{r}$

Let A be the area of the circular sector.

$\therefore$ $A = \frac{\pi r^2\theta}{2\pi} = \frac{r^2}{2}\cdot\left(\frac{20-2r}{r}\right)=r(10-r)$

$\Rightarrow$ $\frac{dA}{dr}=10-2r$

For maximum or minimum value of A, we have

$\frac{dA}{dr}=0\Rightarrow r=5$ and $\frac{d^2A}{dr^2}=-2<0$

$\therefore$ Area is maximum at $r = 5$

$\therefore$ Maximum area, $A = 5\,(10 - 5) = 25\text{ cm}^2$

29. (c): $\because$ Volume of cylinder $= \pi r^2h$

$\therefore$ V = Volume of casted half cylinder $= (1/2)\pi r^2h$

30. (b): Total surface area, $S = \frac{2\pi r(r+h)}{2}+2rh$

$= \pi r^2 + \pi rh + 2rh$

31. (c): Here, $S = \pi r^2+\frac{2V(\pi+2)}{\pi r}$ $\left[\because V=\frac{1}{2}\pi r^2h \Rightarrow \frac{2V}{\pi r}=rh\right]$

32. (a): $\because$ $S = \pi r^2+\frac{2V(\pi+2)}{\pi r}$

$\Rightarrow \frac{dS}{dr}=2\pi r-\frac{2V(\pi+2)}{\pi}\times\frac{1}{r^2}$

For S to be minimum, $\frac{dS}{dr}=0$

$\Rightarrow$ $2\pi r=\frac{2V(\pi+2)}{\pi r^2}$ $\Rightarrow$ $\pi^2r^3=V(\pi+2)$

33. (d): $\because$ $V=\frac{1}{2}\pi r^2h$ ...(i)

and S will be minimum, when $(\pi + 2)\,V = \pi^2r^3$

$\Rightarrow V=\frac{\pi^2r^3}{\pi+2}$ ...(ii)

From (i) and (ii), we get

$\Rightarrow \frac{1}{2}\pi r^2h=\frac{\pi^2r^3}{\pi+2}$ $\Rightarrow \pi r^2h\,(\pi + 2) = 2\pi^2r^3$

$\Rightarrow$ $h(\pi + 2) = 2\pi r$ $\Rightarrow \frac{h}{2r}=\frac{\pi}{\pi+2}$

Thus, required ratio *i.e.*, $h : 2r$ is $\pi : \pi + 2$.

34. Here, $f(x) = 2\sin x$

$\Rightarrow$ $f'(x) = 2\cos x$

Putting $f'(x) = 0$

$\Rightarrow$ $2\cos x = 0$ $\Rightarrow$ $\cos x = 0$

$\Rightarrow$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore$ $\frac{\pi}{2}, \frac{3\pi}{2}$ are the critical points

At $x = \frac{\pi}{2}$, $f(x) = 2 \times 1 = 2$

At $x = \frac{3\pi}{2}$, $f(x) = 2\sin\left(\pi+\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2} = -2$

Hence, absolute minimum value of $f(x)$ is -2.

35. We have, $f(x) = ax+\frac{b}{x}$

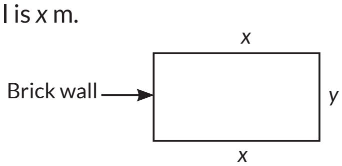
$\therefore$ $f'(x) = a - (b/x^2)$

Putting $f'(x) = 0 \Rightarrow a-\frac{b}{x^2}=0\Rightarrow a=\frac{b}{x^2}\Rightarrow x=\sqrt{\frac{b}{a}}$ (as $x > 0$)

The least value of $f(x)$ is

$f\left(\sqrt{\frac{b}{a}}\right)=a\left(\sqrt{\frac{b}{a}}\right)+\sqrt{ab}=\sqrt{ab}+\sqrt{ab}=2\sqrt{ab}$

36. Given, the length of side of garden perpendicular to the brick wall is x m.



The length of the side parallel to the brick wall is y m.

(i) $2x + y = 200$

We know that area of rectangle is $= l \times b$

$\Rightarrow$ $A(x) = xy = x(200 - 2x) = 200x - 2x^2$

(ii) Since, $A(x) = 200x - 2x^2$ ...(i)

Differentiating (i) w.r.t. x, we get

$\frac{d}{dx}A(x)=200-4x$ ...(ii)

For critical point $\frac{d}{dx}A(x)=0$

$\Rightarrow$ $200 - 4x = 0$ $\Rightarrow$ $4x = 200$ $\Rightarrow$ $x = 50$

Again differentiating (ii) w.r.t. x, we get

$\frac{d^2}{dx^2}A(x)=-4<0$ *i.e.*, area $A(x)$ is maximum at $x = 50$

Hence, maximum area is

$A(50) = 200(50) - 2(50)^2$

$= 10000 - 5000 = 5000\text{ m}^2$

37. Let x be the side of square base and y be the height of the open tank.

$\therefore$ $l = x$, $b = x$ and $h = y$

where l, b and h be the length, breadth and height of tank respectively.

Volume of tank $V = x^2 y \Rightarrow y = \frac{V}{x^2}$

The cost of the material will be least if the total surface area is least.

Total surface area of tank (S) = $x^2 + 4xy$

$$\Rightarrow S = x^2 + 4x\left(\frac{V}{x^2}\right) \qquad \left(\because y = \frac{V}{x^2}\right)$$

$$\Rightarrow S = x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

For maxima or minima, $\frac{dS}{dx} = 0$

$$\Rightarrow 2x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = 2V \Rightarrow x = 2y \qquad (\because V = x^2 y)$$

Also, $\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} > 0$

∴ Cost of material is least, when $y = \frac{x}{2}$

i.e., the depth of the tank is half of its width.

As the cost is borne by nearby settled lower income families it shows that they are spending money on social welfare so that no body will face the water problem in future. It shows social responsibility.

Concept Applied

- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. The value of $f(c)$ is local maximum value of f.

38. Let the side of an equilateral triangle be $2x$ cm, then its median = $\sqrt{3}x$

Let $M = \sqrt{3}x \Rightarrow \frac{dM}{dt} = \sqrt{3}\frac{dx}{dt}$

$$\Rightarrow 2\sqrt{3} = \sqrt{3}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \text{ cm/sec}$$

Now, side of triangle = $2x$

$\Rightarrow \quad s = 2x$

$$\Rightarrow \frac{ds}{dt} = 2\frac{dx}{dt} = 2 \times 2 \text{ cm/sec} = 4 \text{ cm/sec}$$

So, side is increasing at the rate of 4 cm/sec.

39. Let the two numbers be x and y.

According to question, we have

$$x + y = 5 \Rightarrow y = 5 - x \qquad \text{... (i)}$$

Let $p = x^3 + y^3$

$= x^3 + (5 - x)^3$

$= x^3 + 125 - x^3 - 75x + 15x^2$

$\Rightarrow \quad p = 15x^2 - 75x + 125$

Differentiating with respect to x, we get

$$\frac{dp}{dx} = 30x - 75$$

For minimum, $\frac{dp}{dx} = 0 \Rightarrow 30x - 75 = 0 \Rightarrow x = \frac{5}{2}$

Now, $\frac{d^2p}{dx^2} = 30 > 0$

So, $x^3 + y^3$ is minimum at $x = \frac{5}{2}$

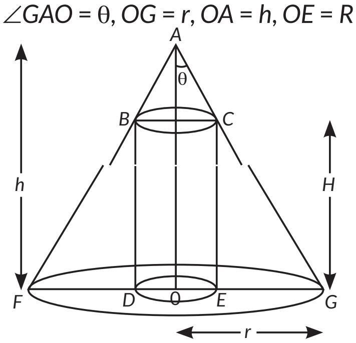
From (i), $y = 5 - \frac{5}{2} = \frac{5}{2}$

So, required value = $x^2 + y^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2$

$$= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} = \frac{25}{2}$$

40. Given the right circular cone of fixed height h and semi-vertical angle θ. Let R be the radius of the base and H be the height of the right circular cylinder that can be inscribed in the right circular cone.

In the figure, $\angle GAO = \theta$, $OG = r$, $OA = h$, $OE = R$ and $CE = H$



We have, $r/h = \tan\theta$

$$\therefore \quad r = h\tan\theta \qquad \text{...(1)}$$

Since, ΔAOG and ΔCEG are similar.

$$\therefore \quad \frac{AO}{OG} = \frac{CE}{EG} = \frac{CE}{OG - OE}$$

$$\therefore \quad \frac{h}{r} = \frac{H}{r - R} \Rightarrow R = r\left(1 - \frac{H}{h}\right)$$

Now, volume of cylinder (V) = $\pi R^2 H = \pi r^2\left(1 - \frac{H}{h}\right)^2 H$

For maximum volume, $\frac{dV}{dH} = 0$

$$\Rightarrow \quad \pi r^2\left[H \times 2\left(1 - \frac{H}{h}\right) \times -\frac{1}{h} + \left(1 - \frac{H}{h}\right)^2\right] = 0$$

$$\Rightarrow \quad \pi r^2\left[-\frac{4H}{h} + \frac{3H^2}{h^2} + 1\right] = 0 \Rightarrow \pi r^2\left(\frac{H}{h} - 1\right)\left(\frac{3H}{h} - 1\right) = 0$$

$$\Rightarrow \quad H = \frac{h}{3} \qquad [\because H \neq h]$$

So, height of cylinder = $\frac{1}{3}$ of height of cone

Also, maximum volume of cylinder

$$= \pi r^2\left(1 - \frac{1}{3}\right)^2 \times \frac{h}{3} = \left(\frac{1}{3}\pi r^2 h\right) \times \frac{4}{9}$$

$= \frac{4}{9}$ of volume of cone.

41. Let $u = ax + by$, where $xy = c^2$

$$\Rightarrow \quad u = ax + b\left(\frac{c^2}{x}\right) \qquad \text{... (i)}$$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} = a - \frac{bc^2}{x^2} \text{ and } \frac{d^2u}{dx^2} = \frac{2bc^2}{x^3}$$

For critical points, $\frac{du}{dx} = 0$

$$\Rightarrow \quad \frac{ax^2 - bc^2}{x^2} = 0 \Rightarrow x^2 = \frac{bc^2}{a}$$

$\therefore \quad x = \pm\sqrt{\frac{b}{a}}c$

At $x=\sqrt{\frac{b}{a}}c,\ \frac{d^2u}{dx^2}=2bc^2\left(\sqrt{\frac{a}{b}}\times\frac{1}{c}\right)^3$

$=\frac{2bc^2}{c^3}\left(\frac{a\sqrt{a}}{b\sqrt{b}}\right)=2\frac{a}{c}\sqrt{\frac{a}{b}}>0$

$\Rightarrow$ u is minimum at $x=c\sqrt{\frac{b}{a}}$

At $x=-\sqrt{\frac{b}{a}}c,\ \frac{d^2u}{dx^2}=-2bc^2\left(\sqrt{\frac{a}{b}}\times\frac{1}{c}\right)<0$

$\Rightarrow$ u is maximum at $x = -\sqrt{\frac{b}{a}}c$

The minimum value of u at $x=\sqrt{\frac{b}{a}}c$ is

$$u=a\left(\sqrt{\frac{b}{a}}c\right)+bc^2\left(\sqrt{\frac{a}{b}}\times\frac{1}{c}\right)=c\sqrt{ab}+\sqrt{bac}=2c\sqrt{ab}$$

42. We have, volume of cylinder = πr^2h

$\pi r^2 h = 125\pi$ (given)

$\Rightarrow r^2h = 125$

$\Rightarrow h=\frac{125}{r^2}$ (1)

Surface Area (S) = $2\pi rh + \pi r^2$

$=2\pi r\cdot\frac{125}{r^2}+\pi r^2$

$\therefore \quad S=\frac{250\pi}{r}+\pi r^2$ (2)

On differentiating (2) w.r. to 'r', we get

$$\frac{dS}{dr}=\frac{d\left(\frac{250\pi}{r}\right)}{dr}+\frac{d(\pi r^2)}{dr}=-250\pi r^{-2}+2\pi r$$

$$\frac{dS}{dr}=\frac{-250\pi}{r^2}+2\pi r \text{ and } \frac{d^2S}{dr^2}=\frac{500\pi}{r^3}+2\pi$$

For maximum or minimum value of surface area, we have to put $\frac{dS}{dr}=0$

$\frac{dS}{dr}=0\Rightarrow\frac{-250\pi}{r^2}+2\pi r=0$

$\Rightarrow \quad 2\pi r=\frac{250\pi}{r^2} \Rightarrow r\cdot r^2=\frac{250\pi}{2\pi}$

$\Rightarrow r^3 = 125 \Rightarrow r = 5$ cm

Now, $\left(\frac{d^2S}{dr^2}\right)_{r=5\text{ cm}}=\frac{500\pi}{125}+2\pi=6\pi>0$

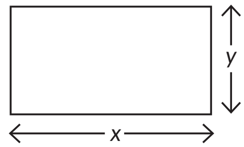
Hence, the surface area is minimum at $r = 5$ cm

Put $r = 5$ in equation (1), we get

$h=\frac{125}{r^2}=\frac{125}{5\times5}=5\text{ cm}$

Hence, the radius and height of the right circular cylindrical box are $r = 5$ cm and $h = 5$ cm respectively.

43. Let length and breadth of rectangle be x and y respectively.



Given, perimeter of rectangle = 36 cm

$\Rightarrow \quad 2x + 2y = 36$

$\Rightarrow \quad x + y = 18$

$\Rightarrow \quad y = 18 - x$ ...(i)

Let rectangle be revolved about its length x.

Then volume of resultant cylinder (V) = $\pi x^2 y$

$\Rightarrow \quad V = \pi x^2(18 - x)$ (from (i))

$\Rightarrow \quad V = \pi[18x^2 - x^3]$ ...(ii)

On differentiating (ii) w.r.t. 'x', we get

$\frac{dV}{dx}=\pi(36x-3x^2)$ ...(iii)

Put $\frac{dV}{dx}=0\Rightarrow 3x^2-36x=0$

$\Rightarrow \quad 3x(x - 12) = 0$

$\Rightarrow \quad x = 0, 12 \quad \therefore\ x = 12$ $(\because x\neq 0)$

Again differentiating (iii) w.r. t 'x', we get

$$\frac{d^2V}{dx^2}=\pi(36-6x) \Rightarrow \left(\frac{d^2V}{dx^2}\right)_{x=12}=\pi(36-6\times12)=-36\pi<0$$

At x = 12, volume of resultant cylinder is the maximum therefore the length and breadth of rectangle are 12 cm and 6 cm respectively.

Hence, maximum volume of resultant cylinder,

$(V)_{x=12} = \pi(12^2\times 6) = \pi\times144\times6 = 864\ \pi\text{ cm}^3$

44. Let 'r' and 'h' be the radius and height of right circular cylinder and 'R' and 'H' be the radius and height of cone.

The curved surface area of cylinder, $S = 2\pi rh$

Since, ΔAOC and ΔFEC are similar.

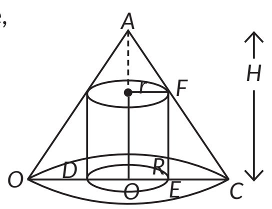
$\therefore \quad \frac{OC}{EC}=\frac{AO}{FE} \Rightarrow \frac{R}{R-r}=\frac{H}{h} \Rightarrow h=\left(\frac{R-r}{R}\right)H$ ...(i)

$\therefore \quad S = 2\pi r\left(\frac{R-r}{R}\right)H$ [From (i)]

$\Rightarrow \quad S=\frac{2\pi H(Rr-r^2)}{R}$ ...(ii)

On differentiating (ii) w.r.t. 'r', we have

$\frac{dS}{dr}=\frac{2\pi H}{R}(R-2r)$ ...(iii)



For maximum and minimum value,

put $\frac{dS}{dr}=0,$

$\Rightarrow \quad \frac{2\pi H}{R}(R-2r)=0$

$\Rightarrow \quad R - 2r = 0$

$\Rightarrow r=\frac{R}{2}$

$\therefore \quad \frac{d^2S}{dr^2}=\frac{d}{dr}\left[\frac{2\pi H}{R}(R-2r)\right]$

$=\frac{2\pi H}{R}(0-2)=-\frac{4\pi H}{R}<0$

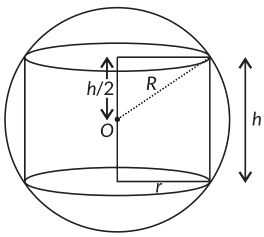
$\therefore \quad \left(\frac{d^2S}{dr^2}\right)_{r=R/2}=\frac{-4\pi H}{R}<0$

Hence for $r=\frac{R}{2}$, surface area is maximum, *i.e.*, surface area is maximum when radius of cylinder is half of that of cone.

45. Let r and h be the base radius and height of cylinder respectively.

$\therefore \quad \left(\frac{h}{2}\right)^2 + r^2 = R^2$ ... (i)

Now, V = Volume of the cylinder inscribed in a sphere
$= \pi r^2 h$



$\Rightarrow \quad V = \pi h\left(R^2 - \frac{h^2}{4}\right)$ [Using (i)]

$\Rightarrow \quad V = \pi\left(R^2 h - \frac{h^3}{4}\right)$

Now differentiating w.r.t. h, *we get*

$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$ and $\frac{d^2V}{dh^2} = \pi\left(0 - \frac{3}{4}\cdot 2h\right)$

For maximum or minimum,

$$\frac{dV}{dh} = 0 \Rightarrow R^2 - \frac{3}{4}h^2 = 0 \Rightarrow h^2 = \frac{4}{3}R^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$$

For this value of h, $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi\cdot\frac{2R}{\sqrt{3}} = -\sqrt{3}\pi R < 0$

$\Rightarrow \quad V$ is maximum

Also maximum value of V

$$= \pi\cdot\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}\cdot\frac{4}{3}R^2\right) = \pi\cdot\frac{2R}{\sqrt{3}}\cdot\frac{2}{3}R^2 = \frac{4\pi}{3\sqrt{3}}R^3 \text{ cu.units}$$

Key Points

- Volume of cylinder = $\pi r^2 h$

46. Let r be the radius of base of circular cylinder and h be its height. Let V be the volume and S be total surface area.

$\therefore \quad S = \pi r^2 + 2\pi r h \Rightarrow h = \frac{S - \pi r^2}{2\pi r}$ ...(i)

$V = \pi r^2 h = \pi r^2\left(\frac{S - \pi r^2}{2\pi r}\right)$ [from (i)]

$\therefore \quad V = \frac{1}{2}(Sr - \pi r^3) \Rightarrow \frac{dV}{dr} = \frac{1}{2}(S - 3\pi r^2)$

Put $\frac{dV}{dr} = 0 \Rightarrow \frac{1}{2}(S - 3\pi r^2) = 0 \Rightarrow r = \sqrt{\frac{S}{3\pi}}$

Now, $\frac{d^2V}{dr^2} = \frac{1}{2}(0 - 6\pi r) = -3\pi r$

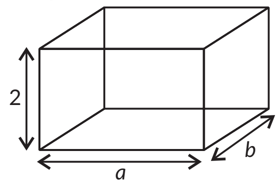
For $r = \sqrt{\frac{S}{3\pi}}, \frac{d^2V}{dr^2} = -3\pi\sqrt{\frac{S}{3\pi}} = -\sqrt{3\pi S} < 0$

$\therefore \quad$ Volume is greatest when $r = \sqrt{\frac{S}{3\pi}}$

From (i), $h = \frac{S - \pi\frac{S}{3\pi}}{2\pi\sqrt{\frac{S}{3\pi}}} = \frac{\frac{2S}{3}}{2\sqrt{\frac{\pi S}{3}}} = \frac{S}{3}\times\frac{\sqrt{3}}{\sqrt{\pi S}} = \sqrt{\frac{S}{3\pi}} = r$

Hence, proved.

47. Let a m and b m be the length and breadth of rectangular tank respectively.



$\therefore \quad$ Volume of tank = $2ab = 8$ [Given]

$\Rightarrow \quad ab = 4 \Rightarrow b = \frac{4}{a}$ ...(i)

If C is the total cost in rupees, then

$C = 70(ab) + 45(2a + 2b) \times 2$

$\Rightarrow \quad C = 70ab + 90(2a + 2b)$

$\Rightarrow \quad C = 70(a)\left(\frac{4}{a}\right) + 180\left(a + \frac{4}{a}\right)$ [Using (i)]

$\Rightarrow \quad C = 280 + 180a + \frac{720}{a}$ ...(ii)

Differentiating (ii) w.r.t. 'a', we get

$$\frac{dC}{da} = 180 - \frac{720}{a^2} \text{ and } \frac{d^2C}{da^2} = \frac{720\times 2}{a^3}$$

For maximum or minimum cost,

$\frac{dC}{da} = 0 \Rightarrow 180 - \frac{720}{a^2} = 0 \Rightarrow a^2 = 4 \Rightarrow a = 2$

For $a = 2, \frac{d^2C}{da^2} > 0 \Rightarrow C$ is least

Using (i), $a = 2$ and $b = 2$

Hence,cost of least expensive tank is

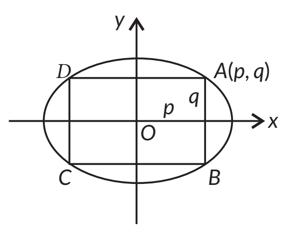
$C = 280 + 360 + 360 =$ ₹ 1000

Key Points

- Remember $\frac{d}{da}\left(\frac{1}{a}\right) = -\frac{1}{a^2}$

48. Let $ABCD$ be a rectangle inscribed in the ellipse,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let $AB = 2q$, $DA = 2p$.

Then coordinates of A are (p, q).

As A lies on the ellipse so

$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \Rightarrow q^2 = b^2\left(1 - \frac{p^2}{a^2}\right)$

Now area A of the rectangle $ABCD = 2p\cdot 2q = 4pq$

$\Rightarrow \quad A^2 = 16p^2q^2$

$$= 16p^2\cdot b^2\left(1 - \frac{p^2}{a^2}\right) = 16b^2\left(p^2 - \frac{p^4}{a^2}\right)$$

$\therefore \quad \frac{dA^2}{dp} = 16b^2\left(2p - \frac{4p^3}{a^2}\right)$ and

$\frac{d^2A^2}{dp^2} = 16b^2\left(2 - \frac{12p^2}{a^2}\right)$

For A to be max. or min.

$\frac{dA^2}{dp} = 0 \Rightarrow 2p - \frac{4p^3}{a^2} = 0 \Rightarrow p^2 = \frac{a^2}{2}$ $\quad (\because p \neq 0)$

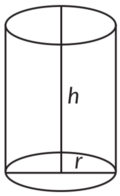
For this value of p^2,

$\frac{d^2A^2}{dp^2} = 16b^2(2 - 12 \times \frac{1}{2}) = 16b^2(-4) < 0$

Hence, A^2 is max. $\Rightarrow A$ is max.

$\Rightarrow$ The area of the greatest rectangle inscribed in the ellipse $= 4pq = 4 \cdot \sqrt{\frac{a^2}{2}} \cdot \sqrt{b^2\left(1 - \frac{1}{2}\right)} = 2ab$ sq.units

49. Let 'r' and 'h' be the radius and height of right circular cylinder.



Surface area of cylinder is given by

$S = 2\pi r^2 + 2\pi rh$

$\Rightarrow \quad 2\pi rh = S - 2\pi r^2$

or $\quad h = \frac{S - 2\pi r^2}{2\pi r}$ $\quad$ ...(i)

Volume of cylinder is given by

$V = \pi r^2 h$ $\quad$ ...(ii)

On substituting the value of h from (i) into (ii), we get

$V = \frac{Sr}{2} - \pi r^3$ $\quad$ ...(iii)

On differentiating (iii) with respect to 'r', we get

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$ $\quad$ ...(iv)

For maximum and minimum value, put $\frac{dV}{dr} = 0$

$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$

$\Rightarrow \quad S = 6\pi r^2$

or $r^2 = \frac{S}{6\pi} \Rightarrow r = \sqrt{\frac{S}{6\pi}}$

Again differentiating (iv) w.r.t. r, we have

$\frac{d^2V}{dr^2} = -6\pi r$

$\left(\frac{d^2V}{dr^2}\right)_{r=\sqrt{\frac{S}{6\pi}}} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0$

$\therefore \quad$ Volume is maximum when

$r^2 = \frac{S}{6\pi}$ or $S = 6\pi r^2$

From (i) $h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r}$

$\Rightarrow \quad h = 2r$

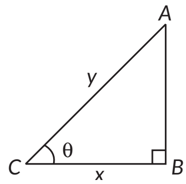
Hence volume is maximum when height is twice the radius, *i.e.*, height is equal to the diameter of base.

50. Let ABC be a right angled triangle with $BC = x$, $AC = y$ such that $x + y = k$, where k is any constant.

Let θ be the angle between the base and the hypotenuse.

Let P be the area of the triangle.

$P = \frac{1}{2} \times BC \times AB = \frac{1}{2} \times x\sqrt{y^2 - x^2} \Rightarrow P^2 = \frac{x^2}{4}(y^2 - x^2)$



$\Rightarrow P^2 = \frac{x^2}{4}[(k - x)^2 - x^2]$

$\Rightarrow P^2 = \frac{k^2x^2 - 2kx^3}{4}$

Let $Q = P^2$ *i.e.* $Q = \frac{k^2x^2 - 2kx^3}{4}$

$\therefore \quad P$ is maximum when Q is maximum.

Differentiating Q w.r.t. x, we get

$\frac{dQ}{dx} = \frac{2k^2x - 6kx^2}{4}$ $\quad$ ...(i)

For maximum or minimum area,

$\frac{dQ}{dx} = 0 \Rightarrow k^2x - 3kx^2 = 0 \Rightarrow x = \frac{k}{3}$

Differentiating (i) w.r.t. x, we get

$\frac{d^2Q}{dx^2} = \frac{2k^2 - 12kx}{4}$

$\therefore \quad \left[\frac{d^2Q}{dx^2}\right]_{x=\frac{k}{3}} = \frac{-k^2}{2} < 0$

Thus, Q is maximum when $x = \frac{k}{3}$

$\Rightarrow \quad P$ is maximum at $x = \frac{k}{3}$

Now, $x = \frac{k}{3} \Rightarrow y = k - \frac{k}{3} = \frac{2k}{3}$ $\quad [\because x + y = k]$

$\therefore \cos\theta = \frac{x}{y} = \frac{k/3}{2k/3} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

Answer Tips

➲ Remember $\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}$

So, the area of ΔABC is maximum when angle between the hypotenuse and base is $\frac{\pi}{3}$.

Concept Applied

➲ Area of triangle $= \frac{1}{2} \times \text{Base} \times \text{Height}$

51. Let V and S be the volume and the surface area of a closed cuboid of length = x units, breadth = x units and height = y units respectively.

Then, $V = x^2y \Rightarrow y = \frac{V}{x^2}$ $\quad$ ...(i)

and $S = 2(x^2 + xy + xy) = 2x^2 + 4xy$ $\quad$ ...(ii)

$\Rightarrow S = 2x^2 + 4x\left(\frac{V}{x^2}\right)$ $\quad$ [From (i)]

$\Rightarrow \; S = 2x^2 + \frac{4V}{x} \quad \Rightarrow \; \frac{dS}{dx} = 4x - \frac{4V}{x^2}$ ...(iii)

For maximum or minimum of S, $\frac{dS}{dx} = 0$

$\therefore \; \frac{dS}{dx} = 0 \quad \Rightarrow \; 4x - \frac{4V}{x^2} = 0 \Rightarrow V = x^3$

$\Rightarrow \quad x^2y = x^3$ $[\because V = x^2y]$

$\Rightarrow \quad x = y$

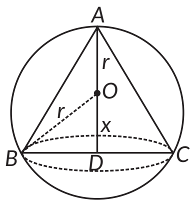
Differentiating (iii) with respect to x, we get

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} = 4 + \frac{8x^2y}{x^3} = 4 + \frac{8y}{x}$$

$$\Rightarrow \left(\frac{d^2S}{dx^2}\right)_{y=x} = 12 > 0.$$

Thus, S is minimum when $x = y$.

52. Let ABC be a cone of maximum volume inscribed in the sphere.



Let $OD = x$

$\therefore \; BD = \sqrt{r^2 - x^2}$

and $AD = AO + OD$

$= r + x$ = altitude of cone.

Let V be the volume of cone.

$$V = \frac{1}{3}\pi(BD)^2(AD) = \frac{1}{3}\pi(r^2 - x^2)(r + x)$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{1}{3}\pi\left[(r^2 - x^2) + (r + x)(-2x)\right] = \frac{\pi}{3}[r^2 - 3x^2 - 2rx]$$

and $\frac{d^2V}{dx^2} = \frac{\pi}{3}[-6x - 2r]$

For maximum or minimum value $\frac{dV}{dx} = 0$

$\Rightarrow \quad r^2 - 3x^2 - 2rx = 0$

$\Rightarrow \quad r^2 - 3rx + rx - 3x^2 = 0$

$\Rightarrow \quad (r - 3x)(r + x) = 0$

$\Rightarrow \quad r = 3x$ $[\because r + x \neq 0]$

$\Rightarrow \quad x = \frac{r}{3}$

Also, $\left(\frac{d^2V}{dx^2}\right)_{x=\frac{r}{3}} = \frac{\pi}{3}\left[-6\left(\frac{r}{3}\right) - 2r\right]$

$$= \frac{\pi}{3}[-2r - 2r] = \frac{-4}{3}r\pi < 0$$

$\Rightarrow \quad V$ is maximum when $x = \frac{r}{3}$

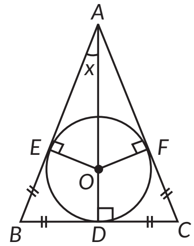
and altitude of cone $= AD = r + x = r + \frac{r}{3} = \frac{4r}{3}$

Also, maximum volume of cone when $x = \frac{r}{3}$

$$= \frac{1}{3}\pi\left(r^2 - \frac{r^2}{9}\right)\left(r + \frac{r}{3}\right) = \frac{\pi}{3}\left(\frac{8}{9}r^2\right)\left(\frac{4}{3}r\right)$$

$= \frac{8}{27}\left(\frac{4}{3}\pi r^3\right) = \frac{8}{27}$ (Volume of sphere) cube units.

53. Let ΔABC be the given triangle and AD is the altitude of the isosceles triangle ABC.



Since, 'r' be the radius of the inscribed circle.

So, $OD = OE = OF = r$, where O is the centre of the inscribed circle.

AB and AC are the equal sides.

$BD = DC$ ...(i)

$BD = BE$ and $CD = CF$ ...(ii)

From (i) and (ii), $BD = BE = DC = CF$ ...(iii)

Similarly, $AE = AF$ ...(iv)

Perimeter of the triangle $ABC = AB + BC + AC$

$= AE + BE + BD + DC + CF + AF$

$= 2AE + 4\,BD$ (Using (iii) and (iv))

In right triangle OEA, $AE = \frac{OE}{\tan x} = \frac{r}{\tan x}$ and $AO = \frac{r}{\sin x}$

In right triangle ABD, $BD = AD \tan x$

$$= (AO + OD)\tan x = \left(\frac{r}{\sin x} + r\right)\tan x$$

Let P be the perimeter of a triangle ABC.

So perimeter, $(P) = 2AE + 4BD$

$$= \frac{2r}{\tan x} + 4\left(\frac{r}{\sin x} + r\right)\tan x$$

$\Rightarrow \quad P(x) = r(2\cot x + 4\sec x + 4\tan x)$

For maximum or minimum perimeter, $\frac{dP(x)}{dx} = 0$

$\Rightarrow \; \frac{dP(x)}{dx} = r(-2\operatorname{cosec}^2 x + 4\sec x\tan x + 4\sec^2 x) = 0$

$$\Rightarrow r\left(-\frac{2}{\sin^2 x} + \frac{4\sin x}{\cos^2 x} + \frac{4}{\cos^2 x}\right) = 0$$

$$\Rightarrow r\left(\frac{-2\cos^2 x + 4\sin^3 x + 4\sin^2 x}{\sin^2 x\cos^2 x}\right) = 0$$

$\Rightarrow \quad -2(1 - \sin^2 x) + 4\sin^3 x + 4\sin^2 x = 0$

$\Rightarrow \quad 2\sin^3 x + 3\sin^2 x - 1 = 0$

$\Rightarrow \quad (\sin x + 1)(2\sin^2 x + \sin x - 1) = 0$

$\sin x$ cannot be -1 because 'x' cannot be more than 90°.

So, $2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x + 1) = 0$

Again, $\sin x$ cannot be -1.

So $2\sin x - 1 = 0$

$\Rightarrow \quad \sin x = \frac{1}{2} \Rightarrow x = 30°$

$$\frac{d^2P(x)}{dx^2} = r[4\operatorname{cosec}^2 x\cot x + 4\sec x\tan^2 x + 4\sec^3 x + 8\sec^2 x\tan x]$$

$$\therefore \quad \left[\frac{d^2P}{dx^2}\right]_{x=30°} > 0$$

So it is a point of minima for $P(x)$

Hence, least perimeter $= [P(x)]_{x=30°} = r(2\cot 30° + 4\sec 30° + 4\tan 30°)$

$$= r\left(2\sqrt{3} + 4 \times \frac{2}{\sqrt{3}} + 4 \times \frac{1}{\sqrt{3}}\right) = r\left(\frac{18}{\sqrt{3}}\right) = 6\sqrt{3}r$$

Key Points

- If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.

54. Surface area of cuboid = $2(lb + bh + hl)$

$$= 2\left(2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right) = 6x^2$$

Let radius of the sphere be r.
Surface area of sphere = $4\pi r^2$
Therefore, $6x^2 + 4\pi r^2 = k$ (constant) ...(i)
Now, sum of volumes of cuboid and sphere is

$$V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 \quad ...(ii)$$

Putting the value of r from (i) into (ii), we get

$$V = \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{k-6x^2}{4\pi}\right)^{3/2} \quad ...(iii)$$

Differentiating (iii) w.r.t. 'x', we get

$$\frac{dV}{dx} = 2x^2 + \frac{4}{3}\pi\left(\frac{1}{4\pi}\right)^{3/2}\frac{3}{2}(k-6x^2)^{1/2}(-12x) \quad ...(iv)$$

For minimum or maximum value, $\frac{dV}{dx} = 0$

$$\Rightarrow \frac{dV}{dx} = 2x^2 + \frac{4}{3}\pi\left(\frac{1}{4\pi}\right)^{3/2}\cdot\frac{3}{2}(k-6x^2)^{1/2}(-12x) = 0$$

$$\Rightarrow 2x^2 = \left(\frac{1}{4\pi}\right)^{1/2}(k-6x^2)^{1/2}(6x)$$

$$\Rightarrow 2x^2 = \left(\frac{1}{4\pi}\right)^{1/2}(4\pi r^2)^{1/2}(6x) \quad \text{[From (i)]}$$

$\Rightarrow x = 3r$

Differentiating (iv) w.r.t. 'x', we get

$$\frac{d^2V}{dx^2} = 4x - \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\left[(6)[k-6x^2]^{1/2} + (6x)\frac{(-12x)}{2(k-6x^2)^{1/2}}\right]$$

Now, $\left[\frac{d^2V}{dx^2}\right]_{x=3r} = \frac{24\pi r^2 + 324r^2}{4\pi r} > 0$

Thus, V is minimum at $x = 3r$.
Further, minimum value of sum of their volume

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{x}{3}\right)^3 \quad \left[\because r = \frac{x}{3}\right]$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\frac{x^3}{27} = \frac{2}{3}x^3\left(1 + \frac{2\pi}{27}\right) = \frac{2}{3}x^3\left(1 + \frac{44}{189}\right)$$

$$= \frac{2}{3}x^3\cdot\frac{233}{189} = \frac{466}{567}x^3$$

55. We have, $f(x) = \sin x - \cos x$

$\Rightarrow f'(x) = \cos x + \sin x$

For maxima or minima, $f'(x) = 0$

$\Rightarrow \cos x + \sin x = 0 \Rightarrow \tan x = -1$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$f''(x) = -\sin x + \cos x$

$$\text{At } x = \frac{3\pi}{4}, f''(x) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

$$\text{At } x = \frac{7\pi}{4}, f''(x) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Since $(f''(x)) < 0$ when $x = \frac{3\pi}{4}$

$\therefore$ $f(x)$ has local maxima at $x = \frac{3\pi}{4}$

Since $(f''(x)) > 0$ when $x = \frac{7\pi}{4}$

$\therefore$ $f(x)$ has local minima at $x = \frac{7\pi}{4}$

$\therefore$ Local maximum value at $x = \frac{3\pi}{4}$ is

$$f(x) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Local minimum value at $x = \frac{7\pi}{4}$ is

$$f(x) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

Concept Applied

- If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a point of local maxima.
- If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.

56. Let $P(h, k)$ be the coordinates of the point on given parabola.

$\therefore$ $k = h^2 + 7h + 2$ ...(i)

The distance S of P from the straight line $-3x + y + 3 = 0$ is

$$S = \left|\frac{-3h+k+3}{\sqrt{10}}\right| = \left|\frac{-3h+h^2+7h+2+3}{\sqrt{10}}\right| \quad \text{[From (i)]}$$

$$= \left|\frac{h^2+4h+5}{\sqrt{10}}\right| \quad \therefore \quad S = \frac{f(h)}{\sqrt{10}}$$

$\Rightarrow$ S will be maximum or minimum according as $f(h)$ is maximum or minimum.

Since, $f(h) = h^2 + 4h + 5$

$f'(h) = 2h + 4$

For maxima or minima, $f'(h) = 0 \Rightarrow 2h + 4 = 0 \Rightarrow h = -2$

Also, $f''(h) = 2 > 0$ when $h = -2$

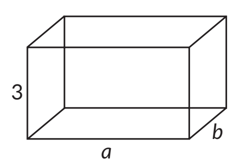
S is minimum at $h = -2$

Putting this value in (i), we get

$k = (-2)^2 + 7(-2) + 2 = 4 - 14 + 2 = -8$

$\therefore$ The required coordinates are $(-2, -8)$

57. Let a m and b m be the sides of the base of the tank.



$\therefore$ Volume of the tank = $a \cdot b \cdot 3 = 75\ m^3$(given)

$$\Rightarrow ab = 25 \Rightarrow b = \frac{25}{a} \quad ...(i)$$

If C is the total cost in rupees, then

$C = a \times b \times 100 + 2 \times 3 \times a \times 50 + 2 \times 3 \times b \times 50$

$= 100\,ab + 300(a + b) = 100 \times 25 + 300\left(a + \dfrac{25}{a}\right)$

$\Rightarrow \quad C = 2500 + 300\left(a + \dfrac{25}{a}\right)$

Differentiating w.r.t. a, we get

$\dfrac{dC}{da} = 300\left(1 - \dfrac{25}{a^2}\right)$ and

$\dfrac{d^2C}{da^2} = 300\left(0 + \dfrac{25 \times 2}{a^3}\right) = \dfrac{300 \times 50}{a^3}$

For maximum or minimum cost,

$\dfrac{dC}{da} = 0 \Rightarrow 1 - \dfrac{25}{a^2} = 0 \Rightarrow a = 5\,\text{m}$

and from (i) $b = 5$ m

At $a = 5$; $\dfrac{d^2C}{da^2} > 0 \Rightarrow C$ is minimum.

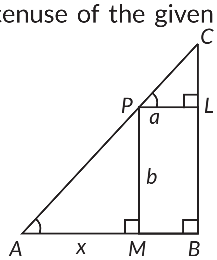
Hence, the least cost of the tank is

$C = \left[2500 + 300\left(5 + \dfrac{25}{5}\right)\right]$ = [2500 + 3000] = ₹5500.

Concept Applied

- Volume of Cuboid = $l \times b \times h$

58. Let P be any point on the hypotenuse of the given right triangle.



Let $PL = a$, $PM = b$ and $AM = x$.

Clearly, ΔCPL and ΔPAM are similar

$\therefore \quad \dfrac{PL}{CL} = \dfrac{AM}{PM} \quad \Rightarrow \quad CL = \dfrac{PL \cdot PM}{AM} = \dfrac{a \cdot b}{x}$

Now $AB = x + a$ and $BC = b + CL = b + \dfrac{ab}{x}$.

From right ΔABC, $AC^2 = AB^2 + BC^2$

Taking $l = AC^2$

$\therefore \quad l = (x + a)^2 + \left(b + \dfrac{ab}{x}\right)^2 = (x + a)^2 + b^2\left(1 + \dfrac{a}{x}\right)^2$

Differentiating w.r.t. x, we get

$\dfrac{dl}{dx} = 2(x + a) + b^2 2\left(1 + \dfrac{a}{x}\right) \cdot \dfrac{-a}{x^2}$

$= 2(x + a) - \dfrac{2ab^2(x + a)}{x^3} = 2(x + a)\left[1 - \dfrac{ab^2}{x^3}\right]$

and $\dfrac{d^2l}{dx^2} = 2 \cdot 1\left[1 - \dfrac{ab^2}{x^3}\right] + 2(x + a) \cdot \dfrac{3ab^2}{x^4}$

$\therefore$ For maximum and minimum value of l,

$\dfrac{dl}{dx} = 0 \Rightarrow x + a = 0$ or $1 - \dfrac{ab^2}{x^3} = 0$

As $x = AM \neq 0$ $\therefore$ Reject $x + a = 0$

$\therefore \quad x^3 = ab^2 \Rightarrow x = a^{1/3}b^{2/3}$

For this value of x, clearly $\dfrac{d^2l}{dx^2} > 0$

$\therefore$ l and consequently the hypotenuse AC is minimum (least).

Hence, the least value of AC is given by

$AC = \sqrt{(x + a)^2 + b^2\left(1 + \dfrac{a}{x}\right)^2}$ where $x = a^{1/3}b^{2/3}$

$= \sqrt{(x + a)^2 + \dfrac{b^2}{x^2}(x + a)^2} = (x + a)\sqrt{1 + \dfrac{b^2}{x^2}}$

$= \left(\dfrac{x + a}{x}\right)\sqrt{b^2 + x^2} = \left(\dfrac{a^{1/3}b^{2/3} + a}{a^{1/3}b^{2/3}}\right)\sqrt{b^2 + a^{2/3}b^{4/3}}$

$= \dfrac{a^{1/3}(b^{2/3} + a^{2/3})}{a^{1/3}b^{2/3}} \cdot b^{2/3}\sqrt{b^{2/3} + a^{2/3}} = (a^{2/3} + b^{2/3})^{3/2}$

59. Let r and h be the radius and height of the cylindrical can respectively.

Therefore, the total surface area of the closed cylinder is given by

$S = 2\pi rh + 2\pi r^2 = 2\pi r(r + h)$ ...(i)

Given volume of the can = 128π cm^3

Also volume $(V) = \pi r^2 h$ ...(ii)

$\Rightarrow \quad \pi r^2 h = 128\pi \Rightarrow h = \dfrac{128}{r^2}$ ...(iii)

Putting the value of h in equation (i), we get

$S = 2\pi r\left(r + \dfrac{128}{r^2}\right) = 2\pi r^2 + \dfrac{256}{r}\pi$ ...(iv)

Differentiating (iv) w.r.t. r, we get

$\dfrac{dS}{dr} = 4\pi r - \dfrac{256\pi}{r^2}$ ...(v)

Substituting $\dfrac{dS}{dr} = 0$ for critical points, we get

$4\pi r - \dfrac{256\pi}{r^2} = 0 \Rightarrow r^3 = 64 \Rightarrow r = 4$ cm

Differentiating (v) w.r.t. r, we get

$\dfrac{d^2S}{dr^2} = 4\pi - 256\pi(-2r^{-3}) = 4\pi + \dfrac{512}{r^3}\pi \qquad \therefore \left[\dfrac{d^2S}{dr^2}\right]_{r=4} > 0$

Thus the total surface area of the cylinder is minimum when $r = 4$.

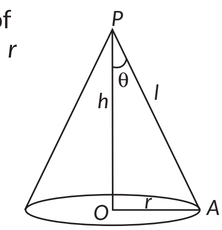
From (iii), we have $h = \dfrac{128}{r^2} = \dfrac{128}{16} = 8$

Thus radius = 4 cm and height = 8 cm.

Commonly Made Mistake

- Remember the difference between first derivative test and second derivative test for finding local maxima and local minima.

60. Let θ be the semi-vertical angle of the cone, V its volume, h its height, r base radius and slant height l.



Then from ΔOAP,

$r = l\sin\theta$, $h = l\cos\theta$

Now, $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi l^2 \sin^2\theta \cdot l\cos\theta$

$= \dfrac{1}{3}\pi l^3 \sin^2\theta \cdot \cos\theta$

$$\Rightarrow \frac{dV}{d\theta}=\frac{1}{3}\pi l^3\,(2\sin\theta\cdot\cos\theta\cdot\cos\theta-\sin^2\theta\cdot\sin\theta)$$

$$=\frac{1}{3}\pi l^3\sin\theta(2\cos^2\theta-\sin^2\theta)$$

and $$\frac{d^2V}{d\theta^2}=\frac{1}{3}\pi l^3\,[\cos\theta(2\cos^2\theta-\sin^2\theta)+\sin\theta(-4\cos\theta\sin\theta-2\sin\theta\cos\theta)]$$

$$=\frac{1}{3}\pi l^3[\cos\theta(2\cos^2\theta-\sin^2\theta)-6\sin^2\theta\cos\theta]$$

For maximum or minimum value of V,

$$\frac{dV}{d\theta}=0\Rightarrow \sin\theta(2\cos^2\theta-\sin^2\theta)=0$$

$\Rightarrow$ $\sin\theta = 0$ or $2\cos^2\theta - \sin^2\theta = 0$

$\Rightarrow$ $2\cos^2\theta - (1-\cos^2\theta) = 0$ [Note : $\sin\theta\neq 0$ as $\theta\neq 0$]

$$\Rightarrow \cos^2\theta=\frac{1}{3}\Rightarrow\cos\theta=\frac{1}{\sqrt{3}}\ \Rightarrow\ \theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

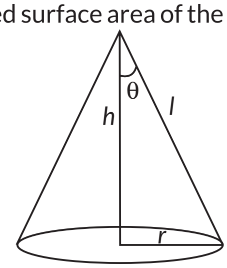
For $\cos\theta=\frac{1}{\sqrt{3}}\ \Rightarrow\ \sin\theta=\frac{\sqrt{2}}{\sqrt{3}}$

$$\therefore\ \frac{d^2V}{d\theta^2}=\frac{1}{3}\pi l^3\left[\frac{1}{\sqrt{3}}\left(2\cdot\frac{1}{3}-\frac{2}{3}\right)-6\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}}\right]$$

$$=\frac{1}{3}\pi l^3\left(-\frac{4}{\sqrt{3}}\right)<0$$

$\therefore$ V is maximum for $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

61. Let r, h, l, V and S be respectively the base radius, height, slant height, volume and curved surface area of the cone. Then,



$l^2 = r^2 + h^2$,

$$V=\frac{1}{3}\pi r^2 h \quad \text{... (i)}$$

and $S=\pi rl=\pi r\sqrt{r^2+h^2}$

$\Rightarrow$ $S^2 = \pi^2 r^2(r^2 + h^2)$

$$=\pi^2\frac{3V}{\pi h}\left(\frac{3V}{\pi h}+h^2\right) \quad \text{[Using (i)]}$$

$$=3\pi V\left(\frac{3V}{\pi h^2}+h\right)$$

For S to be least, S^2 is also least.

$$\therefore\ \frac{dS^2}{dh}=3\pi V.\left(\frac{-6V}{\pi h^3}+1\right) \text{ and}$$

$$\frac{d^2S^2}{dh^2}=3\pi V\left(\frac{-6V}{\pi}\right)\cdot\frac{-3}{h^4}=\frac{54V^2}{h^4}$$

For maximum or minimum S (and so S^2),

$$\frac{dS^2}{dh}=0\Rightarrow 6V=\pi h^3$$

$$\Rightarrow\ h=\left(\frac{6V}{\pi}\right)^{1/3} \quad \text{... (ii)}$$

For this value of h, $\frac{d^2S^2}{dh^2}=\frac{54V^2}{h^4}>0$

$\Rightarrow$ S^2 and therefore S is least.

$$\cot\theta=\frac{h}{r}=\frac{h}{\sqrt{3V/\pi h}}=\sqrt{\frac{\pi}{3V}}\cdot h^{3/2} \quad \text{[From (i)]}$$

$$\Rightarrow\ \cot\theta=\sqrt{\frac{\pi}{3V}}\sqrt{\frac{6V}{\pi}}=\sqrt{2} \quad \text{[From (ii)]}$$

$\Rightarrow$ The semi vertical angle, $\theta=\cot^{-1}\sqrt{2}$

Answer Tips

➲ $\cot\theta=\frac{\text{Base}}{\text{Perpendicular}}$

62. Let a be the side of the given square and r be the radius of the circle.

By hypothesis

$4a + 2\pi r = k$

$$\Rightarrow\ a=\frac{k-2\pi r}{4} \quad \text{...(i)}$$

Let A = Sum of areas of the circle and the square

$$\Rightarrow\ A=\pi r^2+a^2=\pi r^2+\frac{1}{16}(k-2\pi r)^2 \quad \text{[Using (i)]}$$

$$\Rightarrow\ \frac{dA}{dr}=2\pi r+\frac{1}{16}\cdot 2(k-2\pi r)\cdot(-2\pi)$$

$$=2\pi r-\frac{\pi}{4}(k-2\pi r)$$

and $$\frac{d^2A}{dr^2}=2\pi-\frac{\pi}{4}(0-2\pi)=2\pi+\frac{\pi^2}{2}$$

For maxima or minima,

$$\frac{dA}{dr}=0\Rightarrow 2\pi r-\frac{\pi}{4}(k-2\pi r)=0$$

$$\Rightarrow\ 8r-k+2\pi r=0\Rightarrow(8+2\pi)r=k\ \Rightarrow\ r=\frac{k}{2\pi+8}$$

For this value of r, $\frac{d^2A}{dr^2}=2\pi+\frac{\pi^2}{2}>0$

$\therefore$ A is minimum (least), when $r=\frac{k}{2\pi+8}$

$$\text{From (i), } a=\frac{k-2\pi\cdot\frac{k}{2\pi+8}}{4}$$

$$=\frac{k}{4}\cdot\left(\frac{2\pi+8-2\pi}{2\pi+8}\right)=\frac{2k}{2\pi+8}=2r$$

$\therefore$ Area is least, when $a = 2r$.

63. Let r and h be the base radius and height of the cylinder respectively and volume of cylinder, $V = \pi r^2 h$

$$\Rightarrow\ h=\frac{V}{\pi r^2} \quad \text{... (i)}$$

Total surface area of the cylinder, $S = 2\pi rh + \pi r^2$

$$\Rightarrow\ S=2\pi r\left(\frac{V}{\pi r^2}\right)+\pi r^2 \quad \text{[By using (i)]}$$

$$\Rightarrow\ S=\frac{2V}{r}+\pi r^2$$

On differentiating w.r.t. r both sides, $\frac{dS}{dr}=-\frac{2V}{r^2}+2\pi r$

Again differentiating w.r.t. r both sides, $\frac{d^2S}{dr^2}=\frac{4V}{r^3}+2\pi$

For maxima or minima, $\frac{dS}{dr}=0$

$$\Rightarrow \quad -\frac{2V}{r^2}+2\pi r=0 \;\Rightarrow\; 2\pi r=\frac{2V}{r^2}$$

$$\Rightarrow \quad \pi r^3=V \;\Rightarrow\; r=\left(\frac{V}{\pi}\right)^{1/3}$$

$$\therefore \quad \left[\frac{d^2V}{dh^2}\right]_{r=\left(\frac{V}{\pi}\right)^{1/3}}=4V\left(\frac{\pi}{V}\right)+2\pi=6\pi>0$$

So, S is minimum at $r=\left(\frac{V}{\pi}\right)^{1/3}$

Now, $\pi r^3 = V \Rightarrow \pi r^3 = \pi r^2 h \Rightarrow r = h$

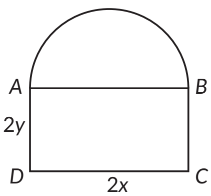
Hence, the cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base.

64. Let $ABCD$ be a rectangle and let the semi-circle is described on the side AB as its diameter.

Let $AB = 2x$ and $AD = 2y$. Let $P = 10$ m be the given perimeter of window.

Therefore, $10 = 2x + 4y + \pi x$

$\Rightarrow \quad 4y = 10 - 2x - \pi x$ ...(i)



Area of the window,

$$A=(2x)(2y)+\frac{1}{2}\pi x^2$$

$$\Rightarrow \quad A=4xy+\frac{1}{2}\pi x^2$$

$$\Rightarrow \quad A=10x-2x^2-\pi x^2+\frac{1}{2}\pi x^2 \quad \text{[using (i)]}$$

$$\Rightarrow \quad A=10x-2x^2-\frac{1}{2}\pi x^2$$

On differentiating w.r.t. x, $\frac{dA}{dx}=10-4x-\pi x$

Again differentiating w.r.t. x, $\frac{d^2A}{dx^2}=-(4+\pi)$

For maxima or minima,

$$\frac{dA}{dx}=0\Rightarrow 10-4x-\pi x=0\Rightarrow x=\frac{10}{4+\pi}$$

$$\left[\frac{d^2A}{dx^2}\right]_{x=\frac{10}{4+\pi}}=-(4+\pi)<0$$

So, A is maximum at $x=\left(\frac{10}{4+\pi}\right)$m.

Now, length of the window is $2x=\left(\frac{20}{4+\pi}\right)$m and width is $2y=\left(\frac{10}{4+\pi}\right)$m.

Concept Applied

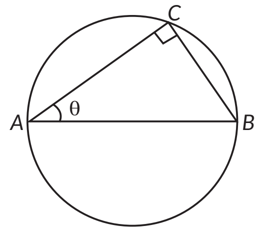
- Area of rectangle = length × breadth

  Area of semicircle $=\frac{1}{2}\pi r^2$

65. Here BA is a diameter of the given circle, of radius = r.

Let $\angle CAB = \theta$

Also $\angle ACB=\frac{\pi}{2}$



Now $AC = AB \cos\theta = 2r \cos\theta$

$BC = AB \sin\theta = 2r\sin\theta$

Let Area of $\Delta ABC=\frac{1}{2}\cdot AC\cdot BC$

$$=\frac{1}{2}\cdot 2r\cos\theta\cdot 2r\sin\theta=r^2\cdot\sin 2\theta$$

$$\Rightarrow \quad \frac{d\Delta}{d\theta}=r^2\cdot 2\cos 2\theta \text{ and } \frac{d^2\Delta}{d\theta^2}=-r^2\cdot 4\sin 2\theta$$

For maxima or minima,

$$\frac{d\Delta}{d\theta}=0\Rightarrow\cos 2\theta=0$$

$$\Rightarrow \quad 2\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{4}$$

and $\left[\frac{d^2\Delta}{d\theta^2}\right]_{\theta=\frac{\pi}{4}}=-4r^2\sin\frac{\pi}{2}=-4r^2<0$

Hence, area of ΔABC is maximum when

$\angle CAB=\theta=\frac{\pi}{4}=\angle ABC$ $\left[\because \quad \angle ACB=\frac{\pi}{2}\right]$

$\Rightarrow \quad \Delta ABC$ is isosceles.

Answer Tips

- Chain rule of derivative : $\frac{dt}{dx}=\frac{dt}{du}\cdot\frac{du}{dx}$

66. The given parabola is

$y^2 = 4ax$ ...(i)

Let $Q(11a, 0)$.

Any point on (i) is $P(at^2, 2at)$

$\therefore \quad PQ^2 = (at^2 - 11a)^2 + (2at - 0)^2$

Let $l = PQ^2 = a^2t^4 - 18\,a^2t^2 + 121a^2$

$$\Rightarrow \quad \frac{dl}{dt}=4a^2t^3-36a^2t \text{ and } \frac{d^2l}{dt^2}=12a^2t^2-36a^2$$

For maximum or minimum value of l,

$$\frac{dl}{dt}=0\Rightarrow 4a^2t(t^2-9)=0\Rightarrow t=0,3,-3$$

For $t=0, \frac{d^2l}{dt^2}=-36a^2<0$

This corresponds to a maximum value of l.

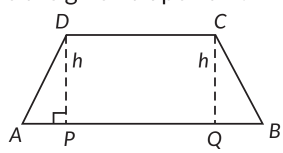
Both for $t = 3$ and -3,

$$\frac{d^2l}{dt^2}=12a^2\cdot 9-36a^2=72a^2>0$$

$\therefore$ This corresponds to a minimum value of l *i.e.*, of PQ^2 and therefore of PQ.

Thus, there are two such points P with coordinates, $(9a, 6a)$and $(9a, -6a)$ nearest to the given point Q.

67. Let $ABCD$ be the given trapezium.



Then $AD = DC = CB = 10$ cm

In ΔAPD and ΔBQC

$DP = CQ = h$

$AD = BC = 10$ cm

$\angle DPA = \angle CQB = 90°$

$\therefore\ \Delta APD \cong \Delta BQC$ (by R.H.S. congruency)

$\Rightarrow\ AP = QB = x$ cm (Say)

$\therefore\ AB = AP + PQ + QB$

$= x + 10 + x = (2x + 10)$cm

Also from ΔAPD, $AP^2 + PD^2 = AD^2$

$\Rightarrow\ x^2 + h^2 = 10^2$

$\Rightarrow\ h = \sqrt{100 - x^2}$ ...(i)

Now, area A of this trapezium is given by

$$A = \frac{1}{2}(AB + DC)\cdot h = \frac{1}{2}(2x + 10 + 10)h$$

$$= (x + 10)\cdot\sqrt{100 - x^2} \quad \text{...(ii)} \quad \text{[Using (i)]}$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 1\cdot\sqrt{100 - x^2} + (x + 10)\cdot\frac{1}{2\sqrt{100 - x^2}}\cdot(-2x)$$

$$= \frac{100 - x^2 - x^2 - 10x}{\sqrt{100 - x^2}} = \frac{-2(x^2 + 5x - 50)}{\sqrt{100 - x^2}}$$

$$= \frac{-2(x + 10)(x - 5)}{\sqrt{100 - x^2}}$$

For maximum or minimum value of A, $\frac{dA}{dx} = 0$

$\Rightarrow\ (x + 10)(x - 5) = 0$

$\Rightarrow\ x = 5$ (Reject $x = -10$ as $x \neq 0$)

For this value of x, $\frac{dA}{dx}$ changes sign from positive to negative.

$\therefore\ A$ is maximum at $x = 5$.

From (ii), the maximum value of $A = (5 + 10)\cdot\sqrt{100 - 5^2}$

$$= 15\sqrt{75} = 75\sqrt{3} \text{ sq.cm.}$$

Key Points

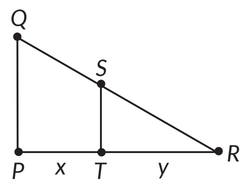
- Area of Trapezium $= \frac{1}{2}(a + b)h$, where a and b are parallel side and h is the height of the trapezium.

CBSE Sample Questions

1. Let PQ represent the height of the street light from the ground. At any time t seconds, let the man represent as ST of height 1.6 m be at a distance of x m from PQ and the length of his shadow TR be y m.

Using similarity of triangles, we have $\frac{4}{1.6} = \frac{x + y}{y}$ **(1/2)**

$\Rightarrow\ 3y = 2x$



Differentiating both sides w.r.t. t, we get $3\frac{dy}{dt} = 2\frac{dx}{dt}$

$\Rightarrow\ \frac{dy}{dt} = \frac{2}{3}\times 0.3 \Rightarrow \frac{dy}{dt} = 0.2$ **(1/2)**

At any time t seconds, the tip of his shadow is at a distance of $(x + y)$ m from PQ.

The rate at which the tip of his shadow moving

$= \left(\frac{dx}{dt} + \frac{dy}{dt}\right)$m/s $= 0.5$ m/s **(1/2)**

The rate at which his shadow is lengthening

$= \frac{dy}{dt}$ m/s $= 0.2$ m/s **(1/2)**

2. **(b):** We have, $f(x) = x^2 - 4x + 6$

$\Rightarrow\ f'(x) = 2x - 4$

$\because\ f(x)$ is strictly increasing.

$\therefore\ f'(x) > 0$

$\Rightarrow\ 2x - 4 > 0 \Rightarrow x > 2$

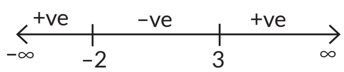
$\Rightarrow\ x \in (2, \infty)$ **(1)**

3. **(b):** We have, $f(x) = 2x^3 - 3x^2 - 36x + 7$

$\Rightarrow\ f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$= 6(x - 3)(x + 2)$

$f'(x) = 0 \Rightarrow (x - 3)(x + 2) = 0, x = -2, 3$



$\therefore\ f(x)$ is strictly increasing in $(-\infty, -2) \cup (3, \infty)$ and strictly decreasing in $(-2, 3)$. **(1)**

4. **(b):** We have, $f(x) = x + \cos x + b$

$\Rightarrow\ f'(x) = 1 - \sin x \Rightarrow f'(x) \geq 0\ \forall\ x \in R$

$\Rightarrow$ No such value of b exists **(1)**

5. We have, $f(x) = \tan x - 4x$

$\Rightarrow\ f'(x) = \sec^2 x - 4$ **(1)**

(a) For $f(x)$ to be strictly increasing, $f'(x) > 0$

$\Rightarrow\ \sec^2 x - 4 = 0 \Rightarrow \sec^2 x > 4$

$$\Rightarrow \cos^2 x < \frac{1}{4} \Rightarrow \cos^2 x < \left(\frac{1}{2}\right)^2$$

$$\Rightarrow -\frac{1}{2} < \cos x < \frac{1}{2} \Rightarrow \frac{\pi}{3} < x < \frac{\pi}{2} \quad \left(\because x \in \left(0, \frac{\pi}{2}\right)\right)$$ **(1)**

(b) For $f(x)$ to be strictly decreasing, $f'(x) < 0$

$\Rightarrow\ \sec^2 x - 4 < 0 \Rightarrow \sec^2 x < 4$

$$\Rightarrow \cos^2 x > \frac{1}{4} \Rightarrow \cos^2 x > \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \cos x > \frac{1}{2}\left[\because x \in \left(0, \frac{\pi}{2}\right)\right] \Rightarrow 0 < x < \frac{\pi}{3}$$ **(1)**

6. **(c):** We have, $f(x) = 2\cos x + x$

$\Rightarrow\ f'(x) = -2\sin x + 1$

$\Rightarrow\ f''(x) = -2\cos x$

For critical points, $f'(x) = 0$

$\Rightarrow\ -2\sin x + 1 = 0$

$$\Rightarrow \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) \quad \left(\because x \in \left[0, \frac{\pi}{2}\right]\right)$$

$\Rightarrow \ x=\dfrac{\pi}{6}$

$f''(x)\left(\text{at } x=\dfrac{\pi}{6}\right)=-2\cos\dfrac{\pi}{6}=-\sqrt{3}<0$

So, $x=\dfrac{\pi}{6}$ is the point of maxima.

Now, $f(0)=2$ and $f\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{2}=1.57$

$\Rightarrow$ Least value of $f(x) = 1.57$ *i.e.*, $\dfrac{\pi}{2}$ **(1)**

7. (c) : We have, $f(x)=(10+x)\sqrt{100-x^2}$

Which will give some real area if $-10 < x < 10$

$\Rightarrow f'(x)=\dfrac{(10+x)\times(-2x)}{2\sqrt{100-x^2}}+\sqrt{100-x^2}\times 1$

$\Rightarrow f'(x)=\dfrac{-2x^2-10x+100}{\sqrt{100-x^2}}$

For critical points, put $f'(x) = 0$

$\Rightarrow x^2 + 5x - 50 = 0$

$\Rightarrow (x + 10)(x - 5) = 0$

$\Rightarrow x = -10$ or $5 \Rightarrow x = 5$ $[\because -10 < x < 10]$

Now, $f''(x)$

$$=\frac{(\sqrt{100-x^2})(-4x-10)+(2x^2+10x-100)\times\frac{1}{2}\frac{(-2x)}{\sqrt{100-x^2}}}{(100-x^2)}$$

$=\dfrac{2x^3-300x-1000}{(100-x^2)^{3/2}}\Rightarrow f''(5)=\dfrac{-30}{\sqrt{75}}<0$

$\therefore$ Maximum area of trapezium

$=(10+5)(\sqrt{75})=75\sqrt{3}\text{ cm}^2$ **(1)**

8. (c) : Let $f(x) = [x(x - 1) + 1]^{1/3}, 0\le x\le 1$

$\Rightarrow f'(x)=\dfrac{2x-1}{3(x^2-x+1)^{2/3}}$

For critical points, put $f'(x) = 0$

$\Rightarrow x=\dfrac{1}{2}\in[0,1]$

Now, $f(0)=1,\ f\left(\dfrac{1}{2}\right)=\left(\dfrac{3}{4}\right)^{1/3}$ and $f(1)=1$

$\therefore$ Maximum value of $f(x)$ is 1. **(1)**

9. (d) : Let F be the fuel cost per hour and v be the speed of train in km/hr.

According to question, we have,

$F\propto v^2\Rightarrow F = kv^2$, where k is proportionality constant

$\Rightarrow 48=k(16)^2\Rightarrow k=\dfrac{3}{16}$ **(1)**

10. (b) : Let total cost of running the train be C.

Then, $C=\dfrac{3}{16}v^2t+1200t$

Now, distance covered = 500 km $\Rightarrow$ Time $=\dfrac{500}{v}$ hrs

$\therefore$ Total cost of running the train for 500 km

$=\dfrac{3}{16}v^2\left(\dfrac{500}{v}\right)+1200\left(\dfrac{500}{v}\right)$

$\Rightarrow C=\dfrac{375}{4}v+\dfrac{600000}{v}$ **(1)**

11. (c) : We have, $\dfrac{dC}{dv}=\dfrac{375}{4}-\dfrac{600000}{v^2}$

Put $\dfrac{dC}{dv}=0\Rightarrow v^2=\dfrac{600000\times 4}{375}=6400$

$\Rightarrow v = 80$ km/h

$\dfrac{d^2C}{dv^2}=\dfrac{2\times 600000}{v^3}>0$, for $v = 80$

$\therefore$ Most economical speed is 80 km/h. **(1)**

12. (c) : Fuel cost for running the train for 500 km

$=\dfrac{3}{16}v^2\left(\dfrac{500}{v}\right)$

$=\dfrac{375}{4}v=\dfrac{375}{4}\times 80=₹7500$ **(1)**

13. (d) : Total cost for running the train for 500 km

$=\dfrac{375}{4}v+\dfrac{600000}{v}$

$=\dfrac{375\times 80}{4}+\dfrac{600000}{80}=₹15000$ **(1)**

14. (i) (b) : We have, perimeter of floor = 200 m

$\Rightarrow 2x+2\pi\left(\dfrac{y}{2}\right)=200$

$\Rightarrow 2x + \pi y = 200$ ... (i) **(1)**

(ii) (a) : Area of rectangular region $(A) = xy$

$=x\left(\dfrac{200-2x}{\pi}\right)$ [Using (i)]

$=\dfrac{2}{\pi}(100x-x^2)$ **(1)**

(iii) (c) : We have, $A=\dfrac{2}{\pi}(100x-x^2)$

$\Rightarrow \dfrac{dA}{dx}=\dfrac{2}{\pi}(100-2x)$

For maximum or minimum, $\dfrac{dA}{dx}=0$

$\Rightarrow 100 - 2x = 0 \Rightarrow x = 50$

Now, $\left[\dfrac{d^2A}{dx^2}\right]_{x=50}=-\dfrac{4}{\pi}<0$

Thus, A is maximum at $x = 50$.

Thus, maximum value of $A=\dfrac{2}{\pi}(5000-2500)$

$=\dfrac{5000}{\pi}\text{m}^2$ **(1)**

(iv) (a) : Let P be the area of the whole floor.

Then, $P=xy+\pi\left(\dfrac{y}{2}\right)^2=xy+\dfrac{\pi}{4}y^2=y\left(x+\dfrac{\pi}{4}y\right)$

$=\left(\dfrac{200-2x}{\pi}\right)\left(\dfrac{200+2x}{4}\right)$ [Using (i)]

$=\frac{40000-4x^2}{4\pi}=\frac{10000-x^2}{\pi}$

$\therefore \frac{dP}{dx}=-\frac{2x}{\pi}$

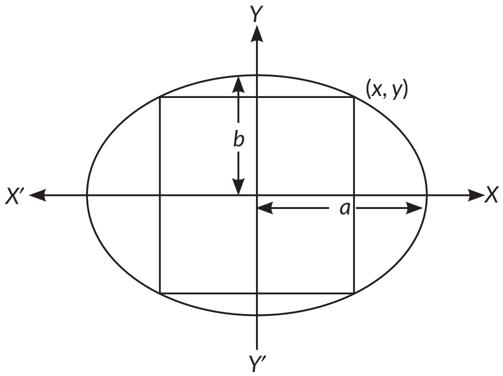
For maximum or minimum, $\frac{dP}{dx}=0 \Rightarrow x=0$

Now, $\frac{d^2P}{dx^2}=\frac{-2}{\pi}<0$

So, P is maximum at x = 0 m. **(1)**

(v) (d) **(1)**

15. (i) Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



(1/2)

Let $(x,y)=\left(x,\frac{b}{a}\sqrt{a^2-x^2}\right)$ be the upper right vertex of the rectangle.

The area function $A=2x\times 2\frac{b}{a}\sqrt{a^2-x^2}$

$=\frac{4b}{a}x\sqrt{a^2-x^2}, x\in(0,a)$ **(1/2)**

(ii) The first derivative of function is

$\frac{dA}{dx}=\frac{4b}{a}\left[x\times\frac{-x}{\sqrt{a^2-x^2}}+\sqrt{a^2-x^2}\right]$

$=\frac{4b}{a}\times\frac{a^2-2x^2}{\sqrt{a^2-x^2}}=-\frac{4b}{a}\times\frac{2\left(x+\frac{a}{\sqrt{2}}\right)\left(x-\frac{a}{\sqrt{2}}\right)}{\sqrt{a^2-x^2}}$ **(1/2)**

To find the critical point, put $\frac{dA}{dx}=0$

$\Rightarrow x=\frac{a}{\sqrt{2}}$

So, $x=\frac{a}{\sqrt{2}}$ is the critical point. **(1/2)**

(iii) For the values of x less than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}, \frac{dA}{dx}>0$ and for the values of x greater than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}, \frac{dA}{dx}<0$. **(1)**

Hence, by the first derivative test, there is a local maximum at the critical point $x=\frac{a}{\sqrt{2}}$.

Since there is only one critical point, therefore, the area of the soccer field is maximum at this critical point $x=\frac{a}{\sqrt{2}}$. **(1)**

Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

OR

$A=2x\times 2\frac{b}{a}\sqrt{a^2-x^2}, x\in(0,a)$

Squaring both sides, we get

$Z=A^2=\frac{16b^2}{a^2}x^2(a^2-x^2)=\frac{16b^2}{a^2}(x^2a^2-x^4), x\in(0,a)$ **(1/2)**

A is maximum when Z is maximum

To find the critical point, put $\frac{dz}{dx}=0$ **(1/2)**

$\frac{dZ}{dx}=\frac{16b^2}{a^2}(2xa^2-4x^3)=\frac{32b^2}{a^2}x(a+\sqrt{2}x)(a-\sqrt{2}x)$

To find the critical point, put $\frac{dZ}{dx}=0 \Rightarrow x=\frac{a}{\sqrt{2}}$

The second derivative is ; $\frac{d^2Z}{dx^2}=\frac{32b^2}{a^2}(a^2-6x^2)$

$\therefore \left(\frac{d^2Z}{dx^2}\right)_{x=\frac{a}{\sqrt{2}}}=\frac{32b^2}{a^2}(a^2-3a^2)=-64b^2<0$ **(1/2)**

Hence, by the second derivative test, there is a local maximum value of Z at the critical point $x=\frac{a}{\sqrt{2}}$. **(1/2)**

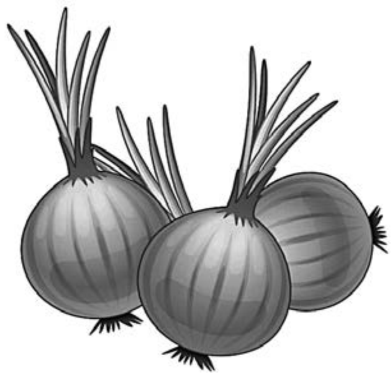
Since there is only one critical point, therefore, Z is maximum at $x=\frac{a}{\sqrt{2}}$, hence, A is maximum at $x=\frac{a}{\sqrt{2}}$.

Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

Self Assessment

Case Based Objective Questions (4 marks)

1. The Government declare that farmers can get ₹ 300 per quintal for their onions on 1st July and after that,the price will be dropped by ₹ 3 per quintal per extra day. Shyam's father has 80 quintal of onions in the field on 1st July and he estimates that crop is increasing at the rate of 1 quintal per day.



Based on the above information, attempt any 4 out of 5 subparts.

(i) If x is the number of days after 1st July, then price and quantity of onion respectively can be expressed as

(a) ₹ $(300 - 3x)$, $(80 + x)$ quintals
(b) ₹ $(300 - 3x)$, $(80 - x)$ quintals
(c) ₹ $(300 + x)$, 80 quintals
(d) None of these

(ii) Revenue R as a function of x can be represented as

(a) $R(x) = 3x^2 - 60x - 24000$
(b) $R(x) = -3x^2 + 60x + 24000$
(c) $R(x) = 3x^2 + 40x - 16000$
(d) $R(x) = 3x^2 - 60x - 14000$

(iii) Find the number of days after 1st July, when Shyam's father attain maximum revenue.

(a) 10 (b) 20
(c) 12 (d) 22

(iv) On which day should Shyam's father harvest the onions to maximise his revenue?

(a) 11th July
(b) 20th July
(c) 12th July
(d) 22nd July

(v) Maximum revenue is equal to

(a) ₹ 20,000 (b) ₹ 24,000
(c) ₹ 24,300 (d) ₹ 24,700

Multiple Choice Questions (1 mark)

2. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metres per hour. Then the depth of the wheat is increasing at the rate of

(a) 1 m/h (b) 0.1 m/h
(c) 1.1 m/h (d) 0.5 m/h

3. The side of an equilateral triangle are increasing at the rate of 2 cm/s. The rate at which the area increases, when the side is 20 cm, is

(a) $\sqrt{3}\ \text{cm}^2/\text{s}$
(b) $20\ \text{cm}^2/\text{s}$
(c) $20\sqrt{3}\ \text{cm}^2/\text{s}$
(d) $\frac{20}{\sqrt{3}}\ \text{cm}^2/\text{s}$

4. The value of b for which the function $f(x) = \sin x - bx + c$ is decreasing for $x \in R$ is given by

(a) $b < 1$ (b) $b \geq 1$ (c) $b > 1$ (d) $b \leq 1$

5. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in

(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ (d) None of these

6. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval [0, 2]. Find the value of a.

(a) 100 (b) 120 (c) 140 (d) 160

7. For what value of a, $f(x) = -x^3 + 4ax^2 + 2x - 5$ is decreasing $\forall\, x$?

(a) ± 5 (b) 3
(c) 0 (d) Cannot say

OR

The distance 's' metres covered by a body in t seconds, is given by $s = 3t^2 - 8t + 5$. The body will stop after

(a) 1 s (b) $\frac{3}{4}$ s
(c) $\frac{4}{3}$ s (d) 4 s

VSA Type Questions (1 mark)

8. Find the rate of change of perimeter of a circle with respect to its radius $r = 6$ cm.

9. Find the set of values of 'b' for which $f(x) = b(x + \cos x) + 4$ is decreasing on R.

OR

If $f(x)$ attains a local minimum at $x = c$, then write the values of $f'(c)$ and $f''(c)$.

10. The values of a for which the function $f(x) = \sin x - ax + b$ increases on R are _____.

11. Show that the function $P(x) = a_0 + a_1x^2 + a_2x^4 + ... a_nx^{2n}$ with all $a_i > 0$, $i = 0, 1, 2, ..., n$, has exactly one minimum.

12. The radius of a circle is increasing at the rate of 0.5 cm/sec. Find the rate of increase of its circumference.

SA I Type Questions (2 marks)

13. If the minimum value of a is $-k/2$, such that the function $f(x) = x^2 + ax + 5$, is increasing in $[1, 2]$. Then the value of k is _____.

14. Divide 64 into two parts such that the sum of the cubes of two parts is minimum.

15. Show that $f(x) = e^{1/x}$, $x \neq 0$ is a strictly decreasing function.

OR

Show that the function $f(x) = x^2 - [x]$, $x \in [1, 2)$ is strictly increasing, where $[\cdot]$ denotes the greatest integer function.

16. A beam of length l is supported at one end. If W is the uniform load per unit length, the bending moment M at a distance x from the end is given by $M = \frac{1}{2}lx - \frac{1}{2}Wx^2$. Find the point on the beam at which the bending moment has the maximum value.

SA II Type Questions (3 marks)

17. A kite is 120 m high and 130 m of string is out. If the kite is moving away horizontally at the rate of 52 m/sec, find the rate at which the string is being pulled out.

OR

Prove that the function given by $f(x) = \cos x$ is
(a) decreasing in $(0, \pi)$
(b) increasing in $(\pi, 2\pi)$ and
(c) neither increasing nor decreasing in $(0, 2\pi)$

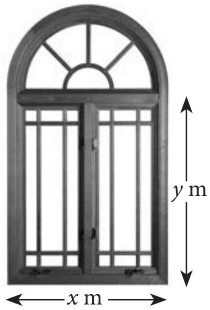
18. If the function $f(x) = \ln\left(\frac{1-x+x^2}{1+x+x^2}\right)$ is decreasing in $\left(-\frac{\alpha}{7}, \frac{\alpha}{7}\right)$, then greatest integral value of α is _____.

19. If $f(x) = a \log|x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find a and b.

20. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.

Case Based Questions (4 marks)

21. Rohan, a student of class XII, visited his uncle's flat with his father. He observe that the window of the house is in the form of a rectangle surmounted by a semicircular opening having perimeter 10 m as shown in the figure.



Based on the given information, answer the following questions.

(i) Find the area (A) of the window.

(ii) Rohan is interested in maximizing the area of the whole window, for this to happen, find the value of x.

LA Type Questions (4/6 marks)

22. An airforce plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is r km, how fast is the area of the earth, visible from the plane, increasing at 3 minutes after it started ascending? Given that the visible area A at height h is given by $A = 2\pi r^2 \frac{h}{r+h}$.

23. Show that $f(x) = 2x + \cot^{-1}x + \log(\sqrt{1+x^2} - x)$ is increasing in R.

24. An open box with square base is to be made of a given quantity of card board of area, c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cu. units.

25. Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.

OR

A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite? (the height of boy is 1.5 m.)

Detailed SOLUTIONS

1. (i) (a) : Let x be the number of extra days after 1st July.

$\therefore$ Price = ₹(300 – 3 × x) = ₹(300 – 3x)

Quantity = 80 quintals + x(1 quintal per day)
= (80 + x) quintals

(ii) (b) : $R(x)$ = Quantity × Price
= $(80 + x)(300 - 3x) = 24000 - 240x + 300x - 3x^2$
= $24000 + 60x - 3x^2$

(iii) (a) : We have, $R(x) = 24000 + 60x - 3x^2$

$\Rightarrow R'(x) = 60 - 6x \Rightarrow R''(x) = -6$

For $R(x)$ to be maximum, $R'(x) = 0$ and $R''(x) < 0$

$\Rightarrow 60 - 6x = 0 \Rightarrow x = 10$

(iv) (a) : Shyam's father will attain maximum revenue after 10 days.
So, he should harvest the onions after 10 days of 1st July *i.e.*, on 11th July.

(v) (c) : Maximum revenue is collected by Shyam's father when $x = 10$

$\therefore$ Maximum revenue = $R(10)$
= $24000 + 60(10) - 3(10)^2 = 24000 + 600 - 300$ = ₹24300

2. (a) : Let h be the depth of the cylindrical tank.

$\therefore$ V be the volume of cylindrical tank
$= \pi r^2 h = \pi(10)^2 h = 100\pi h$

Differentiate V with respect to t, we get

$\therefore$ Rate of change of volume,

$$\frac{dV}{dt} = 100\pi\frac{dh}{dt}$$

$$\Rightarrow 314 = 100\pi\frac{dh}{dt} \quad \left(\frac{dV}{dt} = 314 \text{ (given)}\right)$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100\pi} = \frac{314}{100\times 3.14} = \frac{314}{314} = 1 \text{ m/h.}$$

3. (c) : Let x is the side of an equilateral triangle and A is its area, then

$$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4}2x\frac{dx}{dt}$$

At x = 20 cm and $\frac{dx}{dt} = 2$ cm/s

$$\therefore \frac{dA}{dt} = \frac{\sqrt{3}}{4}2(20)2 = 20\sqrt{3} \text{ cm}^2/\text{s}$$

4. (b) : $f(x) = \sin x - bx + c$

$\Rightarrow f'(x) = \cos x - b \le 0$ for all $x \in R$

Hence, $\cos x \le b \Rightarrow b \ge 1$

5. (d) : Since, $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\therefore f'(x) = \frac{1}{1+(\sin x+\cos x)^2}(\cos x - \sin x)$$

$$= \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

$f(x)$ is increasing, if $f'(x) > 0 \Rightarrow \cos\left(x+\frac{\pi}{4}\right) > 0$

$$\Rightarrow -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

6. (b) : Let $f(x) = x^4 - 62x^2 + ax + 9$.

Then, $f'(x) = 4x^3 - 124x + a$.

It is given that $f(x)$ attains its maximum at $x = 1$

$\therefore f'(1) = 0 \Rightarrow 4 - 124 + a = 0 \Rightarrow a = 120$

7. (d) : $\because f(x) = -x^3 + 4ax^2 + 2x - 5$

$\therefore f'(x) = -3x^2 + 8ax + 2$

Since, $f(x)$ is decreasing, $\forall$ x, therefore

$f'(x) < 0$

$\Rightarrow -3x^2 + 8ax + 2 < 0$

From above, it is clear that decreasingness of $f(x)$ will be depend on the value of a and x.

OR

(c) : Given, $s = 3t^2 - 8t + 5$ and $v = \frac{ds}{dt} = 6t - 8$

The body will be stopped when velocity is zero.

$\Rightarrow 6t - 8 = 0 \Rightarrow t = \frac{4}{3}$ s

8. Let P be the perimeter of the circle.

$\therefore P = 2\pi r$

$$\Rightarrow \frac{dP}{dr} = 2\pi \Rightarrow \left(\frac{dP}{dr}\right)_{r=6} = 2\pi$$

9. Here, $f(x) = b(x + \cos x) + 4 \Rightarrow f'(x) = b(1 - \sin x) < 0$
[$\because f(x)$ is decreasing on R]

$\Rightarrow$ Now, $\sin x \le 1\ \forall x \in R \Rightarrow 1 - \sin x \ge 0\ \forall x \in R$

$\therefore$ b should be negative. Thus, $b \in (-\infty, 0)$

OR

If $f(x)$ attains a local minimum at $x = c$, then $f'(c) = 0$ and $f''(c) > 0$.

10. We have, $f(x) = \sin x - ax + b$

$\Rightarrow f'(x) = \cos x - a$

For increasing function, $f'(x) > 0 \Rightarrow \cos x > a$

Since, $\cos x \in [-1, 1]$

$\Rightarrow a < -1 \Rightarrow a \in (-\infty, -1)$

The values of a for which the function $f(x) = \sin x - ax + b$ increases on R are $(-\infty, -1)$.

11. $P(x) > 0$

$P'(x) = 2a_1 x + 4a_2 x^3 + \ldots = 2x(a_1 + 2a_2 x^2 + \ldots)$

$P'(x) = 0 \Rightarrow x = 0$, since the second factor is positive.

$\therefore$ $P(x)$ has minimum at $x = 0$ and $P_{min} = a_0$.

12. Let r be the radius and C be the circumference of the circle at any time t.

$$\therefore C = 2\pi r \Rightarrow \frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi \times 0.5 \quad \left[\because \frac{dr}{dt} = 0.5 \text{ cm/sec}\right]$$

$$\Rightarrow \frac{dC}{dt} = \pi \text{ cm/sec}$$

13. Given, $f(x) = x^2 + ax + 5 \Rightarrow f'(x) = 2x + a$
Since $x \in [1, 2] \Rightarrow 1 \le x \le 2$
$\Rightarrow \quad 2 \le 2x \le 4 \Rightarrow 2 + a \le 2x + a \le 4 + a$
Since, $f(x)$ is increasing in $[1, 2]$
$\Rightarrow \quad f'(x) \ge 0 \Rightarrow 2 + a \ge 0 \Rightarrow a \ge -2$
$\therefore$ Minimum value of a is $-2 = \frac{-k}{2} \Rightarrow k = 4$

14. Let one number be x and other be $64 - x$.
Now, let $z = x^3 + (64 - x)^3$

$$\Rightarrow \quad \frac{dz}{dx} = 3x^2 + 3(64 - x)^2(-1)$$

For maxima or minima, $\frac{dz}{dx} = 0$

$\Rightarrow \quad 3x^2 = 3(64 - x)^2$
$\Rightarrow \quad x^2 = x^2 + 4096 - 128x$

$$\Rightarrow \quad x = \frac{4096}{128} \Rightarrow x = 32$$

Also, $\frac{d^2z}{dx^2} = 6x + 6(64 - x) \Rightarrow \left.\frac{d^2z}{dx^2}\right|_{x=32} = 384 > 0$

Thus, z is minimum when 64 is divided into two equal parts *i.e.*, 32 and 32.

15. We have, $f(x) = e^{1/x} \quad \therefore \quad f'(x) = e^{1/x}\left(\frac{-1}{x^2}\right)$

For all $x \ne 0$, $e^{1/x} > 0$ and $\frac{-1}{x^2} < 0$

$\therefore \quad f'(x) = (+)(-) = -$ve for all $x \ne 0$
$\therefore \quad f(x)$ is a strictly decreasing function for $x \ne 0$.

OR

We have, $f(x) = x^2 - [x] = x^2 - 1$
$[\because x \in [1, 2) \Rightarrow 1 \le x < 2 \quad \therefore \quad [x] = 1]$
$\therefore \quad f'(x) = 2x$
Now, $x \in [1, 2) \Rightarrow 1 \le x < 2$
$\Rightarrow \quad 2 \le 2x < 4 \Rightarrow 2 \le f'(x) < 4$
Thus, $f'(x) > 0$ for all $x \in [1, 2)$.
Hence, $f(x)$ is strictly increasing on $[1, 2)$.

16. We have, $M = \frac{lx}{2} - \frac{Wx^2}{2}$

$$\Rightarrow \quad \frac{dM}{dx} = \frac{l}{2} - Wx \text{ and } \frac{d^2M}{dx^2} = -W$$

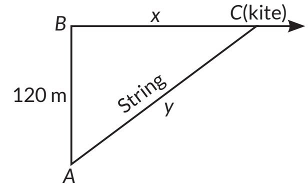
For the critical points of M, $\frac{dM}{dx} = 0.$

$$\Rightarrow \quad \frac{l}{2} - Wx = 0 \Rightarrow x = \frac{l}{2W}$$

Clearly, $\frac{d^2M}{dx^2} = -W < 0$ for all values of x

Thus, M is maximum when $x = l/2W$.
Hence, the required point is at a distance of $l/2W$ from the supporting end.

17. We have, $y^2 = x^2 + (120)^2$



$$\Rightarrow \quad 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = 52\frac{x}{y} \qquad \left[\because \frac{dx}{dt} = 52\right]$$

Putting $y = 130$ in $y^2 = x^2 + (120)^2$, we get $x = 50$.

$$\therefore \quad \frac{dy}{dt} = \frac{52 \times 50}{130} = 20 \text{ m/see}$$

OR

Here, $f(x) = \cos x$
$\Rightarrow \quad f'(x) = -\sin x$
(a) Since for each $x \in (0, \pi)$, $\sin x > 0$
$\Rightarrow \quad -\sin x < 0 \; \forall x \in (0, \pi)$
$\Rightarrow \quad f'(x) < 0$. So, f is decreasing in $(0, \pi)$.
(b) Now for each $x \in (\pi, 2\pi)$ $\sin x < 0$ *i.e.*, $-\sin x > 0$
$\Rightarrow \quad f'(x) > 0$. So, f is increasing in $(\pi, 2\pi)$
(c) Thus, from part (a) and (b), f is neither increasing nor decreasing in $(0, 2\pi)$.

18. Given, $f(x) = \ln\left(\frac{1 - x + x^2}{1 + x + x^2}\right)$

$$\Rightarrow \quad f'(x) = \frac{1 + x + x^2}{1 - x + x^2}\left(\frac{(1 + x + x^2)(2x - 1) - (1 - x + x^2)(2x + 1)}{(1 + x + x^2)^2}\right)$$

$$f'(x) = \frac{1 + x + x^2}{1 - x + x^2}\frac{2(x^2 - 1)}{(1 + x + x^2)^2} = \frac{2(x^2 - 1)}{(1 - x + x^2)(1 + x + x^2)}$$

For decreasing $f'(x) < 0$
$\Rightarrow \quad x^2 - 1 < 0 \Rightarrow x \in (-1, 1)$

Now, $\left(-\frac{\alpha}{7}, \frac{\alpha}{7}\right) = (-1, 1) \Rightarrow \alpha = 7 \Rightarrow [\alpha] = 7$

19. We observe that $f(x)$ is defined for all $x \ne 0$.
Now, $f(x) = a\log|x| + bx^2 + x \Rightarrow f'(x) = \frac{a}{x} + 2bx + 1$
It is given that $f(x)$ has extreme values at $x = -1$ and $x = 2$.
$\therefore \quad f'(-1) = 0$ and $f'(2) = 0$

$$\Rightarrow \quad -a - 2b + 1 = 0 \text{ and } \frac{a}{2} + 4b + 1 = 0$$

$\Rightarrow \quad a + 2b = 1$ and $a + 8b = -2$
Solving these equations, we get: $a = 2$ and $b = -1/2$.

20. Given x and y are the sides of two squares such that $y = x - x^2$.
Let area of the first square $(A_1) = x^2$
and area of the second square $(A_2) = y^2 = (x - x^2)^2$

Now, $\frac{dA_1}{dt} = \frac{d}{dt}x^2 = 2x \cdot \frac{dx}{dt}$

Also, $\frac{dA_2}{dt} = \frac{d}{dt}(x - x^2)^2$

$$= 2(x - x^2)\left(\frac{dx}{dt} - 2x \cdot \frac{dx}{dt}\right) = \frac{dx}{dt}(1 - 2x)2(x - x^2)$$

$\therefore \quad \dfrac{dA_2}{dA_1} = \dfrac{(dA_2/dt)}{(dA_1/dt)} = \dfrac{\dfrac{dx}{dt}\cdot(1-2x)(2x-2x^2)}{2x\cdot\dfrac{dx}{dt}}$

$= \dfrac{(1-2x)2x(1-x)}{2x} = (1-2x)(1-x)$

$= 2x^2 - 3x + 1$

21. (i) Given, perimeter of window = 10 m

$\therefore \quad x + y + y$ + perimeter of semicircle = 10

$\Rightarrow \quad x + 2y + \pi\cdot\dfrac{x}{2} = 10$ ...(i)

$A = x\cdot y + \dfrac{1}{2}\pi\left(\dfrac{x}{2}\right)^2$

$= x\left(5 - \dfrac{x}{2} - \dfrac{\pi x}{4}\right) + \dfrac{1}{2}\dfrac{\pi x^2}{4} \quad \left[\because \text{From (i)}, y = 5 - \dfrac{x}{2} - \dfrac{\pi x}{4}\right]$

$= 5x - \dfrac{x^2}{2} - \dfrac{\pi x^2}{4} + \dfrac{\pi x^2}{8} = 5x - \dfrac{x^2}{2} - \dfrac{\pi x^2}{8}$

(ii) We have, $A = 5x - \dfrac{x^2}{2} - \dfrac{\pi x^2}{8}$

$\Rightarrow \quad \dfrac{dA}{dx} = 5 - x - \dfrac{\pi x}{4}$

Now, $\dfrac{dA}{dx} = 0 \Rightarrow 5 = x + \dfrac{\pi x}{4}$

$\Rightarrow \quad x(4 + \pi) = 20 \Rightarrow x = \dfrac{20}{4+\pi}$

$\left[\text{Clearly, } \dfrac{d^2A}{dx^2} < 0 \text{ at } x = \dfrac{20}{4+\pi}\right]$

22. It is given that the plane is ascending vertically at the constant rate of 100 km/h.

$\therefore \quad \dfrac{dh}{dt} = 100$ km/h

$\Rightarrow$ Height of the plane after 3 minutes

$= 100 \times \dfrac{3}{60} = 5$ km. [Using $h = vt$]

Now, $A = 2\pi r^2 \dfrac{h}{r+h}$

$\Rightarrow \quad \dfrac{dA}{dt} = 2\pi r^2 \dfrac{d}{dt}\left(\dfrac{h}{r+h}\right) = 2\pi r^2 \left\{\dfrac{(r+h)\dfrac{dh}{dt} - h\dfrac{d}{dt}(r+h)}{(r+h)^2}\right\}$

$= 2\pi r^2 \left\{\dfrac{(r+h)\dfrac{dh}{dt} - h\dfrac{dh}{dt}}{(r+h)^2}\right\}$ [$\because$ radius r = constant]

$\Rightarrow \quad \dfrac{dA}{dt} = \dfrac{2\pi r^3}{(r+h)^2}\dfrac{dh}{dt}$

$\Rightarrow \quad \dfrac{dA}{dt} = \dfrac{2\pi r^3}{(r+h)^2} \times 100 = \dfrac{200\pi r^3}{(r+h)^2} \quad \left[\because \dfrac{dh}{dt} = 100 \text{ km/h}\right]$

We have to find $\dfrac{dA}{dt}$ when t = 3 minutes and at $t = \dfrac{3}{60}h$, we have h = 5 km.

$\therefore \quad \left(\dfrac{dA}{dt}\right)_{t=3} = \dfrac{200\pi r^3}{(r+5)^2}$ km²/h

23. We have given,

$f(x) = 2x + \cot^{-1}x + \log(\sqrt{1+x^2} - x)$

$\Rightarrow \quad f'(x) = 2 + \left(\dfrac{-1}{1+x^2}\right) + \dfrac{1}{(\sqrt{1+x^2} - x)} \times \left(\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x - 1\right)$

$= 2 - \dfrac{1}{1+x^2} + \dfrac{1}{(\sqrt{1+x^2} - x)}\cdot\dfrac{(x - \sqrt{1+x^2})}{\sqrt{1+x^2}}$

$= 2 - \dfrac{1}{1+x^2} - \dfrac{1}{\sqrt{1+x^2}}$

$= \dfrac{2 + 2x^2 - 1 - \sqrt{1+x^2}}{1+x^2} = \dfrac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2}$

For increasing function, $f'(x) \ge 0$

$\Rightarrow \quad \dfrac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2} \ge 0 \Rightarrow 1 + 2x^2 \ge \sqrt{1+x^2}$

$\Rightarrow \quad (1 + 2x^2)^2 \ge 1 + x^2 \Rightarrow 1 + 4x^4 + 4x^2 \ge 1 + x^2$

$\Rightarrow \quad 4x^4 + 3x^2 \ge 0 \Rightarrow x^2(4x^2 + 3) \ge 0$

which is true for any real value of x.

Therefore, $f(x)$ is increasing in R. (Hence proved.)

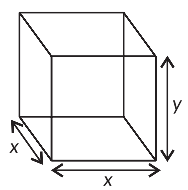
24. Let the length of the side of the square base of an open box be x units and its height be y units.

Then area of the cardboard used = $x^2 + 4xy$

$\Rightarrow \quad x^2 + 4xy = c^2$ (given)

$\Rightarrow \quad y = \dfrac{c^2 - x^2}{4x}$ ...(i)

Since, volume of the box (V) = x^2y



$\Rightarrow \quad V = x^2\cdot\left(\dfrac{c^2 - x^2}{4x}\right)$ (Using (i))

$= \dfrac{1}{4}x(c^2 - x^2)$

$= \dfrac{1}{4}(c^2x - x^3)$ ...(ii)

On differentiating (ii) with respect to x, we get

$\dfrac{dV}{dx} = \dfrac{1}{4}(c^2 - 3x^2)$ ...(iii)

Put $\dfrac{dV}{dx} = 0 \Rightarrow c^2 = 3x^2$

$\Rightarrow \quad x^2 = \dfrac{c^2}{3}$

$\Rightarrow \quad x = \dfrac{c}{\sqrt{3}}$ (taking positive sign)

Differentiating (iii) with respect to x, we get

$\dfrac{d^2V}{dx^2} = \dfrac{1}{4}(-6x) = \dfrac{-3}{2}x$

$\therefore \quad \left(\frac{d^2V}{dx^2}\right)_{\text{at } x=\frac{c}{\sqrt{3}}} = -\frac{3}{2}\cdot\left(\frac{c}{\sqrt{3}}\right) < 0$

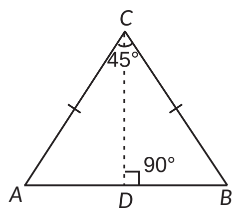
Hence, volume (V) is maximum at $x = \frac{c}{\sqrt{3}}$.

$\therefore$ Maximum volume of the box,

$$(V)_{x=\frac{c}{\sqrt{3}}} = \frac{1}{4}\left(c^2\cdot\frac{c}{\sqrt{3}} - \frac{c^3}{3\sqrt{3}}\right) \quad \text{(Using (ii))}$$

$$= \frac{1}{4}\cdot\frac{(3c^3 - c^3)}{3\sqrt{3}} = \frac{c^3}{6\sqrt{3}} \text{ cu. units}$$

25. Let two men start from the point C with velocity v each at the same time.
Also, $\angle BCA = 45°$



As, A and B are moving with same velocity, so they will cover same distance in same time.

So, ΔABC is an isosceles triangle with $AC = BC$

Also, draw $CD \perp AB$

Let at any instant, the distance between them is AB

Consider $AC = BC = x$ and $AB = y$

In ΔACD and ΔDCB,

$\angle CAD = \angle CBD \quad [\because AC = BC]$

$\angle CDA = \angle CDB = 90°$

$\therefore \quad \angle ACD = \angle DCB$

or $\angle ACD = \frac{1}{2}\times\angle ACB$

$\Rightarrow \angle ACD = \frac{1}{2}\times 45° \Rightarrow \angle ACD = \frac{\pi}{8}$

In ΔACD, $\sin\frac{\pi}{8} = \frac{AD}{AC}$

$\Rightarrow \sin\frac{\pi}{8} = \frac{y/2}{x} \quad [\because AD = y/2]$

$\Rightarrow y = 2x\cdot\sin\frac{\pi}{8}$

On differentiating both sides with respect to t, we get

$\frac{dy}{dt} = 2\cdot\sin\frac{\pi}{8}\cdot\frac{dx}{dt}$

$= 2\cdot\sin\frac{\pi}{8}\cdot v \quad \left[\because v = \frac{dx}{dt}\right]$

$= 2v\cdot\frac{\sqrt{2-\sqrt{2}}}{2} \quad \left[\because \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$

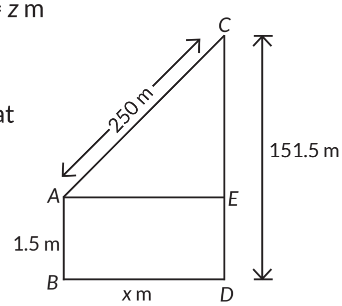
$= \sqrt{2-\sqrt{2}}\,v$ unit/s,

which is the rate at which A and B are being separated.

OR

We have, height (h) = 151.5 m,
Speed of kite (v) = 10 m/s
Let CD be the height of kite and AB be the height of boy.
Let $DB = x$ m $= EA$ and $AC = z$ m

$\therefore \quad \frac{dx}{dt} = 10$ m/s



From the figure, we see that

$EC = CD - DE$

$= 151.5 - 1.5 = 150$ m

and $AE = BD = x$ m

In right angled ΔCEA,

$AE^2 + EC^2 = AC^2$

$\Rightarrow \quad x^2 + (150)^2 = z^2$

Differentiating it with respect to t, we get

$2x\cdot\frac{dx}{dt} + 0 = 2z\cdot\frac{dz}{dt} \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt}$

$\therefore \quad \frac{dz}{dt} = \frac{x}{z}\cdot\frac{dx}{dt}$

When $z = 250$ m, $x^2 + (150)^2 = (250)^2$

$\Rightarrow \quad x^2 = (250)^2 - (150)^2$

$= (250 + 150)(250 - 150) = 400 \times 100$

$\therefore \quad x = 20 \times 10 = 200$

$\Rightarrow \left(\frac{dz}{dt}\right)_{z=250\text{ m}} = \frac{200}{250}\times 10 \quad \left(\because \frac{dx}{dt} = 10 \text{ m/s}\right)$

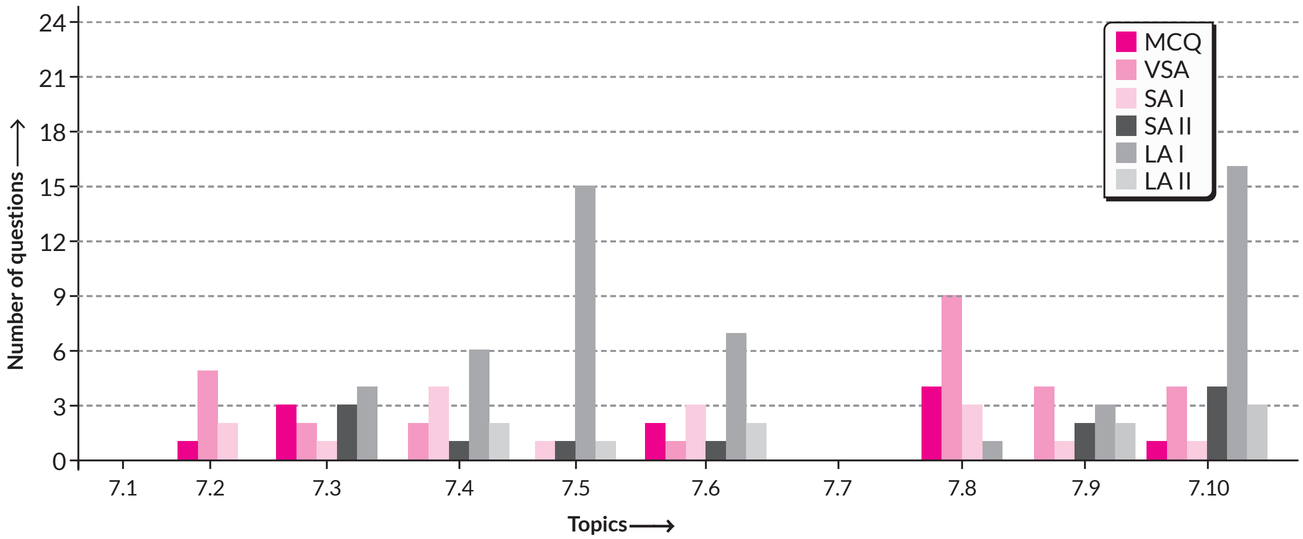
Hence, the required rate at which the string is being let out is 8 m/s.

CHAPTER 7

Integrals

TOPICS

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 7.10 is highly scoring topic.
- Maximum weightage is of Topic 7.10 *Some Properties of Definite Integrals.*
- Maximum VSA type questions were asked from Topic 7.8 *Fundamental Theorem of Calculus.*
- Maximum LA I type questions were asked from Topic 7.10 *Some Properties of Definite Integrals.*

QUICK RECAP

Indefinite Integral

- Integration is the inverse process of differentiation.

  i.e., $\frac{d}{dx}F(x)=f(x) \Rightarrow \int f(x)\,dx=F(x)+C,$ where C is the constant of integration.

  Integrals are also known as antiderivatives.

Some Standard Integrals

- $\int dx = x + C$, where 'C' is the constant of integration
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log_e a} + C$, where $a > 0$
- $\int \frac{1}{x} dx = \log_e |x| + C$, where $x \neq 0$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \text{cosec}^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C$, where $|x| < 1$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C = -\cot^{-1} x + C$
- $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C = -\text{cosec}^{-1} x + C$, where $|x| > 1$
- $\int \tan x \, dx = \log|\sec x| + C = -\log|\cos x| + C$
- $\int \cot x \, dx = \log|\sin x| + C$
- $\int \sec x \, dx = \log|\sec x + \tan x| + C$
  $$= \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$
- $\int \text{cosec}\, x \, dx = \log|\text{cosec}\, x - \cot x| + C$
  $$= \log\left|\tan\frac{x}{2}\right| + C$$

Properties of Indefinite Integral

(i) $\int f'(x) dx = f(x) + C$

(ii) $\int f(x) dx = \int g(x) dx + C$, where f and g are indefinite integrals with the same derivative.

(iii) $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

(iv) $\int k \cdot f(x) dx = k \int f(x) dx$, k being any real number.

Methods of Integration

Integration by Substitution

The given integral $\int f(x) dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Integrals	Substitution
$\int f(ax+b) dx$	$ax + b = t$
$\int f(g(x)) g'(x) dx$	$g(x) = t$
$\int \frac{f'(x)}{f(x)} dx$	$f(x) = t$
$\int (f(x))^n f'(x) dx$	$f(x) = t$
$\int (px+q)\sqrt{cx+d}\, dx$ or $\int \frac{px+q}{\sqrt{cx+d}} dx$	$px + q = A(cx + d) + B$. Find A and B by equating coefficients of like powers of x on both sides.
$\int \frac{1}{(px+q)\sqrt{cx+d}} dx$ or $\int \frac{1}{(px^2+qx+r)\sqrt{cx+d}} dx$	$cx + d = t^2$
$\int \frac{1}{(px+q)\sqrt{cx^2+dx+e}} dx$	$px + q = \frac{1}{t}$
$\int \frac{1}{(px^2+q)\sqrt{cx^2+d}} dx$	$x = \frac{1}{t}$ and then $c + dt^2 = u^2$
$\int \frac{px+q}{ax^2+bx+c} dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$	$(px+q) = A\frac{d}{dx}(ax^2+bx+c) + B$

- **Integration using Trigonometric Identities**

  When the integrand consists of trigonometric functions, we use known identities to convert it into a form which can be easily integrated. Some of the identities useful for this purpose are given below :

  (i) $2\sin^2\left(\frac{x}{2}\right) = (1 - \cos x)$

  (ii) $2\cos^2\left(\frac{x}{2}\right) = (1 + \cos x)$

  (iii) $\cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos x$

  (iv) $2 \sin x \cos y = \sin(x + y) + \sin(x - y)$

  (v) $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$

  (vi) $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$

  (vii) $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

- **Some Special Substitutions**

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
$\sqrt{a^2 + x^2}$ or $(a^2 + x^2)$	$x = a \tan\theta$ or $a \cot\theta$

$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $a\cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $a\cot^2\theta$
$\sqrt{\frac{a-x}{x-b}}$ or $\sqrt{\frac{x-b}{a-x}}$ or $\sqrt{(a-x)(x-b)}$	$x = a\cos^2\theta + b\sin^2\theta$

- **Integrals of Some Particular Functions**

  (i) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

  (ii) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

  (iii) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$

  (iv) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left|x+\sqrt{x^2+a^2}\right| + C$

  (v) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

  (vi) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Integration by Partial Fractions

- If $f(x)$ and $g(x)$ are two polynomials such that $\deg f(x) \geq \deg g(x)$, then we divide $f(x)$ by $g(x)$.

  $\therefore \frac{f(x)}{g(x)} = \text{Quotient} + \frac{\text{Remainder}}{g(x)}$

- If $f(x)$ and $g(x)$ are two polynomials such that the degree of $f(x)$ is less than the degree of $g(x)$, then we can evaluate $\int \frac{f(x)}{g(x)}dx$ by decomposing $\frac{f(x)}{g(x)}$ into partial fraction.

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px+q}{(x-a)^3}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where $x^2 + bx + c$ can not be factorised further	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Integration by Parts

If u and v are two differentiable functions of x, then

$$\int (uv)\,dx = \left[u \cdot \int v dx\right] - \int\left\{\frac{du}{dx} \cdot \int v dx\right\} dx.$$

In order to choose 1st function, we take the letter which comes first in the word ILATE.

I – Inverse Trigonometric Function

L – Logarithmic Function

A – Algebraic Function

T – Trigonometric Function

E – Exponential Function

- **Integral of the type**

  $\int e^x (f(x) + f'(x))dx = e^x f(x) + C$

Integrals of Some More Types

(i) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

(iii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

Definite Integral

- Let $F(x)$ be integral of $f(x)$, then for any two values of the independent variable x, say a and b, the difference $F(b) - F(a)$ is called the definite integral of $f(x)$ from a to b and is denoted by $\int_a^b f(x)dx$.

  Here, $x = a$ is the lower limit and $x = b$ is the upper limit of the integral.

Fundamental Theorem of Calculus

- **First Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.
- **Second Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and $F(x)$ be an integral of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Evaluation Of Definite Integral By Substitution

- When definite integral is to be found by substitution, change the lower and upper limits of integration. If substitution is $t = f(x)$ and lower limit of integration is a and upper limit is b, then new lower and upper limits will be $f(a)$ and $f(b)$ respectively.

Some Properties of Definite Integrals

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

In Particular $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(vi) $\int_{-a}^a f(x)dx = \begin{cases} 0 & , \text{ if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{ if } f(-x) = f(x) \end{cases}$

(vii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

(viii) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

BRAIN MAP

INTEGRALS

Definite Integrals

Definition

For any two values a and b, we have

$\int_a^b f(x)\,dx = [F(x) + c]_a^b = F(b) - F(a)$

Fundamental Theorem of Calculus

First Fundamental Theorem : Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and let $A(x)$ be the area function.

Then $A'(x) = f(x)$, for all $x \in [a, b]$.

Second Fundamental Theorem : Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and $F(x)$ be an integral of $f(x)$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Indefinite Integrals

Definition

If $F(x)$ is any anti-derivative of $f(x)$, then most general antiderivative of $f(x)$ is called an indefinite integral and denoted by $\int f(x)dx = F(x) + c$ or An integral which is not having any upper and lower limit is known as indefinite.

Some Standard Integrals

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log_e a} + C$, where $a > 0$
- $\int \frac{1}{x} dx = \log_e |x| + C$, where $x \neq 0$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \text{cosec}^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \text{cosec } x \cot x \, dx = -\text{cosec } x + C$

Methods for Solving

Using Integration by Parts

If u and v are two differentiable functions of x, then

$\int (uv)\,dx = \left[u \cdot \int v dx\right] - \int \left\{\frac{du}{dx} \cdot \int v dx\right\} dx$

In order to choose 1st function, we take the letter which comes first in the word ILATE.

I – Inverse Trigonometric Function,

L – Logarithmic Function, A – Algebraic Function,

T – Trigonometric Function, E – Exponential Function,

Using Partial Fractions

If $f(x)$ and $g(x)$ are two polynomials such that $\deg f(x) \geq \deg g(x)$, then we divide $f(x)$ by $g(x)$.

$\therefore \frac{f(x)}{g(x)} = \text{Quotient} + \frac{\text{Remainder}}{g(x)}$

Using Substitution

The given integral $\int f(x)\,dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Properties

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$ In particular $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_{-a}^a f(x)dx = \begin{cases} 0 & \text{, if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{, if } f(-x) = f(x) \end{cases}$
- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

- $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + C$, Putting $x = a \sec\theta$
- $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2+a^2}\right| + C$, Putting $x = a \tan\theta$
- $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$, By partial fraction
- $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + C$, $a > x$, By partial fraction
- $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$, Put $x = a\tan\theta$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$, Put $x = a\sin\theta$

Previous Years' CBSE Board Questions

7.2 Integration as an Inverse Process of Differentiation

MCQ

1. $\int \frac{\sec x}{\sec x - \tan x} dx$ equals

(a) $\sec x - \tan x + c$ (b) $\sec x + \tan x + c$
(c) $\tan x - \sec x + c$ (d) $-(\sec x + \tan x) + c$ *(2023)*

VSA (1 mark)

2. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$ *(AI 2017)*

3. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$. *(Delhi 2014)*

4. Evaluate : $\int \cos^{-1}(\sin x) dx$ *(Delhi 2014)*

5. Evaluate : $\int \frac{dx}{\sin^2 x \cos^2 x}$ *(Foreign 2014, Delhi 2014 C)* U

6. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ *(Delhi 2014 C)*

SA I (2 marks)

7. Find: $\int \sqrt{1 - \sin 2x}\, dx, \frac{\pi}{4} < x < \frac{\pi}{2}$ *(Delhi 2019)*

8. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$ *(2018)* Ap

7.3 Methods of Integration

MCQ

9. $\int e^{5\log x} dx$ is equal to

(a) $\frac{x^5}{5} + C$ (b) $\frac{x^6}{6} + C$
(c) $5x^4 + C$ (d) $6x^5 + C$ *(2023)*

10. $\int x^2 e^{x^3} dx$ equals

(a) $\frac{1}{3} e^{x^3} + C$ (b) $\frac{1}{3} e^{x^4} + C$
(c) $\frac{1}{2} e^{x^3} + C$ (d) $\frac{1}{2} e^{x^2} + C$ *(2020)* Ap

11. $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$ is equal to

(a) $\tan(xe^x) + c$ (b) $\cot(xe^x) + c$
(c) $\cot(e^x) + c$ (d) $\tan[e^x(1+x)] + c$ *(2020)* Ap

VSA (1 mark)

12. Find : $\int \frac{2x}{\sqrt[3]{x^2+1}} dx$ *(2020)*

13. Find : $\int \frac{\sin^6 x}{\cos^8 x} dx$ *(AI 2014 C)*

SA I (2 marks)

14. Find : $\int \frac{dx}{\sqrt{4x - x^2}}$ *(Term II, 2021-22)* Ev

SA II (3 marks)

15. Find : $\int \frac{\sin x}{\sin(x - 2a)} dx$ *(Term II, 2021-22 C)*

16. Find : $\int \frac{1}{e^x + 1} dx$ *(Term II, 2021-22)*

17. Find : $\int \frac{2x}{(x^2+1)(x^2+2)} dx$ *(Term II, 2021-22)*

LA I (4 marks)

18. Integrate the function $\frac{\cos(x+a)}{\sin(x+b)}$ with respect to x. *(AI 2019)*

19. Find $\int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$. *(Delhi 2016)*

20. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)} dx$ *(Foreign 2015)*

21. Evaluate : $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$ *(Delhi 2014)*

7.4 Integrals of Some Particular Functions

VSA (1 mark)

22. Find : $\int \frac{dx}{\sqrt{9 - 4x^2}}$ *(2020)*

23. Find : $\int \frac{dx}{9 + 4x^2}$ *(2020)*

SA I (2 marks)

24. Find: $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$ *(Delhi 2019)* Ap

25. Find : $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$ *(AI 2019)*

26. Find : $\int \frac{dx}{x^2 + 4x + 8}$ *(Delhi 2017)*

27. Find : $\int \frac{dx}{5 - 8x - x^2}$ *(AI 2017)*

SA II **(3 marks)**

28. Find : $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$ *(2023)*

LA I **(4 marks)**

29. Find $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx.$ *(Delhi 2016)* Ap

30. Find $\int(\sqrt{\tan x}+\sqrt{\cot x})dx.$ *(AI 2015, 2014)* Ap

31. Evaluate : $\int \frac{x+2}{2x^2+6x+5}dx$ *(Delhi 2015C)*

32. Find $\int \frac{x+3}{\sqrt{5-4x-2x^2}}dx.$ *(AI 2015C)*

33. Evaluate : $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$ *(AI 2014)*

34. Evaluate : $\int \frac{5x-2}{1+2x+3x^2}dx$ *(Delhi 2014C)* An

LA II **(5/6 marks)**

35. Evaluate : $\int \frac{1}{\cos^4 x+\sin^4 x}dx$ *(AI 2014)*

36. Evaluate : $\int \frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx$ *(AI 2014)* An

7.5 Integration by Partial Fractions

SA I **(2 marks)**

37. Find $\int \frac{x+1}{(x+2)(x+3)}dx.$ *(2020)*

SA II **(3 marks)**

38. Find : $\int \frac{x^2}{x^2+6x+12}dx$ *(2023)*

LA I **(4 marks)**

39. Find : $\int \frac{x^2}{(x^2+1)(3x^2+4)}dx$ *(Term II, 2021-22)* An

40. Find : $\int \frac{x^3+1}{x^3-x}dx$ *(2020)* An

41. Find: $\int \frac{3x+5}{x^2+3x-18}dx$ *(Delhi 2019)* Ap

42. Evaluate : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$ *(2019, AI 2015)*

43. Find : $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$ *(2018)*

44. Find : $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx.$ *(Delhi 2017)* Ap

45. Find : $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$ *(AI 2017)*

46. Find : $\int \frac{x^2}{x^4+x^2-2}dx$ *(AI 2016)*

47. Find : $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$ *(Foreign 2016)* Ap

48. Find : $\int \frac{dx}{\sin x+\sin 2x}$ *(Delhi 2015)*

49. Evaluate : $\int \frac{x^2}{(x^2+4)(x^2+9)}dx$ *(Foreign 2015)* Ap

50. Find : $\int \frac{x}{(x-1)^2(x+2)}dx$ *(Delhi 2015C)*

51. Find : $\int \frac{x}{(x^2+1)(x-1)}dx$ *(AI 2015C)*

52. Evaluate : $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$ *(Delhi 2014C)* Ap

53. Find : $\int \frac{x^3}{x^4+3x^2+2}dx$ *(AI 2014C)*

LA II **(5/6 marks)**

54. Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx.$ *(Delhi 2014C)*

7.6 Integration by Parts

MCQ

55. $\int e^x\left(\frac{x\log x+1}{x}\right)dx$ is equal to

(a) $\log(e^x \log x)+c$ (b) $\frac{e^x}{x}+c$

(c) $x\log x+e^x+c$ (d) $e^x\log x+c$

(2020) Ap

56. $\int \frac{e^x}{x+1}[1+(x+1)\log(x+1)]dx$ equals

(a) $\frac{e^x}{x+1}+c$ (b) $e^x\frac{x}{x+1}+c$

(c) $e^x\log(x+1)+e^x+c$

(d) $e^x\log(x+1)+c$ *(2020C)*

VSA **(1 mark)**

57. Find : $\int x^4\log x\,dx$ *(2020)*

SA I **(2 marks)**

58. Find : $\int \frac{\log x-3}{(\log x)^4}dx.$ *(Term II, 2021-22)* An

59. Find: $\int \sin^{-1}(2x)dx$ *(Delhi 2019)*

60. Find : $\int x\cdot\tan^{-1}x\,dx$ *(AI 2019)* Ev

SA II (3 marks)

61. Find : $\int e^x \cdot \sin 2x\, dx$ *(Term II, 2021-22)*

LA I (4 marks)

62. Find : $\int \sec^3 x\, dx$ *(2020)*

63. Find : $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ *(Foreign 2016)*

64. Integrate the following w.r.t. x : $\frac{x^2-3x+1}{\sqrt{1-x^2}}$ *(Delhi 2015)*

65. Find : $\int \frac{\log x}{(x+1)^2}dx$ *(AI 2015)*

66. Evaluate : $\int e^{2x}\cdot\sin(3x+1)dx$ *(Foreign 2015)*

67. Find : $\int \frac{(x^2+1)e^x}{(x+1)^2}dx$ *(Delhi 2015C)* An

68. Evaluate : $\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$ *(Foreign 2014)*

LA II (5/6 marks)

69. Find : $\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$ *(AI 2014C)*

70. Find : $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx, x\in[0,1]$ *(AI 2014C)* An

7.8 Fundamental Theorem of Calculus

MCQ

71. $\int_{-1}^{1}\frac{|x-2|}{x-2}dx, x\neq 2$ is equal to

(a) 1 (b) −1 (c) 2 (d) −2 *(2023)*

72. $\int_0^4 (e^{2x}+x)dx$ is equal to

(a) $\frac{15+e^8}{2}$ (b) $\frac{16-e^8}{2}$

(c) $\frac{e^8-15}{2}$ (d) $\frac{-e^8-15}{2}$ *(2023)*

73. $\int_0^{\pi/8}\tan^2(2x)dx$ is equal to

(a) $\frac{4-\pi}{8}$ (b) $\frac{4+\pi}{8}$

(c) $\frac{4-\pi}{4}$ (d) $\frac{4-\pi}{2}$ *(2020)* Ap

74. $\int_{-\pi/4}^{\pi/4}\sec^2 x\, dx$ is equal to

(a) −1 (b) 0

(c) 1 (d) 2 *(2020)* U

VSA (1 mark)

75. Evaluate : $\int_2^3 3^x\, dx$ *(Delhi 2017)*

76. Evaluate : $\int_0^3 \frac{dx}{9+x^2}$ *(Delhi 2014)* Ap

77. Evaluate : $\int_0^{\pi/2} e^x(\sin x-\cos x)dx$ *(Delhi 2014)*

78. If $f(x)=\int_0^x t\sin t\, dt$, then write the value of $f'(x)$. *(AI 2014)* Ap

79. If $\int_0^a \frac{1}{4+x^2}dx=\frac{\pi}{8}$, find the value of a. *(AI 2014)*

80. Evaluate : $\int_0^{\pi/4}\tan x\, dx$ *(Foreign 2014)*

81. Evaluate : $\int_0^{\pi/4}\sin 2x\, dx$ *(Foreign 2014)*

82. Evaluate : $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ *(AI 2014C)*

83. Evaluate : $\int_1^2 \frac{x^3-1}{x^2}dx$ *(AI 2014C)*

SA I (2 marks)

84. Evaluate : $\int_0^1 x^2e^x\, dx$ *(Term II, 2021-22)*

85. Evaluate $\int_1^2\left[\frac{1}{x}-\frac{1}{2x^2}\right]e^{2x}dx.$ *(2020)* Ap

86. Find the value of $\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx.$ *(2020)*

LA I (4 marks)

87. Evaluate : $\int_0^{\pi/2} x^2\sin x\, dx$ *(Delhi 2014C)*

7.9 Evaluation of Definite Integrals by Substitution

VSA (1 mark)

88. Evaluate : $\int_2^4 \frac{x}{x^2+1}dx$ *(AI 2014)* An

89. Evaluate : $\int_e^{e^2}\frac{dx}{x\log x}.$ *(AI 2014)*

90. Evaluate : $\int_0^1 xe^{x^2}dx$ (Foreign 2014)

91. Evaluate : $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$ (AI 2014C)

SA I (2 marks)

92. Find : $\int_{-\frac{\pi}{4}}^{0} \frac{1+\tan x}{1-\tan x}dx$ (AI 2019) Ev

SA II (3 marks)

93. Evaluate : $\int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ (2023)

94. Evaluate : $\int_0^{\pi/2} \sqrt{\sin x}\cos^5 x\,dx$ (2023)

LA I (4 marks)

95. Evaluate : $\int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$ (2020C, AI 2014C) Ev

96. Evaluate $\int_0^{\pi} e^{2x}\cdot\sin\left(\frac{\pi}{4}+x\right)dx.$ (Delhi 2016)

97. Find : $\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$ (AI 2015)

LA II (5/6 marks)

98. Evaluate : $\int_0^{\pi/4} \frac{\sin x+\cos x}{16+9\sin 2x}dx$ (2018)

99. Evaluate : $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

(Foreign 2014, Delhi 2014C) Ap

7.10 Some Properties of Definite Integrals

MCQ

100. In the following question, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choice

Assertion (A) : $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx = 3$

Reason (R) : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true, but (R) is not the correct explanation of the (A).

(c) (A) is true and (R) is false.

(d) (A) is false, but (R) is true. (2023)

VSA (1 mark)

101. Evaluate : $\int_0^{\pi/2} \frac{1}{1+\cot^{5/2}x}dx$ (Term II, 2021-22C)

102. Evaluate : $\int_1^3 |2x-1|dx$ (2020)

103. Evaluate : $\int_{-2}^{2} |x|\,dx$ (2020)

104. Find the value of $\int_1^4 |x-5|dx.$ (2020) An

SA I (2 marks)

105. Evaluate : $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ (2021C)

SA II (3 marks)

106. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx$ (2023)

107. Evaluate : $\int_{-2}^{2} \frac{x^2}{1+5^x}dx$ (2023)

108. Evaluate : $\int_1^4 \{|x|+|3-x|\}dx$ (Term II, 2021-22)

109. Evaluate : $\int_1^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}dx$ (Term II, 2021-22) Ap

LA I (4 marks)

110. Evaluate : $\int_0^{\pi} \frac{x}{9\sin^2 x+16\cos^2 x}dx$ (Term II, 2021-22)

111. Evaluate : $\int_{-1}^{2} |x^3-x|dx$

(Term II, 2021-22, 2020, Delhi 2016)

112. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx,$ hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx.$ (Delhi 2019)

OR

Evaluate : $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ (Delhi 2017) Ap

113. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and hence evaluate $\int_0^1 x^2(1-x)^n\,dx.$ (AI 2019)

114. Evaluate : $\int_0^{3/2} |x\sin\pi x|\,dx$ *(Delhi 2017)*

115. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx$ *(AI 2017, Foreign 2014, Delhi 2014C)*

116. Evaluate : $\int_1^4 \{|x-1|+|x-2|+|x-4|\}dx$ *(AI 2017)*

117. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}dx$ *(AI 2016)*

OR

Show that : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ *(AI 2014C)* Ap

118. Evaluate : $\int_0^{3/2} |x\cos\pi x|dx$ *(AI 2016)*

119. Evaluate : $\int_0^{\pi} \frac{x}{1+\sin\alpha\sin x}dx$ *(Foreign 2016)*

120. Evaluate : $\int_{-\pi}^{\pi} (\cos ax-\sin bx)^2\,dx$ *(Delhi 2015)* Ev

121. Evaluate : $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$ *(Foreign 2015)*

122. Evaluate : $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ *(Delhi 2015C)* Ap

123. Evaluate : $\int_0^{\pi/4} \log(1+\tan x)dx$ *(AI 2015C)*

124. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x\ \mathrm{cosec}\ x}dx$ *(Delhi 2014)*

125. Evaluate : $\int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$ *(AI 2014)*

LA II **(5/6 marks)**

126. Evaluate : $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ *(Delhi 2014)*

127. Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ *(Delhi 2014)*

128. Evaluate : $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$ *(Foreign 2014)* An

CBSE Sample Questions

7.3 Methods of Integration

SA I **(2 marks)**

1. Find $\int \frac{1}{\cos^2 x(1-\tan x)^2}dx.$ *(2020-21)*

7.4 Integrals of Some Particular Functions

SA I **(2 marks)**

2. Find $\int \frac{\sin 2x}{\sqrt{9-\cos^4 x}}dx$. *(2020-21)*

7.5 Integration by Partial Fractions

SA I **(2 marks)**

3. Find : $\int \frac{x+1}{(x^2+1)x}dx$. *(2021-22)* An

SA II **(3 marks)**

4. Find $\int \frac{(x^3+x+1)}{(x^2-1)}dx$ *(2022- 23)* Ev

5. Evaluate : $\int_0^4 |x-1|dx$ *(2022-23)* Ev

6. Find $\int \frac{x^2+1}{(x^2+2)(x^2+3)}dx.$ *(2020-21)* Ap

7.6 Integration by Parts

VSA **(1 mark)**

7. Find $\int e^x(1-\cot x+\mathrm{cosec}^2 x)dx.$ *(2020-21)*

SA I **(2 marks)**

8. Find $\int \frac{\log x}{(1+\log x)^2}dx$. *(2021-22)* Ev

7.10 Some Properties of Definite Integrals

VSA **(1 mark)**

9. Evaluate : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\sin x\,dx$ *(2020-21)* Ev

SA I **(2 marks)**

10. Evaluate : $\int_0^1 x(1-x)^n\,dx$ *(2020-21)*

LA I **(4 marks)**

11. Evaluate : $\int_{-1}^{2}|x^3-3x^2+2x|dx$ *(2021-22)* An

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. (b): Let $I=\int \frac{\sec x}{\sec x-\tan x}dx$

$=\int \frac{\sec x(\sec x+\tan x)}{(\sec x-\tan x)(\sec x+\tan x)}dx=\int\left(\frac{\sec^2 x+\sec x\tan x}{\sec^2 x-\tan^2 x}\right)dx$

$=\int \sec^2 x\,dx+\int \sec x\tan x\,dx$ $[\because \sec^2 x-\tan^2 x=1]$

= tanx + secx + c

2. We have, $\int \frac{\sin^2 x-\cos^2 x}{\sin x\cos x}dx$

$=\int \frac{\sin^2 x}{\sin x\cos x}dx-\int \frac{\cos^2 x}{\sin x\cos x}dx$

$=\int \tan x\,dx-\int \cot x\,dx$

$=\ln|\sec x|-\ln|\sin x|+C=\ln\left|\frac{1}{\sin x\cos x}\right|+C$

$=\ln\left|\frac{2}{2\sin x\cos x}\right|+C=\ln|2\operatorname{cosec}2x|+C$

3. The antiderivative of $3\sqrt{x}+\frac{1}{\sqrt{x}}$

$=\int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=3\int x^{1/2}dx+\int x^{-1/2}dx$

$=3\cdot\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+C=2x\sqrt{x}+2\sqrt{x}+C=2\sqrt{x}(x+1)+C$

4. We have, $\int \cos^{-1}(\sin x)dx=\int \cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$

$=\int\left(\frac{\pi}{2}-x\right)dx=\frac{\pi}{2}x-\frac{x^2}{2}+C$

5. We have, $\int \frac{dx}{\sin^2 x\cos^2 x}=\int \frac{(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x}dx$

$=\int(\sec^2 x+\operatorname{cosec}^2 x)dx$ = tan x – cot x + C

6. We have, $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$

$=\int \sec^2 x\,dx-\int \operatorname{cosec}^2 x\,dx$

= tan x + cot x + C

7. Let $I=\int(\sqrt{1-\sin 2x})dx$

$=\int\sqrt{\cos^2 x+\sin^2 x-2\sin x\cos x}\,dx=\pm\int(\cos x-\sin x)dx$

Since, $\frac{\pi}{4}<x<\frac{\pi}{2}$, so we get

$I=\int(\sin x-\cos x)dx$

= –(cosx + sinx) + C

8. Let $I=\int \frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$

$=\int \frac{\cos^2 x-\sin^2 x+2\sin^2 x}{\cos^2 x}dx$

$=\int \frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx=\int \frac{1}{\cos^2 x}dx=\int \sec^2 x\,dx$

= tan x + C

9. (b): Let $I=\int e^{5\log x}dx$

$=\int e^{\log x^5}dx=\int x^5 dx$ $[\because e^{\log x}=x]$

$=\frac{x^6}{6}+C$

10. (a) : Let $I=\int x^2 e^{x^3}dx$

Put $x^3=t \Rightarrow 3x^2dx=dt$

$\therefore\quad I=\int e^t\frac{dt}{3}=\frac{1}{3}e^t+C=\frac{1}{3}e^{x^3}+C$

11. (a): Let $I=\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx$

Put $xe^x=t$

$\Rightarrow\ (xe^x+e^x)\,dx=dt$

$\Rightarrow\ e^x(x+1)\,dx=dt$

$\therefore\quad I=\int \frac{dt}{\cos^2 t}=\int \sec^2 t\,dt=\tan t+c=\tan(xe^x)+c$

12. Let $I=\int \frac{2x}{\sqrt[3]{x^2+1}}dx$

Put $x^2+1=z \Rightarrow 2x\,dx=dz$

$\therefore\ I=\int \frac{dz}{(z)^{1/3}}=\int z^{-1/3}dz=\frac{z^{(-1/3)+1}}{(-1/3)+1}+C$

$=\frac{3}{2}(x^2+1)^{2/3}+C$

13. Let $I=\int \frac{\sin^6 x}{\cos^8 x}dx=\int \frac{\sin^6 x}{\cos^6 x\cdot\cos^2 x}dx$

$=\int \tan^6 x\sec^2 x\,dx$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$\therefore\quad I=\int t^6 dt=\frac{t^7}{7}+C=\frac{1}{7}\tan^7 x+C$

14. $\int \frac{dx}{\sqrt{4x-x^2}}=\int \frac{dx}{\sqrt{4-(x-2)^2}}$

Put $x-2=2\sin\theta$ ...(i)

$\Rightarrow\ dx=2\cos\theta\,d\theta$

$=\int \frac{2\cos\theta}{\sqrt{4-(2\sin\theta)^2}}d\theta$

Now, $4-(2\sin\theta)^2=4-4\sin^2\theta=4(1-\sin^2\theta)=4\cos^2\theta$

$=\int \frac{2\cos\theta\,d\theta}{\sqrt{4\cos^2\theta}}=\int \frac{2\cos\theta\,d\theta}{2\cos\theta}=\theta+c$

$=\sin^{-1}\left(\frac{x-2}{2}\right)+c,$ $\left[\text{From (i)}, \theta=\sin^{-1}\left(\frac{x-2}{2}\right)\right]$

where c is an arbitrary constant.

15. Let $I=\int \frac{\sin x}{\sin(x-2a)}dx$

Put $x-2a=t$

$\Rightarrow \quad x = 2a + t \Rightarrow dx = dt$

$\therefore \quad I = \int \frac{\sin(t+2a)}{\sin t} dx$

$= \int \frac{(\sin t \cos 2a + \cos t \sin 2a)}{\sin t} dx = \int (\cos 2a + \cot t \cdot \sin 2a)\, dx$

$= t \cos 2a + \sin 2a \log|\sin t| + c$

$= (x - 2a)\cos 2a + \sin 2a \log|\sin(x-2a)| + c$

16. Let $I = \int \frac{1}{e^x+1} dx$

Put $e^x + 1 = t \Rightarrow e^x dx = dt \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t-1}$

$\therefore \quad I = \int \frac{dt}{t(t-1)} = \int \frac{t-(t-1)}{t(t-1)} dt$

$= \int \frac{1}{t-1} dt - \int \frac{1}{t} dt = \log(t-1) - \log t + C$

$= \log e^x - \log(e^x+1) + c = \log \frac{e^x}{e^x+1} + c$

17. $\int \frac{2x}{(x^2+1)(x^2+2)} dx$

Let $x^2 + 2 = t \Rightarrow 2x\, dx = dt$

$\Rightarrow \quad \int \frac{2x}{(x^2+1)(x^2+2)} dx = \int \frac{dt}{(t-1)t} = \int \frac{(1-t+t)\,dt}{(t-1)t}$

$= -\int \frac{(t-1)dt}{(t-1)t} + \int \frac{t}{(t-1)t} dt = -\int \frac{1}{t} dt + \int \frac{1}{(t-1)} dt$

$= -\log|t| + \log|t-1| + \log C$

$= \log\left|\frac{t-1}{t}\right| + \log C = \log\left|\left(\frac{x^2+1}{x^2+2}\right)C\right|,$

where C is an arbitrary constant.

18. We have, $\int \frac{\cos(x+a)}{\sin(x+b)} dx = \int \frac{\cos(x+b+a-b)}{\sin(x+b)} dx$

$= \int \frac{\cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)}{\sin(x+b)} dx$

$= \cos(a-b) \int \frac{\cos(x+b)}{\sin(x+b)} dx - \sin(a-b) \int dx$

$= \cos(a-b) \log \sin(x+b) - x \sin(a-b) + C$

Commonly Made Mistake

- Using formula for $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\cos(A-B) = \cos A \cos B + \sin A + \sin B$

19. Let $I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$

(Put $\cos^2\theta = 1 - \sin^2\theta$)

$= 3\int \frac{\sin\theta\cos\theta}{4+\sin^2\theta - 4\sin\theta} d\theta - 2\int \frac{\cos\theta}{4+\sin^2\theta-4\sin\theta} d\theta$

$= 3I_1 - 2I_2$ (say) ...(i)

Now, $I_1 = \int \frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta} d\theta$

Put $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta\, d\theta = dt$

$\therefore \quad I_1 = \frac{1}{2}\int \frac{dt}{4+t-4\sqrt{t}} = \frac{1}{2}\int \frac{dt}{(\sqrt{t}-2)^2}$

Put $\sqrt{t} - 2 = u \Rightarrow \sqrt{t} = u + 2$

$\Rightarrow \quad \frac{1}{2\sqrt{t}} dt = du \Rightarrow dt = 2(u+2)du$

$\therefore \quad I_1 = \int \frac{(u+2)}{u^2} du = \int \frac{du}{u} + 2\int \frac{du}{u^2}$

$= \log u - \frac{2}{u} + C_1 = \log(\sqrt{t}-2) - \frac{2}{\sqrt{t}-2} + C_1$

$= \log(\sin\theta - 2) - \frac{2}{\sin\theta - 2} + C_1$ ...(ii)

Also, $I_2 = \int \frac{\cos\theta}{4+\sin^2\theta - 4\sin\theta} d\theta$

Put $\sin\theta = m \Rightarrow \cos\theta\, d\theta = dm$

$\therefore \quad I_2 = \int \frac{dm}{4+m^2-4m} = \int \frac{dm}{(m-2)^2}$

$= \frac{-1}{m-2} + C_2 = \frac{-1}{\sin\theta - 2} + C_2$ ...(iii)

From (i), (ii) and (iii), we get

$I = 3\log(\sin\theta - 2) - \frac{6}{\sin\theta-2} + \frac{2}{\sin\theta-2} + C,$

where $C = 3C_1 - 2C_2$

$\Rightarrow \quad I = 3\log(\sin\theta-2) - \frac{4}{\sin\theta-2} + C$

20. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)} dx = \int \frac{\sin(x+a-2a)}{\sin(x+a)} dx$

$= \int \left(\frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)}\right) dx$

$\Rightarrow \quad I = \cos 2a \int dx - \sin 2a \int \frac{\cos(x+a)}{\sin(x+a)} dx$

Put $\sin(x+a) = t \Rightarrow \cos(x+a)dx = dt$

$\Rightarrow \quad I = \cos 2a \int dx - \sin 2a \int \frac{dt}{t}$

$\Rightarrow \quad I = x\cos 2a - \sin 2a \log|t| + C$

$\Rightarrow \quad I = x\cos 2a - \sin 2a \log|\sin(x+a)| + C$

21. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{\sin^6 x}{\sin^2 x \cdot \cos^2 x} dx + \int \frac{\cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

$= \int \frac{\sin^4 x}{\cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x} dx$

$= \int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x} dx + \int \frac{\cos^2 x(1-\sin^2 x)}{\sin^2 x} dx$

$= \int \tan^2 x\, dx - \int \sin^2 x\, dx + \int \cot^2 x\, dx - \int \cos^2 x\, dx$

$= \int (\sec^2 x - 1)dx - \int \sin^2 x\, dx + \int (\operatorname{cosec}^2 x - 1)\, dx - \int (1 - \sin^2 x)\, dx$

$= \tan x - x + (-\cot x) - x - x + C = \tan x - \cot x - 3x + C$

22. Let $I = \int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{9}{4} - x^2}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{3}{2}}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$

23. Let $I = \int \frac{dx}{9+4x^2} = \frac{1}{4}\int \frac{dx}{x^2 + \frac{9}{4}} = \frac{1}{4}\int \frac{dx}{x^2 + \left(\frac{3}{2}\right)^2}$

$= \frac{1}{4}\cdot\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$

24. Let $I=\int\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore\ I=\int\frac{dt}{\sqrt{t^2+4}}=\log|t+\sqrt{t^2+4}|+C$

$=\log|\tan x+\sqrt{\tan^2 x+4}|+C$

25. Let $I=\int\frac{dx}{\sqrt{5-4x-2x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{7}{2}-1-2x-x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}}$

$=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right)+C=\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right]+C$

26. We have, $\int\frac{dx}{x^2+4x+8}=\int\frac{dx}{x^2+4x+4+4}$

$=\int\frac{dx}{(x+2)^2+(2)^2}=\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C$

27. Let $I=\int\frac{dx}{5-8x-x^2}$

$=\int\frac{dx}{5+16-16-8x-x^2}=\int\frac{dx}{21-(x+4)^2}$

$=\int\frac{dx}{(\sqrt{21})^2-(x+4)^2}=\frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$

$\left[\because\ \int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C\right]$

28. Let $I=\int\frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

Putting $e^x = t \Rightarrow e^x\,dx = dt$, we get

$I=\int\frac{dt}{\sqrt{5-4t-t^2}}=\int\frac{dt}{\sqrt{9-(t^2+4t+4)}}=\int\frac{dt}{\sqrt{3^2-(t+2)^2}}$

$=\sin^{-1}\left(\frac{t+2}{3}\right)+C=\sin^{-1}\left(\frac{e^x+2}{3}\right)+C$

29. Let $I=\int\frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

Put $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}dx=dt$

$\therefore\ I=\frac{2}{3}\int\frac{dt}{\sqrt{a^3-t^2}}=\frac{2}{3}\int\frac{dt}{\sqrt{(a^{3/2})^2-t^2}}$

$=\frac{2}{3}\left[\sin^{-1}\left(\frac{t}{a^{3/2}}\right)\right]+C=\frac{2}{3}\left[\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)\right]+C$

$=\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2}+C$

30. Let $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx$

$=\int\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$

$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x+1-1}}dx$

$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(1-\sin 2x)}}dx$

$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}}dx$

$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

$\therefore\ I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\sin^{-1}t+C$

$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$

31. Let $I=\int\frac{x+2}{2x^2+6x+5}dx$

Let $x+2=A\left[\frac{d}{dx}(2x^2+6x+5)\right]+B$

$\Rightarrow\ x + 2 = A(4x + 6) + B$

Equating coefficients of x and constant terms, we get

$4A = 1 \Rightarrow A = 1/4$ and $6A + B = 2 \Rightarrow B = 1/2$

$\therefore\ I=\frac{1}{4}\int\frac{4x+6}{2x^2+6x+5}dx+\frac{1}{2}\int\frac{dx}{2x^2+6x+5}$

$=\frac{1}{4}\log|2x^2+6x+5|+\frac{1}{4}\int\frac{dx}{x^2+3x+\frac{5}{2}}$

$=\frac{1}{4}\log|2x^2+6x+5|+\frac{1}{4}\int\frac{dx}{x^2+3x+\frac{9}{4}-\frac{9}{4}+\frac{5}{2}}$

$=\frac{1}{4}\log|2x^2+6x+5|+\frac{1}{4}\int\frac{dx}{\left(x+\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}$

$=\frac{1}{4}\log|2x^2+6x+5|+\frac{1}{2}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right)+C$

$=\frac{1}{4}\log|2x^2+6x+5|+\frac{1}{2}\tan^{-1}(2x+3)+C$

32. Let $I=\int\frac{x+3}{\sqrt{5-4x-2x^2}}dx$

Let $x+3=A\frac{d}{dx}(5-4x-2x^2)+B=A(-4-4x)+B$

On comparing the coefficients of like term, we get

$-4A=1\Rightarrow A=-\frac{1}{4}$ and $-4A+B=3\Rightarrow B=2$

$x+3=-\frac{1}{4}(-4-4x)+2$

$\Rightarrow\ I=\int\frac{-\frac{1}{4}(-4-4x)+2}{\sqrt{5-4x-2x^2}}dx$

$= \frac{-1}{4}\int \frac{-4-4x}{\sqrt{5-4x-2x^2}}dx + 2\cdot\int \frac{1}{\sqrt{5-4x-2x^2}}dx$

$= -\frac{1}{4}I_1 + 2I_2$ ...(i)

where $I_1 = \int \frac{-4-4x}{\sqrt{5-4x-2x^2}}dx$

Put $5 - 4x - 2x^2 = t \Rightarrow (-4-4x)dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = 2\sqrt{t} + C_1$

$= 2\sqrt{5-4x-2x^2} + C_1$ ...(ii)

and $I_2 = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{7/2}}\right) + C_2$ ...(iii)

From (i), (ii) and (iii), we get

$I = -\frac{1}{4}\cdot 2\sqrt{5-4x-2x^2} + 2\cdot\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$

where $C = C_1 + C_2$

$\Rightarrow \quad I = -\frac{1}{2}\sqrt{5-4x-2x^2} + \sqrt{2}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$

33. Let $I = \int \frac{x+2}{\sqrt{x^2+5x+6}}dx = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}dx$

$= \frac{1}{2}\int (x^2+5x+6)^{-1/2}(2x+5)dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$

Put $x^2 + 5x + 6 = t \Rightarrow (2x+5)\,dx = dt$

$\Rightarrow \quad I = \frac{1}{2}\int t^{-1/2}dt - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} + C$

$= \frac{1}{2}\frac{t^{1/2}}{\frac{1}{2}} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$

$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x + \frac{5}{2} + \sqrt{x^2+5x+6}\right| + C$

Concept Applied

➲ In $\int \frac{px+q}{ax^2+bx+c}$

substitute $px + q = A\frac{d}{dx}(ax^2+bx+c) + B.$

34. Let $I = \int \frac{5x-2}{1+2x+3x^2}dx$

Let $5x - 2 = A\frac{d}{dx}(1+2x+3x^2) + B = A(2+6x) + B$

On comparing the coefficients of like terms, we get

$6A = 5 \Rightarrow A = \frac{5}{6}$ and $2A + B = -2 \Rightarrow B = \frac{-11}{3}$

$\therefore \quad I = \int \frac{\frac{5}{6}\frac{d}{dx}(1+2x+3x^2) - \frac{11}{3}}{1+2x+3x^2}dx$

$= \frac{5}{6}\int \frac{\frac{d}{dx}(1+2x+3x^2)}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{dx}{1+2x+3x^2}$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\cdot 3}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}}$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\cdot\frac{1}{\sqrt{2}/3}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\sqrt{2}/3}\right) + C$

$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

35. Let $I = \int \frac{1}{\cos^4 x + \sin^4 x}dx = \int \frac{\sec^4 x}{1+\tan^4 x}dx$

$= \int \frac{(\tan^2 x + 1)\sec^2 x}{1+\tan^4 x}dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore \quad I = \int \frac{t^2+1}{t^4+1}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$

Put $t - \frac{1}{t} = y \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dy$

$\therefore \quad I = \int \frac{dy}{y^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right) + C$

36. Let $I = \int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}dx$

$= \int \frac{\sec^4 x\,dx}{\tan^4 x + \tan^2 x + 1} = \int \frac{(1+\tan^2 x)\sec^2 x\,dx}{\tan^4 x + \tan^2 x + 1}$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore \quad I = \int \frac{1+t^2}{t^4+t^2+1}dt = \int \frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3}dt$

Put $t - \frac{1}{t} = y \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dy$

Thus, $I = \int \frac{dy}{y^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right) + C$

$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{3}} + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{3}}\right) + C$

37. Let $I = \int \frac{(x+1)}{(x+2)(x+3)}dx$

Also let, $\frac{(x+1)}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$

$\Rightarrow \quad x + 1 = A(x+3) + B(x+2)$ ...(i)

Putting $x = -3$ in (i), we get
$-B = -3 + 1 = -2 \Rightarrow B = 2$
Putting $x = -2$ in (i), we get
$A = -2 + 1 = -1$

$\therefore \quad I = \int \frac{-1}{(x+2)}dx + 2\int \frac{1}{(x+3)}dx$

$= -\log(x + 2) + 2\log(x + 3) + C$

Answer Tips

➲ Form of rational fraction to the partial fraction.

$$\frac{px+q}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

38. Let $I = \int \frac{x^2}{x^2+6x+12}dx$

$= \int \frac{x^2+6x+12-(6x+12)}{x^2+6x+12}dx = \int dx - \int \frac{6x+12}{x^2+6x+12}dx$

$= x - 3\int \frac{2x}{x^2+6x+12}dx - 12\int \frac{dx}{x^2+6x+12}$

$= x - 3\int \frac{2x+6-6}{x^2+6x+12}dx - 12\int \frac{dx}{x^2+6x+12}$

$= x - 3\int \frac{2x+6}{x^2+6x+12}dx - 18\int \frac{dx}{x^2+6x+12}$

$= x - 3I_1 - 18I_2$

Consider, $I_1 = \int \frac{2x+6}{x^2+6x+12}dx$

Put $x^2 + 6x + 12 = t \Rightarrow (2x + 6)dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| + C = \log|x^2 + 6x + 12| + C_1$

and $I_2 = \int \frac{dx}{x^2+6x+12}$

$= \int \frac{dx}{(x+3)^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C_2$

$\therefore \quad I = x - 3\log|x^2+6x+12| + 6\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C$

39. Let $I = \int \frac{x^2}{(x^2+1)(3x^2+4)}dx$

Let $x^2 = y$

So, $\frac{x^2}{(x^2+1)(3x^2+4)} = \frac{y}{(y+1)(3y+4)}$

$= \frac{A}{y+1} + \frac{B}{3y+4} = \frac{A(3y+4)+B(y+1)}{(y+1)(3y+4)}$

$\Rightarrow \quad y = A(3y + 4) + B(y + 1)$

$\Rightarrow \quad y = (3A + B)y + (4A + B)$

Equating the like coefficients, we get

$3A + B = 1$ and $4A + B = 0$

On solving we get $A = -1, B = 4$

So, $I = \int \left[-\frac{1}{x^2+1} + \frac{4}{3x^2+4}\right]dx$

$= -\int \frac{1}{1+x^2}dx + \frac{4}{3}\int \frac{1}{x^2+\frac{4}{3}}dx$

$= -\tan^{-1}x + \frac{4}{3} \times \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C$

$= -\tan^{-1}x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C$

40. Let $I = \int \frac{x^3+1}{x^3-x}dx = \int \frac{(x+1)(x^2-x+1)}{x(x^2-1)}dx$

$= \int \frac{(x^2-x+1)}{x(x-1)}dx = \int \left[1 + \frac{1}{x(x-1)}\right]dx$ ...(i)

Consider, $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$\Rightarrow \quad \frac{1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)}$

$\Rightarrow \quad 1 = A(x - 1) + Bx$

On solving, we get $A = -1, B = 1$

$\Rightarrow \quad \frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$

$\therefore \quad I = \int \left(1 + \frac{1}{x-1} - \frac{1}{x}\right)dx = x + \log|x-1| - \log|x| + c$

41. Let $I = \int \frac{3x+5}{x^2+3x-18}dx = \int \frac{3x+5}{(x+6)(x-3)}dx$

Let $\frac{3x+5}{(x+6)(x-3)} = \frac{A}{(x+6)} + \frac{B}{(x-3)}$

$\Rightarrow \quad 3x + 5 = A(x - 3) + B(x + 6)$ ...(i)

Putting $x = 3$ in (i), we get $9B = 14 \Rightarrow B = \frac{14}{9}$

Putting $x = -6$ in (i), we get $-9A = -13 \Rightarrow A = \frac{13}{9}$

$\therefore \quad I = \frac{13}{9}\int \frac{1}{(x+6)}dx + \frac{14}{9}\int \frac{1}{(x-3)}dx$

$= \frac{13}{9}\log(x+6) + \frac{14}{9}\log(x-3) + C$

42. Let $I = \int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$ ...(i)

Let $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$ ...(ii)

$\Rightarrow \quad x^2 + x + 1 = (Ax + B)(x + 2) + C(x^2 + 1)$ ...(iii)

Put $x = 0$, 1 and -2 in (iii), We get

$1 = 2B + C$; $3 = 3(A + B) + 2C$ and $3 = 5C$

$\Rightarrow \quad C = \frac{3}{5}, B = \frac{1}{5}$ and $A = \frac{2}{5}$

From (ii), we get

$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{\left(\frac{2}{5}x + \frac{1}{5}\right)}{x^2+1} + \frac{\frac{3}{5}}{x+2}$

$= \frac{1}{5} \cdot \frac{2x+1}{x^2+1} + \frac{3}{5} \cdot \frac{1}{x+2}$

$\therefore \quad I = \frac{1}{5}\int \frac{2x+1}{x^2+1}dx + \frac{3}{5}\int \frac{1}{x+2}dx$

$$=\frac{1}{5}\left[\int\frac{2x}{x^2+1}dx+\int\frac{dx}{x^2+1}\right]+\frac{3}{5}\int\frac{dx}{x+2}$$

$$=\frac{1}{5}[\log|x^2+1|+\tan^{-1}x]+\frac{3}{5}\log|x+2|+C_1$$

43. Let $I=\int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore\ I=2\int\frac{1}{(1-t)(1+t^2)}dt$ ...(i)

Now, $\frac{1}{(1-t)(1+t^2)}=\frac{A}{(1-t)}+\frac{Bt+C}{(1+t^2)}$

$\Rightarrow\ 1 = A(1 + t^2) + (Bt + C)(1 - t)$

$\Rightarrow\ 1 = A + At^2 + Bt - Bt^2 + C - Ct$

$\Rightarrow\ 1 = (A + C) + t^2(A - B) + t(B - C)$

On equating coefficients of like terms on both sides, we get

$A + C = 1,\ A - B = 0 \Rightarrow A = B$

$B - C = 0 \Rightarrow B = C$

$\therefore\ A = B = C$

$\therefore\ A=\frac{1}{2}, B=\frac{1}{2}, C=\frac{1}{2}$

$\therefore\ \frac{1}{(1-t)(1+t^2)}=\frac{1}{2(1-t)}+\frac{1}{2}\frac{t}{(1+t^2)}+\frac{1}{2(1+t^2)}$

Put above equation in (i), we get

$$I=2\left[\int\frac{1}{2(1-t)}dt+\int\frac{1}{2}\frac{t}{(1+t^2)}dt+\int\frac{1}{2(1+t^2)}dt\right]$$

$$=\int\frac{dt}{(1-t)}+\int\frac{t}{(1+t^2)}dt+\int\frac{dt}{(1+t^2)}$$

$$= -\log(1-t)+\frac{1}{2}\log(1+t^2)+\tan^{-1}(t)+C$$

$$= -\log(1-\sin x)+\frac{1}{2}\log(1+\sin^2 x)+\tan^{-1}(\sin x)+C$$

$$= -\log(1-\sin x)+\log(1+\sin^2 x)^{1/2}+\tan^{-1}(\sin x)+C$$

$$=\log\frac{\sqrt{1+\sin^2 x}}{1-\sin x}+\tan^{-1}(\sin x)+C$$

44. Let $I=\int\frac{2x}{(x^2+1)(x^2+2)^2}dx$

Put $x^2 = y \Rightarrow 2x\,dx = dy$

$\therefore\ I=\int\frac{dy}{(y+1)(y+2)^2}$

Let $\frac{1}{(y+1)(y+2)^2}=\frac{A}{y+1}+\frac{B}{y+2}+\frac{C}{(y+2)^2}$

$\Rightarrow\ 1 = A(y + 2)^2 + B(y + 1)(y + 2) + C(y + 1)$ ...(i)

Putting $y = -1$, in (i), we get $1 = A$

Putting $y = -2$, in (i), we get $1 = -C \Rightarrow C = -1$

Putting $y = 0$, in (i), we get $1 = 4A + 2B + C$

$\Rightarrow\ B=\frac{1-4+1}{2}=-1$

$\therefore\ I=\int\left[\frac{1}{y+1}-\frac{1}{y+2}-\frac{1}{(y+2)^2}\right]dy$

$$=\log(y+1)-\log(y+2)+\frac{1}{y+2}+c$$

$$=\log\left(\frac{y+1}{y+2}\right)+\frac{1}{y+2}+c$$

$$=\log\left(\frac{x^2+1}{x^2+2}\right)+\frac{1}{x^2+2}+c \qquad [\because y = x^2]$$

45. $I=\int\frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

$$=\int\frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$\therefore\ I=\int\frac{1}{(4+t^2)(1+4t^2)}dt$

Consider, $\frac{1}{(4+t^2)(1+4t^2)}=\frac{At+B}{4+t^2}+\frac{Ct+D}{1+4t^2}$ (Using partial fractions)

$1 = (At + B)(1 + 4t^2) + (Ct + D)(4 + t^2)$

$= At + B + 4At^3 + 4Bt^2 + 4Ct + Ct^3 + 4D + Dt^2$

$= (4A + C)t^3 + (4B + D)t^2 + (A + 4C)t + (B + 4D)$

On comparing the coefficients of like terms, we get

$4A + C = 0$ ...(i)

$4B + D = 0$ ...(ii)

$A + 4C = 0$ ...(iii)

$B + 4D = 1$ ...(iv)

Solving (i) & (iii), we get

$A = 0$ and $C = 0$

Solving (ii) & (iv), we get

$B=\frac{-1}{15}$ and $D=\frac{4}{15}$

$\therefore\ \frac{1}{(4+t^2)(1+4t^2)}=\frac{-1/15}{4+t^2}+\frac{4/15}{1+4t^2}$

$\therefore\ I=-\frac{1}{15}\int\frac{1}{4+t^2}dt+\frac{4}{15}\times\frac{1}{4}\int\frac{1}{\frac{1}{4}+t^2}dt$

$$=-\frac{1}{15}\times\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+\frac{1}{15}\times\frac{1}{1/2}\tan^{-1}\left(\frac{t}{1/2}\right)+C$$

$$=-\frac{1}{30}\tan^{-1}\left(\frac{t}{2}\right)+\frac{2}{15}\tan^{-1}(2t)+C$$

$$=\frac{2}{15}\tan^{-1}(2\sin\theta)-\frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right)+C$$

46. Let $I=\int\frac{x^2}{x^4+x^2-2}dx=\int\frac{x^2}{(x^2-1)(x^2+2)}dx$

Let $x^2 = z$

$\therefore\ \frac{x^2}{(x^2-1)(x^2+2)}=\frac{z}{(z-1)(z+2)}$

Using partial fractions, we have

$\frac{z}{(z-1)(z+2)}=\frac{A}{z-1}+\frac{B}{z+2}$

$\Rightarrow\ z = A(z + 2) + B(z - 1)$

when $z = 1$, we get $A=\frac{1}{3}$

and when $z = -2$, we get $B=\frac{2}{3}$

$\therefore\ I=\int\frac{x^2}{(x^2-1)(x^2+2)}dx$

$$=\int\frac{1/3}{(x^2-1)}dx+\int\frac{2/3}{(x^2+2)}dx$$

$$=\frac{1}{3}\int\frac{1}{x^2-1}dx+\frac{2}{3}\int\frac{1}{x^2+(\sqrt{2})^2}dx$$

$$=\frac{1}{3}\times\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+\frac{2}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

$$=\frac{1}{6}\log\left|\frac{x-1}{x+1}\right|+\frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

Concept Applied

➲ $\int\frac{1}{x^2-a^2}\cdot dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

47. Let $I=\int\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

Let $x^2 = t$

$$\therefore\quad \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}=\frac{(t+1)(t+4)}{(t+3)(t-5)}$$

$$=\frac{t^2+5t+4}{(t+3)(t-5)}=1+\frac{7t+19}{(t+3)(t-5)}$$

Let $\frac{7t+19}{(t+3)(t-5)}=\frac{A}{t+3}+\frac{B}{t-5}$

$\Rightarrow\quad 7t+19 = A(t-5) + B(t+3)$

Putting $t = 5$, we get $B = \frac{27}{4}$

Putting $t = -3$, we get $A = \frac{1}{4}$

$$\therefore\quad \frac{t^2+5t+4}{(t+3)(t-5)}=1+\frac{1}{4(t+3)}+\frac{27}{4(t-5)}$$

$$\Rightarrow\quad I=\int\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx=\int dx+\frac{1}{4}\int\frac{1}{(x^2+3)}dx+\frac{27}{4}\int\frac{1}{(x^2-5)}dx$$

$$=x+\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{27}{4}\times\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+C$$

$$=x+\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+C$$

48. Let $I=\int\frac{1}{\sin x+\sin 2x}dx$

$$=\int\frac{1}{\sin x+2\sin x\cos x}dx=\int\frac{1}{\sin x(1+2\cos x)}dx$$

$$=\int\frac{\sin x}{\sin^2 x(1+2\cos x)}dx$$

Let $u = \cos x \Rightarrow du = -\sin x\,dx$

Also, $\sin^2 x = 1-\cos^2 x = 1-u^2$

$$\therefore\ I=\int\frac{-1}{(1-u^2)(1+2u)}du$$

$$=\int\frac{-1}{(1+u)(1-u)(1+2u)}du$$

Using partial fractions, we have

$$\frac{-1}{(1+u)(1-u)(1+2u)}=\frac{A}{(1+u)}+\frac{B}{(1-u)}+\frac{C}{(1+2u)}$$

$\Rightarrow\quad -1 = A(1-u)(1+2u) + B(1+u)(1+2u) + C(1+u)(1-u)$

Putting $u = 1$, we get $B = -1/6$

Putting $u = -1$, we get $A = 1/2$

Put $u=-\frac{1}{2}$, we get $C=\frac{-4}{3}$

So, $\frac{-1}{(1+u)(1-u)(1+2u)}=\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}$

$$\Rightarrow\quad I=\int\left[\frac{1}{2(1+u)}-\frac{1}{6(1-u)}-\frac{4}{3(1+2u)}\right]du$$

$$=\frac{1}{2}\log(1+u)+\frac{1}{6}\log(1-u)-\frac{4}{3\times 2}\log(1+2u)+C_1$$

$$=\frac{1}{2}\log(1+\cos x)+\frac{1}{6}\log(1-\cos x)-\frac{2}{3}\log(1+2\cos x)+C_1$$

49. Let $I=\int\frac{x^2}{(x^2+4)(x^2+9)}dx$

Put $x^2 = y$. Then $\frac{x^2}{(x^2+4)(x^2+9)}=\frac{y}{(y+4)(y+9)}$

Let $\frac{y}{(y+4)(y+9)}=\frac{A}{y+4}+\frac{B}{y+9}$ ...(i)

$\Rightarrow\quad y = A(y+9) + B(y+4)$ ...(ii)

Putting $y = -4$ and $y = -9$ successively in (ii), we get $A=\frac{-4}{5}$ and $B=\frac{9}{5}$

Substituting the values of A and B in (i), we get

$$\frac{y}{(y+4)(y+9)}=\frac{-4/5}{(y+4)}+\frac{9/5}{(y+9)}$$

$$\Rightarrow\quad \frac{x^2}{(x^2+4)(x^2+9)}=\frac{-4}{5(x^2+4)}+\frac{9}{5(x^2+9)}$$

$$\therefore\quad I=\int\frac{x^2}{(x^2+4)(x^2+9)}dx$$

$$=\frac{-4}{5}\int\frac{1}{(x^2+4)}dx+\frac{9}{5}\int\frac{1}{(x^2+9)}dx$$

$$=\frac{-4}{5}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{9}{5}\times\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

$$=\frac{-2}{5}\tan^{-1}\left(\frac{x}{2}\right)+\frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right)+C$$

50. Let $I=\int\frac{x}{(x-1)^2(x+2)}dx$. ...(i)

Using partial fraction, we have

$$\frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}\quad ...(ii)$$

$\Rightarrow\quad x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$ ...(iii)

On comparing the coefficients of x^2, x and constant term in (iii), we get $0 = A + C$, $1 = A + B - 2C$, $0 = -2A + 2B + C$

Solving these, we get

$A=\frac{2}{9}, B=\frac{1}{3}, C=-\frac{2}{9}$

From (ii), we get $\frac{x}{(x-1)^2(x+2)}=\frac{2}{9}\cdot\frac{1}{x-1}+\frac{1}{3}\cdot\frac{1}{(x-1)^2}-\frac{2}{9}\cdot\frac{1}{x+2}$

$\therefore \quad I=\frac{2}{9}\int\frac{1}{x-1}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{x+2}dx$

$\Rightarrow \quad I=\frac{2}{9}\log|x-1|-\frac{1}{3}\cdot\frac{1}{x-1}\frac{-2}{9}\log|x+2|$

$=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3}\cdot\frac{1}{x-1}+C_1$

51. Let $I=\int\frac{x}{(x^2+1)(x-1)}dx$

Let $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$ ...(i)

$\Rightarrow \quad x=(Ax+B)(x-1)+C(x^2+1)$ ...(ii)

Comparing coefficients of x^2, x and constant terms, we get

$0=A+C$, $1=B-A$, $0=-B+C$

Solving these, we get

$A=-\frac{1}{2}, C=\frac{1}{2}$ and $B=\frac{1}{2}$

$\therefore$ From (i), we get

$\frac{x}{(x^2+1)(x-1)}=\frac{-\frac{1}{2}(x-1)}{x^2+1}+\frac{1}{2}\cdot\frac{1}{x-1}$

$=-\frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}\cdot\frac{1}{x^2+1}+\frac{1}{2}\cdot\frac{1}{x-1}$

$\therefore \quad I=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\int\frac{dx}{x^2+1}+\frac{1}{2}\int\frac{dx}{x-1}$

$\Rightarrow \quad I=-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C_1$

52. Let $I=\int\frac{x^2}{(x^2+1)(x^2+4)}dx$

Put $x^2=y$

So, $\frac{x^2}{(x^2+1)(x^2+4)}=\frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)}=\frac{A}{y+1}+\frac{B}{y+4}$ ...(i)

$\Rightarrow \quad y=A(y+4)+B(y+1)$ ...(ii)

Putting $y=-4$ and $y=-1$, successively in (ii), we get

$A=\frac{-1}{3}$ and $B=\frac{4}{3}$

From (i), we get

$\frac{y}{(y+1)(y+4)}=\frac{-1}{3(y+1)}+\frac{4}{3(y+4)}$

$\Rightarrow \quad \frac{x^2}{(x^2+1)(x^2+4)}=\frac{-1}{3(x^2+1)}+\frac{4}{3(x^2+4)}$

$\therefore \quad I=\int\frac{x^2}{(x^2+1)(x^2+4)}dx=\frac{-1}{3}\int\frac{dx}{x^2+1}+\frac{4}{3}\int\frac{dx}{x^2+4}$

$=\frac{-1}{3}\times\tan^{-1}x+\frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

$=\frac{-1}{3}\tan^{-1}x+\frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right)+C$

53. Let $I=\int\frac{x^3}{x^4+3x^2+2}dx$

Put $x^2=t\Rightarrow xdx=\frac{1}{2}dt$

$\therefore \quad I=\frac{1}{2}\int\frac{t}{t^2+3t+2}dt=\frac{1}{2}\int\frac{t}{(t+2)(t+1)}dt$ ...(i)

Let $\frac{t}{(t+2)(t+1)}=\frac{A}{(t+2)}+\frac{B}{(t+1)}$

$\Rightarrow \quad t=A(t+1)+B(t+2)$

Put $t=-1, -2$ in it, we get $A=2$, $B=-1$

$\therefore \quad \frac{t}{(t+2)(t+1)}=\frac{2}{t+2}-\frac{1}{t+1}$ ...(ii)

From (i) and (ii), we get $I=\frac{1}{2}\int\left[\frac{2}{(t+2)}-\frac{1}{(t+1)}\right]dt$

$=\frac{1}{2}[2\log|t+2|-\log|t+1|]+C=\frac{1}{2}[2\log|x^2+2|-\log|x^2+1|]+C$

Concept Applied

➲ $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

54. Let $I=\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$

Let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}$ ...(i)

$\Rightarrow \quad x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$

Put $x=-1, -2, 0$ successively in it, we get

$B=1$; $C=3$; $A=-2$

From (i), we get

$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}$

Integrating both sides w.r.t. x, we get

$I=\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx$

$=-2\int\frac{1}{x+1}dx+\int\frac{1}{(x+1)^2}dx+3\int\frac{1}{x+2}dx$

$=-2\log|x+1|-\frac{1}{x+1}+3\log|x+2|+C_1$

55. (d) : Let $I=\int e^x\left(\log x+\frac{1}{x}\right)dx$

$\Rightarrow \quad I=e^x\log x+c$ $\quad \left(\because \int e^x[f(x)+f'(x)]dx=e^xf(x)+c\right)$

56. (d) : Let $I=\int\frac{e^x}{x+1}[1+(x+1)\log(x+1)]dx$

$=\int e^x\left[\frac{1}{x+1}+\log(x+1)\right]dx$

It is of the form $\int e^x[f(x)+f'(x)dx]$,

where $f(x)=\log(x+1)$ and $f'(x)=\frac{1}{x+1}$

So, $I=e^x\log(x+1)+C$

Concept Applied

➲ $\int e^x[f(x)+f'(x)]dx=e^xf(x)+c$

57. Let $I = \int x^4 \log x\, dx = \log x \cdot \frac{x^5}{5} - \int \frac{1}{x} \cdot \frac{x^5}{5} dx$

$= \frac{x^5}{5}\log x - \frac{1}{5}\int x^4 dx = \frac{1}{5}x^5 \log x - \frac{x^5}{25} + C$

58. Let $I = \int \frac{\log x - 3}{(\log x)^4} dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$\therefore \quad I = \int \left[\frac{t-3}{t^4}\right] e^t dt = \int e^t\, t^{-3}\, dt - 3\int t^{-4} e^t dt$

$= t^{-3}e^t + 3\int t^{-4}e^t dt - 3\int t^{-4}e^t dt + c$

$= t^{-3}e^t + c = (\log x)^{-3}\, x + c$

59. Let $I = \int \sin^{-1}(2x)dx = \int 1 \cdot \sin^{-1}(2x)\, dx$

$= \sin^{-1}(2x)x - \int \left(\frac{1}{\sqrt{1-4x^2}} \frac{d}{dx}(2x) \cdot x\right) dx$

$= x\sin^{-1}(2x) - \int \frac{2x}{\sqrt{1-4x^2}} dx$

$= x\sin^{-1}(2x) + \int \frac{dt}{4\sqrt{t}}$ (Putting $1-4x^2 = t \Rightarrow -8x\, dx = dt$)

$= x\sin^{-1}(2x) + \frac{2}{4}(t)^{1/2} + C = x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$

60. Let $I = \int x \cdot \tan^{-1} x\, dx$

On integrating by parts w.r.t. x, we get

$I = \tan^{-1} x \int x dx - \int \left\{\frac{d}{dx}(\tan^{-1} x) \int x dx\right\} dx$

$= (\tan^{-1} x)\frac{x^2}{2} - \int \frac{1}{(1+x^2)} \frac{x^2}{2} dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2} dx$

$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$

$= \frac{1}{2}(1+x^2)\tan^{-1} x - \frac{x}{2} + C$

61. Let $I = \int e^x \sin 2x\, dx$ ...(i)

$= \left[\sin 2x\, e^x - 2\int \cos 2x\, e^x dx\right]$

$= \left[e^x \sin 2x - 2\int e^x \cos 2x\, dx\right]$

$= \left[e^x \sin 2x - 2\left[\cos 2x e^x + 2\int \sin 2x e^x dx\right]\right]$

$\Rightarrow \quad I = e^x \sin 2x - 2e^x \cos 2x - 4I + C$ (From (i))

$\Rightarrow \quad 5I = e^x \sin 2x - 2e^x \cos 2x + C$

$\Rightarrow \quad I = \frac{1}{5}(e^x \sin 2x - 2e^x \cos 2x) + \frac{C}{5}$

$= \frac{e^x}{5}(\sin 2x - 2\cos 2x) + C_1$, where $C_1 = \frac{C}{5}$ is an arbitrary constant.

Answer Tips

- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\,[a\sin bx - b\cos bx] + C$

62. Let $I = \int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx$

$= \sec x \tan x - \int \sec x \tan x \cdot \tan x\, dx$

(Applying integration by parts)

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$= \sec x \tan x - I + \ln|\sec x + \tan x| + C_1$

$\Rightarrow \quad 2I = \sec x \tan x + \ln|\sec x + \tan x| + C_1$

$\therefore \quad I = \frac{1}{2}(\sec x \tan x) + \frac{1}{2}\ln|\sec x + \tan x| + C \quad \left(\text{Where, } C = \frac{C_1}{2}\right)$

Key Points

- $\int \sec x\, dx = \log|\sec x + \tan x| + C$

63. Let $I = \int \frac{x\sin^{-1} x}{\sqrt{1-x^2}} dx$

$= \sin^{-1} x \int \frac{x}{\sqrt{1-x^2}} dx - \int \left[\frac{d}{dx}(\sin^{-1} x) \int \frac{x}{\sqrt{1-x^2}} dx\right] dx$

(Applying integration by parts)

Firstly, let us evaluate the integral $\int \frac{x}{\sqrt{1-x^2}} dx$

Put $t = 1 - x^2 \Rightarrow dt = -2x\, dx$.

$\therefore \quad \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$

$\therefore \quad I = \sin^{-1} x\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right) dx$

$= -\sqrt{1-x^2}\,\sin^{-1} x + \int dx = -\sqrt{1-x^2}\,\sin^{-1} x + x + C$

64. Let $I = \int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} dx$

$= -\int \frac{(-x^2 + 3x - 1)}{\sqrt{1-x^2}} dx = -\int \frac{(1-x^2) + 3x - 2}{\sqrt{1-x^2}} dx$

i.e., $I = -\int \sqrt{1-x^2}\, dx + \int \frac{-3x+2}{\sqrt{1-x^2}} dx$

$= -\int \sqrt{1-x^2}\, dx + \frac{3}{2}\int \frac{-2x}{\sqrt{1-x^2}} dx + 2\int \frac{1}{\sqrt{1-x^2}} dx$

$= -\int \sqrt{1-x^2}\, dx + \frac{3 \times 2}{2}\sqrt{1-x^2} + 2\sin^{-1} x + C$

$= -\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right] + 3\sqrt{1-x^2} + 2\sin^{-1} x + C$

$= -\frac{x}{2}\sqrt{1-x^2} + \frac{3}{2}\sin^{-1} x + 3\sqrt{1-x^2} + C$

Commonly Made Mistake

- Remember the formula for $\int \sqrt{a^2 - x^2}\, dx$ and $\int \frac{1}{\sqrt{a^2 - x^2}} dx$.

65. Let $I=\int \frac{\log x}{(x+1)^2}dx=\int (x+1)^{-2}\cdot \log x\,dx$

On integrating by parts, taking log x as first function, we have

$$I=\frac{(x+1)^{-1}}{-1}\cdot \log x-\int \frac{(x+1)^{-1}}{-1}\cdot\frac{1}{x}dx$$
$$=\frac{-\log x}{x+1}+\int\frac{dx}{x(x+1)}=\frac{-\log x}{x+1}+\int\left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$
$$=-\frac{\log x}{x+1}+\log x-\log(x+1)+C$$
$$=-\frac{\log x}{x+1}+\log\left(\frac{x}{x+1}\right)+C$$

66. Let $I=\int e^{2x}\sin(3x+1)dx$

On integrating by parts, taking e^x as first function, we have

$$=e^{2x}\int\sin(3x+1)dx-\int\left(\frac{d(e^{2x})}{dx}\cdot\int\sin(3x+1)dx\right)dx$$
$$=e^{2x}\frac{[-\cos(3x+1)]}{3}-\int 2e^{2x}\cdot\frac{[-\cos(3x+1)]}{3}dx$$
$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\int e^{2x}\cos(3x+1)dx$$
$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\left[e^{2x}\int\cos(3x+1)dx-\int\left(\frac{d}{dx}(e^{2x})\cdot\int\cos(3x+1)dx\right)dx\right]$$
$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)\ -\frac{4}{9}\int e^{2x}\sin(3x+1)dx$$
$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)-\frac{4}{9}I+C_1$$
$$\Rightarrow\ I+\frac{4}{9}I=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$
$$\Rightarrow\ \frac{13I}{9}=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$
$$\Rightarrow\ I=\frac{9}{13}\left[\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1\right]$$
$$=\frac{9}{13}e^{2x}\left[\frac{2\sin(3x+1)-3\cos(3x+1)}{9}\right]+\frac{9}{13}C_1$$
$$=\frac{1}{13}e^{2x}[2\sin(3x+1)-3\cos(3x+1)]+C \qquad \left(\text{Where, } C=\frac{9C_1}{13}\right)$$

67. Let $I=\int\frac{x^2+1}{(x+1)^2}e^x dx$

$$=\int e^x\cdot\left[\frac{(x^2-1)+2}{(x+1)^2}\right]dx$$
$$=\int e^x\cdot\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right]dx$$
$$=\int e^x\cdot\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right]dx \qquad \text{...(i)}$$

Take $f(x)=\frac{x-1}{x+1}$

$$\Rightarrow\ f'(x)=\frac{(x+1)\cdot1-(x-1)\cdot1}{(x+1)^2}=\frac{2}{(x+1)^2}$$

By using the formula, we get

$$I=\int e^x[f(x)+f'(x)]dx\ =e^xf(x)+C$$
$$I=e^x\cdot\left[\frac{x-1}{x+1}\right]+C$$

68. Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Put $\cos^{-1}x=\theta\Rightarrow x=\cos\theta\Rightarrow dx=-\sin\theta\,d\theta$

$$\Rightarrow\quad I=\int\frac{\cos\theta(\theta)}{\sqrt{1-\cos^2\theta}}(-\sin\theta)d\theta$$
$$\Rightarrow\quad I=-\int\theta\cos\theta d\theta$$
$$\Rightarrow\quad -I=\theta\int\cos\theta\,d\theta-\int\left(\frac{d}{d\theta}\theta\int\cos\theta d\theta\right)d\theta$$

(Applying integration by parts)

$$\Rightarrow\quad -I=\theta\sin\theta-\int\sin\theta d\theta$$
$$\Rightarrow\quad -I=\theta\sin\theta+\cos\theta+C$$
$$\Rightarrow\quad I=-[\theta\sqrt{1-\cos^2\theta}+\cos\theta]+C$$
$$\Rightarrow\quad I=-[\sqrt{1-x^2}\cos^{-1}x+x]+C$$

Key Points

- The value of sin $(-\theta)$ is –ve and value of cos $(-\theta)$ is +ve.

69. Let $I=\int\frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$

$$=\int\frac{\sqrt{x^2+1}\log\left(\frac{x^2+1}{x^2}\right)}{x^4}dx$$

Put $\frac{x^2+1}{x^2}=t\Rightarrow x^2=\frac{1}{t-1}$

$$\Rightarrow\quad 2xdx=\frac{-1}{(t-1)^2}dt\ \Rightarrow\ dx=-\frac{1}{2x}\cdot\frac{1}{(t-1)^2}dt$$
$$=-\frac{1}{2}\cdot\sqrt{t-1}\cdot\frac{1}{(t-1)^2}dt\ =-\frac{dt}{2(t-1)^{3/2}}$$

Also, $\sqrt{x^2+1}=\sqrt{\frac{1}{t-1}+1}=\sqrt{\frac{t}{t-1}}$

$$\therefore\quad I=\int\sqrt{\frac{t}{t-1}}\cdot\log t\cdot\frac{1}{1/(t-1)^2}\times\frac{-dt}{2(t-1)^{3/2}}\ =-\frac{1}{2}\int\sqrt{t}\cdot\log t\,dt$$

On integrating by parts, taking log t as first function, we have

$$=-\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\cdot\log t-\int\frac{t^{3/2}}{3/2}\cdot\frac{1}{t}dt\right]+C$$
$$=-\frac{1}{3}\left[t^{3/2}\log t-\int t^{1/2}dt\right]+C$$
$$=\frac{-1}{3}\left[t^{3/2}\log t-\frac{2}{3}t^{3/2}\right]+C$$
$$=\frac{-1}{3}\left[\left(\frac{x^2+1}{x^2}\right)^{3/2}\log\left(\frac{x^2+1}{x^2}\right)-\frac{2}{3}\left(\frac{x^2+1}{x^2}\right)^{3/2}\right]+C$$

70. Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx, x\in[0,1]$

We know that $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\sqrt{x} = \frac{\pi}{2} - \cos^{-1}\sqrt{x}$

$\therefore \quad I = \int \frac{\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}}{\pi/2} dx$

$= \int 1 \cdot dx - \frac{4}{\pi}\int 1 \cdot \cos^{-1}\sqrt{x}\, dx$

$= x - \frac{4}{\pi}\left[x \cdot \cos^{-1}\sqrt{x} - \int x \cdot \frac{-1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} dx\right] + C$

$= x - \frac{4}{\pi}\left[x\cos^{-1}\sqrt{x} + \frac{1}{2}\int \sqrt{\frac{x}{1-x}}\, dx\right] + C$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$\therefore I = x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \cdot 2\sin\theta\cos\theta d\theta + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int \frac{\sin\theta}{\cos\theta} \cdot 2\sin\theta\cos\theta d\theta + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int (1 - \cos 2\theta) d\theta + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}[\theta - \sin\theta\cos\theta] + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}[\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}] + C$

Concept Applied

- $\cos 2\theta = 1 - 2\sin^2\theta$
  $\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$

71. (d): Let $I = \int_{-1}^{1} \frac{|x-2|}{x-2} dx$

$= \int_{-1}^{1} \frac{-(x-2)}{x-2} dx = \int_{-1}^{1} -1 \cdot dx = [-x]_{-1}^{1} = -[1-(-1)] = -2$

72. (a): Let $I = \int_0^4 (e^{2x} + x)\, dx = \left[\frac{e^{2x}}{2} + \frac{x^2}{2}\right]_0^4$

$= \frac{e^8}{2} + \frac{16}{2} - \frac{e^0}{2} - 0 = \frac{e^8}{2} + \frac{16}{2} - \frac{1}{2} = \frac{e^8 + 15}{2}$

73. (a) : Let $I = \int_0^{\pi/8} \tan^2(2x)\, dx = \int_0^{\pi/8} (\sec^2(2x) - 1)\, dx$

$= \left(\frac{1}{2}\tan 2x - x\right)_0^{\pi/8} = \frac{1}{2}\tan 2\left(\frac{\pi}{8}\right) - \frac{\pi}{8} = \frac{1}{2}\tan\frac{\pi}{4} - \frac{\pi}{8}$

$= \frac{1}{2} - \frac{\pi}{8} = \frac{4-\pi}{8}$

74. (d) : Let $I = \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = [\tan x]_{-\pi/4}^{\pi/4}$

$= \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right) = 1 + 1 = 2$

75. We have, $\int_2^3 3^x\, dx = \left[\frac{3^x}{\log 3}\right]_2^3 = \frac{3^3 - 3^2}{\log 3} = \frac{18}{\log 3}$

76. $\int \frac{dx}{9+x^2} = \int \frac{dx}{x^2+3^2} = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right] = F(x)$

$\therefore$ By second fundamental theorem of integral calculus, we have

$\int_0^3 \frac{dx}{9+x^2} = F(3) - F(0) = \frac{1}{3}[\tan^{-1}1 - \tan^{-1}0] = \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$

Answer Tips

- $\int \frac{1}{a^2+b^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

77. Let $I = \int_0^{\pi/2} e^x(\sin x - \cos x) dx$

$= \underbrace{\int_0^{\pi/2} e^x \sin x\, dx}_{\text{I}} - \underbrace{\int_0^{\pi/2} e^x \cos x\, dx}_{\text{II}}$

On integrating II integral by parts, we get

$I = \int_0^{\pi/2} e^x \sin x\, dx - \left[\{e^x \cos x\}_0^{\pi/2} - \int_0^{\pi/2} e^x(-\sin x) dx\right]$

$= \int_0^{\pi/2} e^x \sin x\, dx - [e^{\pi/2} \cdot 0 - e^0 \cdot 1] - \int_0^{\pi/2} e^x \sin x\, dx = 1$

78. By the first fundamental theorem of integral calculus

$f'(x) = \frac{d}{dx}\int_0^x t\sin t\, dt = x\sin x.$

79. Given, $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$

$\Rightarrow \int_0^a \frac{1}{x^2+2^2} dx = \frac{\pi}{8} \Rightarrow \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$

$\Rightarrow \frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8} \Rightarrow \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$

$\Rightarrow \frac{a}{2} = \tan\frac{\pi}{4} = 1 \Rightarrow a = 2$

80. We have, $\int_0^{\pi/4} \tan x\, dx = [\log|\sec x|]_0^{\pi/4}$

$= \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0| = \log|\sqrt{2}| - \log|1|$

$\therefore \int_0^{\pi/4} \tan x\, dx = \frac{1}{2}\log 2$

81. We have, $\int \sin 2x\, dx = -\frac{1}{2}[\cos 2x] = F(x)$

$\therefore$ By second fundamental theorem of integral calculus, we have $\int_0^{\pi/4} \sin 2x\, dx = F\left(\frac{\pi}{4}\right) - F(0)$

$\Rightarrow \int F(\sin 2x)\, dx = -\frac{1}{2}[\cos(\pi/2) - \cos(0)] = \frac{1}{2}$

82. We have, $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx=[\sin^{-1}x]_0^1$

$=\sin^{-1}1-\sin^{-1}0=\frac{\pi}{2}-0=\frac{\pi}{2}$

83. Here, $\int_1^2 \frac{x^3-1}{x^2}dx=\int_1^2 (x-x^{-2})dx$

$=\left[\frac{x^2}{2}-\frac{x^{-1}}{-1}\right]_1^2=\left[\frac{x^2}{2}+\frac{1}{x}\right]_1^2$

$=\left[\frac{4}{2}+\frac{1}{2}\right]-\left[\frac{1}{2}+1\right]=\frac{5}{2}-\frac{3}{2}=1$

84. $\int_0^1 x^2e^x dx=[x^2e^x]_0^1-2\int_0^1 xe^x dx$

$=e-2[xe^x-e^x]_0^1=e-2(0-(-1))=e-2$

85. Let $I=\int_1^2\left[\frac{1}{x}-\frac{1}{2x^2}\right]e^{2x}dx$

Putting $2x=y \Rightarrow 2dx=dy$
When $x\to 1$, then $y\to 2$
and when $x\to 2$, then $y\to 4$

$\therefore \quad I=\frac{1}{2}\int_2^4\left[\frac{2}{y}-\frac{2}{y^2}\right]e^y dy=\int_2^4\left[\frac{1}{y}-\frac{1}{y^2}\right]e^y dy$

$\Rightarrow \quad I=\left[e^y\cdot\frac{1}{y}\right]_2^4=\frac{1}{4}e^4-\frac{1}{2}e^2=\frac{e^2}{2}\left(\frac{e^2}{2}-1\right)$

86. Let $I=\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx$

$=\int_0^1 \tan^{-1}\left[\frac{(1-x)-x}{1+x(1-x)}\right]dx$

$I=\int_0^1[\tan^{-1}(1-x)-\tan^{-1}x]dx$ ...(i)

$I=\int_0^1[\tan^{-1}x-\tan^{-1}(1-x)]dx$ ...(ii)

$\left[\text{Using property } \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

Adding (i) and (ii), we get

$2I=\int_0^1[\tan^{-1}(1-x)-\tan^{-1}x+\tan^{-1}x-\tan^{-1}(1-x)]dx=0$

$\Rightarrow \quad I=0$

87. Let $I=\int_0^{\pi/2} x^2\sin x\,dx$

On integrating by parts, we have

$I=[x^2(-\cos x)]_0^{\pi/2}-\int_0^{\pi/2}2x(-\cos x)dx$

$=-\frac{\pi^2}{4}.0+0+2\int_0^{\pi/2}x\cos x\,dx=2\int_0^{\pi/2}x\cos x\,dx$

Again integrating by parts, we have

$I=2\left[[x\sin x]_0^{\pi/2}-\int_0^{\pi/2}1\cdot\sin x\,dx\right]$

$=2\left\{\frac{\pi}{2}\cdot 1-0-[-\cos x]_0^{\pi/2}\right\}=2\left[\frac{\pi}{2}+(0-1)\right]=\pi-2$

88. Let $I=\int_2^4\frac{x}{x^2+1}dx$

Put $x^2+1=t \Rightarrow x\,dx=\frac{1}{2}dt$

When $x=2$, then $t=5$ and when $x=4$, then $t=17$

$\therefore \quad I=\frac{1}{2}\int_5^{17}\frac{dt}{t}=\frac{1}{2}[\log t]_5^{17}=\frac{1}{2}[\log 17-\log 5]=\frac{1}{2}\log\left(\frac{17}{5}\right).$

89. Let $I=\int_e^{e^2}\frac{dx}{x\log x}$

Put $\log x=t \Rightarrow \frac{1}{x}dx=dt$

When $x=e$, then $t=\log e=1$
and when $x=e^2$, then $t=\log e^2=2\log e=2$

$\therefore \quad I=\int_1^2\frac{dt}{t}=[\log t]_1^2=\log 2-\log 1=\log 2$

90. Let $I=\int_0^1 xe^{x^2}dx$

Put $x^2=t \Rightarrow x\,dx=\frac{1}{2}dt.$

When $x=0$, then $t=0$ and when $x=1$, then $t=1$.

$\therefore \quad I=\frac{1}{2}\int_0^1 e^t dt\Rightarrow I=\frac{1}{2}[e^t]_0^1=\frac{1}{2}(e-1)$

91. Let $I=\int_0^1\frac{\tan^{-1}x}{1+x^2}dx$

Put $\tan^{-1}x=t \Rightarrow \frac{1}{1+x^2}dx=dt$

When $x=0$, then $t=0$ and when $x=1$, then $t=\frac{\pi}{4}$

$\therefore \quad I=\int_0^{\pi/4}t\,dt=\left[\frac{1}{2}t^2\right]_0^{\pi/4}=\frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2-0\right]=\frac{\pi^2}{32}$

92. Let $I=\int_{-\pi/4}^{0}\frac{(1+\tan x)}{(1-\tan x)}dx=\int_{-\pi/4}^{0}\frac{\left(1+\frac{\sin x}{\cos x}\right)}{\left(1-\frac{\sin x}{\cos x}\right)}dx$

$=\int_{-\pi/4}^{0}\frac{\cos x+\sin x}{\cos x-\sin x}dx$

Put $\cos x-\sin x=t \Rightarrow -(\sin x+\cos x)\,dx=dt$

When $x=0$, then $t=1$, when $x=\frac{-\pi}{4}$, then $t=\sqrt{2}$

$$\therefore\ I=\int_{\sqrt{2}}^{1}-\frac{dt}{t}=\int_{1}^{\sqrt{2}}\frac{dt}{t}=[\log t]_1^{\sqrt{2}}=\log\sqrt{2}-\log 1=\frac{1}{2}\log 2$$

Concept Applied

➲ $\int_b^a f(x)dx=-\int_a^b f(x)dx$

93. Let $I=\int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$

Put $2x=t \Rightarrow dx=\frac{1}{2}dt$

When $x=\frac{\pi}{4},\ t=\frac{\pi}{2}$; When $x=\frac{\pi}{2},\ t=\pi$

$$\therefore\ I=\frac{1}{2}\int_{\pi/2}^{\pi} e^t\left(\frac{1-\sin t}{1-\cos t}\right)dt$$

$$=\frac{1}{2}\int_{\pi/2}^{\pi} e^t\left(\frac{1-2\sin t/2\cos t/2}{2\sin^2 t/2}\right)dt$$

$$=\frac{1}{2}\int_{\pi/2}^{\pi} e^t\left(\frac{1}{2}\operatorname{cosec}^2\frac{t}{2}-\cot\frac{t}{2}\right)dt$$

$$=\frac{1}{2}\int_{\pi/2}^{\pi} e^t\left(-\cot\frac{t}{2}+\frac{1}{2}\operatorname{cosec}^2\frac{t}{2}\right)dt$$

$$\Rightarrow I=\left[\frac{1}{2}e^t\left(-\cot\frac{t}{2}\right)\right]_{\pi/2}^{\pi}\quad\left(\because \int e^x[f(x)+f'(x)]dx=e^x f(x)+C\right)$$

$$\Rightarrow I=\frac{1}{2}\left[e^{\pi}\left(-\cot\frac{\pi}{2}\right)-e^{\pi/2}\left(-\cot\frac{\pi}{4}\right)\right]=\frac{1}{2}(0+e^{\pi/2})=\frac{e^{\pi/2}}{2}$$

94. Let $I=\int_0^{\pi/2}\sqrt{\sin x}\cos^5 x\,dx$

$$\Rightarrow I=\int_0^{\pi/2}\sqrt{\sin x}\,(\cos^2 x)^2\cos x\,dx$$

$$\Rightarrow I=\int_0^{\pi/2}\sqrt{\sin x}\,(1-\sin^2 x)^2\cos x\,dx$$

Put $\sin x=t \Rightarrow \cos x\,dx=dt$

When $x=0$, $t=\sin 0=0$

When $x=\pi/2$, $t=\sin\pi/2=1$

$$\therefore\ I=\int_0^1\sqrt{t}\,(1-t^2)^2\,dt=\int_0^1\sqrt{t}\,(1+t^4-2t^2)\,dt$$

$$=\int_0^1(\sqrt{t}+t^{9/2}-2t^{5/2})\,dt=\left[\frac{t^{3/2}}{3/2}+\frac{t^{11/2}}{11/2}-\frac{2t^{7/2}}{7/2}\right]_0^1$$

$$=\frac{2}{3}+\frac{2}{11}-\frac{4}{7}=\frac{154+42-132}{231}=\frac{64}{231}$$

95. Let $I=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{1-(1-\sin 2x)}}dx$

$$=\int_{\pi/6}^{\pi/3}\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Put $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)dx=dt$

When $x=\frac{\pi}{3}$, then $t=\frac{\sqrt{3}-1}{2}=\alpha$

and when, $x=\frac{\pi}{6}$, then $t=\frac{1-\sqrt{3}}{2}=-\alpha$

$$\therefore\ I=\int_{-\alpha}^{\alpha}\frac{dt}{\sqrt{1-t^2}}=[\sin^{-1}t]_{-\alpha}^{\alpha}=2\sin^{-1}\alpha=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

96. Let $I=\int_0^{\pi}e^{2x}\cdot\sin\left(\frac{\pi}{4}+x\right)dx$

Put $\frac{\pi}{4}+x=t \Rightarrow x=t-\frac{\pi}{4}\quad\Rightarrow dx=dt$

When $x=0$, $t=\frac{\pi}{4}$ and when $x=\pi$, $t=\frac{5\pi}{4}$

$$\therefore\ I=\int_{\pi/4}^{5\pi/4}e^{2\left(t-\frac{\pi}{4}\right)}\sin t\,dt=e^{-\pi/2}\int_{\pi/4}^{5\pi/4}e^{2t}\sin t\,dt$$

$$=e^{-\pi/2}\left[\left(\sin t\frac{e^{2t}}{2}\right)_{\pi/4}^{5\pi/4}-\int_{\pi/4}^{5\pi/4}\cos t\frac{e^{2t}}{2}dt\right]$$

$$=e^{-\pi/2}\left[\frac{1}{2}\left(e^{5\pi/2}\sin\frac{5\pi}{4}-e^{\pi/2}\sin\frac{\pi}{4}\right)-\left(\frac{e^{2t}}{4}\cos t\right)_{\pi/4}^{5\pi/4}-\int_{\pi/4}^{5\pi/4}\frac{e^{2t}}{4}\sin t\,dt\right]$$

$$=e^{-\pi/2}\left[\frac{1}{2}\left(\frac{-1}{\sqrt{2}}e^{5\pi/2}-\frac{1}{\sqrt{2}}e^{\pi/2}\right)-\frac{1}{4}\left(-\frac{1}{\sqrt{2}}e^{5\pi/2}-\frac{1}{\sqrt{2}}e^{\pi/2}\right)\right]-\frac{I}{4}$$

$$\Rightarrow\quad I+\frac{1}{4}I=-\frac{1}{2\sqrt{2}}[e^{2\pi}+1]+\frac{1}{4\sqrt{2}}[e^{2\pi}+1]$$

$$\Rightarrow\quad \frac{5}{4}I=\frac{(e^{2\pi}+1)}{2\sqrt{2}}\left[\frac{1}{2}-1\right]=-\frac{1}{4\sqrt{2}}[e^{2\pi}+1]\Rightarrow I=\frac{-1}{5\sqrt{2}}(1+e^{2\pi})$$

97. Let $I=\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}=\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\cdot 2\sin x\cos x}}$

$$=\frac{1}{2}\int\frac{dx}{\cos^{\left(3+\frac{1}{2}\right)}x\cdot\sin^{\frac{1}{2}}x}=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^{\frac{7}{2}}x\cdot\tan^{\frac{1}{2}}x\cdot\cos^{\frac{1}{2}}x}$$

$$=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^4 x\sqrt{\tan x}}=\frac{1}{2}\int_0^{\pi/4}\frac{\sec^4 x}{\sqrt{\tan x}}dx$$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

When $x=0$, then $t=0$ and when $x=\frac{\pi}{4}$, then $t=1$

$$\therefore\ I=\frac{1}{2}\int_0^1\frac{(1+t^2)dt}{\sqrt{t}}=\frac{1}{2}\int_0^1(t^{-\frac{1}{2}}+t^{\frac{3}{2}})\,dt$$

$$=\frac{1}{2}\left[\frac{t^{1/2}}{1/2}+\frac{t^{5/2}}{5/2}\right]_0^1=\frac{1}{2}\left[2+\frac{2}{5}\right]=1+\frac{1}{5}=\frac{6}{5}$$

98. Let $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x)\,dx = dt$

and $1 - 2\sin x\cos x = t^2 \Rightarrow 1 - \sin 2x = t^2$

When $x=\frac{\pi}{4}, t=\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$

When $x = 0$, $t = \sin 0 - \cos 0 = -1$

$\therefore \int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}dx=\int_{-1}^{0}\frac{dt}{16+9(1-t^2)}$

$=\int_{-1}^{0}\frac{dt}{25-9t^2}=\frac{1}{9}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{3}\right)^2-t^2}=\frac{1}{9}\cdot\frac{1}{2\times\frac{5}{3}}\left[\log\left|\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right|\right]_{-1}^{0}$

$=\frac{1}{30}[\log 1-(\log 1-\log 4)]=\frac{1}{30}\log 4$

Key Points

➲ $\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

99. Let $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x)\,dx = dt$

Also, $1 - 2\sin x\cos x = t^2 == 1 - \sin 2x = t^2$

When $x=\frac{\pi}{4}$, $t = 0$ and when $x = 0$, $t = -1$

$\therefore I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}$

$=\int_{-1}^{0}\frac{dt}{25-16t^2}=\frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$

$=\frac{1}{16}\cdot\frac{4}{2\times 5}\left[\log\left|\frac{5/4+t}{5/4-t}\right|\right]_{-1}^{0}=\frac{1}{40}\left[\log 1-\log\left(\frac{1}{9}\right)\right]$

$=\frac{1}{40}[\log 1-\log 1+\log 9]=\frac{1}{40}\log 9$

100. (a): Let $I=\int_2^8\frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$ ... (i)

$=\int_2^8\frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}dx\quad\left(\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$

$=\int_2^8\frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx$ ... (ii)

Adding (i) and (ii), we get

$$2I=\int_2^8\frac{\sqrt{10-x}+\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}dx=\int_2^8 1\,dx=[x]_2^8$$

$\Rightarrow I=\frac{1}{2}(8-2)=\frac{6}{2}=3$

Hence, both assertion and reason are true and reason is the correct explanation of assertion.

101. Let $I=\int_0^{\pi/2}\frac{1}{1+\cot^{5/2}x}dx$

$\Rightarrow I=\int_0^{\pi/2}\frac{\sin^{5/2}x}{\sin^{5/2}x+\cos^{5/2}x}dx$

$\Rightarrow I=\int_0^{\pi/2}\frac{\sin^{5/2}\left(\frac{\pi}{2}-x\right)}{\sin^{5/2}\left(\frac{\pi}{2}-x\right)+\cos^{5/2}\left(\frac{\pi}{2}-x\right)}dx$ ...(i)

$\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$\Rightarrow I=\int_0^{\pi/2}\frac{\cos^{5/2}x}{\cos^{5/2}x+\sin^{5/2}x}dx$ ...(ii)

Adding (i) and (ii), we get

$2I=\int_0^{\pi/2}\frac{\sin^{5/2}x+\cos^{5/2}x}{\sin^{5/2}x+\cos^{5/2}x}dx$

$\Rightarrow 2I=\int_0^{\pi/2}1.dx=(x)_0^{\pi/2}=\frac{\pi}{2}\Rightarrow 2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$

102. Let $I=\int_1^3|2x-1|dx$

$=\int_1^3(2x-1)dx=\left[\frac{2x^2}{2}-x\right]_1^3$

$= [(3^2 - 3) - (1^2 - 1)] = [(9 - 3) - (1 - 1)] = 6$

103. Let $I=\int_{-2}^{2}|x|dx$

$\therefore I=\int_{-2}^{0}(-x)dx+\int_0^2 x\,dx=\left[-\frac{x^2}{2}\right]_{-2}^{0}+\left[\frac{x^2}{2}\right]_0^2$

$= 2 + 2 = 4$

104. Let $I=\int_1^4|x-5|\,dx$

$=-\int_1^4(x-5)\,dx=\left[-\frac{x^2}{2}+5x\right]_1^4$

$=-\frac{16}{2}+5(4)+\frac{1}{2}-5=-8+20-5+\frac{1}{2}=7+\frac{1}{2}=\frac{15}{2}$

105. Let $I=\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ ...(i)

$\Rightarrow I=\int_0^{\pi/2}\log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$

$\left[\text{Using property }\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$\Rightarrow I=\int_0^{\pi/2}\log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx+\int_0^{\pi/2}\log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$=\int_0^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)\cdot\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$=\int_0^{\pi/2}\log 1\,dx=0\Rightarrow I=0$$

106. Let $I=\int_0^{2\pi}\frac{1}{1+e^{\sin x}}dx$ ...(i)

$$\Rightarrow\quad I=\int_0^{2\pi}\frac{1}{1+e^{\sin(2\pi-x)}}dx\qquad\left[\because\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{2\pi}\frac{1}{1+e^{-\sin x}}dx$$

$$=\int_0^{2\pi}\frac{e^{\sin x}}{e^{\sin x}+1}dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{2\pi}\left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx=\int_0^{2\pi}1\cdot dx$$

$$\Rightarrow\quad I=\frac{1}{2}[x]_0^{2\pi}=\frac{1}{2}\times 2\pi=\pi$$

107. Let $I=\int_{-2}^{2}\frac{x^2}{1+5^x}dx$ ...(i)

Using property : $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$\therefore\quad I=\int_{-2}^{2}\frac{(-2+2-x)^2}{1+5^{-2+2-x}}dx\ \Rightarrow\ I=\int_{-2}^{2}\frac{x^2}{1+5^{-x}}dx$$

$$\Rightarrow\quad I=\int_{-2}^{2}\frac{5^x\cdot x^2}{5^x+1}dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_{-2}^{2}\frac{5^x x^2+x^2}{5^x+1}dx$$

$$\Rightarrow\quad 2I=\int_{-2}^{2}x^2dx=\left[\frac{x^3}{3}\right]_{-2}^{2}\ \Rightarrow\ I=\frac{1}{6}(8+8)=\frac{16}{6}=\frac{8}{3}$$

108. Let $I=\int_1^4\{|x|+|3-x|\}dx$

$$|x|+|3-x|=\begin{cases}x+3-x, & 1\le x<2\\ x+3-x, & 2\le x<3\\ x+x-3, & 3\le x<4\end{cases}=\begin{cases}3, & 1\le x<3\\ 2x-3, & 3\le x<4\end{cases}$$

$$\therefore\quad I=\int_1^3 3dx+\int_3^4(2x-3)dx\ =[3x]_1^3+\left[\frac{2x^2}{2}-3x\right]_3^4$$

$= (9-3)+(16-12-9+9)$

$= 6+4=10$

109. Let $I=\int_1^3\frac{\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}dx$ ...(i)

$$I=\int_1^3\frac{\sqrt{4-x}}{\sqrt{4-x}+\sqrt{x}}dx\ \text{...(ii)}\qquad\left[\because\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

Adding (i) and (ii), we get

$$2I=\int_1^3\frac{\sqrt{x}+\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}dx$$

$$\Rightarrow\quad 2I=\int_1^3 1\,dx\ \Rightarrow\ 2I=[x]_1^3$$

$\Rightarrow\quad 2I=2\Rightarrow I=1$

110. Let $I=\int_0^{\pi}\frac{x}{9\sin^2 x+16\cos^2 x}dx$ ...(i)

$$\Rightarrow\ I=\int_0^{\pi}\frac{(\pi-x)dx}{9\sin^2(\pi-x)+16\cos^2(\pi-x)}$$

$$\left(\text{Using}\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow\ I=\int_0^{\pi}\frac{(\pi-x)dx}{9\sin^2 x+16\cos^2 x}\qquad\text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{\pi dx}{9\sin^2 x+16\cos^2 x}$$

Consider $f(x)=\frac{1}{9\sin^2 x+16\cos^2 x}$

$$f(\pi-x)=\frac{1}{9\sin^2(\pi-x)+16\cos^2(\pi-x)}$$

$$=\frac{1}{9\sin^2 x+16\cos^2 x}=f(x)$$

$$\therefore\ I=\pi\int_0^{\pi/2}\frac{dx}{9\sin^2 x+16\cos^2 x}$$

$$\left(\text{Using}\int_0^{2a}f(x)dx=2\int_0^a f(x)dx,\text{if } f(2a-x)=f(x)\right)$$

$$\Rightarrow\ I=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{9\tan^2 x+16}\quad(\text{Dividing } N^r\ \&\ D^r \text{ by } \cos^2 x)$$

Put $\tan x=t\Rightarrow\sec^2 x\,dx=dt$

When $x=0,\ t=0;\ x=\pi/2,\ t=\infty$

$$\Rightarrow\ I=\pi\int_0^{\infty}\frac{dt}{9t^2+16}=\frac{\pi}{9}\int_0^{\infty}\frac{dt}{t^2+\frac{16}{9}}$$

$$=\frac{\pi}{9}\cdot\frac{3}{4}\left[\tan^{-1}\frac{3t}{4}\right]_0^{\infty}=\frac{\pi}{12}\left(\tan^{-1}\infty-\tan^{-1}0\right)=\frac{\pi^2}{24}$$

111. Let $I=\int_{-1}^{2}|x^3-x|dx=\int_{-1}^{2}|x(x-1)(x+1)|dx$

$$=\int_{-1}^{0}(x^3-x)dx-\int_0^1(x^3-x)dx+\int_1^2(x^3-x)dx$$

$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}-\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_0^1+\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2=\frac{-3}{4}+\frac{3}{2}+2=\frac{11}{4}$$

112. In the R.H.S. integral, put $(a - x) = t$, so that $dx = -dt$.
Now, when $x = 0$, then $t = a$
and when $x = a$, then $t = 0$

$$\therefore \int_0^a f(a-x)dx = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx$$

Hence, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ ...(i)

Let $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$ ... (ii)

$$= \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+[\cos(\pi-x)]^2}dx \quad \text{[Using (i)]}$$

$$\Rightarrow \quad I = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx \quad \text{...(iii)}$$

Adding (ii) and (iii), we get

$$2I = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$
When $x = 0$, then $t = 1$ and when $x = \pi$, then $t = -1$

$$\therefore\ 2I = \int_1^{-1} \frac{-\pi dt}{1+t^2} \Rightarrow I = \frac{\pi}{2}\int_{-1}^1 \frac{dt}{1+t^2}$$

$$\therefore\ I = \frac{\pi}{2}[\tan^{-1} t]_{-1}^1 = \frac{\pi}{2}[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}.$$

113. In the R.H.S. integral, put $(a - x) = t$, so that $dx = -dt$.
Now, when $x = 0$, then $t = a$
and when $x = a$, then $t = 0$

$$\therefore \int_0^a f(a-x)dx = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx$$

Hence, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ ...(i)

Now, $\int_0^1 x^2(1-x)^n dx = \int_0^1 (1-x)^2(1-(1-x))^n dx$ [using (i)]

$$= \int_0^1 (1-x)^2 x^n dx = \int_0^1 (1+x^2-2x)x^n dx = \int_0^1 (x^n + x^{n+2} - 2x^{n+1})dx$$

$$= \int_0^1 x^n dx + \int_0^1 x^{n+2}dx - 2\int_0^1 x^{n+1}dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 + \left[\frac{x^{n+3}}{n+3}\right]_0^1 - 2\left[\frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \frac{1}{n+1} + \frac{1}{n+3} - \frac{2}{n+2}.$$

114. Let $I = \int_0^{3/2} |x\sin\pi x|dx$

When $0 < x < 1 \Rightarrow 0 < \pi x < \pi \Rightarrow \sin\pi x > 0$

When $1 < x < \frac{3}{2} \Rightarrow \pi < \pi x < \frac{3\pi}{2} \Rightarrow \sin\pi x < 0$

$$\therefore\ |x\sin\pi x| = \begin{cases} x\sin\pi x, \text{ if } 0 < x < 1 \\ -x\sin\pi x, \text{ if } 1 < x < \frac{3}{2} \end{cases}$$

$$\therefore\ I = \int_0^1 x\sin\pi x\, dx - \int_1^{3/2} x\sin\pi x\, dx$$

$$= \left[\frac{-x\cos\pi x}{\pi}\right]_0^1 + \int_0^1 \frac{\cos\pi x}{\pi}dx + \left[\frac{x\cos\pi x}{\pi}\right]_1^{3/2} - \int_1^{3/2}\frac{\cos\pi x}{\pi}dx$$

$$= \frac{1}{\pi} - 0 + \left[\frac{\sin\pi x}{\pi^2}\right]_0^1 + 0 - \frac{(-1)}{\pi} - \left[\frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$= \frac{2}{\pi} + 0 - \frac{(-1)}{\pi^2} + 0 = \frac{2}{\pi} + \frac{1}{\pi^2}$$

115. Let $I = \int_0^\pi \frac{x\tan x}{\sec x + \tan x}dx$

$$= \int_0^\pi \frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}dx = \int_0^\pi \frac{x\sin x}{1+\sin x}dx \quad \text{...(i)}$$

$$= \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx$$

$$\left[\text{Using the property } \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^\pi \frac{(\pi-x)\sin x}{1+\sin x}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^\pi \frac{\pi\sin x}{1+\sin x}dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\sin x}dx = \frac{\pi}{2}\int_0^\pi \frac{1+\sin x - 1}{1+\sin x}dx$$

$$= \frac{\pi}{2}\int_0^\pi \left(1 - \frac{1}{1+\sin x}\right)dx$$

$$= \frac{\pi}{2}\left[\int_0^\pi dx - \int_0^\pi \frac{1}{1+\sin x}dx\right] = \frac{\pi}{2}\left[[x]_0^\pi - \int_0^\pi \frac{1-\sin x}{\cos^2 x}dx\right]$$

$$= \frac{\pi}{2}\left[[x]_0^\pi - \int_0^\pi \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx\right]$$

$$= \frac{\pi}{2}\left[(\pi - 0) - \int_0^\pi (\sec^2 x - \tan x \cdot \sec x)dx\right]$$

$$= \frac{\pi}{2}\left[\pi - (\tan x - \sec x)_0^\pi\right] = \frac{\pi}{2}[\pi - [(0-(-1)) - (0-1)]] = \frac{\pi}{2}[\pi - 2]$$

116. Let $I = \int_1^4 (|x-1| + |x-2| + |x-4|)dx$

Also, let $f(x) = |x - 1| + |x - 2| + |x - 4|$
We have three critical points $x = 1, 2$ and 4.

$$f(x) = \begin{cases} (x-1) - (x-2) - (x-4) \text{ if } 1 \le x < 2 \\ (x-1) + (x-2) - (x-4) \text{ if } 2 \le x < 4 \end{cases}$$

$$\therefore\ f(x) = \begin{cases} -x+5 \text{ if } 1 \le x < 2 \\ x+1 \text{ if } 2 \le x < 4 \end{cases}$$

$$\therefore\ I = \int_1^4 f(x)dx = \int_1^2 f(x)dx + \int_2^4 f(x)dx$$

$$=\int_1^2(-x+5)dx+\int_2^4(x+1)dx=\left[-\frac{x^2}{2}+5x\right]_1^2+\left[\frac{x^2}{2}+x\right]_2^4$$

$$=\left(-\frac{4}{2}+10\right)-\left(-\frac{1}{2}+5\right)+\left(\frac{16}{2}+4\right)-\left(\frac{4}{2}+2\right)$$

$$=8-\frac{9}{2}+12-4=16-\frac{9}{2}=\frac{23}{2}$$

117. Let $I=\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$ ...(i)

$$\Rightarrow I=\int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi/2}\left(\frac{\sin^2 x}{\sin x+\cos x}+\frac{\cos^2 x}{\sin x+\cos x}\right)dx \Rightarrow 2I=\int_0^{\pi/2}\frac{1}{\sin x+\cos x}dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\frac{1}{\frac{2\tan(x/2)}{1+\tan^2(x/2)}+\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\frac{1+\tan^2(x/2)}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx=dt$

When $x = 0$, then $t = 0$ and when $x=\frac{\pi}{2}$, then $t = 1$

$$\therefore 2I=\int_0^1\frac{2dt}{2t+1-t^2}=2\int_0^1\frac{1}{(\sqrt{2})^2-(t-1)^2}dt$$

$$\Rightarrow 2I=2\times\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\left\{0-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}\right\}$$

$$\Rightarrow 2I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2=\frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

Key Points

- $\sin 2A=\frac{2\tan A}{1+\tan^2 A}$ and $\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$

118. Let $I=\int_0^{3/2}|x\cos\pi x|dx$

When $0<x<\frac{1}{2} \Rightarrow 0<\pi x<\frac{\pi}{2} \Rightarrow \cos\pi x>0$

When $\frac{1}{2}<x<\frac{3}{2} \Rightarrow \frac{\pi}{2}<\pi x<\frac{3\pi}{2} \Rightarrow \cos\pi x<0$

$$\therefore |x\cos\pi x|=\begin{cases}x\cos\pi x, \text{if } 0<x<\frac{1}{2}\\ -x\cos\pi x, \text{if } \frac{1}{2}<x<\frac{3}{2}\end{cases}$$

$$\therefore I=\int_0^{1/2}x\cos\pi x\,dx+\int_{1/2}^{3/2}-(x\cos\pi x)dx$$

$$\Rightarrow I=\left[\frac{x}{\pi}\sin\pi x+\frac{\cos\pi x}{\pi^2}\right]_0^{1/2}-\left[\frac{x}{\pi}\sin\pi x+\frac{\cos\pi x}{\pi^2}\right]_{1/2}^{3/2}$$

(Applying integration by parts)

$$=\left[\frac{1}{\pi}\left(\frac{1}{2}-0\right)+\frac{1}{\pi^2}(0-1)\right]-\left[\frac{1}{\pi}\left(\frac{3}{2}(-1)-\frac{1}{2}(1)\right)+\frac{1}{\pi^2}(0-0)\right]$$

$$=\left(\frac{1}{2\pi}-\frac{1}{\pi^2}\right)-\left(\frac{-2}{\pi}\right)=\left(\frac{5\pi-2}{2\pi^2}\right)$$

119. Let $I=\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi-x}{1+\sin\alpha\sin(\pi-x)}dx \quad \left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx-\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx-I \Rightarrow 2I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{1}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{1}{1+\sin\alpha\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{1+\tan^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

$$\therefore I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \sec^2\frac{x}{2}dx=2dt$

Also, when $x\to 0, t\to \tan 0 = 0$;

when $x\to\pi, t\to\tan\frac{\pi}{2}=\infty$

$$\therefore I=\frac{\pi}{2}\int_0^{\infty}\frac{2dt}{t^2+2t\sin\alpha+1}$$

$\Rightarrow \quad I=\pi\int_0^{\infty}\frac{1}{(t+\sin\alpha)^2+\cos^2\alpha}dt$

$\Rightarrow \quad I=\frac{\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{t+\sin\alpha}{\cos\alpha}\right)\right]_0^{\infty}$

$\Rightarrow \quad I=\frac{\pi}{\cos\alpha}[\tan^{-1}\infty-\tan^{-1}(\tan\alpha)] \Rightarrow I=\frac{\pi}{\cos\alpha}\left(\frac{\pi}{2}-\alpha\right)$

120. Let $I=\int_{-\pi}^{\pi}(\cos ax-\sin bx)^2\,dx$

$=\int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx-2\cos ax\sin bx)dx$

$=\int_{-\pi}^{\pi}\cos^2 ax\,dx+\int_{-\pi}^{\pi}\sin^2 bx\,dx-2\int_{-\pi}^{\pi}\cos ax\sin bx\,dx$

$=2\left[\int_0^{\pi}\cos^2 ax\,dx+\int_0^{\pi}\sin^2 bx\,dx\right]$

$\left[\because \int_{-a}^{a}f(x)dx=\begin{cases}0, \text{ if } f(x) \text{ is odd}\\ 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even}\end{cases}\right]$

$=2\left[\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}\right)dx+\int_0^{\pi}\left(\frac{1-\cos 2bx}{2}\right)dx\right]$

$=\int_0^{\pi}(1+\cos 2ax)dx+\int_0^{\pi}(1-\cos 2bx)dx$

$=2\int_0^{\pi}1\cdot dx+\int_0^{\pi}\cos 2a\,x\,dx-\int_0^{\pi}\cos 2b\,x\,dx$

$=(2x)_0^{\pi}+\frac{1}{2a}(\sin 2ax)_0^{\pi}-\frac{1}{2b}(\sin 2bx)_0^{\pi}$

$=2\pi+\frac{\sin 2a\pi}{2a}-\frac{\sin 2b\pi}{2b}$

121. Let $I=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{e^{-x}+1}e^{-x}dx$

$=\int_{-\pi/2}^{\pi/2}\frac{\cos x(e^{-x}+1-1)}{e^{-x}+1}dx=\int_{-\pi/2}^{\pi/2}\cos x\,dx-\int_{-\pi/2}^{\pi/2}\frac{\cos x}{e^{-x}+1}dx$

Now, put $x=-z$ in 2nd integral,

$\therefore \quad dx=-dz$

Also,when $x=\frac{-\pi}{2}$, then $z=\frac{\pi}{2}$ and when $x=\frac{\pi}{2}$, then $z=\frac{-\pi}{2}$

$\therefore \quad I=\int_{-\pi/2}^{\pi/2}\cos x\,dx+\int_{\pi/2}^{-\pi/2}\frac{\cos z}{e^z+1}dz$

$\Rightarrow \quad I=[\sin x]_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2}\frac{\cos x}{e^x+1}dx$

$\Rightarrow \quad I=\left[\sin\frac{\pi}{2}+\sin\frac{\pi}{2}\right]-I \Rightarrow 2I=2 \Rightarrow I=1$

122. Let $I=\int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}}=\int_0^{\pi/2}\frac{dx}{1+\sqrt{\frac{\sin x}{\cos x}}}$

$\Rightarrow \quad I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ ...(i)

By the property, $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$I=\int_0^{\pi/2}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx$

$=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ...(ii)

Adding (i) and (ii), we get

$2I=\int_0^{\pi/2}\left[\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}+\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right]dx$

$=\int_0^{\pi/2}1\cdot dx=[x]_0^{\pi/2}=\frac{\pi}{2} \quad\Rightarrow\quad I=\frac{\pi}{4}$

123. Let $I=\int_0^{\pi/4}\log(1+\tan x)dx$

By using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$I=\int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$

$=\int_0^{\pi/4}\log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$

$=\int_0^{\pi/4}[\log 2-\log(1+\tan x)]dx$

$=\log 2\int_0^{\pi/4}1\cdot dx-\int_0^{\pi/4}\log(1+\tan x)dx \;=\log 2[x]_0^{\pi/4}-I$

$\Rightarrow \quad 2I=\log 2\left[\frac{\pi}{4}-0\right]\Rightarrow I=\frac{\pi}{8}\log 2$

Key Points

➲ $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

124. Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x\operatorname{cosec} x}dx$ ...(i)

Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)\operatorname{cosec}(\pi-x)}dx=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x\operatorname{cosec} x}$ ...(ii)

Adding (i) and (ii), we get

$2I=\int_0^{\pi}\left[\frac{x\tan x}{\sec x\operatorname{cosec} x}+\frac{(\pi-x)\tan x}{\sec x\operatorname{cosec} x}\right]dx$

$=\pi\int_0^{\pi}\frac{\tan x}{\sec x\operatorname{cosec} x}dx=\pi\int_0^{\pi}\frac{\sin x/\cos x}{\frac{1}{\cos x}\cdot\frac{1}{\sin x}}dx$

$$=\pi\int_0^{\pi}\sin^2 x\,dx=\pi\int_0^{\pi}\frac{1-\cos 2x}{2}dx=\frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{\pi^2}{2}\Rightarrow I=\frac{\pi^2}{4}$$

125. Let $I=\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}dx$ ... (i)

$I=\int_0^{\pi}\frac{4(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$ $\left[\text{By using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$\Rightarrow\quad I=\int_0^{\pi}\frac{4(\pi-x)\sin x}{1+\cos^2 x}dx$ ... (ii)

Adding (i) and (ii), we get

Hence, $2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

Put $\cos x=t\Rightarrow\sin x\,dx=-dt$

Also, when $x=0$, then $t=1$ and when $x=\pi$, when $t=-1$

$$\therefore\quad I=2\pi\int_1^{-1}\frac{-dt}{1+t^2}=2\pi\int_{-1}^{1}\frac{dt}{t^2+1^2}=2\pi[\tan^{-1}t]_{-1}^{1}$$

$$=2\pi[\tan^{-1}1-\tan^{-1}(-1)]=2\pi\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\pi^2$$

126. Let $I=\int_0^{\pi/2}\frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ ...(i)

$$\therefore\quad I=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\sin x\cos x\,dx}{\cos^4 x+\sin^4 x}$ ...(ii)

Adding (i) and (ii), we get

$$2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cos x\,dx}{\sin^4 x+\cos^4 x}$$

Dividing numerator and denominator by $\cos^4 x$, we get

$$2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\tan x\sec^2 x\,dx}{1+\tan^4 x}$$

Put $\tan^2 x=t\Rightarrow 2\tan x\sec^2 x\,dx=dt$

When $x=0$, then $t=0$ and when $x=\frac{\pi}{2}$, then $t=\infty$

$$\therefore\quad I=\frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2}=\frac{\pi}{8}[\tan^{-1}t]_0^{\infty}=\frac{\pi}{8}\times\frac{\pi}{2}=\frac{\pi^2}{16}$$

127. Let $I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ...(i)

$$\Rightarrow\quad I=\int_{\pi/6}^{\pi/3}\frac{\left[\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)\right]^{1/2}}{[\sin(\pi/3+\pi/6-x)]^{1/2}+[\cos(\pi/3+\pi/6-x)]^{1/2}}$$

$$\Rightarrow\quad I=\int_{\pi/6}^{\pi/3}\frac{[\sin(\pi/2-x)]^{1/2}}{[\sin(\pi/2-x)]^{1/2}+[\cos(\pi/2-x)]^{1/2}}dx$$

$\Rightarrow\quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_{\pi/6}^{\pi/3}\left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\cos x}\right)dx$$

$$\Rightarrow\quad 2I=\int_{\pi/6}^{\pi/3}1\cdot dx=[x]_{\pi/6}^{\pi/3}\Rightarrow I=\frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]\Rightarrow I=\frac{\pi}{12}$$

Concept Applied

➲ $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

128. Let $I=\int_0^{\pi}\frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$ ...(i)

$$\Rightarrow\quad I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$$

$\left[\text{Using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$\Rightarrow\quad I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2 x+b^2\sin^2 x}$ ...(ii)

Adding (i) and (ii), we get $I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$

Let $f(x)=\frac{1}{a^2\cos^2 x+b^2\sin^2 x}$

$$\Rightarrow\quad f(\pi-x)=\frac{1}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$$

$$\Rightarrow\quad f(\pi-x)=\frac{1}{a^2\cos^2 x+b^2\sin^2 x}=f(x)$$

$\left[\therefore\text{ By using }\int_0^{2a}f(x)dx=2\int_0^a f(x)dx,\text{ if }f(2a-x)=f(x)\right]$

$$\therefore\quad I=\frac{\pi}{2}\left(2\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}\right)\Rightarrow I=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}$$

Put $\tan x=t\Rightarrow\sec^2 x\,dx=dt$.

Also, when $x=0$, then $t=\tan 0=0$.

and when $x=\frac{\pi}{2}$, then $t=\tan\frac{\pi}{2}=\infty$

$$\therefore\quad I=\int_0^{\infty}\frac{dt}{a^2+b^2t^2}\Rightarrow I=\frac{\pi}{b^2}\int_0^{\infty}\frac{dt}{\left(\frac{a}{b}\right)^2+t^2}$$

$$\Rightarrow\quad I=\frac{\pi}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}\Rightarrow I=\frac{\pi}{ab}[\tan^{-1}\infty-\tan^{-1}0]=\frac{\pi^2}{2ab}$$

CBSE Sample Questions

1. Let $I=\int\frac{1}{\cos^2 x(1-\tan x)^2}dx$

Put $1-\tan x=t\Rightarrow -\sec^2 x\,dx=dt$ **(1)**

$\therefore\ I=-\int\frac{dt}{t^2}=-\int t^{-2}\,dt=\frac{1}{t}+C=\frac{1}{1-\tan x}+C$ **(1)**

2. Let $I=\int\frac{\sin 2x}{\sqrt{9-\cos^4 x}}dx$

Put $\cos^2 x = t$

$\Rightarrow$ $-2\cos x \sin x\, dx = dt \Rightarrow \sin 2x\, dx = -dt$ **(1)**

$\therefore \quad I = -\int \frac{dt}{\sqrt{3^2 - t^2}} = -\sin^{-1}\frac{t}{3} + c = -\sin^{-1}\left(\frac{\cos^2 x}{3}\right) + c$ **(1)**

3. Let $\frac{x+1}{(x^2+1)x} = \frac{Ax+B}{x^2+1} + \frac{C}{x} = \frac{(Ax+B)x + C(x^2+1)}{(x^2+1)x}$ **(1/2)**

$\Rightarrow$ $x + 1 = (Ax + B)x + C(x^2 + 1)$

By equating the like coefficients, we get

$B = 1, C = 1, A + C = 0$

Hence, $A = -1, B = 1$ and $C = 1$ **(1/2)**

$\therefore$ The given integral $= \int \frac{-x+1}{x^2+1}dx + \int \frac{1}{x}dx$

$= \frac{-1}{2}\int \frac{2x-2}{x^2+1}dx + \int \frac{1}{x}dx \quad = \frac{-1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx + \int \frac{1}{x}dx$ **(1/2)**

$= \frac{-1}{2}\log(x^2+1) + \tan^{-1}x + \log|x| + c$ **(1/2)**

4. Let $I = \int \frac{(x^3+x+1)}{(x^2-1)}dx = \int \left(x + \frac{2x+1}{(x-1)(x+1)}\right)dx$ **(1/2)**

Now resolving $\frac{2x+1}{(x-1)(x+1)}$ into partial fractions as

$\frac{2x+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{x(A+B)+(A-B)}{(x-1)(x+1)}$ **(1/2)**

Calculating A and B we get,

$\frac{2x+1}{(x-1)(x+1)} = \frac{3}{2(x-1)} + \frac{1}{2(x+1)}$

Now, $I = \int \frac{(x^3+x+1)}{(x^2-1)}dx = \int \left(x + \frac{2x+1}{(x-1)(x+1)}\right)dx$

$= \int \left(x + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}\right)dx$ **(1)**

$= \frac{x^2}{2} + \frac{3}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C$

$= \frac{x^2}{2} + \frac{1}{2}\log|(x-1)^3(x+1)| + C$ **(1)**

Concept Applied

➲ $\frac{px+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$

5. We have, $\int_0^4 |x-1|dx = \int_0^1 (1-x)dx + \int_1^4 (x-1)dx$ **(1)**

$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$ **(1)**

$= \left(1 - \frac{1}{2}\right) + (8-4) - \left(\frac{1}{2} - 1\right) = \frac{1}{2} + 4 + \frac{1}{2} = 5$ **(1)**

6. Let $I = \int \frac{x^2+1}{(x^2+2)(x^2+3)}dx$

Now, put $x^2 = y$ to make partial fractions.

i.e., $\frac{x^2+1}{(x^2+2)(x^2+3)} = \frac{y+1}{(y+2)(y+3)} = \frac{A}{y+2} + \frac{B}{y+3}$ **(1/2)**

$\Rightarrow$ $y + 1 = A(y + 3) + B(y + 2)$ ... (i) **(1/2)**

Comparing coefficients of y and constant terms on both sides of (i), we get

$A + B = 1$ and $3A + 2B = 1$

Solving, we get $A = -1, B = 2$ **(1)**

$\therefore \quad I = \int \frac{x^2+1}{(x^2+2)(x^2+3)}dx = \int \frac{-1}{x^2+2}dx + 2\int \frac{1}{x^2+3}dx$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$ **(1)**

7. Let $I = \int e^x(1 - \cot x + \operatorname{cosec}^2 x)dx$

$= \int e^x dx + \int e^x((-\cot x) + \operatorname{cosec}^2 x)dx$

$= e^x + e^x(-\cot x) + C$

$= e^x(1 - \cot x) + C$ **(1)**

8. We have, $\int \frac{\log x}{(1+\log x)^2}dx = \int \frac{\log x + 1 - 1}{(1+\log x)^2}dx$

$= \int \frac{1}{1+\log x}dx - \int \frac{1}{(1+\log x)^2}dx$ **(1/2)**

$= \frac{1}{1+\log x} \times x - \int \frac{-1}{(1+\log x)^2} \times \frac{1}{x} \times x dx$

$- \int \frac{1}{(1+\log x)^2}dx + c = \frac{x}{1+\log x} + c$ **(1½)**

9. $\because$ $f(x) = x^2 \sin x$ is an odd function.

$\therefore \int_{-\pi/2}^{\pi/2} x^2 \sin x dx = 0$ **(1)**

10. Let $I = \int_0^1 x(1-x)^n\, dx$

$\Rightarrow \quad I = \int_0^1 (1-x)[1-(1-x)]^n\, dx \quad \left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$ **(1/2)**

$\Rightarrow \quad I = \int_0^1 (1-x)x^n dx = \int_0^1 (x^n - x^{n+1})dx \quad \Rightarrow \quad I = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$ **(1)**

$\Rightarrow \quad I = \left[\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - 0\right] = \frac{1}{(n+1)(n+2)}$ **(1/2)**

11. The given definite integral

$= \int_{-1}^{2} |x(x-1)(x-2)|dx$

$= \int_{-1}^{0} |x(x-1)(x-2)|dx + \int_0^1 |x(x-1)(x-2)|dx + \int_1^2 |x(x-1)(x-2)|dx$ **(1½)**

$= -\int_{-1}^{0}(x^3 - 3x^2 + 2x)dx + \int_0^1 (x^3 - 3x^2 + 2x)dx - \int_1^2 (x^3 - 3x^2 + 2x)dx$ **(1/2)**

$= -\left[\frac{x^4}{4} - x^3 + x^2\right]_{-1}^{0} + \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$

$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$ **(2)**

Self Assessment

Case Based Objective Questions (4 marks)

1. An Integration is the process of finding the anti-derivative of a function. In this process, we are provided with the derivative of a function and asked to find out the function (*i.e.*, Primitive)
Integration is the inverse process of differentiation.
Let $f(x)$ be a function of x. If there is a function $g(x)$, such that $\frac{d}{dx}(g(x)) = f(x)$, then $g(x)$ is called an integral of $f(x)$ w.r.t. x and is denoted by $\int f(x)dx = g(x) + c$, where c is constant of integration.
Also, the given integral $\int f(x)dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$
Consider, $I = \int f(x)dx = \int f(g(t))\,g'(t)\,dt$
Based on the above information, answer the following questions.

(i) Evaluate : $\int \frac{4x+6}{x^2+3x}dx$

(a) $3\log|x+3x^2|+C$ (b) $3\log|x^2+3x|+C$
(c) $2\log|x^2+3x|+C$ (d) $\log|4x+6|+C$

(ii) Evaluate : $\int \frac{1+\cos x}{x+\sin x}dx$

(a) $\log|x+\operatorname{cosec} x|$ (b) $\log|x+\sec x|+C$
(c) $\log|x+\cos x|+C$ (d) $\log|x+\sin x|+C$

(iii) Evaluate : $\int \frac{(x+1)^2}{x(x^2+1)}dx$

(a) $\log|x| + 2\tan^{-1}x + C$ (b) $2\tan^{-1}x - \log|x| + C$
(c) $\log|x| - 2\tan^{-1}x + C$ (d) None of these

(iv) Evaluate : $\int \tan^2 x\,dx$

(a) $\tan x + x + C$ (b) $\tan x - x + C$
(c) $\tan x + x^2 + C$ (d) None of these

(v) Evaluate : $\int \frac{dx}{\sin^2 x\cos^2 x}$

(a) $-2\cot 2x + C$ (b) $2\cot 2x + C$
(c) $\cot 2x + C$ (d) None of these

Multiple Choice Questions (1 mark)

2. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

3. $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

(a) $\frac{\pi^2}{32}$ (b) $\frac{\pi^2}{32}$ (c) $\frac{\pi}{16}$ (d) $\frac{\pi^2}{16}$

OR

Evaluate : $\int_0^1 \left\{e^x + \sin\frac{\pi x}{4}\right\}dx$

(a) $1-\frac{4}{\pi}+\frac{2\sqrt{2}}{\pi}$ (b) $1+\frac{2}{\pi}-\frac{2\sqrt{2}}{\pi}$
(c) $1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$ (d) None of these

4. $\int_0^{\pi/2} \cos x\, e^{\sin x} dx$ is equal to ________.

(a) $e+1$ (b) $e-1$
(c) e^2-1 (d) e^2+1

5. $\int \frac{x^3}{x+1}dx$ is equal to

(a) $x+\frac{x^2}{2}+\frac{x^3}{3}-\log|1-x|+C$
(b) $x+\frac{x^2}{2}-\frac{x^3}{3}-\log|1-x|+C$
(c) $x-\frac{x^2}{2}-\frac{x^3}{3}-\log|1+x|+C$
(d) $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

6. If $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$, then

(a) $a=\frac{-1}{10}, b=\frac{-2}{5}$ (b) $a=\frac{1}{10}, b=-\frac{2}{5}$
(c) $a=\frac{-1}{10}, b=\frac{2}{5}$ (d) $a=\frac{1}{10}, b=\frac{2}{5}$

7. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to

(a) $\frac{e^x}{1+x^2}+C$ (b) $\frac{-e^x}{1+x^2}+C$
(c) $\frac{e^x}{(1+x^2)^2}+C$ (d) $\frac{-e^x}{(1+x^2)^2}+C$

VSA Type Questions (1 mark)

8. Evaluate : $\int \cos^3 x\, e^{\log \sin x} dx$

9. Evaluate : $\int_1^3 x^2 \log x\, dx$

10. Evaluate : $\int \sin^{-1} x\, dx$

OR

Evaluate : $\int \frac{x}{x^4-1}dx$

11. Evaluate : $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

12. Evaluate : $\int_0^{\pi} \frac{x}{1+\sin x}dx$

SA I Type Questions (2 marks)

13. Evaluate : $\int \frac{\sin x}{3+4\cos^2 x}dx$

14. Evaluate : $\int \frac{1+\cos x}{1-\cos x}dx$

15. Evaluate : $\int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx$

16. Evaluate : $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$

OR

If $\int_0^a \frac{1}{1+4x^2}dx = \frac{\pi}{8}$, then find the value of a.

SA II Type Questions (3 marks)

17. Evaluate : $\int \frac{2x+1}{\sqrt{x^2+4x+3}}dx$

18. Evaluate : $\int_0^1 x\log(1+2x)dx$

19. Evaluate : $\int \frac{\sin(\tan^{-1} x)}{1+x^2}dx$

20. Evaluate : $\int [\sin(\log x)+\cos(\log x)]dx$

OR

Evaluate : $\int \frac{\cos x-\cos 2x}{1-\cos x}dx$

21. Evaluate : $\int \frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4 2x}}dx$

Case Based Questions (4 marks)

22. Let f be a continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f, then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a)$$

This result is very useful as it gives us a method of calculating the definite integral easily. Here, we have no need to write integration constant c because if, we will write $F(x) + c$, instead of $f(x)$, we get

$$\int_a^b f(x)dx = [F(x)+c]_a^b = F(b)+c-F(a)-c = F(b)-F(a)$$

Based on the above information, answer the following questions.

(i) Evaluate : $\int_0^1 xe^x dx$ (ii) Evaluate : $\int_0^{\pi/4} 2\tan^3 x\,dx$

LA Type Questions (4 / 6 marks)

23. Evaluate : $\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

24. Evaluate : $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

25. Evaluate : $\int_0^3 (|x|+|x-1|+|x-2|)dx$

OR

Evaluate : $\int \frac{xe^{2x}}{(2x+1)^2}dx$

Detailed SOLUTIONS

1. (i) (c) : Let $I = \int \frac{4x+6}{x^2+3x}dx$

$= 2\int \frac{2x+3}{x^2+3x}dx$

Put $x^2 + 3x = t$

$\Rightarrow$ $(2x + 3)\,dx = dt$

$\therefore I = \int 2\frac{dt}{t} = 2\log|t| + C$

$= 2\log|x^2 + 3x| + C$

(ii) (d): Let $I = \int \frac{1+\cos x}{x+\sin x}dx$

Put $x + \sin x = t$

$\Rightarrow$ $(1 + \cos x)\,dx = dt$

$\therefore I = \int \frac{dt}{t}$

$= \log|t| + C$

$= \log|x + \sin x| + C$

(iii) (a): Let $I = \int \frac{(x+1)^2}{x(x^2+1)}dx = \int \frac{x^2+1+2x}{x(x^2+1)}dx$

$= \int \left(\frac{1}{x}+\frac{2}{x^2+1}\right)dx = \log|x| + 2\tan^{-1}x + c$

(iv) (b): $\int \tan^2 x\,dx = \int (\sec^2 x-1)\,dx$

$= \tan x - x + c$

(v) (a): Let $I = \int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{4}{4\sin^2 x\cos^2 x}dx$

$= 4\int \mathrm{cosec}^2 2x\,dx = -2\cot 2x + c$

2. (a) : Let $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x} = \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x}$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x dx = \int_0^{\pi/4} \sec^2 x dx$$

$$\left[\begin{array}{l}\text{Using property for even function } f(x), \\ \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\end{array}\right]$$

$$= [\tan x]_0^{\pi/4} = \left[\tan\frac{\pi}{4} - \tan 0\right] = 1$$

3. (b) : We have, $I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

When $x = 0$, $t = 0$ and when $x = 1$, $t = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \int_0^{\pi/4} t dt = \left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{\pi^2}{32}$$

OR

(d) : We have, $I = \int_0^1 \left\{e^x + \sin\frac{\pi x}{4}\right\} dx$

$$= [e^x]_0^1 + \frac{4}{\pi}\left[-\cos\frac{\pi}{4}x\right]_0^1 = e - 1 - \frac{4}{\sqrt{2}\pi} + \frac{4}{\pi}$$

4. (b) : Let $I = \int_0^{\pi/2} \cos x \, e^{\sin x} dx$

Substitute $\sin x = t \Rightarrow \cos x \, dx = dt$

$x \to 0 \Rightarrow t \to 0$

and $x \to \pi/2 \Rightarrow t \to 1$

$$\therefore \quad I = \int_0^1 e^t dt = [e^t]_0^1 = e^1 - e^0 = e - 1$$

5. (d) : Let $I = \int \frac{x^3}{x+1} dx$

$$= \int\left((x^2 - x + 1) - \frac{1}{(x+1)}\right) dx = \int (x^2 - x + 1) dx - \int \frac{dx}{x+1}$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

6. (c) : We have given

$$\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1} x + \frac{1}{5}\log|x+2| + C$$

Taking, $I = \int \frac{dx}{(x+2)(x^2+1)}$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

[using method of partial fraction]

$\Rightarrow$ $1 = A(x^2 + 1) + (Bx + C)(x + 2)$

$\Rightarrow$ $1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$

$\Rightarrow$ $1 = (A + B)x^2 + (2B + C)x + A + 2C$

$\Rightarrow$ $A + B = 0, A + 2C = 1, 2B + C = 0$

By solving all three, we get

$A = \frac{1}{5}, B = -\frac{1}{5}$ and $C = \frac{2}{5}$

$$\therefore \int \frac{dx}{(x+2)(x^2+1)} = \frac{1}{5}\int \frac{1}{x+2} dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1} dx$$

$$= \frac{1}{5}\int \frac{1}{x+2} dx - \frac{1}{5}\int \frac{x}{1+x^2} dx + \frac{1}{5}\int \frac{2}{1+x^2} dx$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{10}\log|1+x^2| + \frac{2}{5}\tan^{-1} x + C$$

$\therefore \quad b = \frac{2}{5}$ and $a = -\frac{1}{10}$ [By comparing]

7. (a) : We have, $I = \int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$

$$= \int e^x \left(\frac{1+x^2-2x}{(1+x^2)^2}\right) dx = \int e^x \left(\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right) dx$$

Above integral is of the type $\int e^x (f(x) + f'(x)) dx$

$\therefore$ Solution is $e^x f(x) + C$

$$= e^x \left(\frac{1}{1+x^2}\right) + C$$

8. We have, $I = \int \cos^3 x \, e^{\log \sin x} dx = \int \cos^3 x \sin x \, dx$

Put $\cos x = t \Rightarrow -\sin x \, dx = dt$

$$\Rightarrow \quad I = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

9. $\int_1^3 x^2 \log x \, dx$

$$= \left[(\log x)\left(\frac{x^3}{3}\right)\right]_1^3 - \int_1^3 \frac{1}{x}\cdot\frac{x^3}{3} dx$$

$$= 9\log 3 - 0 - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^3 = 9\log 3 - \frac{26}{9}$$

10. Let $I = \int 1\cdot\sin^{-1} x dx$

$$= (\sin^{-1} x)x - \int \frac{1}{\sqrt{1-x^2}}\cdot x dx + C'$$

Put $1 - x^2 = t^2 \Rightarrow -2x \, dx = 2t \, dt$

$$= x\sin^{-1} x - \int \frac{(-t dt)}{t} + C = x\sin^{-1} x + \sqrt{1-x^2} + C$$

OR

Let $I = \int \frac{x}{x^4 - 1} dx$

Substitute $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{dt}{t^2-1} = \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$\left[\text{Using } \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{4}[\log|x^2-1| - \log|x^2+1|] + C$$

11. Let $I = \int_0^1 \frac{x}{\sqrt{1+x^2}} dx$

Substitute $1 + x^2 = t^2$

$\Rightarrow \quad 2xdx = 2tdt \quad \Rightarrow xdx = tdt$

$$\Rightarrow \ I=\int_1^{\sqrt{2}}\frac{tdt}{t}=[t]_1^{\sqrt{2}}=\sqrt{2}-1$$

12. Let $I=\int_0^{\pi}\frac{x}{1+\sin x}dx$ ... (i)

Using property $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, we get

$$I=\int_0^{\pi}\frac{\pi-x}{1+\sin(\pi-x)}dx=\int_0^{\pi}\frac{\pi-x}{1+\sin x}dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I=\pi\int_0^{\pi}\frac{1}{1+\sin x}dx \ =\pi\int_0^{\pi}\frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$=\pi\int_0^{\pi}\frac{(1-\sin x)\,dx}{\cos^2 x} \qquad [\because \cos^2 x = 1 - \sin^2 x]$$

$$=\pi\int_0^{\pi}(\sec^2 x-\tan x\cdot\sec x)dx \ =\pi[\tan x]_0^{\pi}-\pi[\sec x]_0^{\pi}$$

$= \pi[\tan \pi - \sec \pi - \tan 0 + \sec 0]$

$\Rightarrow \quad 2I = \pi[0 + 1 - 0 + 1] = 2\pi$

$\therefore \quad I = \pi$

13. Let $I=\int\frac{\sin x}{3+4\cos^2 x}dx$

Substitute $\cos x = t \ \Rightarrow \ -\sin x\, dx = dt$

$$\therefore \ I=-\int\frac{dt}{3+4t^2}=-\frac{1}{4}\int\frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2+t^2}$$

$$=-\frac{1}{4}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2t}{\sqrt{3}}+C \qquad \left[\because \int\frac{1\,dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right]$$

$$=-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$$

14. Let $I=\int\frac{1+\cos x}{1-\cos x}dx$

$$\Rightarrow \ I=\int\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}dx=\int\cot^2\frac{x}{2}dx$$

$$\Rightarrow \ I=\int\left(\operatorname{cosec}^2\frac{x}{2}-1\right)dx=-2\cot\frac{x}{2}-x+C$$

15. Let $I=\int_0^{\pi/2}\sqrt{1-\sin 2x}\,dx$

$$=\int_0^{\pi/4}\sqrt{(\cos x-\sin x)^2}dx+\int_{\pi/4}^{\pi/2}\sqrt{(\sin x-\cos x)^2}dx$$

$$\left[\because \text{When } 0<x<\frac{\pi}{4}, \cos x>\sin x \text{ and when } \frac{\pi}{4}<x<\frac{\pi}{2}, \sin x>\cos x\right]$$

$$=[\sin x+\cos x]_0^{\pi/4}+[-\cos x-\sin x]_{\pi/4}^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-0-1+\left(-0-1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)$$

$$=2\sqrt{2}-2=2(\sqrt{2}-1)$$

16. Let $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}x^3\sin^4 x\,dx$

Now, let $f(x) = x^3\sin^4 x$, then

$f(-x) = (-x)^3 (\sin(-x))^4 = -x^3\sin^4 x = -f(x)$

So, $f(x)$ is an odd function.

Hence, $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}x^3\sin^4 x\,dx=0$

OR

We have, $\int_0^a\frac{1}{1+4x^2}dx=\frac{\pi}{8}$

Taking $\int_0^a\frac{1}{4\left(\frac{1}{4}+x^2\right)}dx=\frac{1}{4}\int_0^a\frac{1}{\left(x^2+\left(\frac{1}{2}\right)^2\right)}dx$

$$=\frac{2}{4}[\tan^{-1}2x]_0^a \ =\frac{1}{2}\tan^{-1}2a-0=\frac{1}{2}\tan^{-1}2a$$

Since, $\frac{1}{2}\tan^{-1}2a=\frac{\pi}{8} \ \Rightarrow \ \tan^{-1}2a=\frac{\pi}{4}$

$\Rightarrow \quad 2a = \tan\frac{\pi}{4} = 1 \ \Rightarrow \ a = 1/2$

17. Let $I=\int\frac{2x+1}{\sqrt{x^2+4x+3}}dx$

Let $2x+1=A\left(\frac{d}{dx}(x^2+4x+3)\right)+B=A(2x+4)+B$

Equating the coefficients of x and constant terms, we get $2A = 2$ and $4A + B = 1 \ \Rightarrow \ A = 1$ and $B = -3$

$$\therefore \ I=\int\frac{2x+1}{\sqrt{x^2+4x+3}}dx$$

$$=\int\frac{2x+4}{\sqrt{x^2+4x+3}}dx-\int\frac{3}{\sqrt{x^2+4x+3}}dx$$

$$=\int\frac{2x+4}{\sqrt{x^2+4x+3}}dx-3\int\frac{dx}{\sqrt{(x+2)^2-(1)^2}}$$

$$=2\sqrt{x^2+4x+3}-3\log\left|(x+2)+\sqrt{x^2+4x+3}\right|+C$$

18. Let $I=\int_0^1 x\log(1+2x)\,dx$

$$=\left[\log(1+2x)\frac{x^2}{2}\right]_0^1-\int_0^1\frac{1}{1+2x}\cdot 2\cdot\frac{x^2}{2}dx$$

[Integration by parts]

$$=\frac{1}{2}[x^2\log(1+2x)]_0^1-\int_0^1\frac{x^2}{1+2x}dx$$

$$=\frac{1}{2}[1\log 3-0]-\left[\int_0^1\left(\frac{x}{2}-\frac{\frac{x}{2}}{1+2x}\right)dx\right]$$

$$=\frac{1}{2}\log 3-\frac{1}{2}\int_0^1 x\,dx+\frac{1}{2}\int_0^1\frac{x}{1+2x}dx$$

$$=\frac{1}{2}\log 3-\frac{1}{2}\left[\frac{x^2}{2}\right]_0^1+\frac{1}{2}\int_0^1\frac{\frac{1}{2}(2x+1-1)}{(2x+1)}dx$$

$$=\frac{1}{2}\log 3-\frac{1}{2}\left[\frac{1}{2}-0\right]+\frac{1}{4}\int_0^1 dx-\frac{1}{4}\int_0^1\frac{1}{1+2x}dx$$

$$=\frac{1}{2}\log 3-\frac{1}{4}+\frac{1}{4}[x]_0^1-\frac{1}{8}[\log|(1+2x)|]_0^1$$

$$=\frac{1}{2}\log 3-\frac{1}{4}+\frac{1}{4}-\frac{1}{8}[\log 3-\log 1]$$

$$=\frac{1}{2}\log 3-\frac{1}{8}\log 3=\frac{3}{8}\log 3$$

19. Let $I=\int\frac{\sin(\tan^{-1}x)}{1+x^2}dx$

Put $\tan^{-1}x=t \Rightarrow \frac{1}{1+x^2}dx=dt$

$\therefore\quad I=\int\frac{\sin(\tan^{-1}x)}{1+x^2}dx=\int\sin t\,dt=-\cos t+C$

$\Rightarrow\quad I=-\cos(\tan^{-1}x)+C$

20. We have, $I=\int[\sin(\log x)+\cos(\log x)]dx$

Put $\log x = t$, $x = e^t \Rightarrow dx = e^t dt$

$\therefore\quad I=\int(\sin t+\cos t)e^t\,dt$

Consider, $f(t) = \sin t$

$\Rightarrow\quad f'(t) = \cos t$

$\therefore\quad$ Integrand is in the form $e^t(f(t) + f'(t))$

$\therefore\quad I=\int e^t(\sin t+\cos t)dt=e^t\sin t+C=x\sin(\log x)+C$

OR

Let $I=\int\frac{\cos x-\cos 2x}{1-\cos x}dx$

$$=\int\frac{2\sin\frac{3x}{2}\cdot\sin\frac{x}{2}}{1-1+2\sin^2\frac{x}{2}}dx\qquad\left[\begin{array}{l}\text{Using }\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\\ \text{and }\cos A=1-2\sin^2\frac{A}{2}\end{array}\right]$$

$$=\int\frac{\sin\frac{3x}{2}}{\sin\frac{x}{2}}dx$$

$$=\int\frac{3\sin\frac{x}{2}-4\sin^3\frac{x}{2}}{\sin\frac{x}{2}}dx\qquad[\because \sin 3A = 3\sin A - 4\sin^3 A]$$

$$=3\int dx-4\int\sin^2\frac{x}{2}dx$$

$$=3\int dx-4\int\frac{1-\cos x}{2}dx$$

$$=3\int dx-2\int dx+2\int\cos x\,dx$$

$$=\int dx+2\int\cos x\,dx=x+2\sin x+C$$

21. We have, $I=\int\frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4 2x}}dx$

Put $\cos^2 2x = t \Rightarrow -4\sin 2x\cos 2x\,dx = dt$

$\therefore\quad I=\frac{-1}{4}\int\frac{dt}{\sqrt{9-t^2}}=\frac{-1}{4}\sin^{-1}\frac{t}{3}+C$

$$=\frac{-1}{4}\sin^{-1}\left(\frac{\cos^2 2x}{3}\right)+C$$

22. **(i)** Here, $\int xe^x dx=x\int e^x\cdot dx-\int\left(\frac{d}{dx}(x)\cdot\int e^x dx\right)dx$

$=xe^x-\int 1\cdot e^x dx$

$= xe^x - e^x = e^x(x - 1)$

$= F(x)$

Now, $\int_0^1 xe^x dx=F(1)-F(0)$

$= e(1 - 1) - e^0(0 - 1) = 0 + 1 = 1$

(ii) We have, $\int 2\tan^3 x\,dx$

$=\int 2\tan x\tan^2 x\,dx$

$=\int 2\tan x(\sec^2 x-1)dx$

$=2\int\tan x\sec^2 x\,dx-2\int\tan x\,dx$

$=2\left[\frac{\tan^2 x}{2}\right]-2[-\log|\cos x|]=\tan^2 x+2\log|\cos x|$

Now, $\int_0^{\pi/4}2\tan^3 x=F(\pi/4)-F(0)$

$$=\left(\tan^2\frac{\pi}{4}+2\log\left|\cos\frac{\pi}{4}\right|\right)-(\tan^2 0+2\log|\cos 0|)$$

$$=\left(1+2\log\frac{1}{\sqrt{2}}\right)-(0+2\log 1)=1+2\left(\frac{-1}{2}\log 2\right)-0$$

$= 1 - \log 2$

23. Let $I=\int\frac{(2x-1)}{(x-1)(x+2)(x-3)}dx$

By using partial fraction, we get

$$\frac{2x-1}{(x-1)(x+2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x-3)}$$

$\Rightarrow\quad 2x - 1 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)$

Substitute $x = 1$, we get

$2 - 1 = A(1 + 2)(1 - 3)$

$\Rightarrow\ 1=-6A\ \Rightarrow\ A=-\frac{1}{6}$

Substitute $x = 3$, we get

$$6 - 1 = C(3-1)(3+2)$$

$$\Rightarrow \ 5 = 10C \ \Rightarrow \ C = \frac{1}{2}$$

Now, substitute $x = -2$, we get

$$-4 - 1 = B(-2-1)(-2-3)$$

$$\Rightarrow \ -5 = 15B \ \Rightarrow \ B = -\frac{1}{3}$$

$$\therefore \ I = -\frac{1}{6}\int\frac{1}{x-1}dx - \frac{1}{3}\int\frac{1}{x+2}dx + \frac{1}{2}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{6}\log|(x-1)| - \frac{1}{3}\log|(x+2)| + \frac{1}{2}\log|(x-3)| + C$$

$$= -\log|(x-1)|^{1/6} - \log|(x+2)|^{1/3} + \log\left|\sqrt{(x-3)}\right| + C$$

$$= \log\left|\sqrt{x-3}\right| - \log\left|(x-1)^{1/6}(x+2)^{1/3}\right| + C$$

$$= \log\left|\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right| + C$$

24. Let $I = \int\frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x}dx$

$$= \int\frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x\cdot\cos^2 x}dx$$

$$= \int\frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x)}{\sin^2 x\cdot\cos^2 x}dx$$

$$\left[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2) \text{ and } \sin^2 x + \cos^2 x = 1\right]$$

$$= \int\frac{\sin^4 x}{\sin^2 x\cos^2 x}dx + \int\frac{\cos^4 x}{\sin^2 x\cdot\cos^2 x}dx - \int\frac{\sin^2 x\cos^2 x}{\sin^2 x\cdot\cos^2 x}dx$$

$$= \int\tan^2 x\,dx + \int\cot^2 x\,dx - \int 1\,dx$$

$$= \int(\sec^2 x - 1)dx + \int(\text{cosec}^2 x - 1)dx - \int 1\,dx$$

$$= \int\sec^2 x\,dx + \int\text{cosec}^2 x\,dx - 3\int dx$$

$$\Rightarrow \ I = \tan x - \cot x - 3x + C$$

25. Let $I = \int_0^3(|x| + |x-1| + |x-2|)\,dx$

$f(x) = |x| + |x-1| + |x-2|$

When $0 \le x < 1$, then

$f(x) = x - (x-1) - (x-2) = x - x + 1 - x + 2 = -x + 3$

When $1 \le x < 2$, then

$f(x) = x + (x-1) - (x-2) = x + x - 1 - x + 2 = x + 1$

When $2 \le x < 3$, then

$f(x) = x + (x-1) + (x-2) = 3x - 3$

$$\therefore \ f(x) = \begin{cases} -x+3, & 0 \le x < 1 \\ x+1, & 1 \le x < 2 \\ 3x-3, & 2 \le x < 3 \end{cases}$$

$$\therefore \ I = \int_0^3(|x| + |x-1| + |x-2|)dx$$

$$\Rightarrow \ I = \int_0^1(-x+3)dx + \int_1^2(x+1)dx + \int_2^3(3x-3)dx$$

$$= \left[\frac{-x^2}{2} + 3x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2 + \left[\frac{3x^2}{2} - 3x\right]_2^3$$

$$= \left(\frac{-1}{2} + 3\right) + \left[\left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} + 1\right)\right] + \left[\left(\frac{27}{2} - 9\right) - \left(\frac{12}{2} - 6\right)\right]$$

$$= \frac{5}{2} + \left[4 - \frac{3}{2}\right] + \left[\frac{9}{2} - 0\right] = \frac{5}{2} + 4 - \frac{3}{2} + \frac{9}{2} = \frac{19}{2}$$

OR

Let $I = \int\frac{xe^{2x}}{(2x+1)^2}dx$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$$\therefore \ I = \frac{1}{4}\int\frac{te^t}{(t+1)^2}dt = \frac{1}{4}\int e^t\left[\frac{t+1}{(t+1)^2} - \frac{1}{(t+1)^2}\right]dt$$

$$= \frac{1}{4}\int e^t\left(\frac{1}{t+1} - \frac{1}{(t+1)^2}\right)dt$$

If $f(t) = \frac{1}{t+1} \Rightarrow f'(t) = \frac{-1}{(t+1)^2}$

So, $I = \frac{1}{4}\frac{e^{2x}}{(2x+1)} + C$

CHAPTER
8

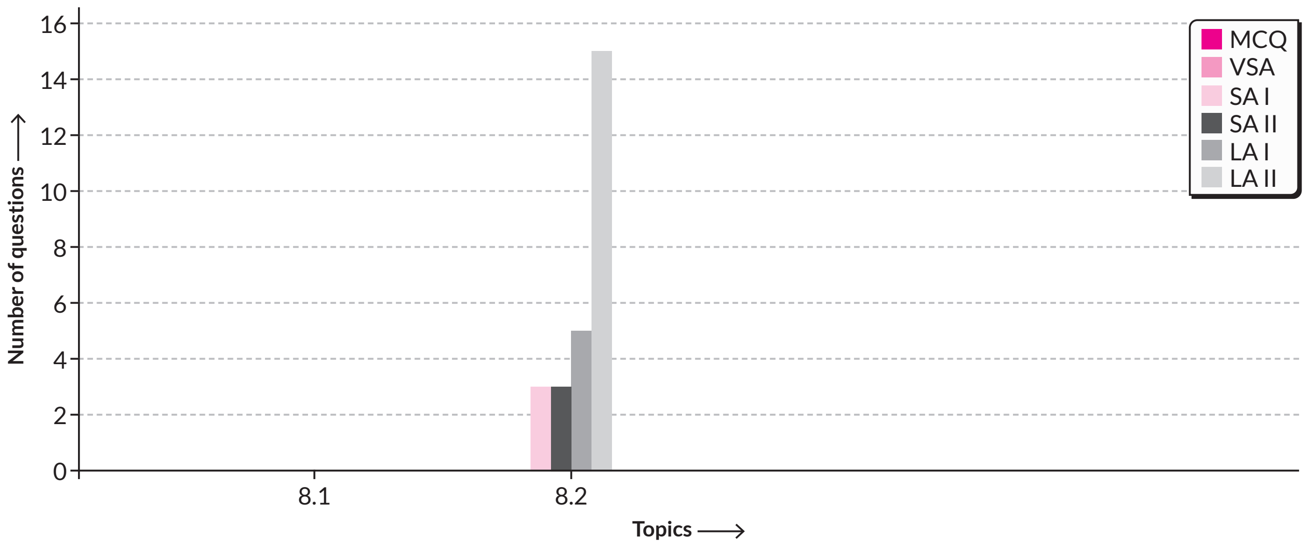
Application of Integrals

TOPICS

8.1 Introduction

8.2 Area under Simple Curves

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

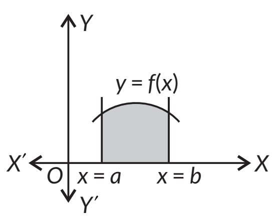
- Maximum weightage is of Topic 8.2 *Area under Simple Curves.*
- Maximum LA II type questions were asked from Topic 8.2 *Area under Simple Curves.*
- No MCQ and VSA type questions were asked till now.

QUICK RECAP

Area under Simple Curves

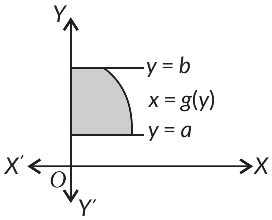
- Area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ ($b > a$) is,

  Area = $\int_a^b y\,dx = \int_a^b f(x)\,dx$



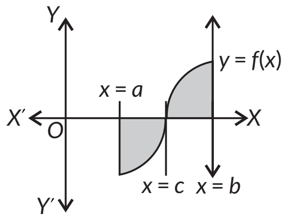
- Area of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = a$ and $y = b$ $(b > a)$ is,

  $$\text{Area} = \int_a^b x\,dy = \int_a^b g(y)\,dy$$



- Area of the region bounded by the curve $y = f(x)$, some portion of which is above the x-axis and some below the x-axis is,

  $$\text{Area} = \left|\int_a^c f(x)dx\right| + \int_c^b f(x)dx$$



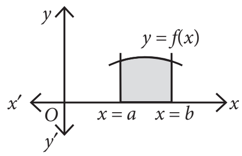
BRAIN MAP

APPLICATION OF INTEGRALS

Area under Simple Curves

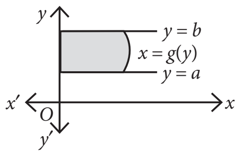
- Area $= \int_a^b y\,dx$

  $= \int_a^b f(x)\,dx$ (where $b > a$)

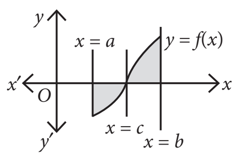


- Area $= \int_a^b x\,dy$

  $= \int_a^b g(y)\,dy$ (where $b > a$)



- Area $= \left|\int_a^c f(x)\,dx\right| + \int_c^b f(x)\,dx$



Points to be Remember

- The area of a region bounded by $y^2 = 4ax$ and $y = mx$ is $\frac{8a^2}{3m^3}$ sq units.
- The area of a region bounded by $y^2 = 4ax$ and its latus rectum is $\frac{8a^2}{3}$ sq units.
- Area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq units.
- The area of a region bounded by $y = ax^2 + bx + c$ and x-axis is $\frac{(b^2 - 4ac)^{3/2}}{6a^2}$ sq units.

Previous Years' CBSE Board Questions

8.2 Area under Simple Curves

SA I (2 marks)

1. Sketch the region bounded by the lines $2x + y = 8$, $y = 2$, $y = 4$ and the y-axis. Hence, obtain its area using integration. *(2023)*

2. Using integration, find the area bounded by the curve $y^2 = 4x$, y-axis and $y = 3$. *(2021C)* Ev

3. Using integration, find the area of the region bounded by the line $2y = -x + 8$, x-axis, $x = 2$ and $x = 4$. *(2021C)* Ev

SA II (3 marks)

4. Find the area of the following region using integration.
$\{(x, y) : y^2 \le 2x \text{ and } y \ge x - 4\}$ *(2023)*

5. Using integration, find the area of the region bounded by $y = mx$ $(m > 0)$, $x = 1$, $x = 2$ and the x-axis. *(2023)*

6. Using integration, find the area of the region $\{(x, y) : y^2 \le x \le y\}$. *(Term II, 2021-22)* Ev

LA I (4 marks)

7. Using integration, find the area of the region $\{(x, y) : 4x^2 + 9y^2 \le 36, 2x + 3y \ge 6\}$. *(Term II, 2021-22)* Ev

8. Using integration, find the area of the region bounded by lines $x - y + 1 = 0$, $x = -2$, $x = 3$ and x-axis. *(Term II, 2021-22)* Ev

9. If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then using integration find the value of a, where $a > 0$. *(Term II, 2021-22)* Ev

10. Find the area of the region bounded by curve $4x^2 = y$ and the line $y = 8x + 12$, using integration. *(Term II, 2021-22)* Ev

11. Find the area bounded by the curves $y = |x - 1|$ and $y = 1$, using integration. *(Term II, 2021-22)* Cr

LA II (5/6 marks)

12. Using integration, find the area of the region bounded by the circle $x^2 + y^2 = 16$, line $y = x$ and y-axis, but lying in the 1st quadrant. *(2023)*

13. Find the area of the following region using integration: $\{(x, y) : y < |x| + 2, y > x^2\}$ *(2020)* Ev

14. Using integration, find the area of a triangle whose vertices are (1, 0), (2, 2) and (3, 1). *(2020)* Ev

15. Using integration, find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$. *(2020)* Ev

16. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. *(2020, NCERT, 2018, Delhi 2014)* Ev

17. If the area between the curves $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, then find the value of a using integration. *(2020C)* Ev

18. Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$. *(2019)* Ev

19. Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration. *(Delhi 2017)* Cr

20. Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$. *(AI 2017, 2015C)*

21. Using integration, find the area of the region bounded by the line $x - y + 2 = 0$, the curve $x = \sqrt{y}$ and y – axis. *(Foreign 2015)*

22. Find the area of the region in the first quadrant enclosed by the y-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$, using integration. *(NCERT, Delhi 2015C)* Cr

23. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$. *(Foreign 2014)* Cr

24. Using integration, find the area of the region bounded by the curves:
$y = |x + 1| + 1$, $x = -3$, $x = 3$, $y = 0$ *(Delhi 2014C)*

25. Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$. *(Delhi 2014C)*

26. Using integration, find the area of the region in the first quadrant enclosed by the x – axis, the line $y = x$ and the circle $x^2 + y^2 = 18$. *(AI 2014C)*

CBSE Sample Questions

8.2 Area under Simple Curves

VSA (1 mark)

1. Find the area bounded by $y = x^2$, the x- axis and the lines $x = -1$ and $x = 1$. *(2020-21)* Ev

SA I (2 marks)

2. Find the area of the region bounded by the parabola $y^2 = 8x$ and the line $x = 2$. *(2020-21)* Ev

SA II (3 marks)

3. Find the area of the region bounded by the curves $x^2 + y^2 = 4$, $y=\sqrt{3}x$ and x-axis in the first quadrant. *(2020-21)*

4. Find the area of the ellipse $x^2 + 9y^2 = 36$ using integration. *(2020-21)* Cr

LA I (4/5 marks)

5. Make a rough sketch of the region $\{(x, y): 0 \le y \le x^2, 0 \le y \le x, 0 \le x \le 2\}$ and find the area of the region using integration. *(2022-23)* Ev

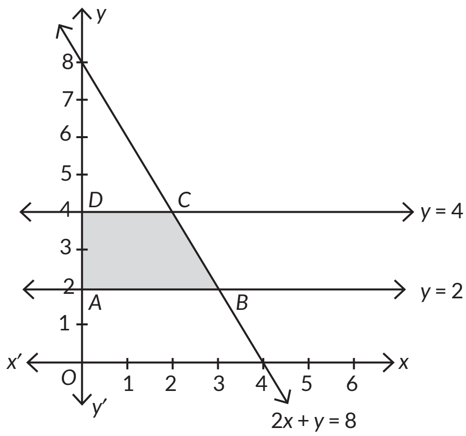
6. Using integration, find the area of the region in the first quadrant enclosed by the line $x + y = 2$, the parabola $y^2 = x$ and the x-axis. *(Term II, 2021-22)* Ev

7. Using integration, find the area of the region $\{(x,y): 0 \le y \le \sqrt{3}x, x^2 + y^2 \le 4\}$. *(Term II, 2021-22)* Cr

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. From the graph, $ABCD$ is the required region.



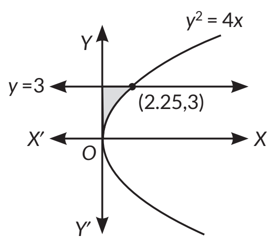
Now, area $= \int_2^4 \left(\frac{8-y}{2}\right) dy = \frac{1}{2}\int_2^4 (8-y)dy$

$= \frac{1}{2}\left[8y - \frac{y^2}{2}\right]_2^4 = \frac{1}{2}\left[\left(32-\frac{16}{2}\right)-\left(16-\frac{4}{2}\right)\right]$

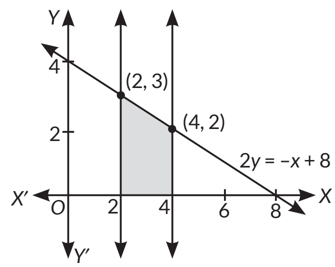
$= \frac{1}{2}\times 10 = 5$ sq.units

2. Required area $= \int_0^3 \frac{y^2}{4} dy$

$= \left[\frac{y^3}{12}\right]_0^3 = \frac{9}{4}$ sq. units.



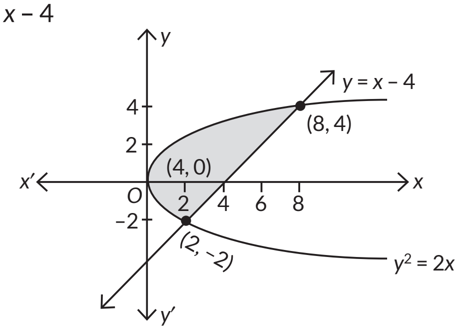
3.



$\therefore$ Required area

$= \int_2^4 \left(\frac{-x+8}{2}\right)dx = \left[\frac{-x^2}{4}+4x\right]_2^4 = 5$ sq. units

4. We have, $y^2 \le 2x$ ... (i)
and $y \ge x - 4$ ... (ii)



Here, the shaded area represents the required area.

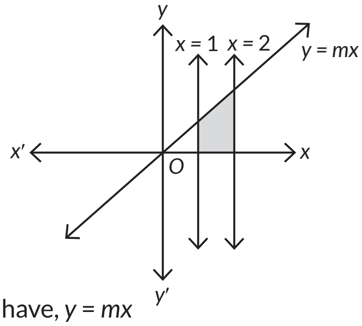
$\therefore$ Required area $= \int_{-2}^{4} (y+4)\,dy - \int_{-2}^{4} \frac{y^2}{2} dy$

$= \left[\frac{y^2}{2}+4y\right]_{-2}^{4} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-2}^{4}$

$= \left[\left(\frac{16}{2}+16\right)-\left(\frac{4}{2}-8\right)\right]-\frac{1}{2}\left[\frac{64}{3}-\left(\frac{-8}{3}\right)\right]$

$= 30 - 12 = 18$ sq. units

5.



We have, $y = mx$

From the figure it is clear that required area is the shaded region.

Required area = $\int_1^2 mx\,dx = \left[\frac{mx^2}{2}\right]_1^2 = \frac{4m}{2} - \frac{m}{2} = \frac{3m}{2}$ sq. units.

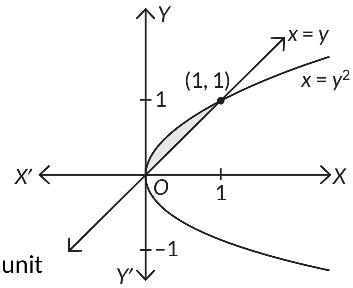
6. On solving $x = y$ and $x = y^2$, we get

$y^2 = y$

$\Rightarrow$ $y(y - 1) = 0$

$\Rightarrow$ $y = 0$ and $y = 1$.

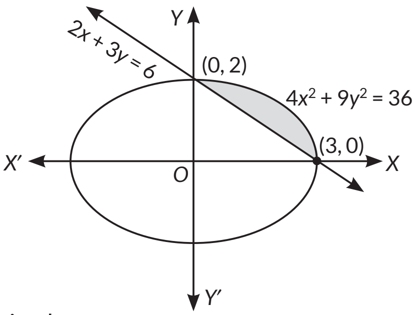
$\therefore$ Required area = area of shaded region



$= \int_0^1 (y - y^2)\,dy$

$= \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. unit

7.



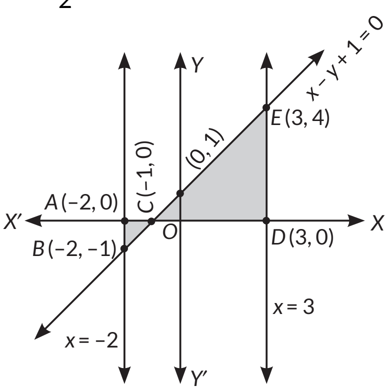
Required area

$= \int_0^3 \left(\frac{\sqrt{36-4x^2}}{3} - \left(\frac{6-2x}{3}\right)\right)dx = \frac{2}{3}\int_0^3 (\sqrt{9-x^2} - (3-x))\,dx$

$= \frac{2}{3}\left[\left(\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right) + \frac{x^2}{2} - 3x\right]_0^3$

$= \frac{3\pi}{2} - 3 = \frac{3}{2}(\pi - 2)$ sq. units

8.



Required area = $ar(\Delta ABC) + ar(\Delta CDE)$

$= \left|\int_{-2}^{-1}(x+1)\,dx\right| + \int_{-1}^{3}(x+1)\,dx = \left|\left[\frac{x^2}{2}+x\right]_{-2}^{-1}\right| + \left[\frac{x^2}{2}+x\right]_{-1}^{3}$

$= \left|\frac{1}{2} - 1 - (2-2)\right| + \left\{\frac{9}{2} + 3 - \left(\frac{1}{2} - 1\right)\right\} = \frac{1}{2} + 8 = \frac{17}{2} = 8.5$ sq. units

9. Given, area $= 2\int_0^{4a}\sqrt{4ax}\,dx = \frac{256}{3}$

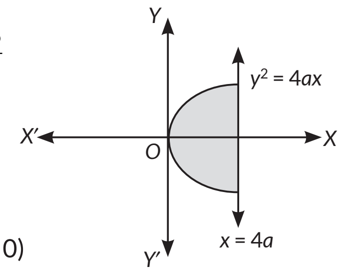
$\Rightarrow$ $4\sqrt{a}\int_0^{4a}\sqrt{x}\,dx = \frac{256}{3}$

$\Rightarrow$ $4\sqrt{a}\,\frac{2}{3}\left[(x)^{3/2}\right]_0^{4a} = \frac{256}{3}$

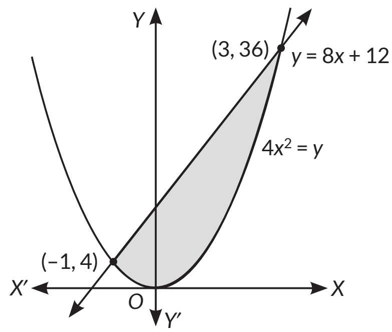
$\Rightarrow$ $\frac{8}{3}\sqrt{a}\,(4a)^{3/2} = \frac{256}{3}$

$\Rightarrow$ $a^2 = \frac{256}{3} \times \frac{3}{8 \times 2 \times 4}$

$\Rightarrow$ $a^2 = 4 \Rightarrow a = 2\ (\because a > 0)$



10. The graph of given region is

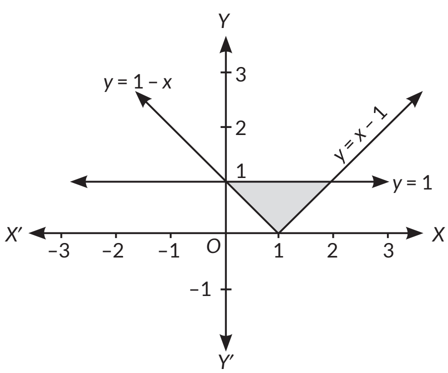


$\therefore$ Required area $= \int_{-1}^{3}(8x + 12 - 4x^2)\,dx$

$= \left[4x^2 + 12x - \frac{4}{3}x^3\right]_{-1}^{3}$

$= 36 + 36 - 36 - \left(4 - 12 + \frac{4}{3}\right) = 44 - \frac{4}{3} = \frac{128}{3}$ sq. units

11. Given curve, $y = |x - 1|$ and line $y = 1$



We have, $y = \begin{cases} x-1, \text{if } x-1 \ge 0 \\ -x+1, \text{if } x-1 < 0 \end{cases}$

Required area = area of shaded region

$$=\int_0^2 1\,dx-\left\{\int_0^1(1-x)dx+\int_1^2(x-1)dx\right\}$$

$$=[x]_0^2-\left(\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^2\right)$$

$$=2-\left(1-\frac{1}{2}+2-2-\frac{1}{2}+1\right)$$

$$=2-\left(\frac{1}{2}+\frac{1}{2}\right)=2-1=1\text{ sq. unit}$$

12. Given, $x^2+y^2=16$ ...(i)

$y=x$ ...(ii) and $x=0$ ...(iii)

From equation (i) and (iii), we get

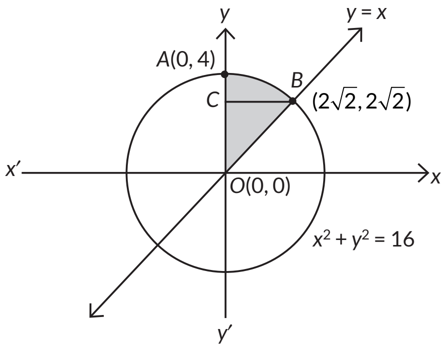
$x=0$ and $y=\pm 4$

∴ Equation (i) and (iii) intersect at (0, 4) and (0, –4).

From equation (i) and (ii), we get

$x=\pm 2\sqrt{2}$ and $y=\pm 2\sqrt{2}$

Required bounded region is shown in the figure.



Let A be the required area.

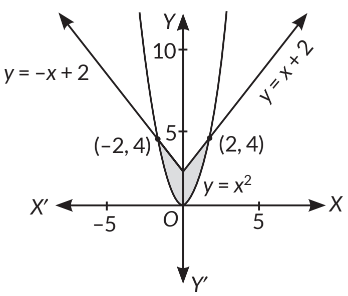
A = Area of ΔOBC + Area of region $CBAC$

$$=\int_0^{2\sqrt{2}} y\,dy+\int_{2\sqrt{2}}^{4}\sqrt{16-y^2}\,dy$$

$$=\left[\frac{y^2}{2}\right]_0^{2\sqrt{2}}+\left[\frac{y}{2}\sqrt{16-y^2}+\frac{16}{2}\sin^{-1}\left(\frac{y}{4}\right)\right]_{2\sqrt{2}}^4$$

$$=4+8\times\frac{\pi}{2}-4-8\times\frac{\pi}{4}=2\pi\text{ sq. units}$$

13. The graph of given region is

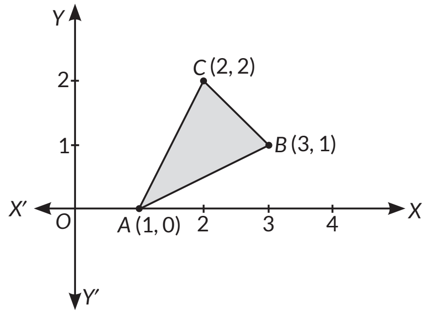


We have, $y=\begin{cases}x+2 & \text{if } x\ge 0\\ -x+2 & \text{if } x<0\end{cases}$

Required area $=2\int_0^2(x+2-x^2)\,dx$

$$=2\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_0^2=4+8-\frac{16}{3}=\frac{20}{3}\text{ sq. units}$$

14. The graph of given region is



Equation of AB, $y-0=\frac{1-0}{3-1}(x-1)$

$\Rightarrow\ 2y=x-1 \Rightarrow\ y=\frac{x-1}{2}$

Equation of BC, $y-2=\frac{1-2}{3-2}(x-2)$

$\Rightarrow\ y=-x+4$

Equation of AC, $y-0=\frac{2-0}{2-1}(x-1)$

$\Rightarrow\ y=2x-2$

Required area

$$=\int_1^2\left[2x-2-\left(\frac{x-1}{2}\right)\right]dx+\int_2^3\left[(-x+4)-\left(\frac{x-1}{2}\right)\right]dx$$

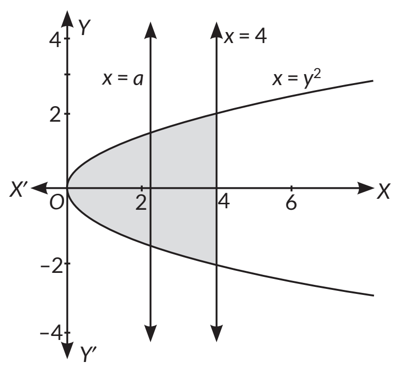
$$=\int_1^2(3x-3)dx+\int_2^3(-3x+9)dx$$

$$=3\left[\frac{x^2}{2}-x\right]_1^2+\left[\frac{-3x^2}{2}+9x\right]_2^3$$

$$=3\left(2-2-\frac{1}{2}+1\right)+\left(-\frac{27}{2}+27+6-18\right)$$

$$=\frac{3}{2}+\frac{3}{2}=3\text{ sq. units}$$

15. The graph of given region is



According to question, $2\int_0^a\sqrt{x}\,dx=2\int_a^4\sqrt{x}\,dx$

$\Rightarrow\ \frac{2}{3}[x^{3/2}]_0^a=\frac{2}{3}[x^{3/2}]_a^4$

$\Rightarrow\ a^{3/2}=8-a^{3/2}$

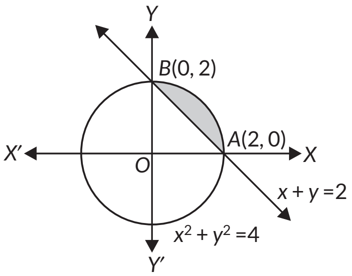
$\Rightarrow \quad 2a^{3/2} = 8$

$\Rightarrow \quad a^{3/2} = 4 \Rightarrow a = (4^2)^{1/3}$

$\Rightarrow \quad a = (16)^{1/3} = (2)^{4/3}$

16. The given curves are

$x^2 + y^2 = 4$ ...(i) and $x + y = 2$ ...(ii)



$\therefore$ Required area $= \int_0^2 \left[\sqrt{4-x^2} - (2-x)\right] dx$

$= \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right]_0^2$

$= 0 + 2\sin^{-1}(1) - 4 + 2 - 0$

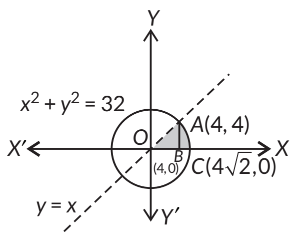
$= 2\cdot\frac{\pi}{2} - 2 = (\pi - 2)$ sq. units.

Key Points

➲ $\int\sqrt{a^2-x^2}\,dx = \frac{x\cdot\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

17. The given equation of the circle is $x^2 + y^2 = 32$ and the line is $y = x$.

These intersect at $A(4, 4)$ in the first quadrant. The required area is shown shaded in the figure. Points $B(4, 0)$ and $C(4\sqrt{2}, 0)$.



$\therefore$ Required area = Area $BACB$ + Area $OABO$

$= \int_4^{4\sqrt{2}} y_1\,dx + \int_0^4 y_2\,dx = \int_4^{4\sqrt{2}} \sqrt{32-x^2}\,dx + \int_0^4 x\,dx$

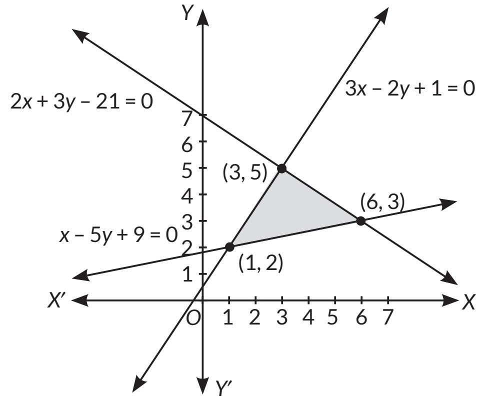
$= \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx + \int_0^4 x\,dx$

$= \left[\frac{x\sqrt{32-x^2}}{2} + \frac{32}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}} + \left[\frac{x^2}{2}\right]_0^4$

$= \frac{4\sqrt{2}\times 0}{2} + 16\sin^{-1}1 - \left(\frac{4\times 4}{2} + 16\sin^{-1}\frac{1}{\sqrt{2}}\right) + \frac{1}{2}(4^2 - 0)$

$= 16\cdot\frac{\pi}{2} - \left(8 + 16\cdot\frac{\pi}{4}\right) + 8 = 16\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 4\pi$ sq.units

18. The graph of given region is



Required area

$= \int_1^3 \left(\frac{3x+1}{2} - \frac{x+9}{5}\right)dx + \int_3^6 \left(\frac{21-2x}{3} - \frac{x+9}{5}\right)dx$

$= \left[\frac{3x^2}{4} + \frac{x}{2} - \frac{x^2}{10} - \frac{9}{5}x\right]_1^3 + \left[7x - \frac{x^2}{3} - \frac{x^2}{10} - \frac{9}{5}x\right]_3^6$

$= \frac{13}{5} + \frac{39}{10} = \frac{65}{10} = \frac{13}{2}$ sq. units

19. We have curves, $y = \frac{1}{\sqrt{3}}x$ ...(i)

and $x^2 + y^2 = 16$ ...(ii)

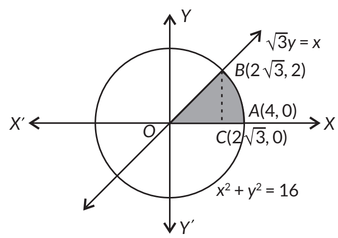
Curves (i) and (ii) intersect at $(2\sqrt{3}, 2)$ and $(-2\sqrt{3}, -2)$.

$\therefore$ Required area = Area of region $OBAO$

= area of ΔOBC + area of region $BCAB$

$= \int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^4 \sqrt{16-x^2}\,dx$

$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{2\sqrt{3}} + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{2\sqrt{3}}^4$



$= 2\sqrt{3} + 8\left(\frac{\pi}{2}\right) - 2\sqrt{3} - \frac{8\pi}{3} = \frac{12\pi - 8\pi}{3} = \frac{4\pi}{3}$ sq.units

Commonly Made Mistake

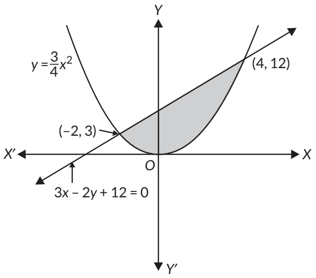
➲ Remember difference between the formula for $\int\sqrt{a^2-x^2}\,dx$ and $\int\sqrt{x^2-a^2}\,dx$.

20. Given equations are $y = \frac{3x^2}{4}$ ...(i)

and $3x - 2y + 12 = 0 \Rightarrow y = \frac{3x+12}{2}$ ...(ii)

Solving equations (i) and (ii), we get

$\frac{3x^2}{4} = \frac{3x+12}{2}$



$\Rightarrow \quad x^2 - 2x - 8 = 0 \Rightarrow (x + 2)(x - 4) = 0 \Rightarrow x = -2, 4$

When $x = -2 \Rightarrow y = 3$

When $x = 4 \Rightarrow y = 12$

$\therefore$ Required area $= \int_{-2}^{4}\left(\frac{3x+12}{2}-\frac{3}{4}x^2\right)dx$

$$=\left[\frac{3}{4}x^2+6x-\frac{x^3}{4}\right]_{-2}^{4}$$

$$=\left[\frac{3\times16}{4}+6\times4-\frac{64}{4}\right]-\left[\frac{3}{4}\times4-6\times2+\frac{8}{4}\right] = 27 \text{ sq. units.}$$

21. We have curves $x - y + 2 = 0$ and $x=\sqrt{y}$.

$\Rightarrow \quad y = x^2$, which is a parabola with vertex at origin.

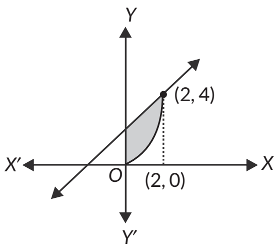
From the given equations, we get

$x - x^2 + 2 = 0 \Rightarrow (x - 2)(x + 1) = 0$

$\Rightarrow x = 2$ or $x = -1 \Rightarrow x = 2$ $\quad$ [$\because x \neq -1$, x is positive]

When $x = 2$, $y = 4$

So, the point of intersection is (2, 4).



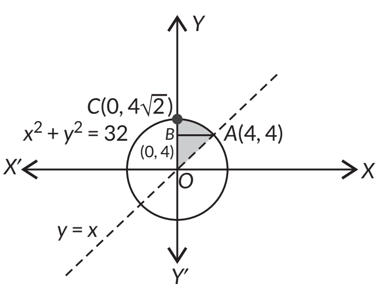
$\therefore$ Required area $=\int_0^2(x+2)dx-\int_0^2 x^2dx \quad =\int_0^2(x+2-x^2)dx$

$$=\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_0^2$$

$$=2+4-\frac{8}{3}=\frac{10}{3}\text{sq.units}$$

22. The given equation of the circle is $x^2 + y^2 = 32$ and the line is $y = x$.

These intersect at $A(4, 4)$ in the first quadrant. The required area is shown shaded in the figure. Points $B(0, 4)$ and $C\left(0,4\sqrt{2}\right)$.



$\therefore$ Required area = Area $BACB$ + Area $OABO$

$$=\int_4^{4\sqrt{2}}x_1dy+\int_0^4 x_2dy$$

$$=\int_4^{4\sqrt{2}}\sqrt{32-y^2}dy+\int_0^4 y\,dy$$

$$=\int_4^{4\sqrt{2}}\sqrt{(4\sqrt{2})^2-y^2}dy+\int_0^4 y\,dy$$

$$=\left[\frac{y\sqrt{32-y^2}}{2}+\frac{32}{2}\sin^{-1}\left(\frac{y}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}+\left[\frac{y^2}{2}\right]_0^4$$

$$=\frac{4\sqrt{2}\times0}{2}+16\sin^{-1}1-\left(\frac{4\times4}{2}+16\sin^{-1}\frac{1}{\sqrt{2}}\right)+\frac{1}{2}(4^2-0)$$

$$=16\cdot\frac{\pi}{2}-\left(8+16\cdot\frac{\pi}{4}\right)+8=16\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=4\pi\text{ sq.units}$$

Concept Applied

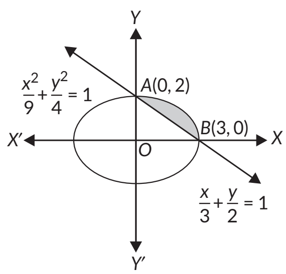
- Area of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = a$ and $y = b$ ($b > a$) is,

  Area $=\int_a^b x\,dy=\int_a^b g(y)\,dy$

23. We have $\frac{x^2}{9}+\frac{y^2}{4}=1$ ...(i) and $\frac{x}{3}+\frac{y}{2}=1$ ...(ii)

Curve (i) is an ellipse of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

That means its major axis is along x – axis. Also this ellipse is symmetrical about the x – axis.



Required area $=\frac{2}{3}\int_0^3\sqrt{(3)^2-x^2}dx-\frac{2}{3}\int_0^3(3-x)dx$

$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3-\frac{2}{3}\left[\frac{(3-x)^2}{-2}\right]_0^3$$

$$=\frac{2}{3}\left[\left(0+\frac{9}{2}\sin^{-1}(1)\right)-\left(0+\frac{9}{2}\sin^{-1}(0)\right)\right]+\frac{1}{3}[0^2-9]$$

$$=\frac{3\pi}{2}-3\text{ sq. units.}$$

24. Here, $y = |x + 1| + 1$

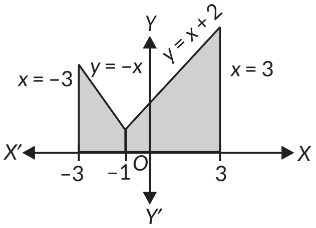
$$y=\begin{cases}x+2 & \text{if} \quad x\geq-1\\ -x & \text{if} \quad x<-1\end{cases}$$

We know draw the lines : $y = 0, x = 3, x = -3$ and

$y = x + 2$ if $x \geq -1$ ...(i)

$y = -x$ if $x < -1$ ...(ii)

Lines (i) and (ii) intersect at $(-1, 1)$



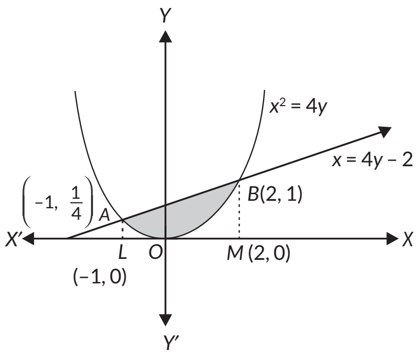
$\therefore$ Required area $= \int_{-3}^{-1}(-x)dx + \int_{-1}^{3}(x+2)dx$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2}+2x\right]_{-1}^{3} = -\frac{1}{2}(1-9)+\frac{1}{2}(9-1)+2(3+1)$$

$= 4 + 4 + 8 = 16$ sq. units.

25. The given curve is $x^2 = 4y$ ...(i)

The given line is $x = 4y - 2$ ...(ii)



Putting $4y = (x + 2)$ from (ii) in (i), we get $(x + 2) = x^2$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2, -1$

Putting $x = 2$ in (i), we get $y = 1$

Putting $x = -1$ in (i), we get $y = \frac{1}{4}$

Thus the points of intersection of the given curve and line are $A\left(-1, \frac{1}{4}\right)$ and $B(2, 1)$.

$\therefore$ Required area

$$= \int_{-1}^{2}\left(\frac{x+2}{4}\right)dx - \int_{-1}^{2}\frac{x^2}{4}dx = \int_{-1}^{2}\left(\frac{x}{4}+\frac{1}{2}-\frac{x^2}{4}\right)dx$$

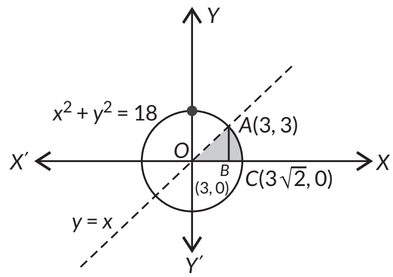
$$= \frac{1}{4}\left[\frac{x^2}{2}\right]_{-1}^{2} + \frac{1}{2}[x]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{8}[4-1]+\frac{1}{2}[2+1]-\frac{1}{12}[8+1]$$

$$= \frac{3}{8}+\frac{3}{2}-\frac{3}{4} = \frac{3}{2}\left[\frac{1}{4}+1-\frac{1}{2}\right] = \frac{3}{2}\left[\frac{3}{4}\right] = \frac{9}{8} \text{ sq. units}$$

26. The given equation of the circle is $x^2 + y^2 = 18$ and the line is $y = x$.

These intersect at $A(3, 3)$ in the first quadrant. The required area is shown shaded in the figure. Points $B(3, 0)$ and $C(3\sqrt{2}, 0)$.



$\therefore$ Required area = Area $BACB$ + Area $OABO$

$$= \int_{3}^{3\sqrt{2}} y_1 dx + \int_0^3 y_2 dx = \int_{3}^{3\sqrt{2}} \sqrt{18-x^2}\,dx + \int_0^3 x\,dx$$

$$= \int_{3}^{3\sqrt{2}} \sqrt{(3\sqrt{2})^2 - x^2}\,dx + \int_0^3 x\,dx$$

$$= \left[\frac{x\sqrt{18-x^2}}{2}+\frac{18}{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right]_3^{3\sqrt{2}} + \left[\frac{x^2}{2}\right]_0^3$$

$$= \frac{3\sqrt{2}\times 0}{2}+9\sin^{-1}1-\left(\frac{3\times 3}{2}+9\sin^{-1}\frac{1}{\sqrt{2}}\right)+\frac{1}{2}(9-0)$$

$$= 9\cdot\frac{\pi}{2}-\left(\frac{9}{2}+9\cdot\frac{\pi}{4}\right)+\frac{9}{2} = 9\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{9\pi}{4} \text{ sq.units}$$

Answer Tips

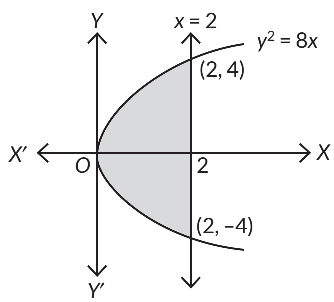
- Area of the region in the first quadrant enclosed by x-axis, the line $y = x$ and circle $x^2 + y^2 = a^2$ is $\frac{\pi a^2}{8}$.

CBSE Sample Questions

1. Required area, $A = \int_{-1}^{1} x^2\,dx$

$$\Rightarrow A = 2\int_0^1 x^2 dx = \frac{2}{3}\left[x^3\right]_0^1 = \frac{2}{3} \text{ sq.units} \quad \textbf{(1)}$$

2. Required area



$= 2\int_0^2 \sqrt{8x}\,dx$ **(1)**

$= 2\times 2\sqrt{2}\int_0^2 x^{1/2}\,dx$

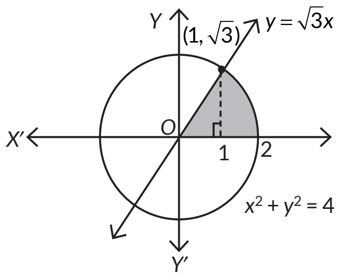
$= 4\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2$ **(1/2)**

$= \frac{8}{3}\sqrt{2}[2^{3/2}-0] = \frac{8\sqrt{2}}{3}\times 2\sqrt{2} = \frac{32}{3}$ sq.units **(1/2)**

3. Solving $y = \sqrt{3}x$ and $x^2 + y^2 = 4$, we get

$x^2 + 3x^2 = 4$

$\Rightarrow x^2 = 1 \Rightarrow x = 1$ **(1/2)**



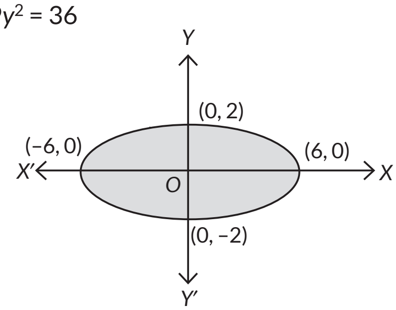
(1/2)

$\therefore$ Required area $= \sqrt{3}\int_0^1 x\,dx + \int_1^2 \sqrt{2^2 - x^2}\,dx$ **(1/2)**

$$= \frac{\sqrt{3}}{2}[x^2]_0^1 + \left[\frac{x}{2}\sqrt{2^2 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$ **(1)**

$$= \frac{\sqrt{3}}{2} + \left[2\times\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2\times\frac{\pi}{6}\right] = \frac{2\pi}{3}\text{ sq. units}$$ **(1/2)**

4. Given equation of ellipse is
$x^2 + 9y^2 = 36$



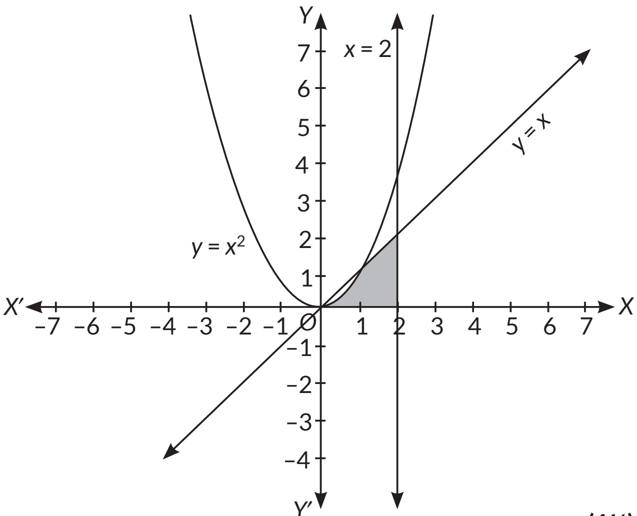
$\therefore$ Required area $= 4\int_0^6 \sqrt{\frac{36 - x^2}{9}}\,dx$ **(1/2)**

$$= \frac{4}{3}\int_0^6 \sqrt{6^2 - x^2}\,dx$$ **(1/2)**

$$= \frac{4}{3}\left[\frac{x}{2}\sqrt{6^2 - x^2} + 18\sin^{-1}\left(\frac{x}{6}\right)\right]_0^6$$ **(1)**

$$= \frac{4}{3}\left[18\times\frac{\pi}{2} - 0\right] = 12\pi\text{ sq. units}$$ **(1)**

5. The graph of given region is;



(1½)

The points of intersection of the parabola $y = x^2$ and the line $y = x$ are

$x^2 = x$

$\Rightarrow$ $x(x - 1) = 0$

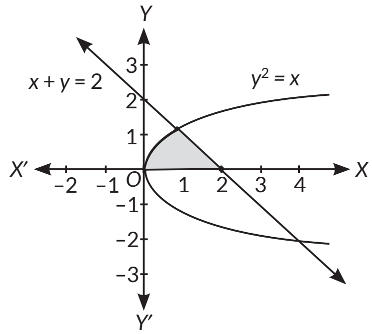
$\Rightarrow$ $x = 0, 1$ **(1)**

So, point of intersection is (0, 0) and (1, 1).

Required area $= \int_0^1 x^2 dx + \int_1^2 x\,dx$ **(1)**

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}\text{ sq. units}$$ **(1½)**

6. By solving $x + y = 2$ and $y^2 = x$ simultaneously, we get the points of intersection as (1, 1) and (4, –2). **(1)**



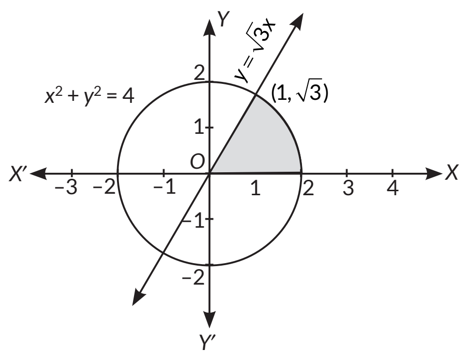
(1)

$\therefore$ Required area = The shaded area

$$= \int_0^1 \sqrt{x}\,dx + \int_1^2 (2 - x)dx$$ **(1)**

$$= \frac{2}{3}[x^{3/2}]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}\text{ square units.}$$ **(1)**

7. By solving $y = \sqrt{3}x$ and $x^2 + y^2 = 4$, we get the points of intersection as $(1, \sqrt{3})$ and $(-1, -\sqrt{3})$. **(1)**



(1)

$\therefore$ Required area = The shaded area

$$= \int_0^1 \sqrt{3}x\,dx + \int_1^2 \sqrt{4 - x^2}\,dx$$

$$= \frac{\sqrt{3}}{2}[x^2]_0^1 + \frac{1}{2}\left[x\sqrt{4 - x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$ **(1)**

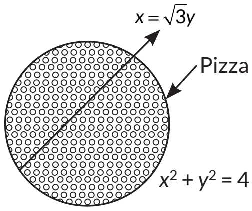
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left[2\pi - \sqrt{3} - 2\times\frac{\pi}{3}\right]$$

$$= \frac{2\pi}{3}\text{ square units.}$$ **(1)**

Self Assessment

Case Based Objective Questions (4 marks)

1. A child cut a pizza with a knife. Pizza is circular in shape which is represented by $x^2 + y^2 = 4$ and sharp edge of knife represents a straight line given by $x = \sqrt{3}y$.



Based on above information, attempt any 4 out of 5 subparts.

(i) The point(s) of intersection of the edge of knife (line) and pizza shown in the figure is (are)

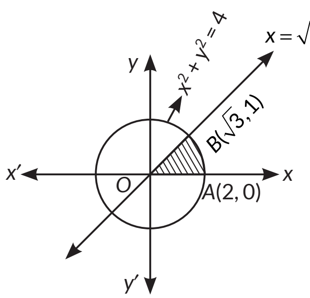
(a) $(1, \sqrt{3}), (-1, -\sqrt{3})$

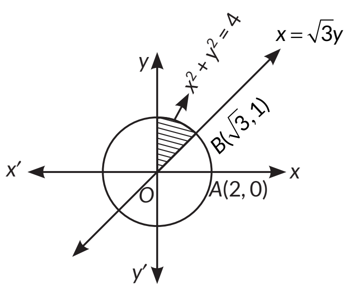
(b) $(\sqrt{3}, 1), (-\sqrt{3}, -1)$

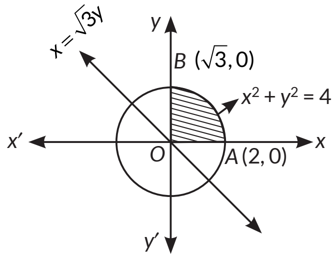
(c) $(\sqrt{2}, 0), (0, \sqrt{3})$

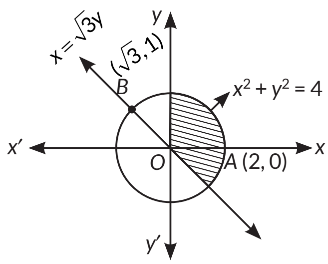
(d) $(-\sqrt{3}, 1), (1, -\sqrt{3})$

(ii) Which of the following shaded portion represent the smaller area bounded by pizza and edge of knife in first quadrant?

(a) 

(b) 

(c) 

(d) 

(iii) Value of area of the region bounded by circular pizza and edge of knife in first quadrant is

(a) $\frac{\pi}{2}$ sq. units (b) $\frac{\pi}{3}$ sq. units

(c) $\frac{\pi}{5}$ sq. units (d) π sq. units

(iv) Area of each slice of pizza when child cut the pizza into 4 equal pieces is

(a) π sq. units (b) $\frac{\pi}{2}$ sq. units

(c) 3π sq. units (d) 2π sq. units

(v) Area of whole pizza is

(a) 3π sq. units (b) 2π sq. units

(c) 5π sq. units (d) 4π sq. units

Multiple Choice Questions (1 mark)

2. Find the area enclosed between the curve $x^2 + y^2 = 16$ and the coordinate axes in the first quadrant.

(a) 4π sq. units (b) 8π sq. units

(c) 2π sq. units (d) 12π sq. units

3. The area between the curve $y = 4 + 3x - x^2$ and x-axis is

(a) $\frac{125}{6}$ sq. units (b) $\frac{125}{3}$ sq. units

(c) $\frac{125}{2}$ sq. units (d) none of these

4. The area bounded by the curve $2x^2 + y^2 = 2$ is

(a) π sq. units (b) $\sqrt{2}\pi$ sq. units

(c) $\frac{\pi}{2}$ sq. units (d) 2π sq. units

5. The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is given by

(a) $\int_{-2}^{1}(3-x^2)dx$

(b) $\int_{-2}^{1}\{(x^2+1)\}-3x\,dx$

(c) $\int_{-1}^{1}\{3-x-(x^2+1)\}dx$

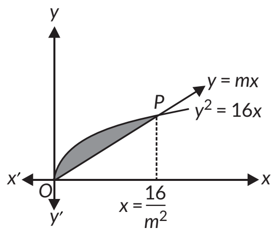
(d) $\int_{-2}^{1}\{3-x-(x^2+1)\}dx$

6. The area bounded by $x = |y|$ and line $x = 2$, is
(a) 4 sq. units (b) 5 sq. units
(c) 1 sq. unit (d) 3 sq. units

7. Area lying between the parabola $y^2 = 4x$ and its latus rectum is
(a) $\frac{1}{3}$ sq. units (b) $\frac{2}{3}$ sq. units
(c) $\frac{5}{3}$ sq. units (d) $\frac{8}{3}$ sq. units

OR

The area by of shaded region in the given figure is $\frac{2}{3}$, then m is equal to



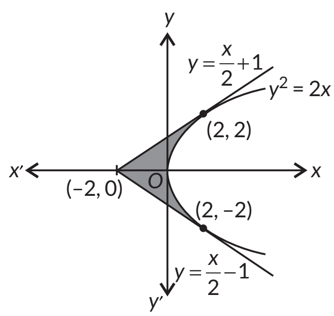
(a) 3 (b) 4 (c) 1 (d) 2

VSA Type Questions (1 mark)

8. Find the area enclosed between the graph of $y = x^3$ and the lines $x = 0, y = 1, y = 8$.

9. Find the area bounded by the curve $x^2 = 4y + 4$ and line $3x + 4y = 0$.

10. Calculate the shaded area of given figure.



OR

The area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = 1, x = b$ is equal to $\sqrt{b^2+1}-\sqrt{2}$ for all $b > 1$, then find the function $f(x)$.

SA I Type Questions (2 marks)

11. Find the area bounded by the curve $y = x|x|$, x-axis and the lines $x = -3$ and $x = 3$.

12. If the area above x-axis, bounded by the curves $y = 2^{kx}, x = 0$ and $x = 2$ is $\frac{3}{\log_e 2}$, then find the value of k.

13. Find the area bounded by the curves $y = \left[\frac{x^2}{64}+2\right]$, $y = x - 1$ and $x = 0$ above x-axis ([·] denotes the greatest integer function).

OR

Find the area bounded by the curve $y = \sec^2 x$, $y = 0$ and $|x| = \frac{\pi}{3}$.

SA II Type Questions (3 marks)

14. Find the area of the region bounded by $y^2 = 2x + 1$ and $x - y = 1$.

15. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is divided into two parts by the line $2x = a$. Find the area of the smaller part.

16. If the line $y = mx$ divides the area enclosed by the lines $x = 0, y = 0, x=\frac{3}{2}$ and curve $y = 1 + 4x - x^2$ into two equal parts, then what is the value of m?

OR

Find the area bounded by the lines $y = ||x| - 1|$ and the x-axis.

Case Based Questions (4 marks)

17. Consider the following equation of curve $y = \cos x$ and the lines $y = x + 1$ and $y = 0$.
Based on the given information, answer the following questions.
(i) Find the point at which the curves $y = \cos x$ and $y = x + 1$ meet.
(ii) What is the area bounded by the given curve and the lines ?

LA Type Questions (4/6 marks)

18. Find the area of the triangle formed by the tangent and normal at the point $(1, \sqrt{3})$ on the circle $x^2 + y^2 = 4$ and the x-axis.

OR

Calculate the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

19. AOB is a positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $OA = a$, $OB = b$. Then, find the area between the arc AB and chord AB of the ellipse.

20. Using integration, find the area of the triangle ABC whose vertices have coordinates $A(2, 5)$, $B(4, 7)$ and $C(6, 2)$.

Detailed SOLUTIONS

1. (i) (b) : We have, $x^2 + y^2 = 4$ ...(i)

and $x = \sqrt{3}y$ ...(ii)

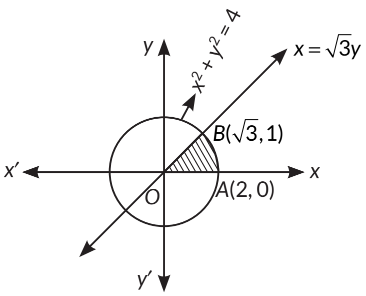
From (i) and (ii), we get

$3y^2 + y^2 = 4 \Rightarrow 4y^2 = 4 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

From (ii), $x = \sqrt{3}, -\sqrt{3}$

$\therefore$ Points of intersection of pizza and edge of knife are $(\sqrt{3}, 1), (-\sqrt{3}, -1)$.

(ii) (a) :



(iii) (b) : Required area $= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\, dx$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{\sqrt{3}}\left[\frac{3}{2} - 0\right] + \left[2\sin^{-1}(1) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{2\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq. units}$$

(iv) (a) : We have, $x^2 + y^2 = 4$

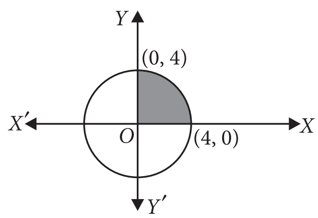
$\Rightarrow (x - 0)^2 + (y - 0)^2 = (2)^2$

$\therefore$ Radius = 2

Area of $\frac{1}{4}$ th slice of pizza $= \frac{1}{4}\pi(2)^2 = \pi$ sq. units

(v) (d) : Area of whole pizza $= \pi(2)^2 = 4\pi$ sq. units

2. (a) : Given curve is a circle with centre (0, 0) and radius 4.



$\therefore$ Required area $= \int_0^4 \sqrt{16 - x^2}\, dx$

$$= \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4 = 4\pi \text{ sq. units}$$

3. (a) : We have, $y = 4 + 3x - x^2$, a parabola with vertex at $\left(\frac{3}{2}, \frac{25}{4}\right)$.

Putting $y = 0$, we get $x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0 \Rightarrow x = -1$ or $x = 4$

$\therefore$ Required area $= \int_{-1}^{4} (4 + 3x - x^2)dx$

$$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4} = \frac{125}{6} \text{ sq. units}$$

4. (b) : We have, $2x^2 + y^2 = 2$

$\Rightarrow \frac{x^2}{1} + \frac{y^2}{2} = 1$, an ellipse

Here, $a = 1$ and $b = \sqrt{2}$

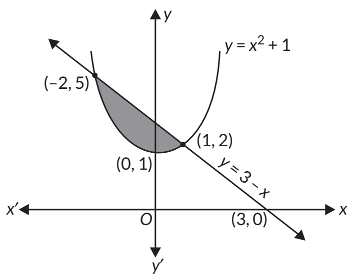
$\because$ Area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

$\therefore$ Required area $= \pi\sqrt{2}$ sq. units.

5. (d) : We have, $y = x^2 + 1$ ...(i)

and $x + y = 3$ ...(ii)

Solving (i) and (ii), we get

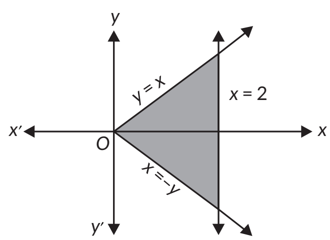


$x^2 + x - 2 = 0 \Rightarrow x = -2, 1$

$\therefore$ Required area = area of shaded region

$$= \int_{-2}^{1} (\text{line} - \text{parabola})dx = \int_{-2}^{1} \{3 - x - (x^2 + 1)\}dx$$

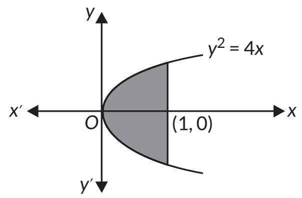
6. (a) : We have, $x = y$, if $y \geq 0$, $x = -y$, if $y < 0$ and $x = 2$



Required area = area of shaded region

$$= 2\int_0^2 x\, dx = 2\left[\frac{x^2}{2}\right]_0^2 = 4 \text{ sq. units}$$

7. (d) : We know that the area of region bounded by the parabola $y^2 = 4ax$ and its latus rectum is $\frac{8}{3}a^2$ sq. units



Here, $a = 1$, therefore required area $= \frac{8}{3}$ sq. units

OR

(b) : Required area = area of shaded region

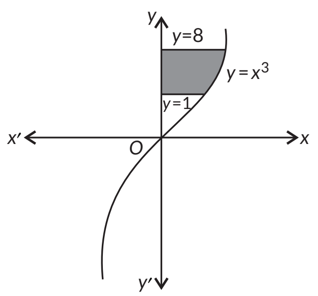
$$\Rightarrow \int_0^{16/m^2} (\sqrt{16x} - mx)dx = \frac{2}{3}$$

$$\Rightarrow \left[4 \times \frac{2}{3} x^{3/2} - m\left(\frac{x^2}{2}\right)\right]_0^{16/m^2} = \frac{2}{3}$$

$$\Rightarrow \frac{8}{3} \times \frac{64}{m^3} - \frac{m}{2}\frac{256}{m^4} = \frac{2}{3} \Rightarrow \frac{1}{m^3}\left[\frac{512}{3} - 128\right] = \frac{2}{3}$$

$\Rightarrow \quad m = 4$

8. Given curve is $y = x^3$ or $x = y^{1/3}$



$\therefore$ Required area = area of shaded region

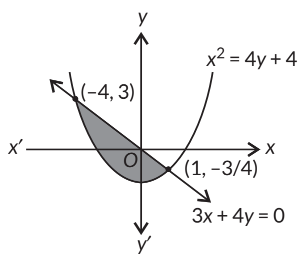
$$= \int_1^8 y^{1/3} dy = \left[\frac{y^{4/3}}{4/3}\right]_1^8 = \frac{3}{4}\left[8^{4/3} - 1^{4/3}\right]$$

$$= \frac{3}{4} \times (16 - 1) = \frac{3}{4} \times 15 = \frac{45}{4} \text{ sq. units}$$

9. We have, $x^2 = 4y + 4$ ...(i)

and $3x + 4y = 0$ ...(ii)

Solving (i) and (ii), we get $x = -4, 1$



$\therefore$ Required area = area of shaded region.

$$= \int_{-4}^{1}\left(-\frac{3x}{4} - \frac{x^2}{4} + 1\right)dx$$

$$= -\frac{3}{8}(1-16) - \frac{1}{12}(1+64) + 5 = \frac{45}{8} - \frac{5}{12} = \frac{125}{24} \text{ sq. units}$$

10. Required area = area of shaded region

$$= 2\int_0^2\left(\frac{y^2}{2} - 2y + 2\right)dy = 2\left[\frac{y^3}{6} - y^2 + 2y\right]_0^2$$

$$= 2\left[\frac{4}{3} - 4 + 4\right] = \frac{8}{3} \text{ sq. units}$$

OR

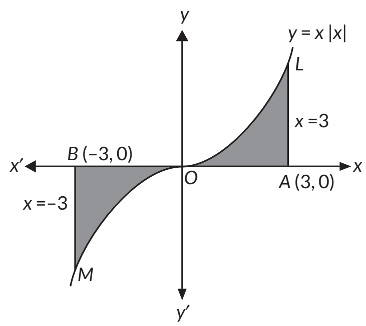
We have, $\int_1^b f(x)dx = \sqrt{b^2+1} - \sqrt{2}$

On differentiating w.r.t. b, we get

$$f(b) = \frac{2b}{2\sqrt{b^2+1}} \Rightarrow f(x) = \frac{x}{\sqrt{x^2+1}}$$

11. The equation of the curve is

$$y = x|x| = \begin{cases} x^2, x \ge 0 \\ -x^2, x < 0 \end{cases}$$



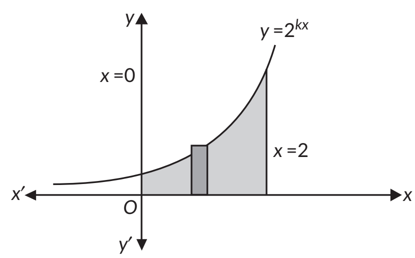
Required area = 2(Area of region shaded in first quadrant)

$$= 2\int_0^3 x^2 dx = 2 \times \left[\frac{x^3}{3}\right]_0^3 = 2 \times 9 = 18 \text{ sq. units}$$

12. It is given that $\int_0^2 y\,dx = \frac{3}{\log_e 2}$

$$\Rightarrow \int_0^2 2^{kx} dx = \frac{3}{\log_e 2} \Rightarrow \left[\frac{2^{kx}}{k\log_e 2}\right]_0^2 = \frac{3}{\log_e 2}$$

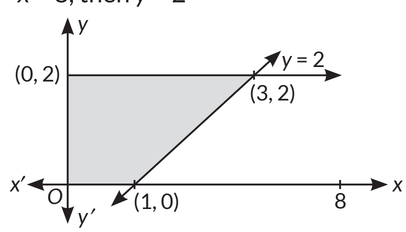
$$\Rightarrow \frac{2^{2k}}{k\log_e 2} - \frac{1}{k\log_e 2} = \frac{3}{\log_e 2} \Rightarrow \frac{4^k - 1}{k} = 3$$



Clearly, $k = 1$ satisfies this equation.

$\therefore \quad k = 1.$

13. If $-8 < x < 8$, then $y = 2$

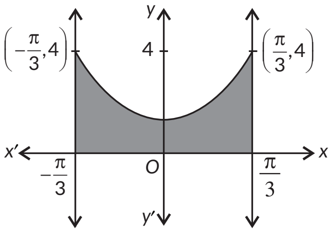


$\therefore$ Required area = area of shaded region

$= \frac{1}{2}(1+3)\times 2 = 4$ sq. units

OR

We have, $y = \sec^2 x$ and $y = 0$ and $x = \frac{\pi}{3}, -\frac{\pi}{3}$

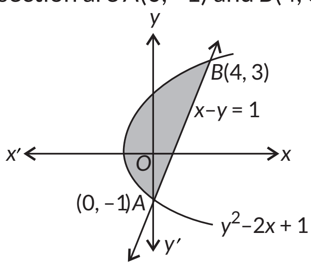


Required area = area of shaded region

$$= \int_{-\pi/3}^{\pi/3} \sec^2 x\, dx = [\tan x]_{-\pi/3}^{\pi/3} = 2\sqrt{3} \text{ sq. units}$$

14. Given $y^2 = 2x + 1$ and $x - y = 1$

Points of intersection are $A(0, -1)$ and $B(4, 3)$.

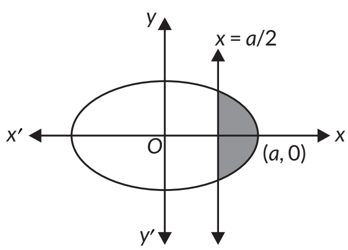


$\therefore$ Required area $= \int_{-1}^{3}(1+y)dy - \int_{-1}^{3}\left(\frac{y^2-1}{2}\right)dy$

$$= \left[y + \frac{y^2}{2}\right]_{-1}^{3} - \left[\frac{1}{2}\left(\frac{y^3}{3} - y\right)\right]_{-1}^{3}$$

$$= \left[3 + \frac{9}{2} - \left(-1 + \frac{1}{2}\right)\right] - \frac{1}{2}\left[9 - 3 - \left(-\frac{1}{3} + 1\right)\right] = \frac{16}{3} \text{ sq. units}$$

15. We have, an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, having centre at (0, 0)



Required area = area of shaded region

$$= 2\int_{a/2}^{a} y\, dx = \frac{2b}{a}\int_{\frac{a}{2}}^{a}\sqrt{a^2 - x^2}\, dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$, Then, we get

Required area $= 2ab\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta d\theta = ab\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1+\cos 2\theta)d\theta$

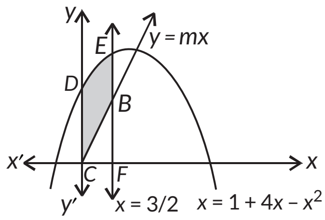
$$= ab\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = ab\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] \text{ sq. units}$$

16. The given curve is $(y - 5) = -(x - 2)^2$

Thus, given curve is a parabola with vertex at (2, 5) and axis is $x = 2$.

Given that area ($CBFC$) = area ($CDEBC$)

So, area ($CDEBFC$) = 2 area ($CBFC$)



Now, area ($CDEBFC$) $= \int_{0}^{3/2}(1 + 4x - x^2)dx$

$$= \left[x + 2x^2 - \frac{x^3}{3}\right]_0^{3/2} = \frac{3}{2} + 2\left(\frac{9}{4}\right) - \frac{9}{8}$$

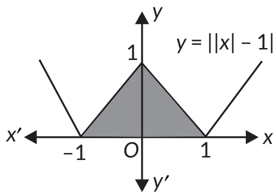
$$= \frac{39}{8} \text{ sq.units}$$

Area $CBFC = \int_{0}^{3/2} mx\,dx = \frac{9m}{8}$

So, we must have $\frac{39}{8} = \frac{18m}{8} \Rightarrow m = \frac{13}{6}$

OR

We have, $y = ||x| - 1|$, if $x \ge 0$



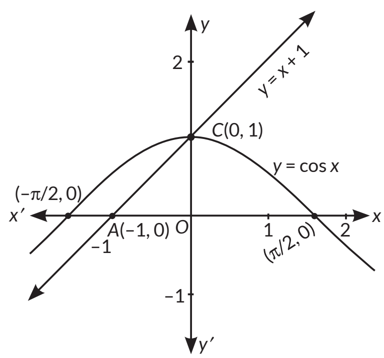
$= \begin{cases}(x-1), & \text{if } x \ge 1\\ -(x-1), & \text{if } x < 1\end{cases}$ and $y = |-x-1| = |-(x+1)| = |x+1|$ if $x < 0$

$= \begin{cases}(x+1), & \text{if } x \ge -1\\ -(x+1), & \text{if } x < -1\end{cases}$

Required area $= 2\int_0^1 (1-x)dx$

$$= 2\left[x - \frac{x^2}{2}\right]_0^1 = 2\times\frac{1}{2} = 1 \text{ sq. unit}$$

17. **(i)** Curves $y = \cos x$ and $y = x + 1$ meet at point $C(0, 1)$ as shown in the graph



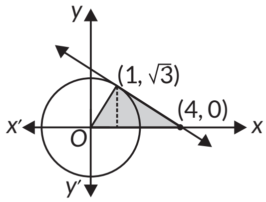
(ii) Required area $= \int_{-1}^{0}(x+1)dx+\int_{0}^{\pi/2}\cos x\,dx$

$$= \frac{1}{2}+1=\frac{3}{2}\text{ sq. units}$$

18. The tangent on $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is $x + \sqrt{3}y = 4$ and equation of normal at $(1, \sqrt{3})$ is $y = x\sqrt{3}$

Required area $= \int_0^1 x\sqrt{3}\,dx+\int_1^4\frac{4-x}{\sqrt{3}}dx$

$$=\sqrt{3}\left[\frac{x^2}{2}\right]_0^1+\frac{1}{\sqrt{3}}\left[4x-\frac{x^2}{2}\right]_1^4$$



$$=\sqrt{3}\times\frac{1}{2}+\frac{1}{\sqrt{3}}\left[4(4-1)-\frac{1}{2}(16-1)\right]$$

$$=\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{2}=2\sqrt{3}\text{ sq. units}$$

OR

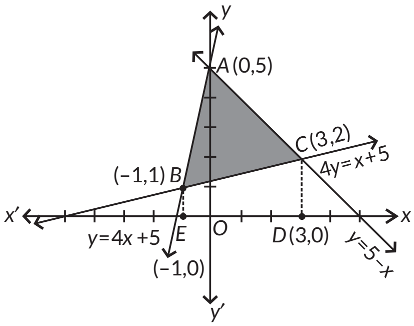
We have, $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$

On solving $y = 4x + 5$ and $y = 5 - x$, we get $x = 0$ and $y = 5$

On solving $y = 4x + 5$ and $4y = x + 5$, we get $x = -1$ and $y = 1$

On solving $y = 5 - x$ and $4y = x + 5$, we get $x = 3$ and $y = 2$

Let us draw a rough sketch of the lines on the graph.



Now, required area = area of shaded region

= Area($EBAOE$) + Area($ACDOA$) – Area($EBCDOE$)

$$=\int_{-1}^{0}(4x+5)\,dx+\int_{0}^{3}(5-x)\,dx-\frac{1}{4}\int_{-1}^{3}(x+5)\,dx$$

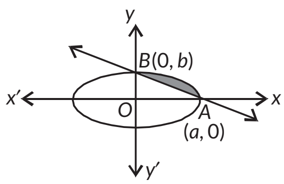
$$=\left[2x^2+5x\right]_{-1}^{0}+\left[5x-\frac{x^2}{2}\right]_0^3-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

$$=[0-(2-5)]+\left[15-\frac{9}{2}\right]-\frac{1}{4}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$$

$$=\frac{27}{2}-\frac{1}{4}\times\frac{48}{2}=\frac{27}{2}-6=\frac{15}{2}\text{ sq. units}$$

19. Required area

$$=\int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}(a-x)\right)dx$$



$$=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx-\frac{b}{a}\int_0^a(a-x)dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a-\frac{b}{a}\left[ax-\frac{x^2}{2}\right]_0^a$$

$$=\frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}1\right]-\frac{b}{a}\left[a^2-\frac{a^2}{2}\right]=\frac{ab}{2}\frac{\pi}{2}-ba\left(\frac{1}{2}\right)=\frac{ab}{4}(\pi-2)\text{ sq.units}$$

20. First we find the equations of the sides of triangle ABC by using $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

The equation of AB is

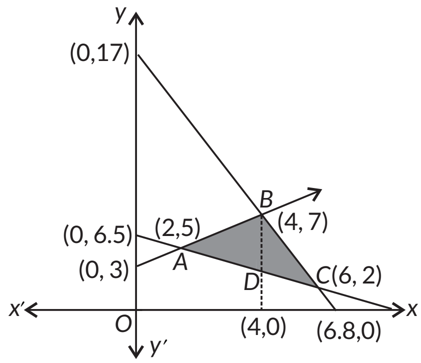
$$y-5=\frac{7-5}{4-2}(x-2)\Rightarrow x-y+3=0 \quad\text{...(i)}$$

The equation of BC is

$$y-7=\frac{2-7}{6-4}(x-4)\Rightarrow 5x+2y-34=0 \quad\text{...(ii)}$$

The equation of side AC is

$$y-5=\frac{2-5}{6-2}(x-2)\Rightarrow 3x+4y-26=0 \quad\text{...(iii)}$$



Clearly, Area of ΔABC = Area (ΔADB) + Area (ΔBDC)

$$\text{Area}(\Delta ADB)=\int_2^4\left\{(x+3)-\left(\frac{26-3x}{4}\right)\right\}dx$$

Similarly, we have

$$\text{Area}(\Delta BDC)=\int_4^6\left\{\left(\frac{34-5x}{2}\right)-\left(\frac{26-3x}{4}\right)\right\}dx$$

$\therefore$ Area of ΔABC

$$=\int_2^4\left\{(x+3)-\left(\frac{26-3x}{4}\right)\right\}dx+\int_4^6\left\{\left(\frac{34-5x}{2}\right)-\left(\frac{26-3x}{4}\right)\right\}dx$$

$$=\frac{1}{4}\int_2^4(7x-14)dx+\frac{1}{4}\int_4^6(42-7x)dx$$

$$=\frac{1}{4}\left[\left[\frac{7x^2}{2}-14x\right]_2^4+\left[42x-\frac{7x^2}{2}\right]_4^6\right]$$

$$=\frac{1}{4}[\{(56-56)-(14-28)\}+\{(252-126)-(168-56)\}]$$

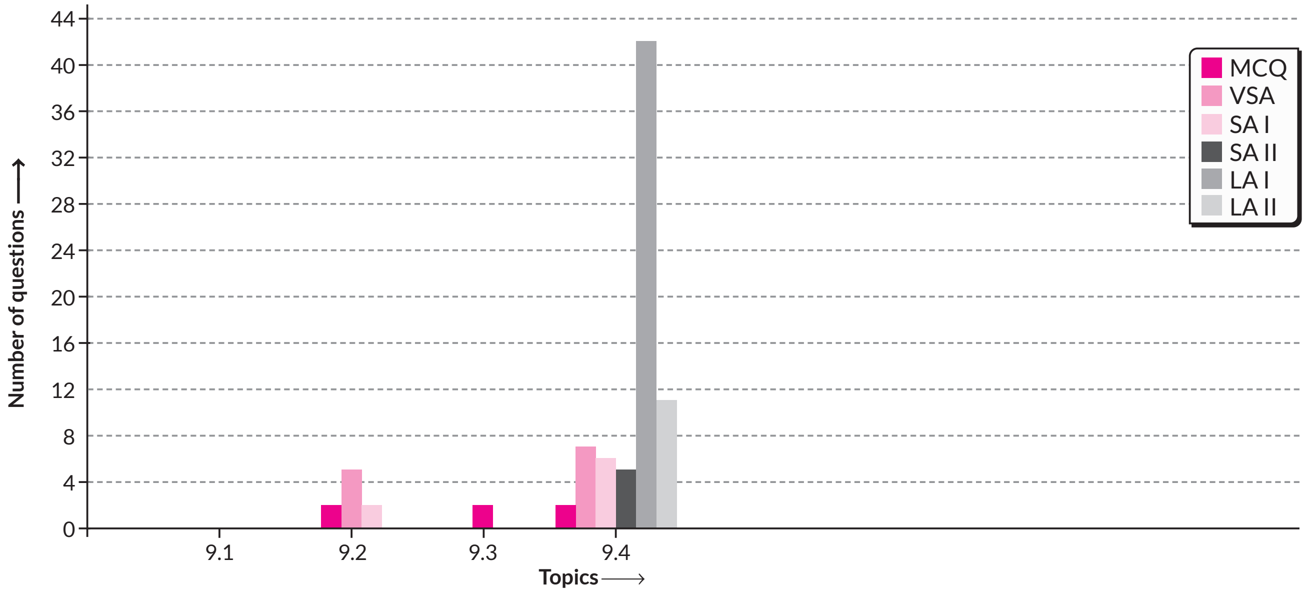
$$=\frac{1}{4}[14+126-112]=7\text{ sq. units}$$

CHAPTER 9

Differential Equations

TOPICS

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 9.4 is highly scoring topic.
- Maximum weightage is from Topic 9.4 *Methods of Solving First Order, First Degree Differential Equations.*
- Maximum LA I and LA II type questions were asked from Topic 9.4 *Methods of Solving First Order, First Degree Differential Equations.*

QUICK RECAP

Differential Equation

- An equation involving an independent variable, a dependent variable and the derivatives of the dependent variable is called differential equation.
  - A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation.
  - A differential equation involving derivatives with respect to more than one independent variables is called a partial differential equation.
- **Order and Degree of a Differential Equation**
  - The order of highest derivative appearing in a differential equation is called order of the differential equation.

- The power of the highest order derivative appearing in a differential equation, after it is made free from radicals and fractions, is called degree of the differential equation.

  Note : Order and degree (if defined) of a differential equation are always positive integers.

Homogeneous Differential Equations

- A differential equation of the form

  $$\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$$

  where, $f(x, y)$ and $g(x, y)$ are homogeneous functions of x and y of the same degree.

Linear Differential Equations

- An equation of the form $\frac{dy}{dx}+Py=Q$ where P and Q are functions of x only (or constants) is called a linear differential equation of the first order.

Solution of a Differential Equation

- Solution of a differential equation is a function of the form $y = f(x) + C$ which satisfies the given differential equation.
- **General Solution :** The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation.
- **Particular Solution :** A solution obtained by giving particular values to arbitrary constants in the general solution.

Methods of Solving Differential Equations

- **Equation in Variable Separable Form :** If the differential equation is of the form $f(x)\,dx = g(y)\,dy$, then the variables are separable and such equations can be solved by integrating on both sides. The solution is given by $\int f(x)\,dx = \int g(y)\,dy + C$, where C is an arbitrary constant.
- **Equation Reducible to Homogeneous Form :** If the equation is of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree in x and y, then put $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ so that the dependent variable y is changed to another variable v, then apply variable separable method.
- **Solution of Linear Differential Equation :** A differential equation of the form $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x (or constants) can be solved as :

  1. Find Integrating Factor $(\text{I.F.})=e^{\int P\,dx}$
  2. The solution of the differential equation is $y(\text{I.F.})=\int Q(\text{I.F.})dx+C$, where C is constant of integration.

BRAIN MAP

DIFFERENTIAL EQUATIONS

Definition

An equation involving dependent variable, independent variable and derivative(s) of dependent variable(s) w.r.t. independent variable(s).

Order

Order is the highest order derivative occurring in a differential equation.

Note : Order of a differential equation is always positive.

Degree

Degree is the highest power of the highest order derivative in the differential equation.

Note : (i) Degree (if defined) of a differential equation is always a positive integer.

(ii) The degree of a differential equation which is not a polynomial equation in derivatives is not defined.

Solution

A function of the form $y = f(x) + c$, which satisfies the given differential equation.

Types of Solutions

General Solution

Contains as many arbitrary constants as order of differential equation.

Particular Solution

Obtained by assigning values to arbitrary constants.

Methods of Solving First Order and First Degree Differential Equations

Homogeneous Differential Equation

If $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, where $f(x,y)$, $g(x, y)$ are homogeneous functions of the same degree in x and y, then put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ then apply variable separable methods

Variable Separable

If $\frac{dy}{dx} = f(x)g(y)$, then

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + c$$

Reducible to Variable Separable

If $dy/dx = f(ax + by + c)$, then put $ax + by + c = z$ and $b\frac{dy}{dx} = \frac{dz}{dx} - a$ and then apply variable separable method.

Linear Differential Equation

If $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x, then $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$, where $e^{\int Pdx}$ is the integrating factor (I.F.).

OR

If $\frac{dx}{dy} + Px = Q$, where P, Q are functions of y, then $xe^{\int Pdy} = \int Qe^{\int Pdy}dy + c$, where $e^{\int Pdy}$ is the integrating factor (I.F.).

Previous Years' CBSE Board Questions

9.2 Basic Concepts

MCQ

1. The sum of the order and the degree of the differential equation $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right)$ is
   (a) 2 (b) 3
   (c) 5 (d) 0
   (2023)

2. The order and the degree of the differential equation $\left(1+3\frac{dy}{dx}\right)^2 = 4\frac{d^3y}{dx^3}$ respectively are
   (a) $1, \frac{2}{3}$ (b) 3, 1 (c) 3, 3 (d) 1, 2
   (2023)

VSA **(1 mark)**

3. The degree of the differential equation $1+\left(\frac{dy}{dx}\right)^2 = x$ is ________. *(2020)*

4. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$. *(Delhi 2019)* U

5. Write the sum of the order and degree of the following differential equation
   $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0.$ *(AI 2015)*

6. Write the sum of the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0$.
   (Foreign 2015) U

7. Write the sum of the order and degree of the differential equation
   $1+\left(\frac{dy}{dx}\right)^4 = 7\left(\frac{d^2y}{dx^2}\right)^3$. *(Delhi 2015C)*

SA I **(2 marks)**

8. Find the product of the order and the degree of the differential equation $\left[\frac{d}{dx}(xy^2)\right]\cdot\frac{dy}{dx} + y = 0.$
   (2022) U

9. Find the value of $(2a - 3b)$, if a and b represent respectively the order and the degree of the differential equation $x\left[y\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx}\right] = 0.$
   (2022 C)

9.3 General and Particular Solutions of a Differential Equation

MCQ

10. The number of solutions of the differential equation $\frac{dy}{dx} = \frac{y+1}{x-1}$, when $y(1) = 2$, is
    (a) zero (b) one (c) two (d) infinite
    (2023)

11. The number of arbitrary constants in the particular solution of a differential equation of second order is (are)
    (a) 0 (b) 1 (c) 2 (d) 3
    (2020) R

9.4 Methods of Solving First Order, First Degree Differential Equations

MCQ

12. The integrating factor for solving the differential equation $x\frac{dy}{dx} - y = 2x^2$ is
    (a) e^{-y} (b) e^{-e} (c) x (d) $\frac{1}{x}$
    (2023)

13. The integrating factor of the differential equation $(x+3y^2)\frac{dy}{dx} = y$ is
    (a) y (b) $-y$ (c) $\frac{1}{y}$ (d) $-\frac{1}{y}$
    (2020)

VSA **(1 mark)**

14. The integrating factor of the differential equation $x\frac{dy}{dx} - y = \log x$ is ______. *(2020 C)* U

15. The integrating factor of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is ________. *(2020)*

16. Find the general solution of the differential equation $e^{y-x}\frac{dy}{dx} = 1.$ *(2020)*

17. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$
    (Delhi 2015, AI 2015C) U

18. Write the integrating factor of the following differential equation :
    $(1+y^2)+(2xy-\cot y)\frac{dy}{dx} = 0$ *(AI 2015)*

19. Write the solution of the differential equation $\frac{dy}{dx} = 2^{-y}$. (Foreign 2015) Ap

20. Find the solution of the differential equation $\frac{dy}{dx} = x^3 e^{-2y}$. (AI 2015C)

SA I (2 marks)

21. Find the general solution of the differential equation : $\log\left(\frac{dy}{dx}\right) = ax + by$. (Term II, 2021-22) Ap

22. Find the general solution of the differential equation $\sec^2 x \cdot \tan y\, dx + \sec^2 y \cdot \tan x\, dy = 0$. (Term II, 2021-22)

23. Find the general solution of the following differential equation : $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ (Term II, 2021-22) Ap

24. Find the integrating factor of $x\frac{dy}{dx} + (1 + x\cot x)y = x$. (2021 C)

25. Solve the following homogeneous differential equation : $x\frac{dy}{dx} = x + y$ (2020 C)

26. Solve the following differential equation: $\frac{dy}{dx} + y = \cos x - \sin x$ (AI 2019)

SA II (3 marks)

27. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$, $y(1) = 0$. (2023)

28. Find the general solution of the differential equation $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$. (2023)

29. Find the particular solution of the differential equation $x\frac{dy}{dx} + x\cos^2\left(\frac{y}{x}\right) = y$; given that when $x = 1, y = \frac{\pi}{4}$. (Term II, 2021-22) Ap

30. Find the general solution of the differential equation $x\frac{dy}{dx} = y(\log y - \log x + 1)$. (Term II, 2021-22)

31. If the solution of the differential equation $\frac{dy}{dx} = \frac{2xy - y^2}{2x^2}$ is $\frac{ax}{y} = b\log|x| + C$, find the value of a and b. (2021C) Ap

LA I (4 marks)

32. **Case study :** An equation involving derivatives of the dependent variable with respect to the independent variables is called a differential equation A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be homogeneous if $F(x, y)$ is a homogeneous function of degree zero, whereas a function $F(x, y)$ is a homogenous function of degree n if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$. To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)$, we make substitution $y = vx$ and then separate the variables.

Based on the above, answer the following questions.

(i) Show that $(x^2 - y^2)\, dx + 2xy\, dy = 0$ is a differential equation of the type $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

(ii) Solve the above equation to find its general solution. (2023)

33. Find the particular solution of the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = \tan x$, given $y(0) = 1$. (Term II, 2021-22C)

34. Find the particular solution of the differential equation $(1 + \sin x)\frac{dy}{dx} = -x - y\cos x$, given $y(0) = 1$. (Term II, 2021-22C)

35. Find the particular solution of the differential equation $x\frac{dy}{dx} + 2y = x^2 \log x$, given $y(1) = 1$. (Term II, 2021-22C)

36. Find the particular solution of the differential equation $x\frac{dy}{dx} + y + \frac{1}{1+x^2} = 0$, given that $y(1) = 0$. (Term II, 2021-22)

37. Find the general solution of the differential equation $x(y^3 + x^3)\, dy = (2y^4 + 5x^3 y)dx$. (Term II, 2021-22)

38. Solve the following differential equation : $(y - \sin^2 x)dx + \tan x\, dy = 0$ (Term II, 2021-22)

39. Find the general solution of the differential equation : $(x^3 + y^3)dy = x^2 y\, dx$ (Term II, 2021-22) Ap

OR

Find the general solution of the differential equation $x^2 y\, dx - (x^3 + y^3)\, dy = 0$. (2020)

40. Find the general solution of the differential equation $ye^y\, dx = (y^3 + 2x\, e^y)dy$. (2020)

41. Solve the following differential equation : $\left(1 + e^{y/x}\right)dy + e^{y/x}\left(1 - \frac{y}{x}\right)dx = 0\ (x \neq 0)$. (2020)

42. Find the particular solution of the differential equation $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$, given that $y = \frac{\pi}{4}$ at $x = 1$. (2020) Ap

43. Find the particular solution of the differential equation $\cos y\, dx + (1 + e^{-x}) \sin y\, dy = 0$ given that $y = \frac{\pi}{4}$ when $x = 0$. (2020)

44. Find the general solution of the differential equation $ye^{x/y}\, dx = (xe^{x/y} + y^2)dy, y \neq 0$ (2020) An

45. Solve the differential equation : $xdy - ydx = \sqrt{x^2 + y^2}\, dx$, given that $y = 0$ when $x = 1$. (Delhi 2019)

46. Solve the differential equation : $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0,$ subject to the initial condition $y(0) = 0$. *(Delhi 2019)* Ap

47. Solve the differential equation : $\frac{dy}{dx}=1+x^2+y^2+x^2y^2$, given that $y = 1$ when $x = 0$. *(AI 2019)*

48. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$, given that $y = 1$ when $x = 0$. *(AI 2019, Delhi 2015)* Ev

49. Solve the following differential equation : $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x;\, x\neq 0.$ *(AI 2019C, 2014C)* Ev

50. Find the particular solution of the differential equation $e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$, given that $y=\frac{\pi}{4}$ when $x = 0$. *(2018)*

51. Find the particular solution of the differential equation $\frac{dy}{dx}+2y\tan x=\sin x,$ given that $y = 0$ when $x=\frac{\pi}{3}$. *(2018, Foreign 2014)* Ev

52. Prove that $x^2 - y^2 = C(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)\, dy$, where C is a parameter. *(NCERT, Delhi 2017)*

53. Solve the differential equation $(\tan^{-1}x - y)dx = (1 + x^2)dy$. *(AI 2017)* Ev

54. Find the general solution of the following differential equation :

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$

(NCERT Exemplar, Delhi 2016)

55. Find the particular solution of the differential equation $(1 - y^2)(1 + \log x)dx + 2xy\, dy = 0$, given that $y = 0$ when $x = 1$. *(Delhi 2016)* An

56. Solve the differential equation : $y+x\frac{dy}{dx}=x-y\frac{dy}{dx}$ *(AI 2016)*

57. Solve the following differential equation $y^2dx + (x^2 - xy + y^2)dy = 0$ *(NCERT, Exemplar, Foreign 2016)*

58. Solve the following differential equation $(\cot^{-1} y + x)dy = (1 + y^2)dx$ *(Foreign 2016)* Ap

59. Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$, given that $y=\frac{\pi}{2}$, when $x = 1$. *(Delhi 2014)* Ev

60. Solve the following differential equation : $(x^2-1)\frac{dy}{dx}+2xy=\frac{2}{x^2-1}, |x|\neq 1$ *(Delhi 2014)*

61. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}dx+\frac{y}{x}dy=0$ given that $y = 1$ when $x = 0$. *(Delhi 2014)* An

62. Solve the following differential equation : $\operatorname{cosec} x \log y\frac{dy}{dx}+x^2y^2=0.$ *(Delhi 2014)*

63. Find the particular solution of the differential equation $\frac{dy}{dx}=1+x+y+xy$, given that $y = 0$ when $x = 1$ *(AI 2014)* An

64. Solve the differential equation $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$ *(AI 2014)*

65. Find the particular solution of the differential equation $x (1 + y^2)\, dx - y (1 + x^2)dy = 0$, given that $y = 1$ when $x = 0$. *(AI 2014)* Ap

66. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right)=3x+4y$, given that $y = 0$ when $x = 0$. *(NCERT, AI 2014)*

67. Solve the differential equation $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$, given that $y = 1$ when $x = 1$. *(Foreign 2014)* Ap

68. Solve the differential equation $\frac{dy}{dx}+y\cot x=2\cos x,$ given that $y = 0$ when $x=\frac{\pi}{2}$. *(Foreign 2014)*

69. Solve the differential equation $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x.$ *(NCERT, Foreign 2014)* Ev

70. If $y(x)$ is a solution of the differential equation $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx}=-\cos x$ and $y(0)=1$, then find the value of $y\left(\frac{\pi}{2}\right)$. *(Delhi 2014C)* Ev

71. Find the general solution of the differential equation $(x-y)\frac{dy}{dx}=x+2y.$ *(Delhi 2014C)*

72. Find the particular solution of the differential equation $x\frac{dy}{dx}-y+x\operatorname{cosec}\left(\frac{y}{x}\right)=0;$ given that $y = 0$ when $x = 1$. *(AI 2014C)*

73. Solve the differential equation $x\frac{dy}{dx}+y=x\cos x+\sin x,$ given $y\left(\frac{\pi}{2}\right)=1.$ *(AI 2014C)*

LA II (5 / 6 marks)

74. Solve the differential equation $x\frac{dy}{dx}+y = x\cos x + \sin x$, given that $y = 1$ when $x=\frac{\pi}{2}$. *(Delhi 2017)*

75. Find the particular solution of the differential equation $(x-y)\frac{dy}{dx}=(x+2y)$, given that $y = 0$ when $x = 1$. *(AI 2017)* Ap

76. Solve the differential equation : $(\tan^{-1}y - x)dy = (1 + y^2)dx$. *(NCERT, Delhi 2015)*

77. Show that the differential equation $\frac{dy}{dx}=\frac{y^2}{xy-x^2}$ is homogeneous and also solve it. *(AI 2015)*

78. Find the particular solution of the differential equation $(\tan^{-1}y-x)\,dy = (1+y^2)dx$, given that $x = 1$ when $y = 0$. *(NCERT, AI 2015)*

79. Solve the following differential equation :

$$\left[y-x\cos\left(\frac{y}{x}\right)\right]dy+\left[y\cos\left(\frac{y}{x}\right)-2x\sin\left(\frac{y}{x}\right)\right]dx=0$$

(Foreign 2015) Ap

80. Solve the following differential equation :

$$\left(\sqrt{1+x^2+y^2+x^2y^2}\right)dx+xy\,dy=0$$ *(Foreign 2015)*

81. Find the particular solution of the differential equation $x\frac{dy}{dx}+y-x+xy\cot x=0$; $x \neq 0$, given that when $x=\frac{\pi}{2}, y=0$. *(NCERT, Delhi 2015C)* An

82. Solve the differential equation $x^2\,dy + (xy+y^2)dx = 0$ given $y = 1$, when $x = 1$ *(Delhi 2015C)*

83. Solve the differential equation

$$\left(x\sin^2\left(\frac{y}{x}\right)-y\right)dx+xdy=0 \text{ given } y=\frac{\pi}{4} \text{ when } x = 1$$

(AI 2015C, 2014C) An

84. Solve the differential equation $\frac{dy}{dx}-3y\cot x = \sin 2x$ given $y = 2$ when $x=\frac{\pi}{2}$. *(AI 2015C)*

CBSE Sample Questions

9.2 Basic Concepts

MCQ

1. If m and n, respectively, are the order and the degree of the differential equation $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)\right]^4=0$, then $m + n =$

(a) 1 (b) 2 (c) 3 (d) 4

(2022-23) U

VSA (1 mark)

2. For what value of n is the following a homogeneous differential equation: $\frac{dy}{dx}=\frac{x^3-y^n}{x^2y+xy^2}$? *(2020-21)*

SA I (2 marks)

3. Write the sum of the order and the degree of the following differential equation $\frac{d}{dx}\left(\frac{dy}{dx}\right)=5$.

(Term II, 2021-22) Ap

9.3 General and Particular Solutions of a Differential Equation

VSA (1 mark)

4. How many arbitrary constants are there in the particular solution of the differential equation $\frac{dy}{dx}=-4xy^2; y(0)=1$? *(2020-21)*

9.4 Methods of Solving First order, First Degree Differential Equations

SA I (2 marks)

5. Solve the following differential equation:

$\frac{dy}{dx}=x^3\operatorname{cosec} y$, given that $y(0) = 0$ *(2020-21)*

SA II (3 marks)

6. Solve the differential equation : $ydx + (x - y^2)dy = 0$ *(2022-23)* Ap

7. Solve the differential equation :

$xdy-ydx=\sqrt{x^2+y^2}\,dx$ *(2022-23)* Ap

8. Find the general solution of the following differential equation.

$x\frac{dy}{dx}=y-x\sin\left(\frac{y}{x}\right)$ *(Term II, 2021-22)*

9. Find the particular solution of the following differential equation, given that $y=0$ when $x=\frac{\pi}{4}$.

$\frac{dy}{dx}+y\cot x=\frac{2}{1+\sin x}$ *(Term II, 2021-22)* Ev

10. Find the general solution of the following differential equation:

$xdy - (y + 2x^2)dx = 0$ *(2020-21)* An

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. **(b) :** [There is error in question, the given differential equation should be $\frac{d}{dx}\left(\frac{dy}{dx}\right)^3 = 0$.]

The given differential equation is,

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right) = 0 \quad \Rightarrow \quad 3\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right) = 0$$

$\therefore$ Order = 2 and degree = 1
So, required sum = 2 + 1 = 3

2. **(b) :** We have, $\left(1+3\frac{dy}{dx}\right)^2 = 4\frac{d^3y}{dx^3}$

Here, order = 3 as highest order derivative is $\frac{d^3y}{dx^3}$.

And degree = 1, as power of highest order derivative *i.e.*, $\frac{d^3y}{dx^3}$ is 1.

3. The degree of the differential equation $1+\left(\frac{dy}{dx}\right)^2 = x$ is 2.

4. The given differential equation is

$$x^2\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^4 \quad \therefore \text{ Its order is 2 and degree is 1.}$$

5. The given differential equation is

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0 \Rightarrow 3.\left(\frac{dy}{dx}\right)^2.\frac{d^2y}{dx^2} = 0$$

Order = 2 and Degree = 1 $\therefore$ Order + Degree = 2 + 1 = 3

Concept Applied

- Highest order derivative appearing in a differential equation is called order of the differential equation.

6. Order = 2, Degree = 2 $\therefore$ Order + Degree = 2 + 2 = 4

7. Order = 2, Degree = 3
$\therefore$ Order + Degree = 2 + 3 = 5

8. The given differential equation is $\left[\frac{d}{dx}(xy^2)\right]\cdot\frac{dy}{dx} + y = 0$

$$\Rightarrow \left[x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right]\frac{dy}{dx} + y = 0 \Rightarrow 2xy\left(\frac{dy}{dx}\right)^2 + y^2\left(\frac{dy}{dx}\right) + y = 0$$

$\therefore$ Its order is 1 and degree is 2.
$\therefore$ Required product = 1 × 2 = 2

Concept Applied

- The degree of a differential equation is the power of the highest ordered derivative, when differential coefficients are made free from radicals and fractions.

9. We have, $x\left[y\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx}\right] = 0$

Its order is 2 and degree is 3.
$\therefore$ $a = 2, b = 3$
$2a - 3b = 2 \times 2 - 3 \times 3 = 4 - 9 = -5$

10. **(b):** Given that; $\frac{dy}{dx} = \frac{y+1}{x-1} \Rightarrow \frac{dy}{y+1} = \frac{dx}{x-1}$

On integrating both sides, we get $\int\frac{dy}{y+1} = \int\frac{dx}{x-1}$

$\Rightarrow$ $\log(y + 1) = \log(x - 1) - \log C$

$\Rightarrow$ $\log(y + 1) + \log C = \log(x - 1) \Rightarrow C = \frac{x-1}{y+1}$

Now, $y(1) = 2 \Rightarrow C = \frac{1-1}{2+1} = 0$

$\therefore$ Required solution is $x - 1 = 0$
Hence, only one solution exist.

11. **(a):** In the particular solution of a differential equation of any order, there is no arbitrary constant because in the particular solution of any differential equation, we remove all the arbitrary constant by substituting some particular values.

12. **(d):** We have, $x\frac{dy}{dx} - y = 2x^2$

i.e., $\frac{dy}{dx} - \frac{y}{x} = 2x \quad \therefore \text{ I.F.} = e^{\int\frac{-1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

$\therefore$ Integrating factor is $\frac{1}{x}$

13. **(c) :** We have, $(x+3y^2)\frac{dy}{dx} = y$

$$\Rightarrow \frac{x+3y^2}{y} = \frac{dx}{dy} \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$$

This is a linear differential equation.

$\therefore$ I.F. $= e^{-\int\frac{dy}{y}} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}$

14. We have, $x\frac{dy}{dx} - y = \log x \Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x}$

Clearly, it is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

$\therefore$ I.F. = $e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

15. We have, $x\frac{dy}{dx} + 2y = x^2 \Rightarrow \frac{dy}{dx} + 2\frac{y}{x} = x$

$\therefore$ I.F. = $e^{\int\frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

16. We have, $e^{y-x}\frac{dy}{dx} = 1 \Rightarrow e^y\cdot e^{-x}\frac{dy}{dx} = 1 \Rightarrow e^y dy = e^x dx$

Integrating both sides, we get
$e^y = e^x + c, \; y = \log(e^x + c)$

17. We have, $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ or $\frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q, \text{where } P=\frac{1}{\sqrt{x}}, Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$\therefore$ I.F. = $e^{\int P dx} \Rightarrow$ I.F. = $e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$

18. The given differential equation is

$(1+y^2)+(2xy-\cot y)\frac{dy}{dx}=0$

$\Rightarrow (1+y^2)\frac{dx}{dy}+2xy-\cot y=0 \Rightarrow \frac{dx}{dy}+\frac{2y}{1+y^2}\cdot x=\frac{\cot y}{1+y^2}$

This is a linear differential equation of the form

$\frac{dx}{dy}$ + Px = Q, where, $P=\frac{2y}{1+y^2}$ and $Q=\frac{\cot y}{1+y^2}$

$\therefore$ I.F. $=e^{\int P dy}=e^{\int \frac{2y}{1+y^2}dy}$ = $e^{\log(1+y^2)}$ = $1+y^2$.

19. We have, $\frac{dy}{dx}=2^{-y} \Rightarrow \frac{dy}{dx}=\frac{1}{2^y} \Rightarrow 2^y dy=dx$ ...(i)

Integrating both sides of (i), we get $\frac{2^y}{\log 2}=x+C$

$\Rightarrow$ $2^y = (C+x)\log 2$

Taking log on both sides to the base 2, we get

$\log_2 2^y = \log_2 [(C+x)\log 2]$

$\Rightarrow$ $y = \log_2[(C+x)\log 2]$

This is the required solution.

Answer Tips

➲ $\int a^x dx = \frac{a^x}{\ln(a)}+C$, where C is arbitrary constant.

20. We have, $\frac{dy}{dx}=x^3 e^{-2y} \Rightarrow e^{2y}\, dy = x^3 dx$

Integrating both sides, we get $\frac{e^{2y}}{2}=\frac{x^4}{4}+C'$

$\Rightarrow$ $2e^{2y} = x^4 + C$, where $C = 4C'$

21.

$\log\left(\frac{dy}{dx}\right) = ax+by$

$\Rightarrow \frac{dy}{dx} = e^{ax+by}$

$\Rightarrow \frac{dy}{dx} = e^{ax}\cdot e^{by}$ $[e^{a+b} = e^a \cdot e^b]$

$\Rightarrow \frac{dy}{e^{by}} = e^{ax}dx$

$\Rightarrow e^{-by}dy = e^{ax}dx$

On integrating both sides.

$\int e^{-by}dy = \int e^{ax}dx$

$\Rightarrow \frac{-e^{-by}}{b} = \frac{e^{ax}}{a}+c$

$\frac{e^{ax}}{a}+\frac{e^{-by}}{b} = c'$ $[c'=-c]$

where c & c' are constants.

Answer: $\frac{e^{ax}}{a}+\frac{e^{-by}}{b} = c'$

[Topper's Answer, 2022]

22. We have, $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$\Rightarrow \quad \frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy \;\Rightarrow\; \frac{d(\tan x)}{\tan x} = -\frac{d(\tan y)}{\tan y}$

$\Rightarrow$ log(tan$x$) = – log(tan y) + log c (integrating on both sides)

$\Rightarrow$ tan$x$ tan$y$ = c

23. We have, $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\Rightarrow \quad dy = \frac{(e^x + x^2)}{e^y}dx \;\Rightarrow\; e^y dy = (e^x + x^2)\,dx$

Integrating on both sides, we get

$\int e^y\, dy = \int (e^x + x^2)\,dx$

$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C,$ which is required solution.

24. We have, $x\frac{dy}{dx} + (1 + x\cot x)y = x$

$\Rightarrow \quad \frac{dy}{dx} + \frac{(1 + x\cot x)}{x}y = 1$

Clearly it is a linear differential equation of the form, $\frac{dy}{dx} + Py = Q,$ where $P = \frac{1 + x\cot x}{x}$ and $Q = 1$

$\therefore \quad$ I.F. = $e^{\int\left(\frac{1}{x} + \cot x\right)dx}$

$= e^{\log x + \log \sin x} = e^{\log(x \sin x)} = x \sin x.$

25. We have, $x\frac{dy}{dx} = x + y$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x + y}{x} = 1 + \frac{y}{x}$

This is a homogeneous differential equation

Put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \;\Rightarrow\; v + x\frac{dv}{dx} = 1 + v$

$\Rightarrow \quad x\frac{dv}{dx} = 1 \;\Rightarrow\; dv = \frac{dx}{x}$

Integrating both sides, we get

$v = \log x + \log c$

$v = \log xc$

$\Rightarrow \quad y = x\log cx$

26. We have, $\frac{dy}{dx} + y = \cos x - \sin x$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q,$ where $P = 1, Q = \cos x - \sin x$

$\therefore \quad$ I.F. = $e^{\int dx} = e^x$

The solution of given differential equation is

$ye^x = \int e^x(\cos x - \sin x)dx + C$

$\Rightarrow \quad ye^x = e^x \cos x + C$

$\Rightarrow \quad y = \cos x + Ce^{-x}$

27. Given differential equation is $\frac{dy}{dx} = \frac{x + y}{x}$

$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 1$

Clearly, it is a linear differential equation of the form $\frac{dy}{dx} + Py = Q,$ where $P = \frac{-1}{x}, Q = 1$

$\text{I.F.} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = \frac{1}{x}$

$\therefore \quad$ Solution is given by $y \cdot \frac{1}{x} = \int 1 \cdot \frac{1}{x}dx + C$

$\Rightarrow \quad \frac{y}{x} = \log x + C$ ...(i)

We have $y(1) = 0$

When $x = 1, y = 0$

$\therefore \quad 0 = 0 + C \Rightarrow C = 0$

$\therefore \quad$ From (i) $\frac{y}{x} = \log x \;\Rightarrow\; y = x\log x$

28. We have, $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

$\Rightarrow \quad e^x \tan y\, dx = (e^x - 1) \sec^2 y\, dy$

$\Rightarrow \quad \frac{e^x}{e^x - 1}dx = \frac{\sec^2 y}{\tan y}dy$

Integrating both sides, we get

$$\int \frac{e^x}{e^x - 1}dx = \int \frac{\sec^2 y}{\tan y}dy \quad \text{... (i)}$$

Put $e^x - 1 = u \Rightarrow e^x dx = du$

and $\tan y = v \Rightarrow \sec^2 y\, dy = dv$

$\therefore \quad$ From (i) $\int \frac{du}{u} = \int \frac{dv}{v} \Rightarrow \log(u) = \log(v) + \log C$

$\Rightarrow \quad \log(e^x - 1) = \log(\tan y) + \log C$

$\Rightarrow \quad \log(e^x - 1) = \log(C \tan y)$

$\Rightarrow \quad e^x - 1 = C\tan y$

29. We have, $x\frac{dy}{dx} + x\cos^2\left(\frac{y}{x}\right) = y$

$\Rightarrow \quad \frac{dy}{dx} + \cos^2\left(\frac{y}{x}\right) = \frac{y}{x}$

This is a homogeneous differential equation.

Now, put $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} + \cos^2 v = v \Rightarrow \frac{xdv}{dx} = -\cos^2 v$

$\Rightarrow \quad \sec^2 v\, dv = -\frac{dx}{x}$

Integrating both sides, we get

$\tan v = -\log x + \log c$

$\Rightarrow \quad \tan v = \log\left|\frac{c}{x}\right| \;\Rightarrow\; \tan\frac{y}{x} = \log\left|\frac{c}{x}\right|$

When $x = 1, y = \frac{\pi}{4}$

$\therefore \quad \tan\frac{\pi}{4} = \log\frac{c}{1} \Rightarrow \log c = 1$

$\Rightarrow \quad c = e$

Particular solution is $\tan\frac{y}{x} = \log\frac{e}{x}$

$\Rightarrow \quad \tan\frac{y}{x} = 1 - \log x$

30.

$x\frac{dy}{dx} = y(\log y - \log x + 1)$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$ $\left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$

on putting $x = \lambda x$, $y = \lambda y$.

$f(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x}\left(\log\left(\frac{\lambda y}{\lambda x}\right) + 1\right)$

$f(\lambda x, \lambda y) = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right) = f(x, y)$

Thus, this equation is homogeneous equation.

Let $\frac{y}{x} = t$ or $\Rightarrow y = tx$

on differentiating with respect to x

$\frac{dy}{dx} = t + x\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$

$t + x\frac{dt}{dx} = t(\log t + 1)$ $\left[\frac{y}{x} = t\right]$

$t + x\frac{dt}{dx} = t\log t + t$

$x\frac{dt}{dx} = t\log t$

$\frac{dt}{t\log t} = \frac{dx}{x}$

on integrating both sides.

$\int \frac{dt}{t\log t} = \int \frac{dx}{x}$

$\int \frac{dt}{t\log t} = \log x + c$

Let $\log t = u$

on differentiating,

$\frac{1}{t}dt = du$

$\int \frac{du}{u} = \ln x + c$ $[\log_e x = \ln x]$

$\ln u = \ln x + c$ [c is integration constant]

$\ln(\ln t) = \ln x + c$ $[u = \log_e t = \ln t]$

$\ln(\ln(y/x)) = \ln x + c$

$\ln(\ln(y/x)) - \ln x = c$

$\ln\left(\frac{\ln(y/x)}{x}\right) = c$

$\left[\log a - \log b = \log(a/b)\right]$

Answer: $\ln\left(\frac{\ln(y/x)}{x}\right) = c$

$[\text{where } \ln x = \log_e x]$

[Topper's Answer, 2022]

31. We have, $\frac{dy}{dx} = \frac{2xy - y^2}{2x^2}$ ...(i)

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2$

This is a homogeneous differential equation.

Now, put $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = v - \frac{1}{2}v^2$

$\Rightarrow x\frac{dv}{dx} = -\frac{1}{2}v^2 \Rightarrow -\frac{1}{v^2}dv = \frac{1}{2x}dx$

Integrating both sides, we get

$\frac{1}{v} = \frac{1}{2}\log|x| + C \Rightarrow \frac{x}{y} = \frac{1}{2}\log|x| + C$

Given, solution of (i) is $\frac{ax}{y} = b\log|x| + C$

On comparing, $a = 1, b = \frac{1}{2}$

32. (i) We have, $(x^2 - y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{(x^2 - y^2)}{2xy} = \frac{y^2 - x^2}{2xy}$

Now, putting $\frac{dy}{dx} = F(x, y)$ and find $F(\lambda x, \lambda y)$,

$\Rightarrow F(x, y) = \frac{y^2 - x^2}{2xy}$

$\therefore F(\lambda x, \lambda y) = \frac{\lambda^2 y^2 - \lambda^2 x^2}{2\lambda^2 xy} = \frac{y^2 - x^2}{2xy} = F(x, y)$

So, $F(x, y)$ is a homogeneous function and the given differential equation is of the type $g\left(\frac{y}{x}\right)$

(ii) We have, $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx} = \frac{v^2 - 1}{2v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = -\left(\frac{v^2 + 1}{2v}\right) \Rightarrow \int \frac{2v}{v^2 + 1}dv = -\int \frac{dx}{x}$

$\Rightarrow \int \frac{2v}{v^2 + 1}dv + \int \frac{dx}{x} = \log C$

$\Rightarrow \log|v^2 + 1| + \log x = \log C$

$$\Rightarrow \log\left|\left(\frac{y^2+x^2}{x^2}\right)\times x\right| = \log C$$

$$\Rightarrow \frac{y^2+x^2}{x} = C \Rightarrow x^2+y^2 = Cx$$

is the required general solution.

33. We have, $(1+x^2)\frac{dy}{dx}+2xy = \tan x$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2} = \frac{\tan x}{1+x^2} \quad \text{...(i)}$$

Clearly, it is a linear differential equation of the form

$\frac{dy}{dx}+Py = Q$, Where $P = \frac{2x}{1+x^2}$ and $Q = \frac{\tan x}{1+x^2}$

$\therefore$ I.F. $= e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$

$\therefore$ Solution of (i) is

$$y(1+x^2) = \int(1+x^2)\frac{\tan x}{(1+x^2)}dx+C$$

$$\Rightarrow y(1+x^2) = \int\tan x\,dx+C$$

$$\Rightarrow y(1+x^2) = \log|\sec x|+C$$

Also given, $y(0) = 1$

$\therefore$ $1 = \log\sec 0 + C \Rightarrow C = 1$

Particular solution is

$y(1+x^2) = \log|\sec x| + 1$

34. We have, $(1+\sin x)\frac{dy}{dx} = -x-y\cos x$

$$\Rightarrow (1+\sin x)\frac{dy}{dx}+y\cos x = -x$$

$$\Rightarrow \frac{dy}{dx}+\frac{\cos x}{1+\sin x}y = \frac{-x}{1+\sin x}$$

This is a linear differential equation of the form $\frac{dy}{dx}+Py = Q$,

where, $P = \frac{\cos x}{1+\sin x}$, $Q = \frac{-x}{1+\sin x}$

I.F. $= e^{\int\frac{\cos x}{1+\sin x}dx} = e^{\log(1+\sin x)} = 1+\sin x$

The solution of given differential equation is

$$y\cdot(1+\sin x) = \int\frac{-x}{(1+\sin x)}(1+\sin x)\,dx+C$$

$$\Rightarrow y\cdot(1+\sin x) = \frac{-x^2}{2}+C$$

Also given $y(0) = 1$

$\therefore$ $1(1+\sin 0) = 0 + C \Rightarrow C = 1$

$\therefore$ Particular solution is $y(1+\sin x) = \frac{-x^2}{2}+1$

35. We have, $\frac{xdy}{dx}+2y = x^2\log x$

$$\Rightarrow \frac{dy}{dx}+\frac{2y}{x} = x\log x \quad \text{...(i)}$$

Clearly, it is a linear differential equation of the form,

$\frac{dy}{dx}+Py = Q$, where $P = \frac{2}{x}$ and $Q = x\log x$.

I.F. $= e^{\int\frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

$\therefore$ Solution of (i) is

$$y\cdot x = \int x^2\cdot x\log x\,dx+C \Rightarrow y\cdot x^2 = \int x^3\cdot\log x\,dx+C$$

$$y\cdot x^2 = \log x\cdot\frac{x^4}{4}-\int\frac{1}{x}\cdot\frac{x^4}{4}dx+C \Rightarrow y\cdot x^2 = \frac{x^4}{4}\log x-\frac{x^4}{16}+C$$

$$\Rightarrow y = \frac{x^2}{4}\log x-\frac{x^2}{16}+\frac{C}{x^2}$$

Also, given $y(1) = 1$

$\therefore$ $1 = 0-\frac{1}{16}+C \Rightarrow C = \frac{17}{16}$

$\therefore$ Particular solution is $y = \frac{x^2}{4}\log x-\frac{x^2}{16}+\frac{17}{16x^2}$

36. We have, $x\frac{dy}{dx}+y+\frac{1}{1+x^2} = 0 \quad \text{...(i)}$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x} = \frac{-1}{(1+x^2)x}$$

Clearly, it is a linear differential equation of the form $\frac{dy}{dx}+Py = Q$, where $P = \frac{1}{x}$ and $Q = \frac{-1}{x(1+x^2)}$

I.F. $= e^{\int\frac{1}{x}dx} = e^{\log x} = x.$

$\therefore$ Solution of (i) is

$$y\cdot x = \int\frac{-1}{x(1+x^2)}\cdot x\,dx+C$$

$$\Rightarrow yx = \int\frac{-1}{1+x^2}dx+C \Rightarrow yx = -\tan^{-1}x+C$$

Also, given $y(1) = 0$

$\therefore$ $0\cdot 1 = -\tan^{-1}1 + C \Rightarrow C = \tan^{-1}1 = \frac{\pi}{4}$

$\therefore$ Particular solution of given differential equation is

$$yx = -\tan^{-1}x+\frac{\pi}{4}$$

37. We have, $x(y^3+x^3)\,dy = (2y^4+5x^3y)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^4+5x^3y}{xy^3+x^4} \Rightarrow \frac{dy}{dx} = \frac{2(y/x)^4+5(y/x)}{(y/x)^3+1}$$

Put $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v+x\frac{dv}{dx}$

$$\therefore v+x\frac{dv}{dx} = \frac{2v^4+5v}{v^3+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v^4+5v}{v^3+1}-v = \frac{v^4+4v}{v^3+1}$$

$$\Rightarrow \frac{v^3+1}{v^4+4v}dv = \frac{dx}{x} \Rightarrow \int\frac{v^3+1}{v^4+4v}dv = \int\frac{dx}{x}$$

Putting $v^4+4v = t \Rightarrow (4v^3+4)\,dv = dt$

$\therefore \frac{1}{4}\int\frac{dt}{t}=\int\frac{dx}{x}$

$\Rightarrow \frac{1}{4}\log(v^4+4v)=\log x+\log C$

$\Rightarrow \frac{1}{4}\log(v^4+4v)=\log x+\log C \Rightarrow \frac{y^4+4yx^3}{x^8}=C$

38. $(y - \sin^2x)\, dx + \tan x\, dy = 0$

$\Rightarrow (y - \sin^2x)dx = - \tan x\, dy$

$\Rightarrow \frac{dy}{dx}=\frac{y-\sin^2 x}{-\tan x} \Rightarrow \frac{dy}{dx}=\frac{\sin^2 x}{\tan x}-\frac{y}{\tan x}$

$\Rightarrow \frac{dy}{dx}=\sin x\cos x-y\cot x$

$\Rightarrow \frac{dy}{dx}+y\cot x=\sin x\cos x$

So, it is a linear differential equation, where $P = \cot x$, $Q = \cos x \sin x$

I.F. $=e^{\int Pdx}=e^{\int \cot x\, dx}$

$= e^{\log_e|\sin x|} = \sin x$

General solution. : $y\,(\text{I.F.}) = \int Q(\text{I.F.})\,dx$

$\Rightarrow y\cdot\sin x=\int\cos x\sin x\cdot\sin x\, dx$

$\Rightarrow y\cdot\sin x=\int\cos x\cdot\sin^2 x\, dx$

$=\int t^2\, dt =\frac{t^3}{3}+c$ $\quad [\because$ Let $\sin x = t \Rightarrow dx = \frac{dt}{\cos x}]$

$\Rightarrow y\sin x=\frac{\sin^3 x}{3}+c \Rightarrow y=\frac{\sin^2 x}{3}+\frac{c}{\sin x}$

39. $(x^3 + y^3)\, dy = x^2y\, dx$ is rearranged as $\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$

Let $\frac{y}{x}=v\Rightarrow y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore v+x\frac{dv}{dx}=\frac{v}{1+v^3}$ $\quad \left[\because \frac{dy}{dx}=\frac{y/x}{1+(y/x)^3}\right]$

$\Rightarrow x\frac{dv}{dx}=\frac{v}{1+v^3}-v \Rightarrow x\frac{dv}{dx}=\frac{v-v-v^4}{1+v^3}$

$\Rightarrow x\frac{dv}{dx}=\frac{-v^4}{1+v^3}$

$\Rightarrow \int\frac{1+v^3}{v^4}dv=-\int\frac{dx}{x}$ $\quad$ [Integrating on both sides]

$\Rightarrow \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv=-\int\frac{dx}{x}$

$\Rightarrow \frac{-1}{3v^3}+\log|v|=-\log|x|+c$

$\Rightarrow \frac{-x^3}{3y^3}+\log\left|\frac{y}{x}\right|=-\log|x|+c \Rightarrow \frac{-x^3}{3y^3}+\log|y|=c$

40. We have, $ye^y dx = (y^3 + 2xe^y)dy$

$\Rightarrow \frac{dx}{dy}=\frac{y^3+2xe^y}{ye^y} \Rightarrow \frac{dx}{dy}-\frac{2}{y}x=y^2e^{-y}$ $\quad$ ...(i)

This is a linear D.E. of the form $\frac{dx}{dy}+Px=Q$

Where $P=-\frac{2}{y}$ and $Q=y^2e^{-y}$

$\therefore$ I.F. $= e^{-\int\frac{2}{y}dy}=e^{-2\log y}=e^{\log y^{-2}}=\frac{1}{y^2}$

So, the solution of (i) is $x\cdot\frac{1}{y^2}=\int\frac{1}{y^2}\cdot y^2e^{-y}dy$

$\Rightarrow \frac{x}{y^2}=-e^{-y}+C \Rightarrow x = -y^2e^{-y} + Cy^2$

41. We have,

$(1+e^{y/x})\,dy+e^{y/x}\left(1-\frac{y}{x}\right)dx=0, x\neq 0$

$\Rightarrow \frac{dy}{dx}+\frac{e^{y/x}}{(1+e^{y/x})}\left(1-\frac{y}{x}\right)=0$ $\quad$ ...(i)

This is a homogeneous differential equation.

Now, put $y = vx$

$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ From (i), $v+x\frac{dv}{dx}+\frac{e^v}{(1+e^v)}(1-v)=0$

$\Rightarrow x\frac{dv}{dx}=-\frac{(1-v)e^v}{1+e^v}-v \Rightarrow x\frac{dv}{dx}=\frac{-e^v+ve^v-v-ve^v}{1+e^v}$

$\Rightarrow x\frac{dv}{dx}=-\frac{(e^v+v)}{1+e^v} \Rightarrow \left(\frac{1+e^v}{e^v+v}\right)dv=\frac{-dx}{x}$

Integrating both sides, we get

$\log(e^v + v) = -\log x + \log c$

$\Rightarrow e^v+v=\frac{c}{x} \Rightarrow e^{\frac{y}{x}}+\frac{y}{x}=\frac{c}{x}$

$\therefore$ Required solution is $e^{\frac{y}{x}}+\frac{y}{x}=\frac{c}{x}$

42. We have, $x\frac{dy}{dx}=y-x\tan\left(\frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x}-\tan\left(\frac{y}{x}\right),$

This is a homogeneous differential equation.

Now, put $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore v+x\frac{dv}{dx}=v-\tan v$

$\Rightarrow x\frac{dv}{dx}=-\tan v \Rightarrow \frac{dv}{\tan v}=-\frac{dx}{x} \Rightarrow \cot v\, dv+\frac{dx}{x}=0$

Integrating both sides,we get

$\log|\sin v| + \log x = \log C$

$\Rightarrow x\sin v = C \Rightarrow x\sin\left(\frac{y}{x}\right)=C$

When $x = 1$, $y = \frac{\pi}{4}$, we get $1\cdot\sin\left(\frac{\pi}{4}\right)=C \Rightarrow C=\frac{1}{\sqrt{2}}$

So, $x\sin\left(\frac{y}{x}\right)=\frac{1}{\sqrt{2}}$ is the required particular solution.

43. We have, $\cos y\, dx + (1 + e^{-x})\sin y\, dy = 0$

$\Rightarrow \quad dx + (1 + e^{-x}) \tan y\, dy = 0$

$\Rightarrow \quad \frac{dx}{1+e^{-x}} + \tan y\, dy = 0$

Integrating both sides, we get

$$\int \frac{e^x}{1+e^x} dx + \int \tan y\, dy = 0$$

$\Rightarrow \quad \log(1 + e^x) + \log|\sec y| = \log C$

$\Rightarrow \quad \sec y\,(1 + e^x) = C$

When, $x = 0$, $y = \frac{\pi}{4}$, we get $C = 2\sqrt{2}$

$\therefore$ Particular solution of the differential equation is,

$\sec y(1 + e^x) = 2\sqrt{2}$

44. We have, $ye^{x/y}\, dx = (xe^{x/y} + y^2)dy,\ y \neq 0$

$\Rightarrow \quad \frac{dx}{dy} = \frac{x}{y} + \frac{y}{e^{x/y}}$ ...(i)

Putting $\frac{x}{y} = t \Rightarrow x = yt \Rightarrow \frac{dx}{dy} = t + y\frac{dt}{dy}$

$\therefore$ Equation (i) becomes, $t + y\frac{dt}{dy} = t + \frac{y}{e^t}$

$\Rightarrow \quad y\frac{dt}{dy} = ye^{-t} \Rightarrow \frac{dt}{dy} = e^{-t} \Rightarrow dy = e^t dt$

Integrating both sides, we get

$y = e^t + C \Rightarrow y = e^{x/y} + C$

Key Points

- $\int e^x dx = e^x + C$

45. We have, $x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

$\Rightarrow x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}$ ...(i)

This is a linear homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ Eq. (i) becomes

$v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$

$\Rightarrow \quad \frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}} \quad \Rightarrow \int\frac{dx}{x} = \int\frac{dv}{\sqrt{1+v^2}}$

$\Rightarrow \quad \log x + \log C_1 = \log|v + \sqrt{1+v^2}|$

$\Rightarrow \quad \log x + \log C_1 = \log\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right|$

$\Rightarrow \quad \log C_1 x = \log|y + \sqrt{x^2+y^2}| - \log x$

$\Rightarrow \quad \pm C_1 x^2 = y + \sqrt{x^2+y^2} \quad \Rightarrow \quad Cx^2 = y + \sqrt{x^2+y^2}$

[where $C = \pm C_1$]

General solution of the given equation is

$Cx^2 = y + \sqrt{x^2+y^2}$ ...(ii)

Now, putting $y = 0$ and $x = 1$ in (ii), we get $C = 1$

$\therefore$ Required solution is $x^2 = y + \sqrt{x^2+y^2}$.

46. We have $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx}$

$= e^{\log(1+x^2)} = 1 + x^2$

Hence, the required solution is

$$y(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)\,dx + C$$

$\Rightarrow \quad y(1+x^2) = 4\int x^2 dx + C$

$\Rightarrow \quad y(1+x^2) = \frac{4x^3}{3} + C$

Given that $y(0) = 0$

$\therefore \quad 0(1 + 0) = 0 + C \Rightarrow C = 0$

Thus, $y = \frac{4x^3}{3(1+x^2)}$ is the required solution.

47. We have, $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$\therefore \frac{dy}{dx} = 1 + x^2 + y^2(1+x^2) = (1 + x^2).(1 + y^2)$

$\Rightarrow \quad \frac{dy}{1+y^2} = (1+x^2)dx$

Integrating both sides, we get $\tan^{-1} y = x + \frac{x^3}{3} + C$

when $x = 0$, $y = 1$

$\tan^{-1} 1 = 0 + 0 + C \Rightarrow C = \frac{\pi}{4}$

$\therefore \tan^{-1} y = x + \frac{1}{3}x^3 + \frac{\pi}{4}$ is the required solution.

48. We have, $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$

This is a homogeneous linear differential equation

$\therefore$ Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{x \cdot vx}{x^2 + v^2x^2}$

$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1+v^2} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^2} - v$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^3}{1+v^2} \quad \Rightarrow \quad \frac{dx}{x} = -\left(\frac{1+v^2}{v^3}\right)dv$

Integrating both sides, we get

$$\int\frac{dx}{x}=-\int v^{-3}dv-\int\frac{1}{v}dv$$

$$\Rightarrow \log x=\frac{1}{2v^2}-\log v+C$$

$$\Rightarrow \log x=\frac{x^2}{2y^2}-\log y+\log x+C$$

$$\Rightarrow \log y=\frac{x^2}{2y^2}+C$$

When $y = 1, x = 0 \Rightarrow \log 1 = 0 + C \Rightarrow C = 0$

$\therefore$ Particular solution is $y=e^{\frac{x^2}{2y^2}}$

49. We have, $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x;\ x\neq 0$

$$\Rightarrow \cos\left(\frac{y}{x}\right)\frac{dy}{dx}=\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)+1 \quad \text{...(i)}$$

This is a linear homogeneous differential equation

Put $y = vx \Rightarrow \frac{dy}{dx}=v\cdot 1+x\cdot\frac{dv}{dx}$

Now (i) becomes

$$\cos v.\left[v+x\frac{dv}{dx}\right]=v\cos v+1$$

$$\Rightarrow x\cos v\frac{dv}{dx}=1 \Rightarrow \cos v\,dv=\frac{dx}{x}$$

Integrating both sides, we get

$\sin v = \log x + C \Rightarrow \sin\left(\frac{y}{x}\right)=\log x+C$

is the required solution.

50. The given differential equation is,

$e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$

$\Rightarrow (2 - e^x)\sec^2 y\, dy = -e^x \tan y\, dx$

$$\Rightarrow \frac{\sec^2 y}{\tan y}dy=\frac{-e^x}{2-e^x}dx$$

Integrating both sides, we get

$$\int\frac{\sec^2 y}{\tan y}dy=\int\frac{-e^x}{2-e^x}dx$$

$\Rightarrow \log \tan y = \log(2 - e^x) + C$

When $y=\frac{\pi}{4}, x=0$

$$\therefore \log\tan\frac{\pi}{4}=\log(2-e^0)+C$$

$\Rightarrow 0 = \log 1 + C \Rightarrow C = 0$

$\therefore$ Particular solution is $\log \tan y = \log (2 - e^x)$

$\Rightarrow e^x + \tan y - 2 = 0$

51. We have, $\frac{dy}{dx}+2y\tan x=\sin x$

It is linear differential equation of the form $\frac{dy}{dx}+Py=Q$

where $P = 2 \tan x$, and $Q = \sin x$

Now, I.F. $=e^{\int 2\tan x\,dx}=e^{2\log|\sec x|}=\sec^2 x$

$\therefore y(\sec^2 x)=\int(\sec^2 x)(\sin x)dx$

$\Rightarrow y(\sec^2 x)=\int\sec x\tan x\,dx$

$\Rightarrow y(\sec^2 x)=\sec x+C$

When $x=\frac{\pi}{3}, y = 0$

$(0)[\sec^2(\pi/3)] = \sec(\pi/3) + C \Rightarrow C = -2$

$\therefore y(\sec^2 x) = \sec x - 2$ *i.e.*, $y = \cos x - 2\cos^2 x$ is the required solution.

52. We have, $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

$$\Rightarrow \frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y} \quad \text{...(i)}$$

Put, $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ (i) becomes

$$v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2-v^4+3v^2}{v^3-3v}=\frac{1-v^4}{v(v^2-3)}$$

$$\Rightarrow \frac{v(v^2-3)dv}{1-v^4}=\frac{dx}{x}$$

$$\Rightarrow \int\frac{(v^3-3v)dv}{(1-v^2)(1+v^2)}=\int\frac{dx}{x} \quad \text{...(ii)}$$

$$\text{Now, let } \frac{v^3-3v}{(1-v^2)(1+v^2)}=\frac{Av+B}{1-v^2}+\frac{Cv+D}{1+v^2} \quad \text{...(iii)}$$

$\Rightarrow v^3 - 3v = (Av + B)(1 + v^2) + (Cv + D)(1 - v^2)$

Comparing coeff. of like powers, we get

$A - C = 1, A + C = -3, B - D = 0$ and $B + D = 0$

Solving these equations, we get $A = -1, B = 0, C = -2, D = 0$

From (ii) and (iii), we have

$$\int\frac{-v}{1-v^2}dv-\int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log(1-v^2)-\log(1+v^2)=\log x+\log C_1$$

$$\Rightarrow \frac{\sqrt{1-v^2}}{1+v^2}=C_1x \Rightarrow x\frac{\left(\sqrt{x^2-y^2}\right)}{x^2+y^2}=C_1x$$

$\Rightarrow x^2 - y^2 = C_1^2(x^2 + y^2)^2$

i.e., $x^2 - y^2 = C(x^2 + y^2)^2$ (where $C_1^2 = C$)

which is the required solution.

Commonly Made Mistake

- Remember the difference between $y = vx$ and $x = vy$ while solving differential equation.

53. We have, $\frac{dy}{dx}=\frac{(\tan^{-1}x-y)}{1+x^2}$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{1+x^2}=\frac{\tan^{-1}x}{1+x^2}$$

which is a linear differential equation

where $P=\frac{1}{1+x^2}$, $Q=\frac{\tan^{-1}x}{1+x^2}$

$\therefore$ $\text{I.F.}=e^{\int Pdx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$

$\therefore$ Solution is $y.(\text{I.F})=\int Q(\text{I.F})dx$

$\Rightarrow$ $ye^{\tan^{-1}x}=\int\frac{\tan^{-1}x}{1+x^2}\cdot e^{\tan^{-1}x}dx$ ...(i)

Let $I=\int\frac{\tan^{-1}x}{1+x^2}e^{\tan^{-1}x}dx$

Put $\tan^{-1}x = t$ $\Rightarrow$ $\frac{dx}{1+x^2}=dt$

$\therefore$ $I=\int t\cdot e^t dt=t\int e^t dt-\int\left(\frac{d}{dt}(t)\int e^t dt\right)dt$

$\Rightarrow$ $I=te^t-\int e^t dt = te^t - e^t + C$

$\Rightarrow$ $I = e^t(t - 1) + C$

$\Rightarrow$ $I=e^{\tan^{-1}x}(\tan^{-1}x-1)+C$ ...(ii)

Putting (ii) in (i), we get

$ye^{\tan^{-1}x}=e^{\tan^{-1}x}(\tan^{-1}x-1)+C$

$\Rightarrow$ $y=\tan^{-1}x-1+Ce^{-\tan^{-1}x}$

54. We have, $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

$\Rightarrow$ $(x-e^{\tan^{-1}y})\frac{dy}{dx}=-(1+y^2)$

$\Rightarrow$ $\frac{dx}{dy}=\frac{x-e^{\tan^{-1}y}}{-(1+y^2)}$ $\Rightarrow$ $\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$

This is a linear differential equation of the form $\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{1+y^2}$ and $Q=\frac{e^{\tan^{-1}y}}{1+y^2}$

$\therefore$ $\text{I.F.}=e^{\int\frac{dy}{1+y^2}}=e^{\tan^{-1}y}$

$\therefore$ Solution is $x\cdot e^{\tan^{-1}y}=\int\frac{(e^{\tan^{-1}y})^2}{1+y^2}dy+C$

$=\int\frac{e^{2\tan^{-1}y}}{1+y^2}dy+C$

$\Rightarrow$ $x\cdot e^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+C_1$ $\Rightarrow$ $x=\frac{e^{\tan^{-1}y}}{2}+C_1e^{-\tan^{-1}y}$

55. We have, $(1 - y^2)(1 + \log x)\, dx + 2xy\, dy = 0$

$\Rightarrow$ $(1 - y^2)(1 + \log x)\, dx = - 2xy\, dy$

$\Rightarrow$ $\frac{(1+\log x)}{x}dx=-\frac{2y}{1-y^2}dy$

On integrating both sides, we get

$\frac{(1+\log x)^2}{2}=\log|1-y^2|+C$

When $x = 1, y = 0$

$\therefore$ $\frac{(1+\log 1)^2}{2}=\log(1)+C$ $\Rightarrow$ $C=\frac{1}{2}$

$\Rightarrow$ $\frac{(1+\log x)^2}{2}=\log|1-y^2|+\frac{1}{2}$

$\Rightarrow$ $(1 + \log x)^2 = 2 \log |1 - y^2| + 1$ is the required solution.

Answer Tips

➲ $\int\mu^n d\mu=\frac{\mu^{n+1}}{n+1}$ with $n \neq - 1$

56. We have, $y+x\frac{dy}{dx}=x-y\frac{dy}{dx}$

$\Rightarrow$ $x\frac{dy}{dx}+y\frac{dy}{dx}=x-y$ $\Rightarrow$ $\frac{dy}{dx}=\frac{x-y}{x+y}$ ...(i)

This is a linear homogeneous D.E.

$\therefore$ Put $y = vx$ $\Rightarrow$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$v+x\frac{dv}{dx}=\frac{x-vx}{x+vx}=\frac{1-v}{1+v}$

$\Rightarrow$ $x\frac{dv}{dx}=\frac{1-v}{1+v}-v=\frac{1-v-v^2-v}{1+v}=\frac{1-2v-v^2}{1+v}$

$\Rightarrow$ $\frac{(1+v)}{v^2+2v-1}dv=-\frac{dx}{x}$

Integrating both sides, we get

$\frac{1}{2}\log|v^2+2v-1|=-\log|x|+\log C$

$\Rightarrow$ $\frac{1}{2}\log|v^2+2v-1|+\log|x|=\log C$

$\Rightarrow$ $\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{2y}{x}-1\right|+\log|x|=\log C$

$\Rightarrow$ $\log\left|\frac{y^2}{x^2}+\frac{2y}{x}-1\right|+2\log|x|=2\log C$

$\Rightarrow$ $\log\left|\frac{y^2+2xy-x^2}{x^2}\times x^2\right|=\log C^2$

$\Rightarrow$ $y^2 + 2xy - x^2 = \pm C^2$

$\Rightarrow$ $y^2 + 2xy - x^2 = C_1$ (where $C_1 = \pm C^2$)

57. We have, $y^2\, dx + (x^2 - xy + y^2)\, dy = 0$

$\Rightarrow$ $\frac{dy}{dx}=\frac{-y^2}{x^2-xy+y^2}$

This is homogeneous differential equation.

$\therefore$ Put $y = vx$ $\Rightarrow$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$, we get

$v+x\frac{dv}{dx}=\frac{-v^2x^2}{x^2-vx^2+v^2x^2}$

$\Rightarrow$ $v+x\frac{dv}{dx}=\frac{-v^2}{1-v+v^2}$ $\Rightarrow$ $x\frac{dv}{dx}=\frac{-v^2}{1-v+v^2}-v$

$\Rightarrow$ $x\frac{dv}{dx}=\frac{-v-v^3}{1-v+v^2}$ $\Rightarrow$ $\frac{1-v+v^2}{v(1+v^2)}dv=-\frac{1}{x}dx$

Integrating both sides, we get

$$\int \frac{1+v^2}{v(1+v^2)}dv - \int \frac{v}{v(1+v^2)}dv = -\int \frac{1}{x}dx$$

$\Rightarrow \int \frac{1}{v}dv - \int \frac{1}{1+v^2}dv = -\int \frac{1}{x}dx$

$\Rightarrow \log|v| - \tan^{-1}v = -\log|x| + \log C$

$\Rightarrow \log\left|\frac{vx}{C}\right| = \tan^{-1}v \Rightarrow \left|\frac{vx}{C}\right| = e^{\tan^{-1}v}$

$\Rightarrow |y| = Ce^{\tan^{-1}(y/x)}$ is the required solution.

Concept Applied

- A differential equation of the form $f(x, y)dy = g(x, y)dx$ is said to be homogeneous differential equation if the degree of $f(x, y)$ and $g(x, y)$ is same.

58. We have, $(\cot^{-1}y + x)\,dy = (1+y^2)dx$

$\Rightarrow \frac{dx}{dy} = \frac{\cot^{-1}y + x}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \left(-\frac{1}{1+y^2}\right)x = \frac{\cot^{-1}y}{1+y^2}$

This is a linear differential equation of the form

$\frac{dx}{dy} + Px = Q$, where, $P = -\frac{1}{1+y^2}$ and $Q = \frac{\cot^{-1}y}{1+y^2}$

$\therefore$ I.F. $= e^{-\int \frac{1}{1+y^2}dy} = e^{\cot^{-1}y}$

$\therefore$ Solution is,

$$xe^{\cot^{-1}y} = \int \frac{\cot^{-1}y}{(1+y^2)}e^{\cot^{-1}y}dy$$

[Put $t = \cot^{-1}y \Rightarrow dt = -\frac{1}{1+y^2}dy$]

$xe^{\cot^{-1}y} = -\int te^t dt$

$\Rightarrow xe^{\cot^{-1}y} = -e^t(t-1) + C$

$\Rightarrow xe^{\cot^{-1}y} = e^{\cot^{-1}y}(1-\cot^{-1}y) + C$

59. We have, $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

$\Rightarrow (\sin y + y\cos y)dy = x(2\log x + 1)dx$

On integrating both sides, we get

$-\cos y + y\sin y - (-\cos y)$

$$= 2\left[\log x \times \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$$

$\Rightarrow y\sin y = x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + C$

$\Rightarrow y\sin y = x^2\log x + C$

when $x = 1, y = \frac{\pi}{2}$

$\therefore \frac{\pi}{2}\sin\frac{\pi}{2} = 1\cdot\log(1) + C \Rightarrow \frac{\pi}{2} = C$

$\therefore y\sin y = x^2\log x + \pi/2$ is the required solution.

60. We have, $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}, |x| \neq 1$

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{x^2-1} = \frac{2}{(x^2-1)^2}$

This is a linear differential equation of the form,

$\frac{dy}{dx} + Py = Q$, where $P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{2x}{x^2-1}dx} = e^{\log(x^2-1)} = x^2-1$

Hence, solution of differential equation is given by

$$y(x^2-1) = \int \frac{2(x^2-1)}{(x^2-1)^2}dx$$

$\Rightarrow y(x^2-1) = 2\int \frac{dx}{x^2-1}$

$\Rightarrow y(x^2-1) = 2 \times \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$

$\Rightarrow y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$

Answer Tips

- $\int e^{\log x} = x + C$ where C is an arbitary constant.

61. We have, $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$

$\Rightarrow x\,e^x dx + \frac{y}{\sqrt{1-y^2}}dy = 0$

Integrating both sides, we get

$$x\cdot e^x - \int 1.e^x dx - \frac{1}{2}\int (1-y^2)^{-\frac{1}{2}}(-2y)dy = C$$

$\Rightarrow xe^x - e^x - \frac{1}{2}\frac{(1-y^2)^{\frac{1}{2}}}{1/2} = C$

$\Rightarrow e^x(x-1) - \sqrt{1-y^2} = C$

When $x = 0, y = 1, e^0(0-1) - \sqrt{1-1} = C$

$\Rightarrow C = -1$

$\therefore e^x(x-1) - \sqrt{1-y^2} = -1$ is the required solution.

62. We have, $\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0$

$\Rightarrow \frac{\log y}{y^2}dy + \frac{x^2}{\operatorname{cosec} x}dx = 0$

Integrating both sides, we get

$\int \frac{\log y}{y^2}dy + \int x^2 \sin x\,dx = 0$

[Put $\log y = t \Rightarrow \frac{1}{y}dy = dt$ and $y = e^t$]

$\Rightarrow \int t.e^{-t}dt + \int x^2\sin x\,dx = 0$

$\Rightarrow\ t.\frac{e^{-t}}{-1}-\int 1.\frac{e^{-t}}{-1}dt+x^2(-\cos x)-\int 2x(-\cos x)dx=C$

$\Rightarrow\ -t\,e^{-t}-e^{-t}-x^2\cos x+2x\sin x-2\int 1.\sin x\,dx=C$

$\Rightarrow -\frac{1+\log y}{y}-x^2\cos x+2x\sin x\ +2\cos x=C$

This is the required solution.

63. We have, $\frac{dy}{dx}=1+x+y+xy$

$\Rightarrow \frac{dy}{dx}=(1+x)+(1+x)y\ =(1+x)(1+y)$

$\Rightarrow \frac{dy}{1+y}=(1+x)dx$

Integrating both sides, we get

$\int\frac{dy}{1+y}=\int(1+x)dx+C$

$\Rightarrow \log(1+y)=x+\frac{x^2}{2}+C$ ...(i)

When $x = 1, y = 0$

$\therefore\ \log 1=1+\frac{1}{2}+C\ \Rightarrow C=-\frac{3}{2}$

$\therefore$ The particular solution of (i) is $\log(1+y)=x+\frac{x^2}{2}-\frac{3}{2}$.

64. We have,

$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$

$\Rightarrow \frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{e^{\tan^{-1}x}}{1+x^2}$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{1+x^2}$ and $Q=\frac{e^{\tan^{-1}x}}{1+x^2}$

$\therefore\ \text{I.F.}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$

So, the required solution is,

$y\cdot e^{\tan^{-1}x}=\int\frac{e^{2\tan^{-1}x}}{1+x^2}dx+C$ ...(i)

Put $\tan^{-1}x = t$

$\Rightarrow \frac{1}{1+x^2}dx=dt$

$\therefore$ (i) becomes,

$y\cdot e^{\tan^{-1}x}=\int e^{2t}dt+C$

$\Rightarrow\ y\cdot e^{\tan^{-1}x}=\frac{e^{2\tan^{-1}x}}{2}+C$ is the required solution.

65. We have, $x(1+y^2)\,dx-y(1+x^2)\,dy=0$

$\Rightarrow \frac{x}{1+x^2}dx-\frac{y}{1+y^2}dy=0\ \Rightarrow\ \frac{2x}{1+x^2}dx=\frac{2y}{1+y^2}dy$

Integrating both sides, we get $\log(1+y^2)=\log(1+x^2)+\log C$

$\Rightarrow\ 1+y^2=C(1+x^2)$

When $x = 0, y = 1$

$\therefore\ 1+1=C(1+0)\Rightarrow C=2$

$\therefore\ 1+y^2=2(1+x^2)$ is the required particular solution.

66. We have, $\log\left(\frac{dy}{dx}\right)=3x+4y$

$\frac{dy}{dx}=e^{3x+4y}=e^{3x}.e^{4y}\ \Rightarrow\ e^{-4y}\,dy=e^{3x}\,dx$

Integrating both sides, we get

$\int e^{3x}dx-\int e^{-4y}dy=0\ \Rightarrow\ \frac{e^{3x}}{3}-\frac{e^{-4y}}{-4}=C$

When $x = 0, y = 0$

$\therefore\ \frac{1}{3}+\frac{1}{4}=C\ \Rightarrow C=\frac{7}{12}$

$\Rightarrow\ \frac{e^{3x}}{3}+\frac{e^{-4y}}{4}=\frac{7}{12}$

$\Rightarrow\ 4e^{3x}+3e^{-4y}=7$ is the required particular solution.

67. We have, $(x^2-yx^2)dy+(y^2+x^2y^2)dx=0$

$\Rightarrow\ x^2(1-y)dy+y^2(1+x^2)dx=0$

$\Rightarrow\int\frac{(1-y)}{y^2}dy+\int\frac{(1+x^2)}{x^2}dx=0$

$\Rightarrow\int\left(\frac{1}{y^2}-\frac{1}{y}\right)dy+\int\left(\frac{1}{x^2}+1\right)dx=0$

$\Rightarrow-\frac{1}{y}-\log|y|-\frac{1}{x}+x=C$

$\Rightarrow\ -x-xy\log|y|-y+x^2y=C(xy)$ ...(i)

when $x = 1, y = 1$

$\therefore\ -(1)-(1)(1)\log|1|-(1)+(1)^2(1)=C(1)$

$\Rightarrow\ C=-1$

$\therefore$ Equation (i) becomes

$x^2y=x+xy\log|y|+y-xy$

68. We have, $\frac{dy}{dx}+y\cot x=2\cos x$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\cot x$, $Q=2\cos x$

$\therefore\ \text{I.F.}=e^{\int\cot x\,dx}=e^{\log|\sin x|}=|\sin x|$

$\therefore\ y|\sin x|=\int|\sin x|(2\cos x)dx$

$\Rightarrow y|\sin x|=\int\sin 2x\,dx$

$\Rightarrow y(\sin x)=-\frac{1}{2}\cos 2x+C$

when $x=\frac{\pi}{2}, y=0$

$\therefore\ 0(\sin\frac{\pi}{2})=-\frac{1}{2}\cos 2\left(\frac{\pi}{2}\right)+C\Rightarrow C=-\frac{1}{2}$

$\therefore\ y(\sin x)=-\frac{1}{2}\cos 2x-\frac{1}{2}$

i.e., $2y\sin x+\cos 2x+1=0$ is the required solution.

Concept Applied

- $\int\sin 2x\,dx=\frac{-1}{2}\cos 2x+C$

69. We have, $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

$\Rightarrow \frac{dy}{dx}+\frac{1}{x\log x}y=\frac{2}{x^2}$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q,$ where $P=\frac{1}{x\log x}, Q=\frac{2}{x^2}$

$\therefore$ $\text{I.F.}=e^{\int\frac{1}{x\log x}dx}=e^{\log\log x}=\log x$

$\therefore$ $y(\log x)=\int(\log x)\frac{2}{x^2}dx$

$\Rightarrow y(\log x)=\log x\int\frac{2}{x^2}dx-\int\left(\frac{d}{dx}(\log x)\int\frac{2}{x^2}dx\right)dx$

$\Rightarrow y(\log x)=\log x\left(-\frac{2}{x}\right)+\int\frac{2}{x^2}dx$

$\Rightarrow y(\log x)=\log x\left(-\frac{2}{x}\right)-\frac{2}{x}+C$

70. We have, $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx}=-\cos x$

$\Rightarrow \frac{dy}{1+y}=-\frac{\cos x}{2+\sin x}dx$

Integrating both sides, we get

$\log(y+1)=-\log(2+\sin x)+\log C$

$\Rightarrow \log(y+1)=\log\frac{C}{2+\sin x}$

$\Rightarrow$ $y+1=C/(2+\sin x)$ $\Rightarrow$ $(y+1)(2+\sin x)=C$

Given : $y(0)=1 \Leftrightarrow x=0, y=1$

$\therefore$ $(1+1).(2+\sin 0)=C$ $\Rightarrow$ $C=4$

$\therefore$ $(y+1)(2+\sin x)=4$

$\Rightarrow y=\frac{4}{2+\sin x}-1$ ...(i)

Put $x=\frac{\pi}{2}$ in (i), $y\left(\frac{\pi}{2}\right)=\frac{4}{2+1}-1=\frac{1}{3}.$

71. We have, $(x-y)\frac{dy}{dx}=x+2y$

$\Rightarrow \frac{dy}{dx}=\frac{x+2y}{x-y}$ ...(i)

This is a linear homogeneous differential equation.

$\therefore$ Put $y=vx \Rightarrow \frac{dy}{dx}=v.1+x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}=\frac{1+2v}{1-v}$

$\Rightarrow x\frac{dv}{dx}=\frac{1+2v}{1-v}-v=\frac{1+v+v^2}{1-v}$

$\Rightarrow \frac{1-v}{1+v+v^2}dv=\frac{dx}{x}$

Integrating both sides, we get

$$\int\frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}dv=\log x+C$$

$$\Rightarrow -\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv+\frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$=\log x+C$

$$\Rightarrow -\frac{1}{2}\log(v^2+v+1)+\frac{3}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\left[\frac{v+\frac{1}{2}}{\sqrt{3}/2}\right]=\log x+C$$

$$\Rightarrow -\frac{1}{2}\log\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}\cdot x}\right)=\log x+C$$

$$\Rightarrow -\frac{1}{2}\log(y^2+xy+x^2)+\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}\cdot x}\right)=C.$$

72. We have, $x\frac{dy}{dx}-y+x\,\text{cosec}\left(\frac{y}{x}\right)=0$

$\Rightarrow$ $\frac{dy}{dx}-\frac{y}{x}=-\text{cosec}\left(\frac{y}{x}\right)$ ...(i)

Put $y=vx \Rightarrow \frac{dy}{dx}=v.1+x.\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$v+x\frac{dv}{dx}-v=-\text{cosec}\,v$ $\Rightarrow$ $x\frac{dv}{dx}=-\text{cosec}\,v$

$\Rightarrow -\sin v\,dv=\frac{dx}{x}$

Integrating both sides, we get $\cos v=\log x+C$

$\Rightarrow \cos\left(\frac{y}{x}\right)=\log x+C$

When $x=1, y=0$

$\Rightarrow \cos\left(\frac{0}{1}\right)=\log 1+C \Rightarrow C=1$

$\therefore \cos\left(\frac{y}{x}\right)=\log x+1$

This is the required particular solution.

73. We have, $x\frac{dy}{dx}+y=x\cos x+\sin x$

$\Rightarrow$ $\frac{dy}{dx}+\frac{1}{x}.y=\frac{x\cos x+\sin x}{x}$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q,$ where $P=\frac{1}{x},$ $Q=\frac{x\cos x+\sin x}{x}$

$\therefore$ $\text{I.F.}=e^{\int P dx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x.$

$\therefore$ $y\cdot x=\int\frac{x\cos x+\sin x}{x}\cdot x\,dx+C$

$\Rightarrow xy=\int x\cos x\,dx+\int\sin x\,dx+C$

$=x\cdot\sin x-\int 1\cdot\sin x\,dx+\int\sin x\,dx+C$

$=x\sin x+C$

Given $y\left(\frac{\pi}{2}\right)=1$

$\therefore$ $\frac{\pi}{2}.1=\frac{\pi}{2}\sin\frac{\pi}{2}+C \Rightarrow C=0$

$\therefore \quad xy = x \sin x$

$\Rightarrow \quad y = \sin x$ is the required solution.

Commonly Made Mistake

- Remember the difference of differential equations of the form $\frac{dy}{dx}+Py=Q$ and $\frac{dx}{dy}+Px=Q$

74. We have, $x\frac{dy}{dx}+y=x\cos x+\sin x$

$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=\cos x+\frac{\sin x}{x}$

It is a linear differential equation.

$\text{I.F.}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$

$\therefore\ y\cdot x=\int x\left(\cos x+\frac{\sin x}{x}\right)dx+c\ =\int (x\cos x+\sin x)dx+c$

$=x\sin x-\int \sin x dx+\int \sin x dx+c\ = x\sin x + c$

$\Rightarrow\ y=\sin x+\frac{c}{x}$

Given that, $y=1$ when $x=\frac{\pi}{2}\ \therefore\ 1=1+\frac{c}{\pi/2}\ \Rightarrow\ c=0$

$\therefore \quad y = \sin x$ is the required solution.

75. We have, $(x-y)\frac{dy}{dx}=x+2y$

$\Rightarrow \quad \frac{dy}{dx}=\frac{x+2y}{x-y}$ ...(i)

Put $y = vx \quad \Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$

Putting $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$v+x\frac{dv}{dx}=\frac{x+2vx}{x-vx}=\frac{1+2v}{1-v}$

$\Rightarrow\ x\frac{dv}{dx}=\frac{1+2v}{1-v}-v=\frac{1+2v-v+v^2}{1-v}\ \Rightarrow\ \int\frac{1-v}{v^2+v+1}dv=\int\frac{1}{x}dx$

$\Rightarrow\ \int\frac{2-2v}{v^2+v+1}dv=2\log|x|+c\ \Rightarrow\ \int\frac{3-(2v+1)}{v^2+v+1}dv=2\log|x|+c$

$\Rightarrow\ \int\frac{3}{v^2+v+1}dv-\int\frac{2v+1}{v^2+v+1}dv=\log|x|^2+c$

$\Rightarrow\ 3\int\frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv-\log|v^2+v+1| = \log|x^2| + c$

$\Rightarrow\ \frac{3}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \log|x^2(v^2+v+1)| + c$

$\Rightarrow\ 2\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=\log|x^2(v^2+v+1)|+c$ ...(ii)

Substituting $v=\frac{y}{x}$ in (ii), we get

$2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=\log\left|x^2\frac{(y^2+yx+x^2)}{x^2}\right|+c$ ...(iii)

Now, at $y = 0$ and $x = 1$, we have

$2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\log|1|+c$

$\Rightarrow\ c=2\sqrt{3}\cdot\frac{\pi}{6}=\frac{\pi}{\sqrt{3}}$

Substituting $c=\frac{\pi}{\sqrt{3}}$ in (iii), we get

$2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=\log|y^2+xy+x^2|+\frac{\pi}{\sqrt{3}}$

$\Rightarrow\ 6\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=\sqrt{3}\log(x^2+xy+y^2)+\pi$

76. We have, $(\tan^{-1}y - x)dy = (1 + y^2)dx$

$\Rightarrow\ \frac{dx}{dy}=\frac{\tan^{-1}y-x}{1+y^2}\ \Rightarrow\ \frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{\tan^{-1}y}{1+y^2}$

This is a linear differential equation of the form

$\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{1+y^2}$ and $Q=\frac{\tan^{-1}y}{1+y^2}$

$\text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

$\therefore$ Required solution is,

$x\cdot e^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}\cdot\tan^{-1}y}{1+y^2}dy+C$ ...(i)

Put $\tan^{-1}y=t\Rightarrow\left(\frac{1}{1+y^2}\right)dy=dt$

$\therefore$ (i) becomes, $x\cdot e^{\tan^{-1}y}=\int e^t\cdot t\,dt+C$

$\Rightarrow\ x\cdot e^{\tan^{-1}y}=t\cdot e^t-\int 1\cdot e^t dt+C\ \Rightarrow\ x\cdot e^{\tan^{-1}y}=t\cdot e^t-e^t+C$

$\Rightarrow\ x\cdot e^{\tan^{-1}y}=\tan^{-1}y\cdot e^{\tan^{-1}y}-e^{\tan^{-1}y}+C$

$\Rightarrow\ x=\tan^{-1}y-1+Ce^{-\tan^{-1}y}$

Answer Tips

- $\int\frac{1}{1+x^2}dx=\tan^{-1}x+C$, where C is arbitrary constant.

77. We have, $\frac{dy}{dx}=\frac{y^2}{xy-x^2}=\frac{y^2/x^2}{(xy-x^2)/x^2}$ ... (i)

This is a homogeneous differential equation

$\therefore$ Put $y=vx\Rightarrow\frac{dy}{dx}=v\cdot1+x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$v+x\frac{dv}{dx}=\frac{v^2}{v-1}\ \Rightarrow\ x\frac{dv}{dx}=\frac{v^2}{v-1}-v=\frac{v}{v-1}\ \Rightarrow\ \frac{v-1}{v}dv=\frac{dx}{x}$

$\Rightarrow\ \left(1-\frac{1}{v}\right)dv=\frac{dx}{x}$

Integrating, we get

$v - \log v = \log x + C \quad \Rightarrow \quad v = \log vx + C$

$\Rightarrow\ \frac{y}{x}=\log y+C$

$\Rightarrow\ y = x(\log y + C)$ is the required solution.

78. We have, $(\tan^{-1}y - x)dy = (1+y^2)dx$

$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} \Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{\tan^{-1}y}{1+y^2}$

This is a linear differential equation of the form

$\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1}y}{1+y^2}$

I.F. $= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$\therefore$ Required solution is,

$x\cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}\cdot\tan^{-1}y}{1+y^2}dy + C$ ...(i)

Put $\tan^{-1}y = t \Rightarrow \left(\frac{1}{1+y^2}\right)dy = dt$

$\therefore$ (i) becomes, $x\cdot e^{\tan^{-1}y} = \int e^t\cdot t\,dt + C$

$\Rightarrow x\cdot e^{\tan^{-1}y} = t\cdot e^t - \int 1\cdot e^t dt + C \Rightarrow x\cdot e^{\tan^{-1}y} = t\cdot e^t - e^t + C$

$\Rightarrow x\cdot e^{\tan^{-1}y} = \tan^{-1}y\cdot e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$

$\Rightarrow x = \tan^{-1}y - 1 + C\,e^{-\tan^{-1}y}$

We get the solution as

$x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}$ ...(ii)

Now, putting $x = 1, y = 0$ in (ii), we get

$1 = \tan^{-1}0 - 1 + Ce^{-\tan^{-1}0} \Rightarrow C = 2$

So, required particular solution is $x = \tan^{-1}y - 1 + 2e^{-\tan^{-1}y}$.

79. We have,

$\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{y - x\cos\left(\frac{y}{x}\right)}$

$\Rightarrow \frac{dy}{dx} = \frac{2\sin\left(\frac{y}{x}\right) - \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\frac{y}{x} - \cos\left(\frac{y}{x}\right)}$ ...(i)

This is a homogeneous differential equation.

$\therefore$ Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} \Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{2\sin v - v^2}{v - \cos v} \Rightarrow \frac{v - \cos v}{2\sin v - v^2}dv = \frac{dx}{x}$

$\Rightarrow \frac{-1}{2}\frac{(2\cos v - 2v)}{2\sin v - v^2}dv = \frac{dx}{x}$

Integrating both sides, we get

$\frac{-1}{2}\log(2\sin v - v^2) = \log x + C_1$

$\Rightarrow \log x^2 + 2C_1 + \log\left(2\sin\frac{y}{x} - \frac{y^2}{x^2}\right) = 0$

$\Rightarrow \log\left[x^2\left(2\sin\frac{y}{x} - \frac{y^2}{x^2}\right)\right] = -2C_1$

$\Rightarrow 2x^2\sin\frac{y}{x} - y^2 = e^{-2C_1} = C$ (say)

which is the required solution.

80. We have, $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{(1+x^2)(1+y^2)}}{xy} \Rightarrow \int\frac{y}{\sqrt{1+y^2}}dy = -\int\frac{\sqrt{1+x^2}}{x^2}xdx$

Putting $1 + x^2 = v^2 \Rightarrow 2xdx = 2vdv$

$\Rightarrow \frac{1}{2}\int\frac{2y}{\sqrt{1+y^2}}dy = -\int\frac{v^2}{v^2-1}dv \Rightarrow \sqrt{1+y^2} = -\int\left(1 + \frac{1}{v^2-1}\right)dv$

$\Rightarrow \sqrt{1+y^2} = -v - \frac{1}{2}\log\left|\frac{v-1}{v+1}\right| + C$

$\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| = C$

81. We have, $x\frac{dy}{dx} + y - x + xy\cot x = 0, (x \neq 0)$

$\Rightarrow x\frac{dy}{dx} + (1 + x\cot x).y = x$

$\Rightarrow \frac{dy}{dx} + \frac{1 + x\cot x}{x}.y = 1$ ...(i)

This is linear D.E. of the form $\frac{dy}{dx} + Py = Q$

where $P = \frac{1 + x\cot x}{x} = \frac{1}{x} + \cot x$ and $Q = 1$

$\therefore$ Now I.F. $= e^{\int P dx} = e^{\log(x\sin x)} = x\sin x$

$\therefore$ The solution. of (i) is $y\cdot x\sin x = \int 1\cdot x\sin x\,dx + C$

$= x(-\cos x) + \int 1\cdot\cos x\,dx + C$. $\Rightarrow xy\sin x = -x\cos x + \sin x + C$

The required solution is

$y\cdot x\sin x = x(-\cos x) + \sin x + C$ ...(i)

Putting $x = \frac{\pi}{2}, y = 0$ in (i), we get

$0 = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + C \Rightarrow C = -1$

$xy\sin x = \sin x - x\cos x - 1$ is the required particular solution.

82. We have, $x^2dy + (xy + y^2)dx = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{xy + y^2}{x^2}$ ...(i)

This is a homogeneous linear differential equation

$\therefore$ Put $y = vx \Rightarrow \frac{dy}{dx} = v\cdot 1 + x\cdot\frac{dv}{dx}$

$\therefore$ (i) becomes $v + x\frac{dv}{dx} = -\frac{x\cdot vx + v^2x^2}{x^2} \Rightarrow x\frac{dv}{dx} = -(2v + v^2)$

Separating the variables, we get

$\frac{dv}{2v + v^2} + \frac{dx}{x} = 0 \Rightarrow \frac{dv}{v(v+2)} + \frac{dx}{x} = 0$

$\Rightarrow \quad \frac{1}{2}\left[\frac{1}{v}-\frac{1}{v+2}\right]dv+\frac{dx}{x}=0$

Integrating, we get

$\frac{1}{2}[\log v-\log(v+2)]+\log x=\log C \Rightarrow \quad \log\left(\frac{v}{v+2}\right)+2\log x=\log C$

$\Rightarrow \quad \log\left(\frac{v}{v+2}\right)+\log x^2=\log C \Rightarrow \quad \log\left(\frac{vx^2}{v+2}\right)=\log C$

$\Rightarrow \quad \frac{vx^2}{v+2}=C$

$\Rightarrow \quad \frac{\frac{y}{x}.x^2}{\frac{y}{x}+2}=C \quad \Rightarrow x^2y = C(2x + y)$ ...(ii)

Putting $x = 1, y = 1$ in (ii), we get

$1^2.1=C(2\cdot1+1)\Rightarrow C=\frac{1}{3}$

∴ The required particular solution is

$3x^2y=2x+y\Leftrightarrow y=\frac{2x}{3x^2-1}$

83. We have, $\left(x\sin^2\left(\frac{y}{x}\right)-y\right)dx+x\,dy=0$

$\Rightarrow \quad \frac{dy}{dx}+\sin^2\left(\frac{y}{x}\right)-\frac{y}{x}=0$ ...(i)

Put $y = vx \Rightarrow \frac{dy}{dx}=v\cdot1+x\frac{dv}{dx}$ ∴ (i) becomes

$v+x\frac{dv}{dx}+\sin^2 v-v=0$

$\Rightarrow \quad x\frac{dv}{dx}+\sin^2 v=0 \Rightarrow \quad \text{cosec}^2 v\,dv+\frac{dx}{x}=0$

Integrating both sides, we get

$\int \text{cosec}^2 v\,dv+\int\frac{dx}{x}=C \quad \Rightarrow -\cot v + \log x = C$

$\Rightarrow \quad -\cot\left(\frac{y}{x}\right)+\log x=C$ ...(ii)

Put $x = 1, y = \pi/4$ in (ii), we get

$-\cot\frac{\pi}{4}+\log 1=C \Rightarrow C=-1$

∴ $-\cot\left(\frac{y}{x}\right)+\log x+1=0$ is the required particular solution.

Answer Tips

➲ $\int \text{cosec}^2 x\,dx=-\cot(x)+C$

84. We have, $\frac{dy}{dx}-3y\cot x=\sin 2x$ ...(i)

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$ where $P = -3\cot x, Q = \sin 2x$

∴ $\quad \text{I.F.}=e^{\int P dx}=e^{-3\int \cot x dx} = e^{-3\log|\sin x|} = |\sin^{-3} x|$

∴ $\quad y\cdot\sin^{-3}x=\int \sin 2x\cdot\sin^{-3}x\,dx+C$

$\Rightarrow \quad \frac{y}{\sin^3 x}=\int\frac{2\sin x\cos x}{\sin^3 x}dx+C$

$=\int\frac{2\cos x}{\sin^2 x}dx+C$ (Put $\sin x = t \Rightarrow \cos x\,dx = dt$)

$= 2\int\frac{dt}{t^2}+C = -\frac{2}{t}+C=-\frac{2}{\sin x}+C$

$\Rightarrow y=-2\sin^2 x+C\sin^3 x$ ...(ii)

Put $x=\frac{\pi}{2}, y=2$ in (ii), we get $2 = -2\cdot1 + C\cdot 1 \Rightarrow C = 4$

∴ $y = 4\sin^3 x - 2\sin^2 x$ is the required particular solution.

CBSE Sample Questions

1. **(c) :** The given differential equation is $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)\right]^4$

$\Rightarrow \quad 4\left(\frac{dy}{dx}\right)^3\frac{d^2y}{dx^2}=0.$

Here, $m = 2$ and $n = 1$

Hence, $m + n = 3$ **(1)**

2. For $n = 3$, the given differential equation becomes homogeneous. **(1)**

3. We have, $\frac{d}{dx}\left(\frac{dy}{dx}\right)=5\Rightarrow\frac{d^2y}{dx^2}=5$

∴ Order = 2 **(1/2)**

Degree = 1 **(1/2)**

∴ Required sum = 3 **(1)**

4. There is no arbitrary constant in a particular solution of differential equation. **(1)**

5. The given differential equation is

$\frac{dy}{dx}=x^3\text{cosec}\,y \Rightarrow \int\frac{dy}{\text{cosec}\,y}=\int x^3 dx$ **(1/2)**

$\Rightarrow \int \sin y\,dy=\int x^3 dx \Rightarrow -\cos y=\frac{x^4}{4}+C$ **(1)**

Now, $y(0) = 0 \Rightarrow -1 = C$

So, the required solution is, $\cos y=1-\frac{x^4}{4}$ **(1/2)**

6. Given, $ydx + (x - y^2)dy = 0$

Reducing the given differential equation to the form

$\frac{dx}{dy}+Px=Q$ we get, $\frac{dx}{dy}+\frac{x}{y}=y$ **(1/2)**

Integrating factor (I.F.) = $e^{\int P dy}=e^{\int\frac{1}{y}dy}=e^{\log y}=y$ **(1/2)**

Thus, solution is given by, $xy=\int y^2 dy+C$ **(1)**

$\Rightarrow xy=\frac{y^3}{3}+C$, which is the required general solution. **(1)**

7. We have, $xdy - ydx = \sqrt{x^2+y^2}\,dx$

$\Rightarrow \quad x\frac{dy}{dx}-y=\sqrt{x^2+y^2} \Rightarrow \quad x\frac{dy}{dx}=y+\sqrt{x^2+y^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x} \quad ...(i)$$

Put $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Putting in (i), we get

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{x^2+v^2x^2}}{x} \quad (1)$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{x\left(v+\sqrt{1+v^2}\right)}{x}$$

$$\Rightarrow x\frac{dv}{dx}=v+\sqrt{1+v^2}-v$$

$$\Rightarrow x\frac{dv}{dx}=\sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x} \quad (1)$$

Integrating both sides, we get

$$\int\frac{dv}{\sqrt{1+v^2}}=\int\frac{dx}{x}$$

$$\Rightarrow \log\left(v+\sqrt{1+v^2}\right)=\log x+\log c$$

$$\Rightarrow \log\left(v+\sqrt{1+v^2}\right)=\log cx \Rightarrow \left(v+\sqrt{1+v^2}\right)=cx$$

$$\Rightarrow \left(\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}\right)=cx \Rightarrow y+\sqrt{x^2+y^2}=cx^2 \quad (1)$$

8. The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y}{x}-\sin\left(\frac{y}{x}\right) \quad ...(i)$$

Since, the equation is a homogeneous differential equation. Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$ **(1)**

From (i), we get

$$v+x\frac{dv}{dx}=v-\sin v$$

$$\Rightarrow \frac{dv}{\sin v}=-\frac{dx}{x} \Rightarrow \operatorname{cosec} v\, dv=-\frac{dx}{x} \quad (1/2)$$

Integrating both sides, we get

$\log|\operatorname{cosec} v - \cot v| = -\log|x| + \log K$, $K > 0$ (Here, $\log K$ is a constant) **(1/2)**

$\Rightarrow \log|(\operatorname{cosec} v - \cot v)x| = \log K \Rightarrow |(\operatorname{cosec} v - \cot v)x| = K$

$\Rightarrow (\operatorname{cosec} v - \cot v)x = \pm K \Rightarrow \left(\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right)x=C,$

which is the required general solution. **(1)**

9. The differential equation is a linear differential equation.

$$\therefore \text{ I.F.}=e^{\int\cot x\,dx}=e^{\log\sin x}=\sin x \quad (1)$$

The general solution is given by

$$y\sin x=\int 2\frac{\sin x}{1+\sin x}dx+c$$

$$\Rightarrow y\sin x=2\int\frac{\sin x+1-1}{1+\sin x}dx+c$$

$$\Rightarrow y\sin x=2\int[1-\frac{1}{1+\sin x}]dx+c \quad (1/2)$$

$$\Rightarrow y\sin x=2\int\left[1-\frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]dx+c$$

$$\Rightarrow y\sin x=2\int[1-\frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}]dx+c$$

$$\left[\because 1+\cos\theta=2\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow y\sin x=2\int[1-\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)]dx+c$$

$$\Rightarrow y\sin x=2\left[x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]+c \quad ...(i) \quad (1)$$

Now, $y=0$, when $x=\frac{\pi}{4}$ $\therefore$ From (i), we get

$$0=2\left[\frac{\pi}{4}+\tan\frac{\pi}{8}\right]+c \Rightarrow c=-\frac{\pi}{2}-2\tan\frac{\pi}{8}$$

Hence, the particular solution is given by

$$y=\operatorname{cosec}x\left[2\left\{x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}-\left(\frac{\pi}{2}+2\tan\frac{\pi}{8}\right)\right] \quad (1/2)$$

10. The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y+2x^2}{x} \Rightarrow \frac{dy}{dx}-\frac{1}{x}y=2x \quad (1/2)$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Px=Q, \text{ where } P=-\frac{1}{x} \text{ and } Q=2x$$

$$\therefore \text{ I.F.}=e^{\int Pdx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x} \quad (1)$$

$\therefore$ Required solution is

$$y\times\frac{1}{x}=\int\left(2x\times\frac{1}{x}\right)dx \Rightarrow \frac{y}{x}=2x+C \quad (1)$$

$$\Rightarrow y=2x^2+Cx \quad (1/2)$$

Self Assessment

Case Based Objective Questions (4 marks)

1. In a college hostel accommodating 1000 students, one of the hostellers came in carrying Corona virus, and the hostel was isolated. The rate at which the virus spreads is assumed to be proportional to the product of the number of infected students and remaining students. There are 50 infected students after 4 days.

Based on the above information, attempt any 4 out of 5 subparts.

(i) If $n(t)$ denote the number of students infected by Corona virus at any time t, then maximum value of $n(t)$ is

(a) 50 (b) 100 (c) 500 (d) 1000

(ii) $\frac{dn}{dt}$ is proportional to

(a) $n(1000 - n)$ (b) $n(100 + n)$
(c) $n(100 - n)$ (d) $n(100 + n)$

(iii) The value of $n(4)$ is

(a) 1 (b) 50 (c) 100 (d) 1000

(iv) The most general solution of differential equation formed in given situation is

(a) $\frac{1}{1000}\log\left(\frac{1000-n}{n}\right)=\lambda t+c$

(b) $\log\left(\frac{n}{100-n}\right)=\lambda t+c$

(c) $\frac{1}{1000}\log\left(\frac{n}{1000-n}\right)=\lambda t+c$

(d) None of these

(v) The value of n at any time is given by

(a) $n(t)=\frac{1000}{1+999e^{-0.9906t}}$ (b) $n(t)=\frac{1000}{1-999e^{-0.9906t}}$

(c) $n(t)=\frac{100}{1-999e^{-0.996t}}$ (d) $n(t)=\frac{100}{999+e^{1000t}}$

Multiple Choice Questions (1 mark)

2. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=x\sin\left(\frac{dy}{dx}\right)$ is

(a) 1 (b) 2
(c) 3 (d) not defined

3. Solution of differential equation $xdy - ydx = 0$ represents
(a) a rectangular hyperbola
(b) parabola whose vertex is at origin
(c) straight line passing through origin
(d) a circle whose centre is at origin

4. The number of solutions of $\frac{dy}{dx}=\frac{y+1}{x-1}$, when $y(1) = 2$ is
(a) none (b) one (c) two (d) infinite

OR

$\tan^{-1}x + \tan^{-1}y = c$ is the general solution of the differential equation

(a) $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$ (b) $\frac{dy}{dx}=\frac{1+x^2}{1+y^2}$

(c) $(1 + x^2)dy + (1 + y^2)dx = 0$
(d) $(1 + x^2)dx + (1 + y^2)dy = 0$

5. The order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2-3\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^4=y^4$ are

(a) 1, 4 (b) 3, 4 (c) 2, 4 (d) 3, 2

6. The differential equation satisfied by $y = \frac{A}{x} + B$ is (A, B are parameters)
(a) $x^2y_1 = y$ (b) $xy_1 + 2y_2 = 0$
(c) $xy_2 + 2y_1 = 0$ (d) none of these

7. Integrating factor of the differential equation $\cos x\frac{dy}{dx}+y\sin x=1$ is

(a) cos$x$ (b) tan$x$ (c) sec$x$ (d) sin$x$

VSA Type Questions (1 mark)

8. General solution of the differential equation of the type $\frac{dx}{dy}+P_1x=Q_1$ is given by ________ .

OR

The differential equation whose solution is $y = Ae^{3x} + Be^{-3x}$ is given by ________ .

9. Integrating factor of the differential equation $\frac{dy}{dx}+y\tan x-\sec x=0$ is ________ .

10. Find the order and degree of the differential equation $\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$.

11. The differential equation having solution as $y = 17e^x + ae^{-x}$ is

12. The order of the differential equation whose general solution is given by

$y=(C_1+C_2)\cos(x+C_3)-C_4e^{x+C_5}$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants, is ________ .

SA I Type Questions (2 marks)

13. Find the degree of the differential equation satisfying $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$.

14. Find the solution of $\frac{dy}{dx}=2^{y-x}$.

OR

Solve the differential equation $(1 + y^2) \tan^{-1} x dx + 2y(1 + x^2)dy = 0$.

15. Find the differential equation for $y = A \cos \alpha x + B \sin \alpha x$, where A and B are arbitrary constants.

16. Find the solution of the differential equation $\cos x \sin y \, dx + \sin x \cos y \, dy = 0$.

SA II Type Questions (3 marks)

17. Solve the differential equation

$$(x^2-1)\frac{dy}{dx}+2xy=\frac{1}{x^2-1}.$$

18. Form the differential equation having $y = (\sin^{-1}x)^2 + A \cos^{-1}x + B$, where A and B are arbitrary constants, as its general solution.

OR

Solve: $(x + y)(dx - dy) = dx + dy$.

19. Solve: $x\frac{dy}{dx}=y(\log y-\log x+1)$

20. Find the general solution of $y^2dx + (x^2 - xy + y^2)dy = 0$.

Case Based Questions (4 marks)

21. If the equation is of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ or $\frac{dy}{dx}=F\left(\frac{y}{x}\right)$, where $f(x, y)$, $g(x, y)$ are homogeneous functions of the same degree in x and y, then put $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$, so that the dependent variable y is changed to another variable v and then apply variable separable method.

Based on the above information, answer the following questions.

(i) Find the general solution of $x^2\frac{dy}{dx}=x^2+xy+y^2$.

(ii) What is the solution of the differential equation

$$2xy\frac{dy}{dx}=x^2+3y^2?$$

LA Type Questions (4 / 6 marks)

22. If $y(t)$ is a solution of $(1+t)\frac{dy}{dt}-ty=1$ and $y(0) = -1$, then show that $y(1)=-\frac{1}{2}$.

23. Find the equation of a curve passing through origin and satisfying the differential equation $(1+x^2)\frac{dy}{dx}+2xy=4x^2$.

OR

Solve: $\frac{dy}{dx}=\cos(x+y)+\sin(x+y)$.

24. Solve: $y+\frac{d}{dx}(xy)=x(\sin x+\log x)$.

25. Find the general solution of $(1 + \tan y)(dx - dy) + 2x\,dy = 0$

Detailed SOLUTIONS

1. (i) (d) : Since, maximum number of students in hostel is 1000.

$\therefore$ Maximum value of $n(t)$ is 1000.

(ii) (a): Clearly, according to given information,

$\frac{dn}{dt} = \lambda n(1000 - n)$, where λ is constant of proportionality.

(iii) (b): Since, 50 students are infected after 4 days.

$\therefore$ $n(4) = 50$.

(iv) (c): We have, $\frac{dn}{dt}=\lambda n(1000-n) \Rightarrow \int\frac{dn}{n(1000-n)}=\lambda\int dt$

$$\Rightarrow \frac{1}{1000}\int\left(\frac{1}{1000-n}+\frac{1}{n}\right)dn=\lambda\int dt$$

$$\Rightarrow \frac{1}{1000}\left[\frac{\log(1000-n)}{-1}+\log n\right]=\lambda t+C$$

$$\Rightarrow \frac{1}{1000}\log\left(\frac{n}{1000-n}\right)=\lambda t+C$$

(v) (a): When, $t = 0, n = 1$

This condition is satisfied by option (a) only.

2. (d) : Since, the given differential equation is not a polynomial in $\frac{dy}{dx}$. Therefore, its degree is not defined.

3. (c) : We have, $xdy - ydx = 0 \Rightarrow \frac{dy}{y}=\frac{dx}{x}$

Integrating both sides, we get

$\log y = \log x + \log C \Rightarrow y = Cx$

which is equation of a straight line passing through origin.

4. (b) : We have, $\frac{dy}{dx}=\frac{y+1}{x-1} \Rightarrow \frac{dy}{y+1}=\frac{dx}{x-1}$

Integrating both sides, we get

$\log(y + 1) = \log(x - 1) - \log C \Rightarrow C(y + 1) = (x - 1)$

$y(1) = 2$ *i.e.*, $x = 1 \Rightarrow y = 2 \therefore 3C = 0 \Rightarrow C = 0$

$\therefore$ Required solution, $x - 1 = 0 \Rightarrow x = 1$

Hence, only one solution exist.

OR

(c) : We have, $\tan^{-1}x + \tan^{-1}y = c$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}\cdot\frac{dy}{dx}=0 \Rightarrow (1+x^2)dy+(1+y^2)dx=0$$

5. (d) : We have, $\left(\frac{d^3y}{dx^3}\right)^2 - 3\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^4 = y^4$

Clearly, order = 3 and degree = 2

6. (c) : Given relation is $y = \frac{A}{x} + B$

Differentiating w.r.t. x, we get $y_1 = -\frac{A}{x^2} \Rightarrow x^2 y_1 = -A$

Again, differentiating w.r.t. x, we get
$x^2y_2 + y_1 2x = 0 \Rightarrow xy_2 + 2y_1 = 0$

7. (c) : We have, $\cos x\frac{dy}{dx} + y\sin x = 1$

$\Rightarrow \frac{dy}{dx} + y\frac{\sin x}{\cos x} = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} + y\tan x = \sec x$

It is linear differential equation of the type $\frac{dy}{dx} + Py = Q$ with $P = \tan x$ and $Q = \sec x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

8. We have, $\frac{dx}{dy} + P_1 x = Q_1$

which is linear differential equation $\therefore$ I.F. $= e^{\int P_1 dy}$

Now, solution is $x \cdot (\text{I.F.}) = \int Q_1 (\text{I.F.}) dy + c$

i.e., $xe^{\int P_1 dy} = \int Q_1 \{e^{\int P_1 dy}\} dy + c$

OR

We have, $y = Ae^{3x} + Be^{-3x}$
Differentiating w.r.t. x, we get, $y_1 = 3Ae^{3x} - 3Be^{-3x}$
Again, differentiating w.r.t. x, we get
$y_2 = 9Ae^{3x} + 9Be^{-3x} = 9(Ae^{3x} + Be^{-3x}) = 9y \Rightarrow y_2 - 9y = 0$

9. $\frac{dy}{dx} + y\tan x - \sec x = 0$
I.F. $= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

10. The given equation can be expressed as a polynomial in derivatives as

$$\rho^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

Clearly, it is a second order differential equation of degree 2.

11. We have $y = 17e^x + ae^{-x} \Rightarrow y' = 17e^x - ae^{-x}$
$\Rightarrow y'' = 17e^x + ae^{-x} = y \Rightarrow y'' - y = 0$

12. The given equation can be written as
$y = A\cos(x + C_3) - Be^x$
where, $A = C_1 + C_2$ and $B = C_4 e^{C_5}$
So, there are three independent variables, (A, B, C_3).
Hence, the differential equation will be of order 3.

13. Given, $a(x - y) = \sqrt{1 - y^2} + \sqrt{1 - x^2}$
Putting $x = \sin A$, $y = \sin B$, we get
$\cos A + \cos B = a(\sin A - \sin B)$

$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2a\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\Rightarrow \cot\left(\frac{A-B}{2}\right) = a \Rightarrow A - B = 2\cot^{-1} a$

$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

On differentiating w.r.t. x, we get

$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-y^2}{1-x^2} \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 1 - y^2$

Clearly, it is differential equation of first order and second degree.

14. We have, $\frac{dy}{dx} = 2^{y-x} \Rightarrow \frac{dy}{dx} = \frac{2^y}{2^x} \Rightarrow \frac{dy}{2^y} = \frac{dx}{2^x}$

Integrating both sides, we get $\frac{-2^{-y}}{\log 2} = \frac{-2^{-x}}{\log 2} + C$

$\Rightarrow -2^{-y} + 2^{-x} = C\log 2 = k(\text{say}) \Rightarrow 2^{-x} - 2^{-y} = k$

OR

We have, $(1 + y^2)\tan^{-1} x dx + 2y(1 + x^2)dy = 0$

$\Rightarrow \frac{\tan^{-1} x dx}{1+x^2} = -\frac{2y}{1+y^2}dy$

Integrating both sides, we get $\frac{1}{2}(\tan^{-1} x)^2 = -\log(1 + y^2) + c$

$\Rightarrow \frac{1}{2}(\tan^{-1} x)^2 + \log(1 + y^2) = c$

15. We have, $y = A\cos\alpha x + B\sin\alpha x$ ...(i)
Differentiating both sides w.r.t. x, we get
$\frac{dy}{dx} = -\alpha A\sin\alpha x + \alpha B\cos\alpha x$
Again, differentiating both sides w.r.t. x, we get

$\frac{d^2y}{dx^2} = -\alpha^2 A\cos\alpha x - \alpha^2 B\sin\alpha x$

$= -\alpha^2 (A\cos\alpha x + B\sin\alpha x) = -\alpha^2 y$ [using (i)]

$\Rightarrow \frac{d^2y}{dx^2} + \alpha^2 y = 0$

16. We have, $\cos x \sin y\, dx + \sin x \cos y\, dy = 0$
$\Rightarrow \cos x \sin y\, dx = -\sin x \cos y\, dy \Rightarrow \cot x\, dx = -\cot y\, dy$
Integrating both sides, we get
$\log \sin x = -\log \sin y + \log c$
$\Rightarrow \log \sin x + \log \sin y = \log c \Rightarrow \sin x \cdot \sin y = c$

17. We have, $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{2x}{x^2-1}\right)y = \frac{1}{(x^2-1)^2}$

which is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

where, $P = \frac{2x}{x^2-1}$, $Q = \frac{1}{(x^2-1)^2}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int\left(\frac{2x}{x^2-1}\right)dx} = e^{\log(x^2-1)} = (x^2 - 1)$

Now, solution is $y \cdot \text{I.F.} = \int Q \cdot \text{I.F.}\, dx + k$

$\Rightarrow y \cdot (x^2 - 1) = \int \frac{1}{(x^2-1)^2} \cdot (x^2 - 1)dx + k$

$= \int \frac{dx}{(x^2-1)} + k = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + k$

$\Rightarrow y \cdot (x^2 - 1) = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + k$

18. We have, $y = (\sin^{-1} x)^2 + A\cos^{-1}x + B$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}+\frac{(-A)}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\frac{dy}{dx}=2\sin^{-1}x-A$$

Again, differentiating both sides w.r.t. x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2}-\frac{x}{\sqrt{1-x^2}}\cdot\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2 \Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0$$

This is the required differential equation.

OR

We have, $(x + y)(dx - dy) = dx + dy$

$$\Rightarrow (x+y)\left(1-\frac{dy}{dx}\right)=1+\frac{dy}{dx} \quad \text{...(i)}$$

Put $x + y = z \Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ From (i), we get $z\left(1-\frac{dz}{dx}+1\right)=\frac{dz}{dx}$

$$\Rightarrow z\left(2-\frac{dz}{dx}\right)=\frac{dz}{dx} \Rightarrow 2z-z\frac{dz}{dx}-\frac{dz}{dx}=0 \Rightarrow \left(1+\frac{1}{z}\right)dz=2dx$$

Integrating both sides, we get, $z + \log z = 2x - \log c$

$\Rightarrow (x + y) + \log (x + y) = 2x - \log c \quad [\because z=x+y]$

$\Rightarrow 2x - x - y = \log c + \log (x + y)$

$\Rightarrow x - y = \log c + \log(x + y) \Rightarrow e^{x-y} = c(x + y)$

$\Rightarrow (x+y)=\frac{1}{c}e^{x-y} \Rightarrow x+y=k\,e^{x-y}$ where, $k=\frac{1}{c}$

19. We have, $x\frac{dy}{dx}=y(\log y-\log x+1)$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}\left(\log\frac{y}{x}+1\right) \quad \text{...(i)}$$

Since R.H.S. is of the form $g(y/x)$, and so it is a homogeneous function of degree zero. Therefore equation (i) is a homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$ $\therefore$ From (i), we get

$$v+x\frac{dv}{dx}=v(\log v+1) \Rightarrow x\frac{dv}{dx}=v\log v \Rightarrow \frac{dv}{v\log v}=\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{dv}{v\log v}=\int\frac{dx}{x} \quad \text{...(ii)}$$

On putting $\log v = t$ in L.H.S. integral, we get $\frac{1}{v}\cdot dv=dt$

$\therefore$ From (ii), $\int\frac{dt}{t}=\int\frac{dx}{x} \Rightarrow \log t = \log x + \log C \Rightarrow t = Cx$

$\Rightarrow \log v = Cx \Rightarrow \log\left(\frac{y}{x}\right)=Cx$

20. We have, $y^2dx + (x^2 - xy + y^2)dy = 0$

$\Rightarrow y^2dx = -(x^2 - xy + y^2)dy$

$$\Rightarrow \frac{dx}{dy}=-\frac{x^2}{y^2}+\frac{x}{y}-1 \quad \text{...(i)}$$

Since R.H.S. is of the form $g(x/y)$, and so it is a homogeneous function of degree zero. Therefore equation (i) is a homogeneous differential equation.

Put $x = vy \Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$

From (i), we get $v+y\frac{dv}{dy}=-v^2+v-1$

$$\Rightarrow y\frac{dv}{dy}=-v^2-1 \Rightarrow \frac{dv}{v^2+1}=-\frac{dy}{y}$$

Integrating both sides, we get

$\tan^{-1}(v) = -\log y + c$

$$\Rightarrow \tan^{-1}\left(\frac{x}{y}\right)+\log y=c \quad \left[\because v=\frac{x}{y}\right]$$

21. (i) We have, $\frac{dy}{dx}=\frac{x^2+xy+y^2}{x^2}$

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \quad v+x\frac{dv}{dx}=\frac{x^2+x\cdot vx+v^2x^2}{x^2}=1+v+v^2$$

$$\Rightarrow \quad x\frac{dv}{dx}=1+v^2 \Rightarrow \int\frac{dv}{1+v^2}=\int\frac{dx}{x}$$

$$\Rightarrow \quad \tan^{-1}v=\log|x|+c \Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\log|x|+c$$

(ii) We have,

$$2xy\frac{dy}{dx}=x^2+3y^2 \Rightarrow \frac{dy}{dx}=\frac{x^2+3y^2}{2xy}$$

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \quad v+x\frac{dv}{dx}=\frac{x^2+3v^2x^2}{2vx^2} \Rightarrow x\frac{dv}{dx}=\frac{1+3v^2}{2v}-v$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{1+v^2}{2v} \Rightarrow \int\frac{2v}{1+v^2}dv=\int\frac{dx}{x}$$

$\Rightarrow \log|1+v^2| = \log|x| + \log|c| \Rightarrow \log|v^2 + 1| = \log|xc|$

$$\Rightarrow \quad v^2+1=xc \Rightarrow \frac{y^2}{x^2}+1=xc \Rightarrow x^2+y^2=x^3c$$

22. We have, $(1+t)\frac{dy}{dt}-ty=1$

$$\Rightarrow \frac{dy}{dt}-\left(\frac{t}{1+t}\right)y=\frac{1}{1+t}$$

which is linear differential equation of the type $\frac{dy}{dt}+Py=Q$

where, $P=-\left(\frac{t}{1+t}\right)$, $Q=\frac{1}{1+t}$

$$\therefore \quad \text{I.F.}=e^{\int P\,dt}=e^{-\int\frac{t}{1+t}dt}=e^{-\int\left(1-\frac{1}{1+t}\right)dt}=e^{-[t-\log(1+t)]}$$

$$=e^{-t}\cdot e^{\log(1+t)}=e^{-t}(1+t)$$

Now, solution is

$$y\cdot(1+t)e^{-t}=\int\frac{(1+t)\cdot e^{-t}}{(1+t)}dt+C$$

$$\Rightarrow \frac{y(1+t)}{e^t}=\int e^{-t}dt+C \Rightarrow \frac{y(1+t)}{e^t}=\frac{e^{-t}}{(-1)}+C$$

$$\Rightarrow y=\frac{e^{-t}}{(-1)}\cdot\frac{e^t}{1+t}+\frac{Ce^t}{1+t} \Rightarrow y=-\frac{1}{1+t}+\frac{e^t}{1+t}C \quad \text{...(i)}$$

$y(0) = -1$ *i.e.*, $t = 0 \Rightarrow y = -1$

$\therefore$ From (i), $-1 = -1 + C \Rightarrow C = 0$

Now, from (i) $y = -\frac{1}{1+t} \quad \therefore y(1) = -\frac{1}{2}$

23. We have, $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{4x^2}{1+x^2}$

which is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, with $P = \frac{2x}{1+x^2}$, $Q = \frac{4x^2}{1+x^2}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Now, solution is

$y\cdot(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)dx + C \Rightarrow y\cdot(1+x^2) = \int 4x^2\, dx + C$

$\Rightarrow y\cdot(1+x^2) = \frac{4x^3}{3} + C$ ...(i)

Since, the curve passes through origin, then substituting $x = 0$ and $y = 0$ in (i), we get

$C = 0$

$\therefore$ Required equation of curve is

$y(1+x^2) = \frac{4x^3}{3} \Rightarrow y = \frac{4x^3}{3(1+x^2)}$

OR

We have, $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ ...(i)

Put $x + y = z \Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

From (i), we get, $\left(\frac{dz}{dx} - 1\right) = \cos z + \sin z$

$\Rightarrow \frac{dz}{dx} = (\cos z + \sin z + 1) \Rightarrow \frac{dz}{\cos z + \sin z + 1} = dx$

Integrating both sides, we get $\int \frac{dz}{\cos z + \sin z + 1} = \int 1\, dx$

$\Rightarrow \int \frac{dz}{\frac{1-\tan^2 z/2}{1+\tan^2 z/2} + \frac{2\tan z/2}{1+\tan^2 z/2} + 1} = \int dx$

$\Rightarrow \int \frac{(1+\tan^2 z/2)\,dz}{2(1+\tan z/2)} = \int dx$

$\Rightarrow \int \frac{\sec^2 z/2\, dz}{2(1+\tan z/2)} = \int dx$ ...(ii)

Put $1 + \tan z/2 = u$

$\Rightarrow \left(\frac{1}{2}\sec^2 z/2\right)dz = du$

From (ii), $\int \frac{du}{u} = \int dx \Rightarrow \log|u| = x + c$

$\Rightarrow \log|1+\tan z/2| = x + c \Rightarrow \log\left|1 + \tan\frac{(x+y)}{2}\right| = x + c$

24. We have, $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

$\Rightarrow y + x\frac{dy}{dx} + y = x(\sin x + \log x)$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = \sin x + \log x$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{2}{x}$, $Q = \sin x + \log x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = x^2$

Now, solution is $y\cdot x^2 = \int(\sin x + \log x)\, x^2 dx + c$

$\Rightarrow y\cdot x^2 = \int x^2 \sin x\, dx + \int x^2 \log x\, dx + c$

$= x^2(-\cos x) + \int 2x\cos x\, dx + \log x\cdot\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}dx + c$

$= -x^2\cos x + [2x(\sin x) - \int 2\sin x\, dx]$

$+ \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx + c$

$= -x^2\cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{1}{9}x^3 + c$

$\Rightarrow y\cdot x^2 = -x^2\cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{1}{9}x^3 + c$

$\therefore y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{x}{3}\log x - \frac{x}{9} + cx^{-2}$

25. We have, $(1 + \tan y)(dx - dy) + 2x dy = 0$

$\Rightarrow (1+\tan y)\left(\frac{dx}{dy} - 1\right) + 2x = 0$

$\Rightarrow (1+\tan y)\frac{dx}{dy} - (1+\tan y) + 2x = 0$

$\Rightarrow (1+\tan y)\frac{dx}{dy} + 2x = (1+\tan y) \Rightarrow \frac{dx}{dy} + \frac{2x}{1+\tan y} = 1$

which is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$, where $P = \frac{2}{1+\tan y}$, $Q = 1$

$\therefore$ I.F. $= e^{\int \frac{2}{1+\tan y}dy} = e^{\int \frac{2\cos y}{\cos y + \sin y}dy}$

$= e^{\int \frac{\cos y + \sin y + \cos y - \sin y}{\cos y + \sin y}dy}$

$= e^{\int \left(1 + \frac{\cos y - \sin y}{\cos y + \sin y}\right)dy} = e^{y + \log(\cos y + \sin y)} = e^y\cdot(\cos y + \sin y)$

Now, solution is

$x\cdot e^y(\cos y + \sin y) = \int 1\cdot e^y(\cos y + \sin y)dy + C$

$\Rightarrow x\cdot e^y(\cos y + \sin y) = \int e^y(\sin y + \cos y)dy + C$

$\Rightarrow x\cdot e^y(\cos y + \sin y) = e^y \sin y + C$

$[\because \int e^x\{f(x) + f'(x)\}dx = e^x f(x)]$

$\Rightarrow x(\sin y + \cos y) = \sin y + Ce^{-y}$

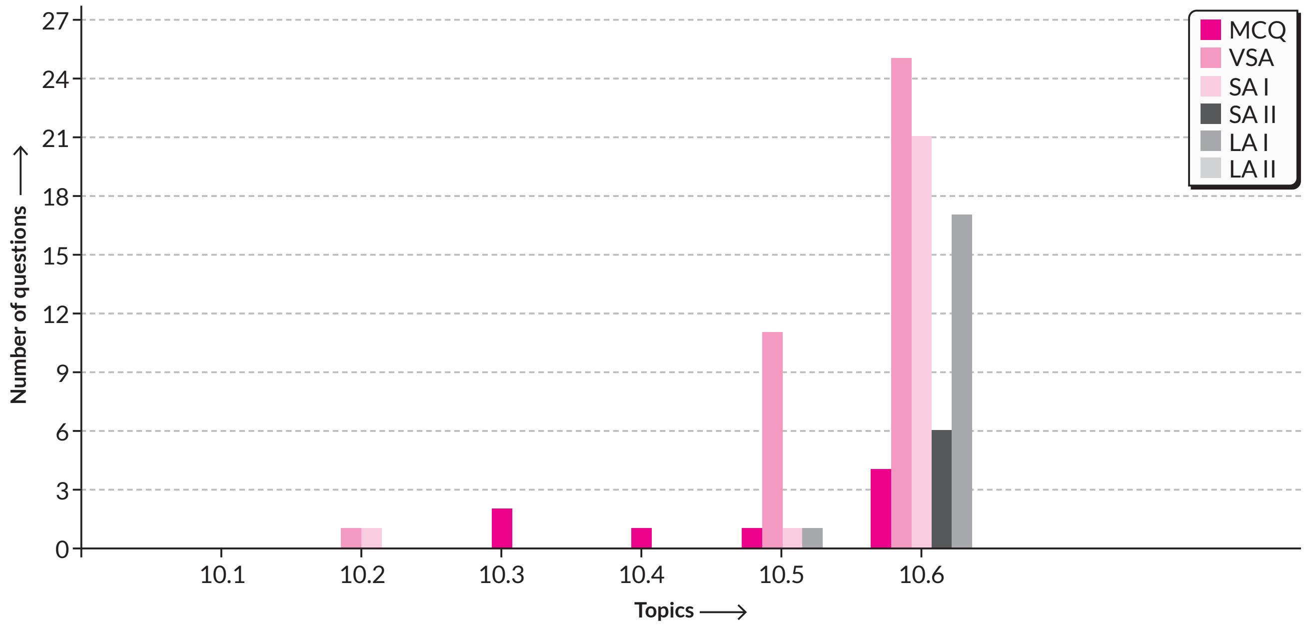
CHAPTER 10

Vector Algebra

TOPICS

10.1 Introduction
10.2 Some Basic Concepts
10.3 Types of Vectors
10.4 Addition of Vectors
10.5 Multiplication of a Vector by a Scalar
10.6 Product of Two Vectors

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



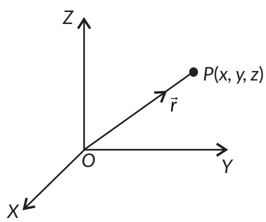
Weightage Xtract

- Topic 10.5 and 10.6 are highly scoring topics.
- Maximum total weightage is of Topic 10.6 *Product of Two Vectors.*
- Maximum VSA, SA I, SA II, LA I type questions were asked from Topic 10.6 *Product of Two Vectors.*
- No LA II type questions were asked till now.

QUICK RECAP

Vector

- A physical quantity having magnitude as well as direction is called a vector. A vector is represented by a line segment, denoted as $\overrightarrow{AB}$ or $\vec{a}$. Here, point A is the initial point and B is the terminal point of the vector $\overrightarrow{AB}$.
  - **Magnitude :** The distance between the points A and B is called the magnitude of the directed line segment $\overrightarrow{AB}$. It is denoted by $|\overrightarrow{AB}|$.



- **Position Vector :** Let P be any point in space, having coordinates (x, y, z) with respect to O (0, 0, 0) as origin, then the vector $\overrightarrow{OP}$ having O as its initial point and P as its terminal point is called the position vector of the point P with respect to O. The vector $\overrightarrow{OP}$ is usually denoted by $\vec{r}$.

  Magnitude of $\overrightarrow{OP}$ is, $|\overrightarrow{OP}|=\sqrt{x^2+y^2+z^2}$

  i.e., $|\vec{r}|=\sqrt{x^2+y^2+z^2}$.

  In general, the position vectors of points A, B, C, etc. with respect to the origin O are denoted by $\vec{a},\vec{b},\vec{c}$, etc. respectively.

Direction Cosines and Direction Ratios

Direction Cosines and Direction Ratios : The angles α, β and γ made by the vector $\vec{r}$ with the positive directions of x, y and z-axes respectively are called its direction angles. The cosine values of these angles, *i.e.*, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the vector $\vec{r}$, and usually denoted by l, m and n respectively.

Direction cosines of $\vec{r}$ are given as

$$l=\frac{x}{\sqrt{x^2+y^2+z^2}}, m=\frac{y}{\sqrt{x^2+y^2+z^2}} \text{ and } n=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

The numbers lr, mr and nr, proportional to the direction cosines of vector $\vec{r}$ are called direction ratios of the vector $\vec{r}$ and denoted as a, b and c respectively.

i.e., $a = lr$, $b = mr$ and $c = nr$

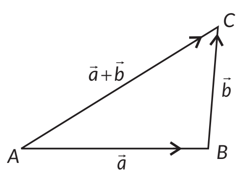
Note : $l^2 + m^2 + n^2 = 1$ and $a^2 + b^2 + c^2 \neq 1$, (in general).

Types of Vectors

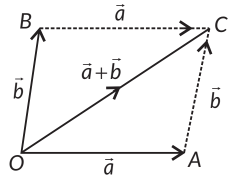
- **Zero Vector :** A vector whose initial and terminal points coincide is called a zero (or null) vector. It cannot be assigned a definite direction as it has zero magnitude and it is denoted by the $\vec{0}$.
- **Unit Vector :** A vector whose magnitude is unity *i.e.*, $|\vec{a}|=1$. It is denoted by $\hat{a}$.
- **Equal Vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a}=\vec{b}$, iff they have same magnitudes and direction regardless of the positions of their initial points.
- **Coinitial Vectors :** Vectors having same initial point are called co-initial vectors.
- **Collinear Vectors :** Two or more vectors are called collinear if they have same or parallel supports, irrespective of their magnitudes and directions.
- **Negative of a Vector :** A vector having the same magnitude as that of a given vector but directed in the opposite sense is called negative of the given vector *i.e.*, $\overrightarrow{BA}=-\overrightarrow{AB}$.

Addition of Vectors

- **Triangle law :** Let the vectors be $\vec{a}$ and $\vec{b}$ so positioned such that initial point of one coincides with terminal point of the other. If $\vec{a}=\overrightarrow{AB}, \vec{b}=\overrightarrow{BC}$. Then the vector $\vec{a}+\vec{b}$ is represented by the third side of ΔABC *i.e.*, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.



- **Parallelogram law :** If the two vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides OA and OB of a parallelogram $OACB$, then their sum $\vec{a}+\vec{b}$ is represented in magnitude and direction by the diagonal OC of parallelogram $OACB$ through their common point O *i.e.*, $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$.



- **Properties of Vector Addition**

  (i) Vector addition is commutative *i.e.*, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

  (ii) Vector addition is associative *i.e.*, $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$.

  (iii) Existence of additive identity : The zero vector acts as additive identity *i.e.*, $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$, for any vector $\vec{a}$.

  (iv) Existence of additive inverse : The negative of $\vec{a}$ *i.e.*, $-\vec{a}$ acts as additive inverse *i.e.*, $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$, for any vector $\vec{a}$.

Multiplication of a Vector by a Scalar

- Let $\vec{a}$ be a given vector and λ be a scalar (a real number), then $\lambda\vec{a}$ is called as the multiplication of vector $\vec{a}$ by the scalar λ. Its magnitude is $|\lambda|$ times the magnitude of $\vec{a}$ *i.e.*, $|\lambda\vec{a}|=|\lambda||\vec{a}|$.

  Direction of $\lambda\vec{a}$ is same as that of $\vec{a}$ if $\lambda>0$ and opposite to that of $\vec{a}$ if $\lambda<0$.

  Note : If $\lambda=\frac{1}{|\vec{a}|}$, provided that $\vec{a}\neq 0$, then $\lambda\vec{a}$ represents the unit vector in the direction of $\vec{a}$ *i.e.* $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$

Components of a Vector

- Let O be the origin and $P(x, y, z)$ be any point in space. Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be unit vectors along the X-axis, Y-axis and Z-axis respectively. Then $\overrightarrow{OP}=x\hat{i}+y\hat{j}+z\hat{k}$, is called the component form of $\overrightarrow{OP}$. Here x, y and z are scalar components of $\overrightarrow{OP}$ and $x\hat{i}$, $y\hat{j}$, $z\hat{k}$ are vector components of $\overrightarrow{OP}$.

  If $\vec{a}$ and $\vec{b}$ are two given vectors as $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and λ be any scalar, then

  (i) $\vec{a}+\vec{b}=(a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$

  (ii) $\vec{a}-\vec{b}=(a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}+(a_3-b_3)\hat{k}$

  (iii) $\lambda\vec{a}=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$

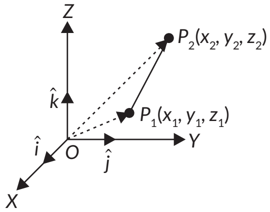
  (iv) $\vec{a}=\vec{b}\Leftrightarrow a_1=b_1, a_2=b_2$ and $a_3=b_3$

  (v) $\vec{a}$ and $\vec{b}$ are collinear if $\frac{b_1}{a_1}=\frac{b_2}{a_2}=\frac{b_3}{a_3}=\lambda$.

Vector Joining Two Points

- If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points in the space, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$.

Applying triangle law in ΔOP_1P_2, we get



$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$

$\Rightarrow \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\therefore |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formula

- Let A, B be two points such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.
- The position vector $\vec{r}$ of the point P which divides the line segment AB internally in the ratio $m : n$ is given by $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.
- The position vector $\vec{r}$ of the point P which divides the line segment AB externally in the ratio $m : n$ is given by $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$.
- The position vector $\vec{r}$ of the mid-point of the line segment AB is given by $\vec{r} = \dfrac{\vec{a} + \vec{b}}{2}$.

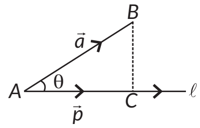
Product of Two Vectors

- **Scalar (or dot) product of two vectors :** The scalar (or dot) product of two (non-zero) vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$ (read as $\vec{a}$ dot $\vec{b}$), is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta$, where, $a = |\vec{a}|, b = |\vec{b}|$ and $\theta (0 \le \theta \le \pi)$ is the angle between $\vec{a}$ and $\vec{b}$.
- **Properties of Scalar Product :**
  (i) Scalar product is commutative : $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
  (ii) $\vec{a} \cdot \vec{0} = 0$
  (iii) Scalar product is distributive over addition :
  - $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
  - $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

  (iv) $\lambda(\vec{a} \cdot \vec{b}) = (\lambda\vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda\vec{b})$, λ be any scalar.
  (v) If $\hat{i}, \hat{j}$ and $\hat{k}$ are three unit vectors along three mutually perpendicular lines, then
  $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$
  (vi) Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$ i.e., $\theta = \cos^{-1}\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$
  (vii) Two non-zero vectors $\vec{a}$ and $\vec{b}$ are mutually perpendicular if and only if $\vec{a} \cdot \vec{b} = 0$
  (viii) If $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$
  (ix) If $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

- **Projection of a vector on a line :**

Let the vector $\overrightarrow{AB}$ makes an angle θ with directed line ℓ in anti clockwise direction.

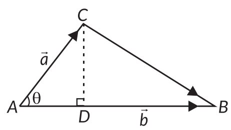


Projection of $\overrightarrow{AB}$ on $\ell = |\overrightarrow{AB}|\cos\theta = \overrightarrow{AC} = \vec{p}$.
The vector $\vec{p}$ is called the projection vector. Its magnitude is $|\vec{p}|$, which is known as projection of vector $\overrightarrow{AB}$ on ℓ. Projection of a vector $\vec{a}$ on $\vec{b}$, is given as $\vec{a} \cdot \hat{b}$ i.e., $\dfrac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b})$.

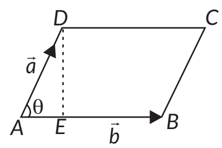
- **Vector (or Cross) Product :** The vector (or cross) product of two (non-zero) vectors $\vec{a}$ and $\vec{b}$ (in an assigned order), denoted by $\vec{a} \times \vec{b}$ (read as $\vec{a}$ cross $\vec{b}$), is defined as $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$, where $\theta (0 \le \theta \le \pi)$ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.
- **Properties of Vector Product :**
  (i) Vector product is non-commutative :
  $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
  (ii) Vector product is distributive over addition :
  $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
  (iii) $\lambda(\vec{a} \times \vec{b}) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b})$, λ be any scalar.
  (iv) $(\lambda_1\vec{a}) \times (\lambda_2\vec{b}) = \lambda_1\lambda_2(\vec{a} \times \vec{b})$
  (v) $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$
  (vi) Two non-zero vectors $\vec{a}, \vec{b}$ are collinear if and only if $\vec{a} \times \vec{b} = \vec{0}$.
  Similarly, $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times (-\vec{a}) = \vec{0}$, since in the first situation $\theta = 0$ and in the second one, $\theta = \pi$, making the value of $\sin\theta$ to be 0.
  (vii) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle as given in the figure.
  Then, area of triangle $ABC = \dfrac{1}{2}AB \cdot CD$

  $= \dfrac{1}{2}|\vec{b}||\vec{a}|\sin\theta = \dfrac{1}{2}|\vec{a} \times \vec{b}|$

  $\left[\because \sin\theta = \dfrac{CD}{|\vec{a}|}\right]$



  (viii) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram as given in the figure.



  Then, area of parallelogram
  $ABCD = AB \cdot DE = |\vec{b}||\vec{a}|\sin\theta = |\vec{a} \times \vec{b}|$
  (ix) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

  then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

  $= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$
  (x) Angle between two vectors $\vec{a}$ and $\vec{b}$ is given by
  $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$ i.e., $\theta = \sin^{-1}\left(\dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right)$

BRAIN MAP

VECTOR ALGEBRA

Vector
A quantity that has magnitude as well as direction is called a vector.

Multiplication of Vectors

Scalar (or Dot Product)
- The scalar product of $\vec{a}$ and $\vec{b}$ written as $\vec{a}\cdot\vec{b}$ and is defined as $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.
- Projection of a vector $\vec{a}$ on other vector $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Properties of Scalar Product
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}\perp\vec{b}$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$,
  $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
  then, $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- $\vec{a}\cdot(\alpha\vec{b}) = \alpha(\vec{a}\cdot\vec{b})$
- $\vec{a}\cdot(\vec{b}\pm\vec{c}) = \vec{a}\cdot\vec{b}\pm\vec{a}\cdot\vec{c}$

Vector (or cross) Product
Vector product of vectors is given by

$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|(\sin\theta)\hat{n}$, where $\hat{n}$ is the unit vector normal to the plane determined by $\vec{a}$ and $\vec{b}$.

Properties of Vector Product
- $\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$
- $\vec{a}\times\vec{a} = 0$
- $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a}\parallel\vec{b}$
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0, \hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$,
  $\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}, \hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

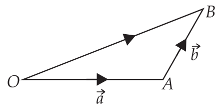
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- $(\alpha\vec{a})\times(\beta\vec{b}) = (\alpha\beta)(\vec{a}\times\vec{b})$
- $\vec{a}\times(\vec{b}\pm\vec{c}) = \vec{a}\times\vec{b}\pm\vec{a}\times\vec{c}$
- $(\vec{a}\cdot\vec{b})^2 + (\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$
- If $\vec{a}$ and $\vec{b}$ are adjacent sides of a triangle, then area of triangle $= \dfrac{1}{2}|\vec{a}\times\vec{b}|$.

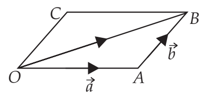
Components of a Vector
Let O be the origin and let $P(x, y, z)$ be any point in space. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along the x-axis, y-axis and z-axis respectively.

Then, $\overrightarrow{OP} = (x\hat{i} + y\hat{j} + z\hat{k})$. Also, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

Addition of Two Vectors
- **Triangle Law :** In ΔOAB, let position vectors of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ be $\vec{a}$ and $\vec{b}$ respectively. Then, $\overrightarrow{OB} = \vec{a} + \vec{b}$



- **Parallelogram Law :** In parallelogram $OABC$, let position vectors of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ be $\vec{a}$ and $\vec{b}$ respectively. Then, $\overrightarrow{OB} = \vec{a} + \vec{b}$



Properties
- **Commutative Law :** $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- **Associative Law :** $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- **Existence of Additive Identity :** $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$
  i.e., $\vec{0}$ is the additive identity.
- **Existence of Additive Inverse :** $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$
  i.e., $(-\vec{a})$ is the additive inverse of $\vec{a}$.

Algebra of Vectors
For any two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ be any scalar, we have

Addition	$\vec{a}+\vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$
Subtraction	$\vec{a}-\vec{b} = (a_1-b_1)\hat{i} + (a_2-b_2)\hat{j} + (a_3-b_3)\hat{k}$
Equality	$\vec{a}$ and $\vec{b}$ are equal if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$
Multiplication by a Scalar	$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

Vector Joining Two Points
Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points, then the vector joining P and Q is given by

$\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

Also, $|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formula
If the point C divides AB in the ratio $m : n$, then its position vector is $\vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m + n}$

Previous Years' CBSE Board Questions

10.2 Some Basic Concepts

VSA (1 mark)

1. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis. *(AI 2014)* Ev

SA I (2 marks)

2. Find a vector $\vec{r}$ equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units. *(2020)* An

10.3 Types of Vectors

MCQ

3. Two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are collinear if
(a) $a_1b_1+a_2b_2+a_3b_3=0$ (b) $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$
(c) $a_1=b_1, a_2=b_2, a_3=b_3$
(d) $a_1+a_2+a_3=b_1+b_2+b_3$ *(2023)*

4. The value of p for which $p(\hat{i}+\hat{j}+\hat{k})$ is a unit vector is
(a) 0 (b) $\frac{1}{\sqrt{3}}$ (c) 1 (d) $\sqrt{3}$ *(2020)* Ap

10.4 Addition of Vectors

MCQ

5. *ABCD* is a rhombus, whose diagonals intersect at *E*. Then $\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$ equals
(a) $\vec{0}$ (b) $\overrightarrow{AD}$ (c) $2\overrightarrow{BC}$ (d) $2\overrightarrow{AD}$ *(2020)* Ap

10.5 Multiplication of a Vector by a Scalar

MCQ

6. A unit vector along the vector $4\hat{i}-3\hat{k}$ is
(a) $\frac{1}{7}(4\hat{i}-3\hat{k})$ (b) $\frac{1}{5}(4\hat{i}-3\hat{k})$
(c) $\frac{1}{\sqrt{7}}(4\hat{i}-3\hat{k})$ (d) $\frac{1}{\sqrt{5}}(4\hat{i}-3\hat{k})$ *(2023)*

VSA (1 mark)

7. The position vector of two points *A* and *B* are $\overrightarrow{OA}=2\hat{i}-\hat{j}-\hat{k}$ and $\overrightarrow{OB}=2\hat{i}-\hat{j}+2\hat{k}$, respectively. The position vector of a point *P* which divides the line segment joining *A* and *B* in the ratio 2 : 1 is ________. *(2020)* An

8. Find the position vector of a point which divides the join of points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio 2 : 1. *(Delhi 2016)* An

9. Write the position vector of the point which divides the join of points with position vectors $3\vec{a}-2\vec{b}$ and $2\vec{a}+3\vec{b}$ in the ratio 2 : 1. *(AI 2016)*

10. Find the unit vector in the direction of the sum of the vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $4\hat{i}-3\hat{j}+2\hat{k}$. *(Foreign 2015)*

11. Find a vector in the direction of $\vec{a}=\hat{i}-2\hat{j}$ that has magnitude 7 units. *(Delhi 2015C)*

12. Write the direction ratios of the vector $3\vec{a}+2\vec{b}$ where $\vec{a}=\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=2\hat{i}-4\hat{j}+5\hat{k}$. *(AI 2015C)* An

13. Write a unit vector in the direction of the sum of the vectors $\vec{a}=2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-7\hat{k}$. *(Delhi 2014)*

14. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel. *(AI 2014)* An

15. Find a vector in the direction of vector $2\hat{i}-3\hat{j}+6\hat{k}$ which has magnitude 21 units. *(Foreign 2014)*

16. Write a unit vector in the direction of vector $\overrightarrow{PQ}$, where *P* and *Q* are the points(1, 3, 0) and (4, 5, 6) respectively. *(Foreign 2014)*

17. Write a vector in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ that has magnitude 9 units. *(Delhi 2014C)* An

SA I (2 marks)

18. *X* and *Y* are two points with position vectors $3\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively. Write the position vector of a point *Z* which divides the line segment *XY* in the ratio 2 : 1 externally. *(AI 2019)* Cr

LA I (4 marks)

19. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides *AB* and *AC*, respectively of a ΔABC. Find the length of the median through *A*. *(Delhi 2016, Foreign 2015)*

10.6 Product of Two Vectors

MCQ

20. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b}\geq 0$ only when
(a) $0<\theta<\frac{\pi}{2}$ (b) $0\leq\theta\leq\frac{\pi}{2}$
(c) $0<\theta<\pi$ (d) $0\leq\theta\leq\pi$ *(2023)*

21. The magnitude of the vector $6\hat{i}-2\hat{j}+3\hat{k}$ is
(a) 1 (b) 5 (c) 7 (d) 12 *(2023)*

22. If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is zero, then the value of λ is
(a) 0 (b) 1
(c) $\frac{-2}{3}$ (d) $\frac{-3}{2}$ *(2020)* An

23. If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along three mutually perpendicular directions, then
(a) $\hat{i} \cdot \hat{j} = 1$ (b) $\hat{i} \times \hat{j} = 1$
(c) $\hat{i} \cdot \hat{k} = 0$ (d) $\hat{i} \times \hat{k} = 0$ *(2020)* Ap

VSA (1 mark)

24. Find the magnitude of vector $\vec{a}$ given by $\vec{a} = (\hat{i} + 3\hat{j} - 2\hat{k}) \times (-\hat{i} + 3\hat{k})$. *(2021C)*

25. If $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 5\hat{i} - 3\hat{j} - 4\hat{k}$ then find the ratio $\frac{\text{projection of vector } \vec{a} \text{ on vector } \vec{b}}{\text{projection of vector } \vec{b} \text{ on vector } \vec{a}}$. *(2020C)*

26. The area of the parallelogram whose diagonals are $2\hat{i}$ and $-3\hat{k}$ is ________ square units. *(2020)*

27. The value of λ for which the vectors $2\hat{i} - \lambda\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}$ are orthogonal is ________. *(2020)*

28. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. *(2018)* An

29. Write the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. *(AI 2016)*

30. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then write the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. *(NCERT, Foreign 2016)*

31. If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 400$ and $|\vec{a}| = 5$ then write the value of $|\vec{b}|$. *(Foreign 2016)*

32. If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$. *(Delhi 2015)*

33. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find the value of $|2\hat{a} + \hat{b} + \hat{c}|$. *(AI 2015)* Ev

34. Write a unit vector perpendicular to both the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. *(AI 2015)*

35. Find the area of a parallelogram whose adjacent sides are represented by the vectors $2\hat{i} - 3\hat{k}$ and $4\hat{j} + 2\hat{k}$. *(Foreign 2015)*

36. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ so that $\sqrt{2}\,\vec{a} - \vec{b}$ is a unit vector ? *(Delhi 2015C)* An

37. Find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$. *(AI 2015C)*

38. Find the projection of vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$. *(Delhi 2014)* U

39. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)* Ap

40. If vectors $\vec{a}$ and $\vec{b}$ are such that, $|\vec{a}| = 3, |\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)* Cr

41. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. *(AI 2014)* An

42. Write the projection of the vector $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$. *(Foreign 2014)*

43. Write the value of $\hat{i} \times (\hat{j} + \hat{k}) + \hat{j} \times (\hat{k} + \hat{i}) + \hat{k} \times (\hat{i} + \hat{j})$. *(Foreign 2014)*

44. Write the projection of the vector $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$. *(Delhi 2014C)* Ap

45. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $(\sqrt{3}\vec{a} - \vec{b})$ is a unit vector. *(Delhi 2014C)*

46. Write the value of cosine of the angle which the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ makes with *y*-axis. *(Delhi 2014C)* Ap

47. If $|\vec{a}| = 8, |\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$, find the angle between $\vec{a}$ and $\vec{b}$. *(AI 2014C)*

48. Find the angle between *x*-axis and the vector $\hat{i} + \hat{j} + \hat{k}$. *(AI 2014C)* Cr

SA I (2 marks)

49. If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector along the vector $\vec{a} \times \vec{b}$. *(2023)*

50. If the vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3, |\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. *(2023)*

51. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$. *(2023)*

52. Write the projection of the vector $(\vec{b} + \vec{c})$ on the vector $\vec{a}$ where, $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. *(Term II, 2021-22)* Ap

53. If $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$ are three vectors, then find a vector perpendicular to both the vectors $(\vec{a} + \vec{b})$ and $(\vec{b} - \vec{c})$. *(Term II, 2021-22C)*

54. $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|2\vec{a}+3\vec{b}| = |3\vec{a}-2\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$. *(Term II, 2021-22)*

55. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=400$ and $|\vec{b}|=5$, then find the value of $|\vec{a}|$. *(Term II, 2021-22)* Ap

56. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then find $|\vec{b}|$. *(Term II, 2021-22)*

57. If $\vec{a},\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$. *(Term II, 2021-22C)*

58. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{b}|$, then prove that $(\vec{a}+2\vec{b})$ is perpendicular to $\vec{a}$. *(Term II, 2021-22)*

59. If the sides *AB* and *BC* of a parallelogram *ABCD* are represented as vectors $\overrightarrow{AB}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\overrightarrow{BC}=\hat{i}+2\hat{j}+3\hat{k}$, then find the unit vector along diagonal *AC*. *(2021C)*

60. Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i}+6\hat{j}-2\hat{k}$ and $\vec{b} = 7\hat{i}+6\hat{j}+2\hat{k}$. *(2020)*

61. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$. *(2020 C)*

62. Show that for any two non-zero vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ iff $\vec{a}$ and $\vec{b}$ are perpendicular vectors. *(2020)*

63. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+7\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+2\hat{k}$ form the sides of a right-angled triangle. *(2020)*

64. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. *(Delhi 2019)*

65. Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$ be two vectors. Show that the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular to each other. *(AI 2019)*

66. Find a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ where $\vec{a}=4\hat{i}-\hat{j}+8\hat{k}$ and $\vec{b}=-\hat{j}+\hat{k}$ *(2019 C)*

67. If $|\vec{a}|=2, |\vec{b}|=7$ and $\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$. *(2019)* Ap

68. For any two vectors, $\vec{a}$ and $\vec{b}$, prove that $(\vec{a}\times\vec{b})^2=\vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2$ *(2019)* An

69. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find $\sin\theta$. *(2018)*

SA II (3 marks)

70. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$, then find a unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$. *(2023)*

71. Three vectors $\vec{a},\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=2$. *(2023)*

72. The two adjacent sides of a parallelogram are represented by $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram also. *(Term II, 2021-22)*

73. If $\vec{a},\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitude, then prove that the vector $(2\vec{a}+\vec{b}+2\vec{c})$ is equally inclined to both $\vec{a}$ and $\vec{c}$. Also, find the angle between $\vec{a}$ and $(2\vec{a}+\vec{b}+2\vec{c})$. *(Term II, 2021-22)*

74. If $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=4$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$ *(Term II, 2021-22)*

75. If $\vec{a}$ and $\vec{b}$ are two vectors of equal magnitude and α is the angle between them, then prove that $\dfrac{|\vec{a}+\vec{b}|}{|\vec{a}-\vec{b}|}=\cot\left(\dfrac{\alpha}{2}\right)$. *(Term II, 2021-22)*

LA I (4 marks)

76. If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram. *(2020)*

77. Using vectors, find the area of the triangle *ABC* with vertices *A*(1, 2, 3), *B*(2, –1, 4) and *C*(4, 5, –1) *(NCERT Exemplar, 2020)*

78. Prove that three points *A*, *B* and *C* with position vectors $\vec{a},\vec{b}$ and $\vec{c}$ respectively are collinear if and only if $(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})=\vec{0}$. *(2020 C)*

79. The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$ *(2019, AI 2014)*

80. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points *A*, *B*, *C* and *D*, then find the angle between the straight lines *AB* and *CD*. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not. *(Delhi 2019)*

81. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}|=1, |\vec{b}|=2, |\vec{c}|=3$. If the projection of $\vec{b}$ along $\vec{a}$ is equal to the projection of $\vec{c}$ along $\vec{a}$; and $\vec{b},\vec{c}$ are perpendicular to each other, then find $|3\vec{a}-2\vec{b}+2\vec{c}|$. *(2019)*

82. Let $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a}=21$. *(2018)*

83. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. *(Delhi 2017)*

84. Show that the points *A*, *B*, *C* with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle. *(AI 2017)*

85. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram. *(AI 2016)*

86. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$. *(Foreign 2016)*

87. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy$. *(Delhi 2015)*

88. If $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$, then find a unit vector perpendicular to both of the vectors $(\vec{a}-\vec{b})$ and $(\vec{c}-\vec{b})$. *(AI 2015)*

89. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)*

90. Find a unit vector perpendicular to both of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$. *(Foreign 2014)*

91. If $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+\hat{k}$, $\vec{c}=2\hat{j}-\hat{k}$ are three vectors, find the area of the parallelogram having diagonals $(\vec{a}+\vec{b})$ and $(\vec{b}+\vec{c})$. *(Delhi 2014C)*

92. Find the vector $\vec{p}$ which is perpendicular to both $\vec{\alpha}=4\hat{i}+5\hat{j}-\hat{k}$ and $\vec{\beta}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{p}\cdot\vec{q}=21$, where $\vec{q}=3\hat{i}+\hat{j}-\hat{k}$. *(AI 2014C)*

CBSE Sample Questions

10.5 Multiplication of a Vector by a Scalar

VSA (1 mark)

1. Find a unit vector in the direction opposite to $-\frac{3}{4}\hat{j}$. *(2020-21)*

2. Vector of magnitude 5 units and in the direction opposite to $2\hat{i}+3\hat{j}-6\hat{k}$ is _____. *(2020-21)* U

10.6 Product of Two Vectors

MCQ

3. The scalar projection of the vector $3\hat{i}-\hat{j}-2\hat{k}$ on the vector $\hat{i}+2\hat{j}-3\hat{k}$ is
   (a) $\frac{7}{\sqrt{14}}$ (b) $\frac{7}{14}$ (c) $\frac{6}{13}$ (d) $\frac{7}{2}$ *(2022-23)*

4. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$, then $|\vec{a}-2\vec{b}|$ is equal to
   (a) $\sqrt{2}$ (b) $2\sqrt{6}$ (c) 24 (d) $2\sqrt{2}$ *(2022-23)*

VSA (1 mark)

5. Find the area of the triangle whose two sides are represented by the vectors $2\hat{i}$ and $-3\hat{j}$. *(2020-21)* Ap

6. Find the angle between the unit vectors $\hat{a}$ and $\hat{b}$, given that $|\hat{a}+\hat{b}|=1$. *(2020-21)*

SA I (2 marks)

7. Find $|\vec{x}|$, if $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$, where $\vec{a}$ is a unit vector. *(2022-23)* An

8. If $\vec{a}$ and $\vec{b}$ are unit vectors, then prove that $|\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$, where θ is the angle between them. *(Term II, 2021-22)* Ev

9. Find the area of the parallelogram whose one side and a diagonal are represented by coinitial vectors $\hat{i}-\hat{j}+\hat{k}$ and $4\hat{i}+5\hat{k}$ respectively. *(2020-21)*

SA II (3 marks)

10. If $\vec{a}\neq\vec{0}, \vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}, \vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, then show that $\vec{b}=\vec{c}$. *(Term II, 2021-22)* An

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. Here, $l=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, $m=\cos\frac{\pi}{2}=0$, $n=\cos\theta$

Since, $l^2+m^2+n^2=1$

$\Rightarrow \frac{1}{2}+0+\cos^2\theta=1 \Rightarrow \cos^2\theta=1-\frac{1}{2}=\frac{1}{2}$

$\Rightarrow \cos\theta=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4}$

Commonly Made Mistake

➲ $\cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4}, \frac{3\pi}{4}$ $\quad \therefore \theta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ The vector of magnitude $5\sqrt{2}$ is

$\vec{a} = 5\sqrt{2}(l\hat{i} + m\hat{j} + n\hat{k})$

$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = 5(\hat{i} + \hat{k})$ $\quad [\because \theta$ is an acute angle]

2. We have, $|\vec{r}| = 3\sqrt{3}$

Since, $\vec{r}$ is equally inclined to three axes, so direction cosine of unit vector $\vec{r}$ will be same. *i.e.*, $l = m = n$

As we know that $l^2 + m^2 + n^2 = 1$ ...(i)

$l^2 + l^2 + l^2 = 1 \Rightarrow 3l^2 = 1$

$\Rightarrow l = \pm\frac{1}{\sqrt{3}} = m = n$ ...(ii)

We have $\overrightarrow{OP} = \pm\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$ $\quad \left\{\because \hat{p} = \frac{\vec{r}}{|\vec{r}|}\right\}$

$\vec{r} = |\vec{r}|\overrightarrow{OP}$ $\quad \{\because |\vec{r}| = 3\sqrt{3}$ (given)$\}$

$= \pm 3\sqrt{3} \times \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) \Rightarrow \vec{r} = \pm 3(\hat{i} + \hat{j} + \hat{k})$

Answer Tips

➲ If a vector is equally inclined to axes, then its direction cosines are equal.

3. (b)

4. (b) : Let $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$

So, unit vector of $\vec{a} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

$\therefore$ The value of p is $\frac{1}{\sqrt{3}}$.

5. (a) : $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED} = \overrightarrow{EA} + \overrightarrow{EB} - \overrightarrow{EA} - \overrightarrow{EB}$

[As diagonals of a rhombus bisect each other]

$= \vec{0}$

6. (b): Let $\vec{v} = 4\hat{i} - 3\hat{k}$

$\therefore |\vec{v}| = \sqrt{4^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

Now, $\hat{v}$ = unit vector along $\vec{v}$

$= \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}(4\hat{i} - 3\hat{k})$

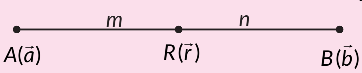
7. Required position vector of point P

$= \frac{1(2\hat{i} - \hat{j} - \hat{k}) + 2(2\hat{i} - \hat{j} + 2\hat{k})}{2+1} = \frac{2\hat{i} - \hat{j} - \hat{k} + 4\hat{i} - 2\hat{j} + 4\hat{k}}{3}$

$= \frac{1}{3}(6\hat{i} - 3\hat{j} + 3\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$

Concept Applied

➲ If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B respectively, then the position vector of $R(\vec{r})$ which divides $\overline{AB}$ internally in the ratio $m : n$ is $\frac{m\vec{b} + n\vec{a}}{m+n}$



8. Required position vector

$= \frac{2\cdot(2\vec{a} + \vec{b}) - 1(\vec{a} - 2\vec{b})}{2-1} = \frac{4\vec{a} + 2\vec{b} - \vec{a} + 2\vec{b}}{1} = 3\vec{a} + 4\vec{b}$

9. Required position vector

$= \frac{2(2\vec{a} + 3\vec{b}) + 1(3\vec{a} - 2\vec{b})}{2+1} = \frac{7\vec{a} + 4\vec{b}}{3} = \frac{7}{3}\vec{a} + \frac{4}{3}\vec{b}$

10. Let $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + 2\hat{k}$.

Then, the sum of the given vectors is

$\vec{c} = \vec{a} + \vec{b} = (2+4)\hat{i} + (3-3)\hat{j} + (-1+2)\hat{k} = 6\hat{i} + \hat{k}$

and $|\vec{c}| = |\vec{a} + \vec{b}| = \sqrt{6^2 + 1^2} = \sqrt{36+1} = \sqrt{37}$

$\therefore$ Unit vector, $\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{6\hat{i} + \hat{k}}{\sqrt{37}} = \frac{6}{\sqrt{37}}\hat{i} + \frac{1}{\sqrt{37}}\hat{k}$

11. A unit vector in the direction of $\vec{a} = \hat{i} - 2\hat{j}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$= \frac{\hat{i} - 2\hat{j}}{\sqrt{1^2 + (-2)^2}} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j})$

$\therefore$ The required vector of magnitude 7 in the direction of $\vec{a} = 7 \cdot \hat{a} = \frac{7}{\sqrt{5}}(\hat{i} - 2\hat{j})$.

12. $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$; $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

$\therefore 3\vec{a} + 2\vec{b} = 3(\hat{i} + \hat{j} - 2\hat{k}) + 2(2\hat{i} - 4\hat{j} + 5\hat{k})$

$= (3\hat{i} + 3\hat{j} - 6\hat{k}) + (4\hat{i} - 8\hat{j} + 10\hat{k}) = 7\hat{i} - 5\hat{j} + 4\hat{k}$

$\therefore$ The direction ratios of the vector $3\vec{a} + 2\vec{b}$ are $7, -5, 4$.

13. We have, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$

Sum of the given vectors is

$\vec{c} = \vec{a} + \vec{b} = (2+2)\hat{i} + (2+1)\hat{j} + (-5-7)\hat{k} = 4\hat{i} + 3\hat{j} - 12\hat{k}$

and $|\vec{c}| = \sqrt{(4)^2 + (3)^2 + (-12)^2} = \sqrt{169} = 13$

$\therefore$ Unit vector, $\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{13} = \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$

14. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} - 2p\hat{j} + 3\hat{k}$

For $\vec{a}$ and $\vec{b}$ to be parallel, $\vec{b} = \lambda\vec{a}$.

$\Rightarrow \hat{i} - 2p\hat{j} + 3\hat{k} = \lambda(3\hat{i} + 2\hat{j} + 9\hat{k}) = 3\lambda\hat{i} + 2\lambda\hat{j} + 9\lambda\hat{k}$

$\Rightarrow 1 = 3\lambda, -2p = 2\lambda, 3 = 9\lambda \Rightarrow \lambda = \frac{1}{3}$ and $p = -\lambda = -\frac{1}{3}$

Concept Applied

➲ Two vectors $\vec{a}$ and $\vec{b}$ are parallel iff one of them is scalar multiple of other.

15. Let $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

A vector in the direction of $\vec{a}$ with a magnitude of $21 = 21\,\hat{a}$

$\therefore$ Required vector $= 21 \times \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + (-3)^2 + 6^2}}$

$= 21 \times \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7} = 6\hat{i} - 9\hat{j} + 18\hat{k}$

16. We have, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 3\hat{j}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$

Required unit vector $= \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{7}$

Concept Applied

- Unit vector of $\overrightarrow{PQ} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

17. Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

The vector in the direction of $\vec{a}$ with magnitude 9 units $= 9\hat{a}$

$\therefore$ Required vector $= 9 \times \dfrac{\vec{a}}{|\vec{a}|} = 9 \times \dfrac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}}$

$= \dfrac{9}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) = 3\hat{i} - 6\hat{j} + 6\hat{k}$

Answer Tips

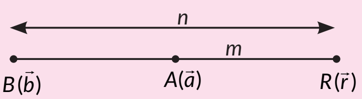
- The vector in direction of $\vec{a} = |\vec{a}| \cdot \hat{a}$

18. Position vector which divides the line segment joining points with position vectors $3\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ in the ratio 2 : 1 externally is given by

$\dfrac{2(\vec{a} - 3\vec{b}) - 1(3\vec{a} + \vec{b})}{2 - 1} = \dfrac{2\vec{a} - 6\vec{b} - 3\vec{a} - \vec{b}}{1} = -\vec{a} - 7\vec{b}$

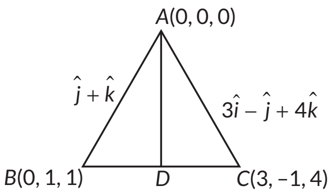
Concept Applied

- If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B respectively, then the position vector of $R(\vec{r})$ which divides $\overline{AB}$ externally in the ratio $m : n$ is $\dfrac{m\vec{b} - n\vec{a}}{m - n}$.



19. Take A to be as origin (0, 0, 0).

$\therefore$ Coordinates of B are (0, 1, 1) and coordinates of C are (3, –1, 4).



Let D be the mid point of BC and AD is a median of ΔABC.

$\therefore$ Coordinates of D are $\left(\dfrac{3}{2}, 0, \dfrac{5}{2}\right)$

So, length of $AD = \sqrt{\left(\dfrac{3}{2} - 0\right)^2 + (0)^2 + \left(\dfrac{5}{2} - 0\right)^2}$

$= \sqrt{\dfrac{9}{4} + \dfrac{25}{4}} = \dfrac{\sqrt{34}}{2}$ units

Concept Applied

- Position vector of mid point of $AB = \dfrac{\vec{a} + \vec{b}}{2}$

20. (b): Given, $\vec{a} \cdot \vec{b} \geq 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

Assuming $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$

$\Rightarrow \cos\theta \geq 0 \quad [\because |\vec{a}| \geq 0, |\vec{b}| \geq 0] \Rightarrow \theta \in \left[0, \dfrac{\pi}{2}\right]$

21. (c): Given vector is $6\hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore$ Its magnitude $= \sqrt{6^2 + (-2)^2 + 3^2}$

$= \sqrt{36 + 4 + 9} = \sqrt{49} = 7$ units

22. (c): Here, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \lambda\hat{k}$

Since, projection of $\vec{a}$ on $\vec{b} = 0$

$\Rightarrow \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 0 \Rightarrow \dfrac{(\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (2\hat{i} + \lambda\hat{k})}{\sqrt{2^2 + \lambda^2}} = 0$

$\Rightarrow \dfrac{2 + 3\lambda}{\sqrt{4 + \lambda^2}} = 0 \Rightarrow 2 + 3\lambda = 0 \Rightarrow \lambda = -\dfrac{2}{3}$

23. (c): Since, $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular to each other.

$\therefore \hat{i} \cdot \hat{k} = 0$

24. Let $\vec{a} = (\hat{i} + 3\hat{j} - 2\hat{k}) \times (-\hat{i} + 0\hat{j} + 3\hat{k})$

$\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} = (9 - 0)\hat{i} - (3 - 2)\hat{j} + (0 + 3)\hat{k} = 9\hat{i} - \hat{j} + 3\hat{k}$

$\therefore |\vec{a}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{91}$

Answer Tips

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

25. Here $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 5\hat{i} - 3\hat{j} - 4\hat{k}$

Since projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

and projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$\therefore \vec{a} \cdot \vec{b} = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (5\hat{i} - 3\hat{j} - 4\hat{k})$

$\vec{a} \cdot \vec{b} = 10 + 3 - 8 = 13 - 8 = 5$

$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

$|\vec{b}| = \sqrt{5^2 + (-3)^2 + (-4)^2} = \sqrt{25 + 9 + 16} = \sqrt{50} = 5\sqrt{2}$

Required ratio $= \dfrac{5/5\sqrt{2}}{5/3} = \dfrac{5}{5\sqrt{2}} \times \dfrac{3}{5} = \dfrac{3\sqrt{2}}{10}$

26. Given, two diagonals $\vec{d}_1$ and $\vec{d}_2$ are $2\hat{i}$ and $-3\hat{k}$ respectively.

$\therefore \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 0 & -3 \end{vmatrix} = \hat{i}(0) - \hat{j}(-6 - 0) + \hat{k}(0) = 6\hat{j}$

So, area of the parallelogram = $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

$= \frac{1}{2} \times 6 = 3$ sq. units

27. Let $\vec{a} = 2\hat{i} - \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

We know, $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow (2\hat{i} - \lambda\hat{j} + \hat{k})\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 0$

$\Rightarrow 2 - 2\lambda - 1 = 0 \Rightarrow 1 - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$

28. Given, $|\vec{a}| = |\vec{b}|, \theta = 60°$ and $\vec{a}\cdot\vec{b} = \frac{9}{2}$

Now, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \cos 60° = \frac{9/2}{|\vec{a}|^2} \Rightarrow \frac{1}{2} = \frac{9/2}{|\vec{a}|^2}$

$\Rightarrow |\vec{a}|^2 = 9 \Rightarrow |\vec{a}| = 3 \therefore |\vec{a}| = |\vec{b}| = 3$

Answer Tips

- $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$

29. Given, $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$

Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are $\pm\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$.

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{i} - 2\hat{j} + 2\hat{k}$

$\therefore$ Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are

$\pm\frac{(-\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{(-1)^2 + (-2)^2 + (2)^2}} = \pm\left(-\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

So, there are two unit vectors perpendicular to the given vectors.

30. We have $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors.

Therefore, $|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{c}| = 1$

Also, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ (given)

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 = 0 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\Rightarrow 1 + 1 + 1 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\Rightarrow 3 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0 \Rightarrow (\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -\frac{3}{2}$

31. $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = 400 \Rightarrow \{|\vec{a}||\vec{b}|\sin\theta\}^2 + \{|\vec{a}||\vec{b}|\cos\theta\}^2 = 400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2\sin^2\theta + |\vec{a}|^2|\vec{b}|^2\cos^2\theta = 400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2 = 400 \Rightarrow 25 \times |\vec{b}|^2 = 400$ $[\because |\vec{a}| = 5]$

$\Rightarrow |\vec{b}|^2 = 16 \Rightarrow |\vec{b}| = 4$

32. Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{(7\hat{i} + \hat{j} - 4\hat{k})\cdot(2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = \frac{14 + 6 - 12}{7} = \frac{8}{7}$

33. Here $\hat{a}, \hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors.

$\Rightarrow |\hat{a}| = |\hat{b}| = |\hat{c}| = 1$ and $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = 0$ ...(1)

$\therefore |2\hat{a} + \hat{b} + \hat{c}|^2 = (2\hat{a} + \hat{b} + \hat{c}).(2\hat{a} + \hat{b} + \hat{c})$

$= 4\hat{a}\cdot\hat{a} + 2\hat{a}\cdot\hat{b} + 2\hat{a}\cdot\hat{c} + 2\hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{b}\cdot\hat{c} + 2\hat{c}\cdot\hat{a} + \hat{c}\cdot\hat{b} + \hat{c}\cdot\hat{c}$

$= 4|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 + 4\hat{a}\cdot\hat{b} + 2\hat{b}\cdot\hat{c} + 4\hat{a}\cdot\hat{c}$

$(\because \hat{b}\cdot\hat{a} = \hat{a}\cdot\hat{b}, \hat{c}\cdot\hat{a} = \hat{a}\cdot\hat{c}, \hat{c}\cdot\hat{b} = \hat{b}\cdot\hat{c})$

$= 4\cdot1^2 + 1^2 + 1^2$ [Using (1)]

$= 6$

$\therefore |2\hat{a} + \hat{b} + \hat{c}| = \sqrt{6}$.

34. Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\hat{i} + \hat{j} + 0\hat{k} = -\hat{i} + \hat{j}$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$= \pm\frac{-\hat{i} + \hat{j}}{\sqrt{(-1)^2 + 1^2}} = \pm\frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$.

Key Points

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ $= \pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

35. Let $\vec{a} = 2\hat{i} - 3\hat{k}$ and $\vec{b} = 4\hat{j} + 2\hat{k}$

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a}\times\vec{b}|$.

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{vmatrix} = 12\hat{i} - 4\hat{j} + 8\hat{k}$

$\therefore |\vec{a}\times\vec{b}| = \sqrt{(12)^2 + (-4)^2 + (8)^2} = \sqrt{144 + 16 + 64}$

$= \sqrt{224} = 4\sqrt{14}$ sq. units.

36. Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$.

$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \vec{a}\cdot\vec{b}$ $(\because |\vec{a}| = 1 = |\vec{b}|)$ ...(i)

Now, $1 = |\sqrt{2}\vec{a} - \vec{b}|$

$\Rightarrow 1 = |\sqrt{2}\vec{a} - \vec{b}|^2 = (\sqrt{2}\vec{a} - \vec{b})\cdot(\sqrt{2}\vec{a} - \vec{b})$

$= 2|\vec{a}|^2 - \sqrt{2}\vec{a}\cdot\vec{b} - \vec{b}\cdot\sqrt{2}\vec{a} + |\vec{b}|^2$

$= 2 - 2\sqrt{2}\vec{a}\cdot\vec{b} + 1$ $(\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$= 3 - 2\sqrt{2}\vec{a}\cdot\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b} = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ [By using (i)]

$\therefore \theta = \pi/4$

37. Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{(2\hat{i}+3\hat{j}+2\hat{k})\cdot(2\hat{i}+2\hat{j}+\hat{k})}{\sqrt{(2)^2+(2)^2+(1)^2}} = \frac{4+6+2}{\sqrt{9}} = \frac{12}{3} = 4$

38. Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$, $\vec{b} = 2\hat{i}-3\hat{j}+6\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{(\hat{i}+3\hat{j}+7\hat{k})\cdot(2\hat{i}-3\hat{j}+6\hat{k})}{\sqrt{(2)^2+(-3)^2+(6)^2}} = \frac{2-9+42}{\sqrt{49}} = \frac{35}{7} = 5$

39. Given $|\vec{a}|=1=|\vec{b}|, |\vec{a}+\vec{b}|=1$

$\Rightarrow |\vec{a}+\vec{b}|^2=1 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=1 \Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=1$

$\Rightarrow |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=1 \Rightarrow 1+2\vec{a}\cdot\vec{b}+1=1$

$\Rightarrow 2\vec{a}\cdot\vec{b}=-1 \Rightarrow 2|\vec{a}|\cdot|\vec{b}|\cos\theta=-1 \Rightarrow 2\cdot1\cdot1\cos\theta=-1$

$\Rightarrow \cos\theta=-\frac{1}{2} \Rightarrow \theta=120°$

40. Given, $|\vec{a}|=3, |\vec{b}|=\frac{2}{3}$ and $|\vec{a}\times\vec{b}|=1$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta=1 \Rightarrow 3.\frac{2}{3}\sin\theta=1 \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$

41. Given: $\vec{a}\perp\vec{b} \Rightarrow \vec{a}\cdot\vec{b}=0$

Also, $|\vec{a}|=5$ and $|\vec{a}+\vec{b}|=13$

$\Rightarrow |\vec{a}+\vec{b}|^2=13^2 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=169$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=169 \Rightarrow |\vec{a}|^2+0+0+|\vec{b}|^2=169$

$\Rightarrow |\vec{b}|^2=169-|\vec{a}|^2=169-5^2=144 \Rightarrow |\vec{b}|=12$

42. The projection of the vector $(\hat{i}+\hat{j}+\hat{k})$ along the vector $\hat{j}$ is $(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{j}}{\sqrt{0^2+1^2+0^2}}\right)=1$

43. We have, $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$

$=\hat{i}\times\hat{j}+\hat{i}\times\hat{k}+\hat{j}\times\hat{k}+\hat{j}\times\hat{i}+\hat{k}\times\hat{i}+\hat{k}\times\hat{j}$

$=\hat{k}-\hat{j}+\hat{i}-\hat{k}+\hat{j}-\hat{i}=\vec{0}$.

44. Projection of $\vec{a}$ on $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{(1)^2+(2)^2+(2)^2}} = \frac{2-2+2}{\sqrt{9}} = \frac{2}{3}$

45. Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$

$\therefore \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\vec{a}\cdot\vec{b}$ ...(i) $(\because |\vec{a}|=|\vec{b}|=1)$

Now, $|\sqrt{3}\vec{a}-\vec{b}|=1 \Rightarrow |\sqrt{3}\vec{a}-\vec{b}|^2=1$

$\Rightarrow 3|\vec{a}|^2+|\vec{b}|^2-2\sqrt{3}\,\vec{a}\cdot\vec{b}=1 \Rightarrow 3+1-2\sqrt{3}|\vec{a}||\vec{b}|\cos\theta=1$

$\Rightarrow 3=2\sqrt{3}\cos\theta \Rightarrow \cos\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{6}$

46. Let θ be the angle between the vector

$\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and y-axis *i.e.*, $\vec{b}=\hat{j}$

$\therefore \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$= \frac{(\hat{i}+\hat{j}+\hat{k}).\hat{j}}{|\hat{i}+\hat{j}+\hat{k}||\hat{j}|} = \frac{1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$

47. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

Given: $|\vec{a}|=8, |\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=12$

$\Rightarrow |\vec{a}||\vec{b}||\sin\theta|=12 \Rightarrow 8\times3\times\sin\theta=12$

$\Rightarrow \sin\theta=\frac{12}{24}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$.

48. Here, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and vector along x-axis is $\hat{i}$

$\therefore$ Angle between $\vec{a}$ and $\hat{i}$ is given by

$\cos\theta = \frac{\vec{a}.\hat{i}}{\sqrt{1^2+1^2+1^2}.\sqrt{1^2}} = \frac{(\hat{i}+\hat{j}+\hat{k})\cdot\hat{i}}{\sqrt{3}\cdot1} = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

49. We have, $\vec{a}=4\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 2 & -2 & 1\end{vmatrix}$$

$=\hat{i}(-1+2)-\hat{j}(4-2)+\hat{k}(-8+2) = \hat{i}-2\hat{j}-6\hat{k}$

$|\vec{a}\times\vec{b}|=\sqrt{1^2+(-2)^2+(-6)^2}=\sqrt{41}$

Unit vector along $\vec{a}\times\vec{b}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$=\frac{\hat{i}-2\hat{j}-6\hat{k}}{\sqrt{41}}=\frac{1}{\sqrt{41}}\hat{i}-\frac{2}{\sqrt{41}}\hat{j}-\frac{6}{\sqrt{41}}\hat{k}$

50. We know that, $|\vec{a}\times\vec{b}|=|\vec{a}|\,|\vec{b}|\sin\theta$

$\Rightarrow 1=3\times\frac{2}{3}\sin\theta$ $(\because \vec{a}\times\vec{b}$ is a unit vector$)$

$\Rightarrow \sin\theta=\frac{1}{2}=\sin30° \Rightarrow \theta=30°$

So, angle between $\vec{a}$ and $\vec{b}$ is 30°.

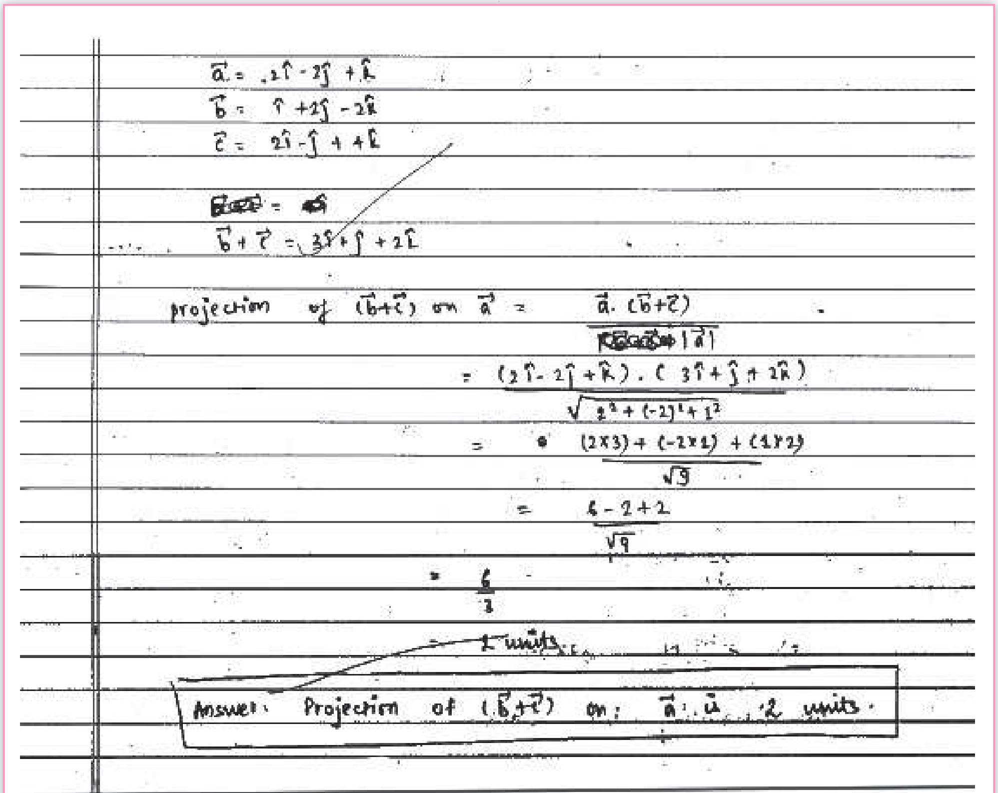
51. Area of parallelogram $=\vec{a}\times\vec{b} = \left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{matrix}\right\|$

$=|(-1+21)\hat{i}-(1-6)\hat{j}+(-7+2)\hat{k}|=|20\hat{i}+5\hat{j}-5\hat{k}|$

$=\sqrt{(20)^2+(5)^2+(-5)^2}$

$=\sqrt{400+25+25}=\sqrt{450}=15\sqrt{2}$ sq.units

52.



$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$
$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$
$\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

$\vec{b} + \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

projection of $(\vec{b}+\vec{c})$ on $\vec{a}$ = $\dfrac{\vec{a}\cdot(\vec{b}+\vec{c})}{|\vec{a}|}$

$= \dfrac{(2\hat{i} - 2\hat{j} + \hat{k})\cdot(3\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{2^2 + (-2)^2 + 1^2}}$

$= \dfrac{(2\times 3) + (-2\times 1) + (1\times 2)}{\sqrt{9}}$

$= \dfrac{6 - 2 + 2}{\sqrt{9}}$

$= \dfrac{6}{3}$

$= 2$ units

Answer: Projection of $(\vec{b}+\vec{c})$ on $\vec{a}$ = 2 units.

[Topper's Answer, 2022]

53. Here $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$

$\therefore\ \vec{a} + \vec{b} = (\hat{i} + \hat{j} - 2\hat{k}) + (-\hat{i} + 2\hat{j} + 2\hat{k}) = 3\hat{j}$

Now $(\vec{b} - \vec{c}) = (-\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 2\hat{j} - \hat{k}) = 3\hat{k}$

Vector perpendicular to both $(\vec{a} + \vec{b})$ and $(\vec{b} - \vec{c})$ is

$$(\vec{a} + \vec{b}) \times (\vec{b} - \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \hat{i}(9-0) - \hat{j}(0-0) + \hat{k}(0-0) = 9\hat{i}$$

$\therefore$ Unit vector perpendicular to both $(\vec{a} + \vec{b})$ and $(\vec{b} - \vec{c})$

$$= \frac{9\hat{i}}{\sqrt{9^2 + 0^2 + 0^2}} = \frac{9\hat{i}}{9} = \hat{i} + 0\hat{j} + 0\hat{k}$$

54. Given $\vec{a}$ and $\vec{b}$ are unit vectors

$\therefore\ |\vec{a}| = |\vec{b}| = 1$ ...(i)

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Also, $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$ (Given)

$\Rightarrow\ |2\vec{a} + 3\vec{b}|^2 = |3\vec{a} - 2\vec{b}|^2$

$\Rightarrow\ (2\vec{a} + 3\vec{b})\cdot(2\vec{a} + 3\vec{b}) = (3\vec{a} - 2\vec{b})\cdot(3\vec{a} - 2\vec{b})$

$\Rightarrow\ 4(\vec{a}\cdot\vec{a}) + 6(\vec{a}\cdot\vec{b}) + 6(\vec{b}\cdot\vec{a}) + 9(\vec{b}\cdot\vec{b})$

$= 9(\vec{a}\cdot\vec{a}) - 6(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{a}) + 4(\vec{b}\cdot\vec{b})$

$\Rightarrow\ 4|\vec{a}|^2 + 12(\vec{a}\cdot\vec{b}) + 9|\vec{b}|^2 = 9|\vec{a}|^2 - 12(\vec{a}\cdot\vec{b}) + 4|\vec{b}|^2$

$\Rightarrow\ 5|\vec{a}|^2 - 5|\vec{b}|^2 - 24|\vec{a}||\vec{b}|\cos\theta = 0$

$\Rightarrow\ 5\cdot 1 - 5\cdot 1 - 24\cos\theta = 0$ $(\because |\vec{a}| = |\vec{b}| = 1)$

$\Rightarrow\ \cos\theta = 0 \ \Rightarrow\ \theta = \dfrac{\pi}{2}$

Answer Tips

- If $\vec{a}$ is a unit vector, then $|\vec{a}| = 1$

55. We have, $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = 400$

$\Rightarrow\ |\vec{a}|^2|\vec{b}|^2\sin^2\theta + |\vec{a}|^2|\vec{b}|^2\cos^2\theta = 400$

$\Rightarrow\ |\vec{a}|^2|\vec{b}|^2 = 400$ $(\because \sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow\ |\vec{a}|^2 \times 25 = 400 \ \Rightarrow\ |\vec{a}|^2 = \dfrac{400}{25} = 16 \ \Rightarrow\ |\vec{a}| = 4$

56. Given : $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{a}\cdot\vec{b} = 1$ and $\vec{a}\times\vec{b} = \hat{j} - \hat{k}$

To find $|\vec{b}|$.

Let $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$

Since, $\vec{a}\cdot\vec{b} = 1$

$\Rightarrow\ (\hat{i} + \hat{j} + \hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = 1 \Rightarrow x + y + z = 1$

and $\vec{a}\times\vec{b} = \hat{j} - \hat{k}$

$\Rightarrow\ (\hat{i} + \hat{j} + \hat{k})\times(x\hat{i} + y\hat{j} + z\hat{k}) = \hat{j} - \hat{k}$

$$\Rightarrow\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k} \ \Rightarrow\ \hat{i}(z-y) - \hat{j}(z-x) + \hat{k}(y-x) = \hat{j} - \hat{k}$$

$\Rightarrow\ x - z = 1,\ y - x = -1,\ z - y = 0$

$\Rightarrow\ z = y$ ...(1), $x - z = 1$ ...(2)

and $y - x = -1$ ...(3)

From equation (1), (2) and (3), we get

$x = 1,\ y = z = 0$

So $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k} = \hat{i}$ $|\vec{b}| = 1$.

57. Given, $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors.

$\therefore\ |\vec{a}| = 1 = |\vec{b}| = |\vec{c}|$

Also, given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow\ |\vec{a} + \vec{b} + \vec{c}|^2 = 0 \Rightarrow\ (\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} + \vec{b} + \vec{c}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(a\cdot b+b\cdot c+c\cdot a)=0$

$\Rightarrow 1+1+1+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0 \Rightarrow (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=\frac{-3}{2}$

58. Given that $|\vec{a}+\vec{b}|=|\vec{b}|$

To prove : $(\vec{a}+2\vec{b})$ is perpendicular to $\vec{a}$.

i.e., $(\vec{a}+2\vec{b})\cdot\vec{a}=0$

Since, $|\vec{a}+\vec{b}|=|\vec{b}|$

Squaring both sides, we get $|\vec{a}+\vec{b}|^2=|\vec{b}|^2$

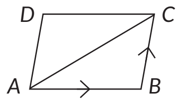
$\Rightarrow \quad (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{b}|^2 \Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{b}|^2$

$\Rightarrow \quad |\vec{a}|^2+2\vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}\cdot(\vec{a}+2\vec{b})=0 \Rightarrow (\vec{a}+2\vec{b})\cdot\vec{a}=0$

$\therefore \quad \vec{a}+2\vec{b}$ is perpendicular to $\vec{a}$.

59. Given, $\overrightarrow{AB}=2\hat{i}+4\hat{j}-5\hat{k},\ \overrightarrow{BC}=\hat{i}+2\hat{j}+3\hat{k}$

Diagonal $\overrightarrow{AC}$ of parallelogram



$ABCD=\overrightarrow{AB}+\overrightarrow{BC}$

$\overrightarrow{AC}=(2\hat{i}+4\hat{j}-5\hat{k})+(\hat{i}+2\hat{j}+3\hat{k})=3\hat{i}+6\hat{j}-2\hat{k}$

Unit vector along diagonal $\overrightarrow{AC}=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{|3\hat{i}+6\hat{j}-2\hat{k}|}$

$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}}=\pm\frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$

60. Here, $\vec{a}=5\hat{i}+6\hat{j}-2\hat{k}$ and $\vec{b}=7\hat{i}+6\hat{j}+2\hat{k}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 6 & -2\\ 7 & 6 & 2\end{vmatrix}$$

$=\hat{i}(12+12)-\hat{j}(10+14)+\hat{k}(30-42)$

$=24\hat{i}-24\hat{j}-12\hat{k}=12(2\hat{i}-2\hat{j}-\hat{k})$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$=\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\pm\frac{24\hat{i}-24\hat{j}-12\hat{k}}{\sqrt{576+576+144}}=\pm\frac{12(2\hat{i}-2\hat{j}-\hat{k})}{\sqrt{1296}}$

$=\pm\frac{12(2\hat{i}-2\hat{j}-\hat{k})}{36}=\pm\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$

61. Let $\vec{p}=|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $\vec{q}=|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$

Then we have $\vec{p}\cdot\vec{q}=\left[(|\vec{a}|\vec{b}+|\vec{b}|\vec{a})\cdot(|\vec{a}|\vec{b}-|\vec{b}|\vec{a})\right]$

$=|\vec{a}|^2(\vec{b}\cdot\vec{b})-|\vec{a}||\vec{b}|(\vec{b}\cdot\vec{a})+|\vec{b}||\vec{a}|(\vec{a}\cdot\vec{b})-|\vec{b}|^2(\vec{a}\cdot\vec{a})$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})+|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})-|\vec{b}|^2|\vec{a}|^2=0 \quad (\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a})$

$\Rightarrow \quad \vec{p}\cdot\vec{q}=0$

Hence, $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$.

62. For any two non-zero vectors $\vec{a}$ and $\vec{b}$, we have

$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$

$\Leftrightarrow \quad |\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2 \Leftrightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$

$\Leftrightarrow \quad 4\vec{a}\cdot\vec{b}=0 \quad \Leftrightarrow \quad \vec{a}\cdot\vec{b}=0$

So, $\vec{a}$ and $\vec{b}$ are perpendicular vectors.

63. Let $A(2\hat{i}-\hat{j}+\hat{k}), B(3\hat{i}+7\hat{j}+\hat{k})$ and $C(5\hat{i}+6\hat{j}+2\hat{k})$.

Then, $\overrightarrow{AB}=(3-2)\hat{i}+(7+1)\hat{j}+(1-1)\hat{k}=\hat{i}+8\hat{j}$

$\overrightarrow{AC}=(5-2)\hat{i}+(6+1)\hat{j}+(2-1)\hat{k}=3\hat{i}+7\hat{j}+\hat{k}$

$\overrightarrow{BC}=(5-3)\hat{i}+(6-7)\hat{j}+(2-1)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$

Now, angle between $\overrightarrow{AC}$ and $\overrightarrow{BC}$ is given by

$\Rightarrow \quad \cos\theta=\frac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|}=\frac{6-7+1}{\sqrt{9+49+1}\sqrt{4+1+1}}$

$\Rightarrow \quad \cos\theta=0 \Rightarrow \theta=90^\circ \Rightarrow AC\perp BC$

So, *A*, *B*, *C* are the vertices of right angled triangle.

64. Given, $\hat{a}+\hat{b}=\hat{c}$

$\Rightarrow (\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b})=\hat{c}\cdot\hat{c} \Rightarrow \hat{a}\cdot\hat{a}+\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}=\hat{c}\cdot\hat{c}$

$\Rightarrow 1+\hat{a}\cdot\hat{b}+1+\hat{a}\cdot\hat{b}=1 \qquad (\because\ \hat{a}\cdot\hat{b}=\hat{b}\cdot\hat{a})$

$\Rightarrow 2\ \hat{a}\cdot\hat{b}=-1$ ...(i)

Now $|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$=\hat{a}\cdot\hat{a}-\hat{a}\cdot\hat{b}-\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}=1-\hat{a}\cdot\hat{b}-\hat{a}\cdot\hat{b}+1$

$=2-2\hat{a}\cdot\hat{b}=2-(-1)$ [Using (i)]

$=3$

$\therefore |\hat{a}-\hat{b}|=\sqrt{3}$

65. Given, $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$

Now, $\vec{a}+\vec{b}=4\hat{i}+\hat{j}-\hat{k}$

Also, $\vec{a}-\vec{b}=-2\hat{i}+3\hat{j}-5\hat{k}$

Now, $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=(4\hat{i}+\hat{j}-\hat{k})\cdot(-2\hat{i}+3\hat{j}-5\hat{k})$

$=(4)(-2)+(1)(3)+(-1)(-5)=-8+3+5=0$

Hence, $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular to each other.

66. Here $\vec{a}=4\hat{i}-\hat{j}+8\hat{k}$ and $\vec{b}=-\hat{j}+\hat{k}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 8\\ 0 & -1 & 1\end{vmatrix}$$

$=\hat{i}(-1+8)-\hat{j}(4-0)+\hat{k}(-4-0)=7\hat{i}-4\hat{j}-4\hat{k}$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\pm\frac{7\hat{i}-4\hat{j}-4\hat{k}}{\sqrt{49+16+16}}=\pm\frac{1}{9}(7\hat{i}-4\hat{j}-4\hat{k})$

67. We have, $|\vec{a}|=2, |\vec{b}|=7$

$\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$

$\therefore \quad |\vec{a}\times\vec{b}|=|3\hat{i}+2\hat{j}+6\hat{k}|=\sqrt{3^2+2^2+6^2}$

$\therefore |\vec{a}\times\vec{b}|=\sqrt{49}=7$

Let 'θ' be the angle between $\vec{a}$ and $\vec{b}$, then we have

$\therefore \sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}=\frac{7}{2\times 7}=\frac{1}{2}$

$\sin\theta=\frac{1}{2} \quad \therefore \theta=\frac{\pi}{6}$

68. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

We have, $(\vec{a}\times\vec{b})^2=|\vec{a}\times\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2\cdot\sin^2\theta=|\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$(\because |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta)$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{a}|^2|\vec{b}|^2\cos^2\theta=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$

Hence, $(\vec{a}\times\vec{b})^2=\vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2$

69. Let $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$

Now, $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow (\hat{i}-2\hat{j}+3\hat{k})\cdot(3\hat{i}-2\hat{j}+\hat{k})=\sqrt{(1)^2+(-2)^2+(3)^2}\times\sqrt{(3)^2+(-2)^2+(1)^2}\cos\theta$$

$$\Rightarrow 3+4+3=\sqrt{14}\times\sqrt{14}\cos\theta \Rightarrow \cos\theta=\frac{10}{14}$$

$$\therefore \sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{100}{196}}=\sqrt{\frac{96}{196}}$$

$$\Rightarrow \sin\theta=\frac{4\sqrt{6}}{14}=\frac{2\sqrt{6}}{7}$$

70. We have, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

$\therefore$ $\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{a}-\vec{b}=0\hat{i}-\hat{j}-2\hat{k}$

A vector which is perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}$$

$$=-2\hat{i}+4\hat{j}-2\hat{k}=\vec{c}\text{ (say)}$$

Now, $|\vec{c}|=\sqrt{(-2)^2+(4)^2+(-2)^2}=\sqrt{24}=2\sqrt{6}$

Required unit vector, $\hat{c}=\frac{\vec{c}}{|\vec{c}|}$

$$=\frac{1}{2\sqrt{6}}(-2\hat{i}+4\hat{j}-2\hat{k})=-\frac{\hat{i}}{\sqrt{6}}+\frac{2\hat{j}}{\sqrt{6}}-\frac{\hat{k}}{\sqrt{6}}$$

71. We have, $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$

$\Rightarrow |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$ ... (i)

Similarly, $\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}=0$

$\Rightarrow \vec{a}\cdot\vec{b}+|\vec{b}|^2+\vec{b}\cdot\vec{c}=0$ ... (ii)

And, $\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=0$

$\Rightarrow \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+|\vec{c}|^2=0$ ... (iii)

Adding (i), (ii) and (iii), we get

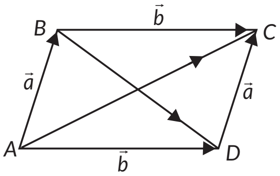
$|\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{a}\cdot\vec{b}+|\vec{b}|^2+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+|\vec{c}|^2=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{c})=0$

$\Rightarrow (3)^2+(4)^2+(2)^2+2\mu=0$

$$\Rightarrow \mu=\frac{-(9+16+4)}{2}=\frac{-29}{2}$$

72. Let $\vec{a}=2\hat{i}-4\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+2\hat{j}+3\hat{k}$



Then diagonal $\overrightarrow{AC}$ of the parallelogram is $\vec{p}=\vec{a}+\vec{b}$

$\Rightarrow \vec{p}=2\hat{i}-4\hat{j}-5\hat{k}+2\hat{i}+2\hat{j}+3\hat{k}=4\hat{i}-2\hat{j}-2\hat{k}$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}}{|\vec{p}|}=\frac{4\hat{i}-2\hat{j}-2\hat{k}}{\sqrt{16+4+4}}=\frac{2\hat{i}-\hat{j}-\hat{k}}{\sqrt{6}}$$

Now, diagonal $\overrightarrow{BD}$ of the parallelogram is

$\vec{p}'=\vec{b}-\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}-2\hat{i}+4\hat{j}+5\hat{k}=6\hat{j}+8\hat{k}$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}'}{|\vec{p}'|}=\frac{6\hat{j}+8\hat{k}}{\sqrt{36+64}}=\frac{6\hat{j}+8\hat{k}}{10}=\frac{3\hat{j}+4\hat{k}}{5}$$

$$\text{Now, } \vec{p}\times\vec{p}'=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -2 & -2\\ 0 & 6 & 8\end{vmatrix}$$

$=\hat{i}(-16+12)-\hat{j}(32-0)+\hat{k}(24-0)$

$=-4\hat{i}-32\hat{j}+24\hat{k}$

$\therefore$ Area of parallelogram $=\frac{|\vec{p}\times\vec{p}'|}{2}$

$$=\frac{\sqrt{16+1024+576}}{2}=2\sqrt{101}\text{ sq. units.}$$

Concept Applied

➲ Area of parallelogram $=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$ where $\vec{d}_1$ and $\vec{d}_2$ diagonals of parallelograms.

73. Given, $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular

$\therefore$ $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$ ...(i)

Also, $|\vec{a}|=|\vec{b}|=|\vec{c}|$

Let α be the angle between $(2\vec{a}+\vec{b}+2\vec{c})$ and $\vec{a}$

$$\therefore \cos\alpha=\frac{(2\vec{a}+\vec{b}+2\vec{c})\cdot\vec{a}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{a}|}$$

$$\Rightarrow \cos\alpha=\frac{2\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+2\vec{c}\cdot\vec{a}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{a}|} \quad \text{[From (i)]}$$

$$\cos\alpha=\frac{2|\vec{a}|^2}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{a}|}=\frac{2|\vec{a}|}{|2\vec{a}+\vec{b}+2\vec{c}|} \quad \text{...(ii)}$$

Let β be the angle between $(2\vec{a}+\vec{b}+2\vec{c})$ and $\vec{c}$

$$\therefore \cos\beta=\frac{(2\vec{a}+\vec{b}+2\vec{c})\cdot\vec{c}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{c}|}$$

$$\Rightarrow \cos\beta=\frac{2\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{c}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{c}|}$$

$$\Rightarrow \cos\beta=\frac{2|\vec{c}|^2}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{c}|} \quad \text{[From (i)]}$$

$$\Rightarrow \cos\beta=\frac{2|\vec{c}|}{|2\vec{a}+\vec{b}+2\vec{c}|} \quad \text{...(iii)}$$

As $|\vec{a}|=|\vec{c}|$

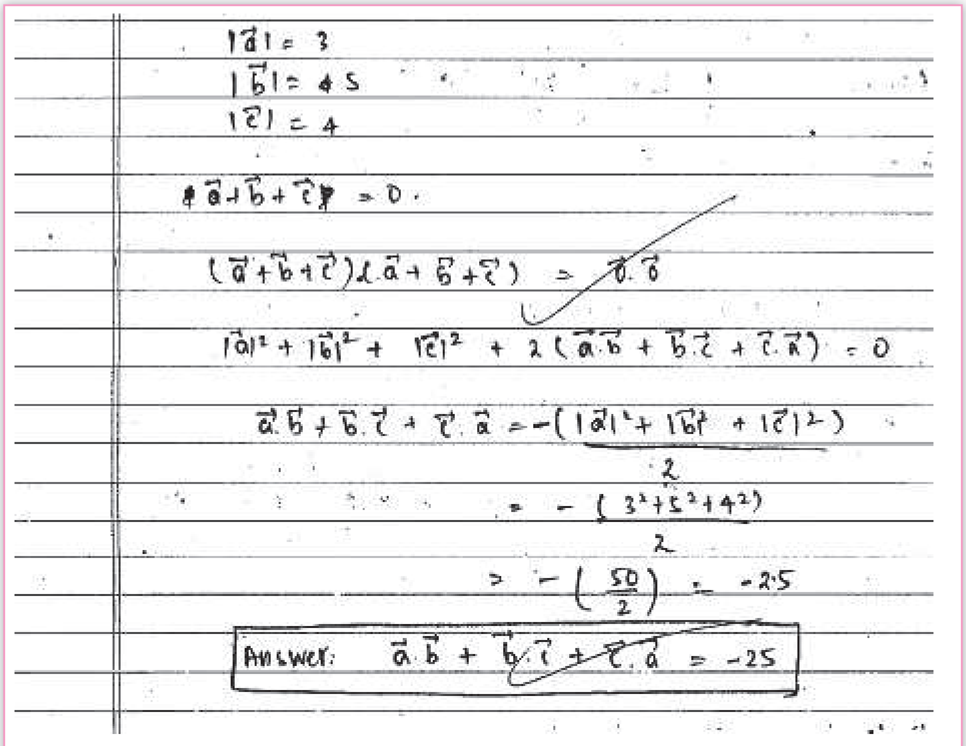
From (ii) & (iii), $\cos\alpha=\cos\beta \Rightarrow \alpha=\beta$

Hence, $(2\vec{a}+\vec{b}+2\vec{c})$ is equally inclined to both $\vec{a}$ and $\vec{c}$.

Angle between $\vec{a}$ and $(2\vec{a}+\vec{b}+2\vec{c})$ is

$$\alpha=\cos^{-1}\left(\frac{2|\vec{a}|}{|2\vec{a}+\vec{b}+2\vec{c}|}\right)$$

74.


$|\vec{a}| = 3$

$|\vec{b}| = 5$

$|\vec{c}| = 4$

$\vec{a} + \vec{b} + \vec{c} = 0$

$(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0}$

$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\dfrac{(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2)}{2}$

$= -\dfrac{(3^2 + 5^2 + 4^2)}{2}$

$= -\left(\dfrac{50}{2}\right) = -25$

Answer: $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -25$

[Topper's Answer, 2022]

75. We have, $|\vec{a}| = |\vec{b}|$

To prove, $\dfrac{|\vec{a}+\vec{b}|}{|\vec{a}-\vec{b}|} = \cot\dfrac{\alpha}{2}$

i.e., $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|\cot\dfrac{\alpha}{2}$

i.e., $|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2 \cot^2\dfrac{\alpha}{2}$

L.H.S. $= |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$= 2|\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\alpha \quad (\because |\vec{a}| = |\vec{b}|) = 2|\vec{a}|^2 + 2|\vec{a}|^2\cos\alpha$

$= 2|\vec{a}|^2(1+\cos\alpha) = 2|\vec{a}|^2\, 2\cos^2\dfrac{\alpha}{2}$

$= 4|\vec{a}|^2\cos^2\dfrac{\alpha}{2}$ ... (i)

R.H.S. $= |\vec{a}-\vec{b}|^2\cot^2\dfrac{\alpha}{2} = \left(|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}\right)\cot^2\left(\dfrac{\alpha}{2}\right)$

$= (2|\vec{a}|^2 - 2|\vec{a}||\vec{b}|\cos\alpha)\cdot\cot^2\left(\dfrac{\alpha}{2}\right)$

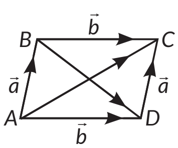
$= 2|\vec{a}|^2(1-\cos\alpha)\cot^2\dfrac{\alpha}{2} = 2|\vec{a}|^2\cdot 2\sin^2\dfrac{\alpha}{2}\cdot\dfrac{\cos^2\alpha/2}{\sin^2\alpha/2}$

$= 4|\vec{a}|^2\cos^2\alpha/2$ ...(ii)

$\therefore$ From (i) and (ii)
L.H.S. = R.H.S.

76. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Then diagonal $\overline{AC}$ of the parallelogram is



$\vec{p} = \vec{a} + \vec{b}$

$= \hat{i} + 2\hat{j} + 3\hat{k} + 2\hat{i} + 4\hat{j} - 5\hat{k}$

$= 3\hat{i} + 6\hat{j} - 2\hat{k}$

Therefore unit vector parallel to it is

$$\frac{\vec{p}}{|\vec{p}|} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{9+36+4}} = \frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$$

Now, diagonal $\overline{BD}$ of the parallelogram is

$$\vec{p}' = \vec{b} - \vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k} = \hat{i} + 2\hat{j} - 8\hat{k}$$

Therefore unit vector parallel to it is

$$\frac{\vec{p}'}{|\vec{p}'|} = \frac{\hat{i} + 2\hat{j} - 8\hat{k}}{\sqrt{1+4+64}} = \frac{1}{\sqrt{69}}(\hat{i} + 2\hat{j} - 8\hat{k})$$

Concept Applied

- In a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$, one of the diagonal is $\vec{a} + \vec{b}$ and the other is $\vec{b} - \vec{a}$.

77. Given, ΔABC with vertices

$A(1,2,3) = \hat{i} + 2\hat{j} + 3\hat{k}$, $B(2,-1,4) = 2\hat{i} - \hat{j} + 4\hat{k}$,

$C(4,5,-1) = 4\hat{i} + 5\hat{j} - \hat{k}$

Now $\overline{AB} = \overline{OB} - \overline{OA} = (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} - 3\hat{j} + \hat{k}$.

$\overline{AC} = \overline{OC} - \overline{OA} = (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} - 4\hat{k}$.

$$\therefore \quad (\overline{AB} \times \overline{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} = 9\hat{i} + 7\hat{j} + 12\hat{k}$$

Hence, area of $\Delta ABC = \dfrac{1}{2}|\overline{AB} \times \overline{AC}| = \dfrac{1}{2}|9\hat{i} + 7\hat{j} + 12\hat{k}|$

$= \dfrac{1}{2}\sqrt{9^2 + 7^2 + 12^2} = \dfrac{1}{2}\sqrt{81+49+144} = \dfrac{1}{2}\sqrt{274}$ sq. units

78. Since $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vector of *A*, *B* and *C* respectively.

Then $\overrightarrow{BC}$ = position vector of C – position vector of B

$= \vec{c} - \vec{b}$ ...(i)

and $\overrightarrow{CA}$ = position vector of A – position vector of C

$= \vec{a} - \vec{c}$ ...(ii)

A, B and C are collinear if and only if $\overrightarrow{BC} \times \overrightarrow{CA} = \vec{0}$

if and only if $(\vec{c} - \vec{b}) \times (\vec{a} - \vec{c}) = \vec{0}$ {From (i) and (ii)}

if and only if $(\vec{c} \times \vec{a}) - (\vec{c} \times \vec{c}) - (\vec{b} \times \vec{a}) + (\vec{b} \times \vec{c}) = \vec{0}$

if and only if $(\vec{c} \times \vec{a}) + (\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) = \vec{0}$

$\{\because \vec{c} \times \vec{c} = \vec{0}$ and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}\}$

iff $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = \vec{0}$

$\therefore$ A, B and C are collinear iff $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = \vec{0}$

79. Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$\Rightarrow \vec{b} + \vec{c} = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

The unit vector along $\vec{b} + \vec{c}$ is $\vec{p} = \dfrac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|}$

$= \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 6^2 + (-2)^2}} = \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$

Also, $\vec{a}.\vec{p} = 1$ (Given)

$\Rightarrow \dfrac{(2+\lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1 \Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6$

$\Rightarrow$ $\lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36 \Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1$

$\therefore$ The required unit vector

$\vec{p} = \dfrac{(2+1)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{1 + 4 + 44}} = \dfrac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$.

80. Given, position vector of A = $\hat{i} + \hat{j} + \hat{k}$

Position vector of B = $2\hat{i} + 5\hat{j}$

Position vector of C = $3\hat{i} + 2\hat{j} - 3\hat{k}$

Position vector of D = $\hat{i} - 6\hat{j} - \hat{k}$

$\therefore$ $\overrightarrow{AB} = (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}$

and $\overrightarrow{CD} = (\hat{i} - 6\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{i} - 8\hat{j} + 2\hat{k}$

Now, $|\overrightarrow{AB}| = \sqrt{(1)^2 + (4)^2 + (-1)^2} = \sqrt{18}$

$|\overrightarrow{CD}| = \sqrt{(-2)^2 + (-8)^2 + (2)^2} = \sqrt{4 + 64 + 4} = \sqrt{72} = 2\sqrt{18}$

$= \sqrt{72} = 2\sqrt{18}$

Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

$\therefore \cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \dfrac{(\hat{i} + 4\hat{j} - \hat{k}) \cdot (-2\hat{i} - 8\hat{j} + 2\hat{k})}{(\sqrt{18})(2\sqrt{18})}$

$= \dfrac{-2 - 32 - 2}{36} = \dfrac{-36}{36} = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

Since, angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is 180°.

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

81. We have, $|\vec{a}| = 1, |\vec{b}| = 2$ and $|\vec{c}| = 3$ ...(i)

Given, Projection of $\vec{b}$ along $\vec{a}$ = Projection of $\vec{c}$ along $\vec{a}$

$\Rightarrow \dfrac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \dfrac{\vec{c} \cdot \vec{a}}{|\vec{a}|} \Rightarrow \vec{b} \cdot \vec{a} = \vec{c} \cdot \vec{a}$

$\Rightarrow \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ ...(ii)

Given, $\vec{b}$ and $\vec{c}$ are perpendicular to each other

$\therefore \vec{b} \cdot \vec{c} = 0$ ...(iii)

Now, $|3\vec{a} - 2\vec{b} + 2\vec{c}|^2 = (3\vec{a} - 2\vec{b} + 2\vec{c}) \cdot (3\vec{a} - 2\vec{b} + 2\vec{c})$

$= 9(\vec{a} \cdot \vec{a}) - 6(\vec{a} \cdot \vec{b}) + 6(\vec{a} \cdot \vec{c}) - 6(\vec{b} \cdot \vec{a}) + 4(\vec{b} \cdot \vec{b}) - 4(\vec{b} \cdot \vec{c})$
$+ 6(\vec{c} \cdot \vec{a}) - 4(\vec{c} \cdot \vec{b}) + 4(\vec{c} \cdot \vec{c})$

$= 9|\vec{a}|^2 + 4|\vec{b}|^2 + 4|\vec{c}|^2 + 2\{-6(\vec{a} \cdot \vec{b}) - 4(\vec{b} \cdot \vec{c}) + 6(\vec{a} \cdot \vec{c})\}$

$= 9 \times 1^2 + 4 \times 2^2 + 4 \times 3^2 + 2\{-6(\vec{a} \cdot \vec{b}) - 4 \times 0 + 6(\vec{a} \cdot \vec{b})\}$

[From eqn (i) and (iii)]

$= 9 + 16 + 36 + 2 \times 0 = 61$ [From eqn. (ii)]

$\therefore |3\vec{a} - 2\vec{b} + 2\vec{c}| = \sqrt{61}$

82. Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

Now, it is given that, $\vec{d}$ is perpendicular to

$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$ $\therefore$ $\vec{d} \cdot \vec{b} = 0$ and $\vec{d} \cdot \vec{c} = 0$

$\Rightarrow$ $x - 4y + 5z = 0$ ...(i)

and $3x + y - z = 0$ ...(ii)

Also, $\vec{d} \cdot \vec{a} = 21$, where $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$

$\Rightarrow$ $4x + 5y - z = 21$ ...(iii)

Eliminating z from (i) and (ii), we get $16x + y = 0$ ...(iv)

Eliminating z from (ii) and (iii), we get $x + 4y = 21$ ...(v)

Solving (iv) and (v), we get $x = \dfrac{-1}{3}, y = \dfrac{16}{3}$

Putting the values of x and y in (i), we get $z = \dfrac{13}{3}$

$\therefore$ $\vec{d} = \dfrac{-1}{3}\hat{i} + \dfrac{16}{3}\hat{j} + \dfrac{13}{3}\hat{k}$ is the required vector.

Concept Applied

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$.

83. Given, $|\vec{a}| = |\vec{b}| = |\vec{c}|$ ...(i)

and $\vec{a} \cdot \vec{b} = 0, \vec{b} \cdot \vec{c} = 0, \vec{c} \cdot \vec{a} = 0$ ...(ii)

Let $(\vec{a} + \vec{b} + \vec{c})$ be inclined to vectors $\vec{a}, \vec{b}$ and $\vec{c}$ by angles α, β and γ respectively. Then

$\cos\alpha = \dfrac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|} = \dfrac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|}$

$= \dfrac{|\vec{a}|^2 + 0 + 0}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|}$ [Using (ii)]

$= \dfrac{|\vec{a}|}{|\vec{a} + \vec{b} + \vec{c}|}$ ...(iii)

Similarly, $\cos\beta = \dfrac{|\vec{b}|}{|\vec{a} + \vec{b} + \vec{c}|}$ ...(iv)

and $\cos\gamma = \dfrac{|\vec{c}|}{|\vec{a} + \vec{b} + \vec{c}|}$ ...(v)

From (i), (iii), (iv) and (v), we get

$\cos\alpha = \cos\beta = \cos\gamma \Rightarrow \alpha = \beta = \gamma$

Hence, the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to the vector $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Also, the angle between them is given as

$$\alpha=\cos^{-1}\left(\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}\right),\beta=\cos^{-1}\left(\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$$

$$\gamma=\cos^{-1}\left(\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$$

84. We have, $A(2\hat{i}-\hat{j}+\hat{k}), B(\hat{i}-3\hat{j}-5\hat{k})$,

and $C(3\hat{i}-4\hat{j}-4\hat{k})$

Then, $\overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$,

$\overrightarrow{AC}=(3-2)\hat{i}+(-4+1)\hat{j}+(-4-1)\hat{k}=\hat{i}-3\hat{j}-5\hat{k}$

and $\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$

Now angle between $\overrightarrow{AC}$ and $\overrightarrow{BC}$ is given by

$$\cos\theta=\frac{(\overrightarrow{AC})\cdot(\overrightarrow{BC})}{|\overrightarrow{AC}||\overrightarrow{BC}|}=\frac{2+3-5}{\sqrt{1+9+25}\,.\sqrt{4+1+1}}$$

$$\Rightarrow \quad \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2} \quad \Rightarrow BC\perp AC$$

So, A, B, C are vertices of right angled triangle.

Now area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BC}|$

$$=\frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & -5\\ 2 & -1 & 1\end{matrix}\right\|=\frac{1}{2}|(-3-5)\hat{i}-(1+10)\hat{j}+(-1+6)\hat{k}|$$

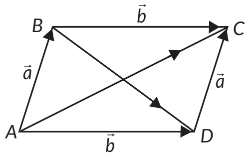
$$=\frac{1}{2}|-8\hat{i}-11\hat{j}+5\hat{k}|=\frac{1}{2}\sqrt{64+121+25}=\frac{\sqrt{210}}{2}\text{ sq. units.}$$

Concept Applied

➲ If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

85. Let $\vec{a}=2\hat{i}-4\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+2\hat{j}+3\hat{k}$



Then diagonal $\overrightarrow{AC}$ of the parallelogram is $\vec{p}=\vec{a}+\vec{b}$

$\Rightarrow \quad \vec{p}=2\hat{i}-4\hat{j}-5\hat{k}+2\hat{i}+2\hat{j}+3\hat{k}=4\hat{i}-2$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}}{|\vec{p}|}=\frac{4\hat{i}-2\hat{j}-2\hat{k}}{\sqrt{16+4+4}}=\frac{2\hat{i}-\hat{j}-\hat{k}}{\sqrt{6}}$$

Now, diagonal $\overrightarrow{BD}$ of the parallelogram is

$\vec{p}\,'=\vec{b}-\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}-2\hat{i}+4\hat{j}+5\hat{k}=6\hat{j}+8\hat{k}$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}\,'}{|\vec{p}\,'|}=\frac{6\hat{j}+8\hat{k}}{\sqrt{36+64}}=\frac{6\hat{j}+8\hat{k}}{10}=\frac{3\hat{j}+4\hat{k}}{5}$$

$$\text{Now, } \vec{p}\times\vec{p}\,'=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -2 & -2\\ 0 & 6 & 8\end{vmatrix}$$

$$=\hat{i}(-16+12)-\hat{j}(32-0)+\hat{k}(24-0)=-4\hat{i}-32\hat{j}+24\hat{k}$$

$$\therefore \quad \text{Area of parallelogram}=\frac{|\vec{p}\times\vec{p}\,'|}{2}$$

$$=\frac{\sqrt{16+1024+576}}{2}=2\sqrt{101}\text{ sq. units.}$$

Concept Applied

➲ Area of parallelogram $=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$ where $\vec{d}_1$ and $\vec{d}_2$ diagonals of parallelograms.

86. Two non zero vectors are parallel if and only if their cross product is zero vector.

So, we have to prove that cross product of $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ is zero vector.

$(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})-(\vec{d}\times\vec{b})+(\vec{d}\times\vec{c})$

Since, it is given that $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$.

And, $\vec{d}\times\vec{b}=-\vec{b}\times\vec{d}$, $\vec{d}\times\vec{c}=-\vec{c}\times\vec{d}$

Therefore, $(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=(\vec{c}\times\vec{d})-(\vec{b}\times\vec{d})+(\vec{b}\times\vec{d})-(\vec{c}\times\vec{d})=\vec{0}$

Hence, $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

87. $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy=[(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}]\cdot[(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j})]+xy$

$=(-y\hat{k}+z\hat{j})\cdot(x\hat{k}-z\hat{i})+xy=-xy+xy=0$

88. Here, $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$

$\therefore \quad \vec{a}-\vec{b}=(\hat{i}+2\hat{j}+\hat{k})-(2\hat{i}+\hat{j})=-\hat{i}+\hat{j}+\hat{k}$

and $\vec{c}-\vec{b}=(3\hat{i}-4\hat{j}-5\hat{k})-(2\hat{i}+\hat{j})=\hat{i}-5\hat{j}-5\hat{k}$

Vector perpendicular to both $\vec{a}-\vec{b}$ and $\vec{c}-\vec{b}$ is

$$(\vec{a}-\vec{b})\times(\vec{c}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 1\\ 1 & -5 & -5\end{vmatrix}$$

$$=(-5+5)\hat{i}-(5-1)\hat{j}+(5-1)\hat{k}=-4\hat{j}+4\hat{k}$$

$\therefore \quad$ Unit vector perpendicular to both $\vec{a}-\vec{b}$ and $\vec{c}-\vec{b}$

$$=\pm\frac{-4\hat{j}+4\hat{k}}{|-4\hat{j}+4\hat{k}|}=\pm\frac{-4\hat{j}+4\hat{k}}{\sqrt{(-4)^2+4^2}}=\pm\frac{-4\hat{j}+4\hat{k}}{4\sqrt{2}}=\pm\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k}).$$

89. Given $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$

We have $\vec{a}+\vec{b}+\vec{c}=0$

$$\Rightarrow \vec{a}+\vec{b}=-\vec{c} \Rightarrow |\vec{a}+\vec{b}|^2=|-\vec{c}|^2 \Rightarrow |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=|\vec{c}|^2$$

$$\Rightarrow 9+25+2|\vec{a}||\vec{b}|\cos\theta=49 \Rightarrow 2\times3\times5\times\cos\theta=49-34=15$$

$$\Rightarrow \cos\theta=\frac{15}{30}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}=60^\circ$$

90. We have, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

Let $\quad \vec{r}=\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{p}=\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$

A unit vector perpendicular to both $\vec{r}$ and $\vec{p}$ is given as $\pm\frac{\vec{r}\times\vec{p}}{|\vec{r}\times\vec{p}|}$.

$$\text{Now, } \vec{r}\times\vec{p}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}=-2\hat{i}+4\hat{j}-2\hat{k}$$

So, the required unit vector is

$$=\pm\frac{(-2\hat{i}+4\hat{j}-2\hat{k})}{\sqrt{(-2)^2+4^2+(-2)^2}}=\mp\frac{(\hat{i}-2\hat{j}+\hat{k})}{\sqrt{6}}.$$

91. Here, $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$ and $\vec{c} = 2\hat{j} - \hat{k}$

$\therefore \quad \vec{a} + \vec{b} = (2\hat{i} - 3\hat{j} + \hat{k}) + (-\hat{i} + \hat{k}) = \hat{i} - 3\hat{j} + 2\hat{k}$

and $\vec{b} + \vec{c} = (-\hat{i} + \hat{k}) + (2\hat{j} - \hat{k}) = -\hat{i} + 2\hat{j}$

$$\therefore (\vec{a} + \vec{b}) \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix} = -4\hat{i} - 2\hat{j} - \hat{k}$$

$\therefore$ Area of a parallelogram whose diagonals are $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$

$$= \frac{1}{2}\left|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})\right| = \frac{1}{2}\left|-4\hat{i} - 2\hat{j} - \hat{k}\right|$$

$$= \frac{1}{2}\sqrt{(-4)^2 + (-2)^2 + (-1)^2} = \frac{\sqrt{21}}{2} \text{ sq.units.}$$

92. Let $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$

Now, it is given that $\vec{p}$ is perpendicular to both $\vec{\alpha}$ and $\vec{\beta}$

$\therefore \quad \vec{p} \cdot \vec{\alpha} = 0$ and $\vec{p} \cdot \vec{\beta} = 0$

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} + 5\hat{j} - \hat{k}) = 0$

and $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} - 4\hat{j} + 5\hat{k}) = 0$

$\Rightarrow \quad 4x + 5y - z = 0$ ...(i)

and $x - 4y + 5z = 0$ ...(ii)

Also, we have, $\vec{p} \cdot \vec{q} = 21 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 21$

$\Rightarrow \quad 3x + y - z = 21$ ...(iii)

Eliminating z from (i) and (ii), we get

$21x + 21y = 0 \Rightarrow x + y = 0$ ...(iv)

Eliminating z from (i) and (iii), we get $x + 4y = -21$ ...(v)

Solving (iv) and (v), we get $x = 7$, $y = -7$

Now, from (i), we get $z = -7$

So, $\vec{p} = 7\hat{i} - 7\hat{j} - 7\hat{k}$.

CBSE Sample Questions

1. Let $\vec{a}$ be the unit vector in the direction opposite to the given vector $\left(-\frac{3}{4}\hat{j}\right)$.

Then, $\vec{a} = \dfrac{-1}{\sqrt{\left(\frac{3}{4}\right)^2}}\left(-\frac{3}{4}\hat{j}\right) = \hat{j}$ **(1)**

2. A vector in the direction opposite to $2\hat{i} + 3\hat{j} - 6\hat{k}$ is $-2\hat{i} - 3\hat{j} + 6\hat{k}$.

Its magnitude is $|\sqrt{4+9+36}| = |\sqrt{49}| = 7$

So, a vector in the direction opposite to $2\hat{i} + 3\hat{j} - 6\hat{k}$ of magnitude 5 units is $\frac{5}{7}(-2\hat{i} - 3\hat{j} + 6\hat{k})$ **(1)**

3. **(a):** Scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

Scalar projection of $3\hat{i} - \hat{j} - 2\hat{k}$ on vector $\hat{i} + 2\hat{j} - 3\hat{k}$

$$= \frac{(3\hat{i} - \hat{j} - 2\hat{k}) \cdot (\hat{i} + 2\hat{j} - 3\hat{k})}{|\hat{i} + 2\hat{j} - 3\hat{k}|} = \frac{7}{\sqrt{14}}$$ **(1)**

4. **(b):** $|\vec{a} - 2\vec{b}|^2 = (\vec{a} - 2\vec{b}) \cdot (\vec{a} - 2\vec{b})$

$= \vec{a} \cdot \vec{a} - 4\vec{a} \cdot \vec{b} + 4\vec{b} \cdot \vec{b} = |\vec{a}|^2 - 4\vec{a} \cdot \vec{b} + 4|\vec{b}|^2 = 4 - 16 + 36 = 24$

$\therefore \quad |\vec{a} - 2\vec{b}| = 2\sqrt{6}$ **(1)**

5. Area of the triangle

$= \frac{1}{2}|2\hat{i} \times (-3)\hat{j}| = \frac{1}{2}|-6\hat{k}| = 3$ sq. units **(1)**

6. We have, $|\hat{a} + \hat{b}|^2 = 1 \Rightarrow \hat{a}^2 + \hat{b}^2 + 2\hat{a} \cdot \hat{b} = 1$

$\Rightarrow \quad 2\hat{a} \cdot \hat{b} = 1 - 1 - 1 \qquad (\because |\hat{a}| = |\hat{b}| = 1)$

$\Rightarrow \quad \hat{a} \cdot \hat{b} = \frac{-1}{2} \Rightarrow |\hat{a}||\hat{b}|\cos\theta = \frac{-1}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\Rightarrow \quad \theta = \pi - \frac{\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$ **(1)**

7. Since $\vec{a}$ is a unit vector, $\therefore |\vec{a}| = 1$

Now, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

$\Rightarrow \quad \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 12$ **(1/2)**

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12 \quad (\because \vec{a} \cdot \vec{x} = \vec{x} \cdot \vec{a})$ **(1/2)**

$\Rightarrow \quad |\vec{x}|^2 - 1 = 12 \Rightarrow |\vec{x}|^2 = 13 \Rightarrow |\vec{x}| = \sqrt{13}$ **(1)**

8. Since, $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b})$ **(1)**

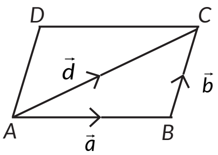
$\therefore \quad |\vec{a} + \vec{b}|^2 = 1 + 1 + 2\cos\theta \quad$ [As $|\vec{a}| = |\vec{b}| = 1$]

$= 2(1 + \cos\theta) = 4\cos^2\frac{\theta}{2} \qquad \left[\because 1 + \cos\theta = 2\cos^2\frac{\theta}{2}\right]$ **(1/2)**

$\therefore \quad |\vec{a} + \vec{b}| = 2\cos\frac{\theta}{2}$ **(1/2)**

9. Let *ABCD* is a parallelogram such that

$\vec{a} = \overrightarrow{AB} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \overrightarrow{BC}$, and $\vec{d} = \overrightarrow{AC} = 4\hat{i} + 5\hat{k}$.



Now, $\vec{a} + \vec{b} = \vec{d}$ (By triangle law)

$\Rightarrow \quad \vec{b} = \vec{d} - \vec{a}$

$\Rightarrow \quad \vec{b} = (4\hat{i} + 5\hat{k}) - (\hat{i} - \hat{j} + \hat{k})$

$= 3\hat{i} + \hat{j} + 4\hat{k}$ **(1/2)**

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = -5\hat{i} - \hat{j} + 4\hat{k}$$ **(1)**

$\therefore \quad$ Area of parallelogram $= |\vec{a} \times \vec{b}|$

$= \sqrt{25 + 1 + 16} = \sqrt{42}$ sq.units **(1/2)**

10. We have, $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \Rightarrow \vec{a} \cdot (\vec{b} - \vec{c}) = 0$

$\Rightarrow \quad (\vec{b} - \vec{c}) = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow \quad \vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$ **(1)**

Also, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$

$\Rightarrow \quad \vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \parallel (\vec{b} - \vec{c})$

$\Rightarrow \quad \vec{b} = \vec{c}$ or $\vec{a} \parallel (\vec{b} - \vec{c})$ **(1)**

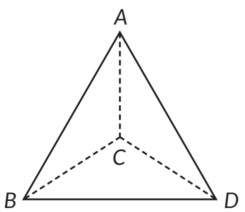
Since, $\vec{a}$ can not be both perpendicular to $(\vec{b} - \vec{c})$ and parallel to $(\vec{b} - \vec{c})$.

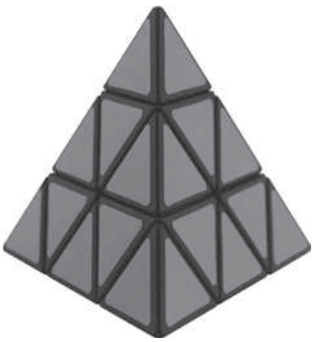
Hence, $\vec{b} = \vec{c}$. **(1)**

Self Assessment

Case Based Objective Questions (4 marks)

1. **Read the passage given below and answer the following questions :**
A building is to be constructed in the form of a triangular pyramid, *ABCD* as shown in the figure.





Let its angular points are $A(0, 1, 2)$, $B(3, 0, 1)$, $C(4, 3, 6)$ and $D(2, 3, 2)$ and G be the point of intersection of the medians of ΔBCD.
Based on the above information, attempts any 4 out of 5 subparts.

(i) The coordinates of point G are
(a) (2, 3, 3) (b) (3, 3, 2)
(c) (3, 2, 3) (d) (0, 2, 3)

(ii) The length of vector $\overrightarrow{AG}$ is
(a) $\sqrt{17}$ units (b) $\sqrt{11}$ units
(c) $\sqrt{13}$ units (d) $\sqrt{19}$ units

(iii) Area of ΔABC (in sq. units) is
(a) $\sqrt{10}$ (b) $2\sqrt{10}$ (c) $3\sqrt{10}$ (d) $5\sqrt{10}$

(iv) The sum of lengths of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is
(a) 5 units (b) 9.32 units
(c) 10 units (d) 11 units

(v) The length of the perpendicular from the vertex D on the opposite face is
(a) $\frac{6}{\sqrt{10}}$ units (b) $\frac{2}{\sqrt{10}}$ units
(c) $\frac{3}{\sqrt{10}}$ units (d) $8\sqrt{10}$ units

Multiple Choice Questions (1 mark)

2. The vector in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ that has magnitude 9 is
(a) $\hat{i}-2\hat{j}+2\hat{k}$ (b) $\frac{\hat{i}-2\hat{j}+2\hat{k}}{3}$
(c) $3(\hat{i}-2\hat{j}+2\hat{k})$ (d) $9(\hat{i}-2\hat{j}+2\hat{k})$

3. Find the value of λ such that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$ are orthogonal.
(a) 0 (b) 1 (c) $\frac{3}{2}$ (d) $\frac{-5}{2}$

OR

The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is
(a) $-\hat{i}+12\hat{j}+4\hat{k}$ (b) $5\hat{i}+2\hat{j}-4\hat{k}$
(c) $-5\hat{i}+2\hat{j}+4\hat{k}$ (d) $\hat{i}+\hat{j}+\hat{k}$

4. The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 4, respectively and $\vec{a}\cdot\vec{b}=2\sqrt{3}$ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{5\pi}{2}$

5. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}|=2, |\vec{b}|=3, |\vec{c}|=5$, then the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ is
(a) 0 (b) 1 (c) –19 (d) 38

6. If $\vec{u}=\hat{i}+2\hat{j}, \vec{v}=-2\hat{i}+\hat{j}$ and $\vec{w}=4\hat{i}+3\hat{j}$. Find scalars x and y respectively such that $\vec{w}=x\vec{u}+y\vec{v}$.
(a) 4, –2 (b) 2, –1
(c) 3, 5 (d) –5, 2

7. If $\vec{a}$ and $\vec{b}$ are unit vectors enclosing an angle θ and $|\vec{a}+\vec{b}|<1$, then
(a) $\theta=\frac{\pi}{2}$ (b) $\theta<\frac{\pi}{3}$
(c) $\pi\geq\theta>\frac{2\pi}{3}$ (d) $\frac{\pi}{3}<\theta<\frac{2\pi}{3}$

VSA Type Questions (1 mark)

8. Find the unit vector in the direction of vector $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$.

9. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having same magnitude and such that the angle between them is 60° and their scalar product is 8.

10. If $\overrightarrow{AB}\times\overrightarrow{AC}=2\hat{i}-4\hat{j}+4\hat{k}$, then find the area of ΔABC.

OR

If $\vec{r}\cdot\vec{a}=0, \vec{r}\cdot\vec{b}=0$ and $\vec{r}\cdot\vec{c}=0$ for some non-zero vector $\vec{r}$, then find the value of $\vec{a}\cdot(\vec{b}\times\vec{c})$.

11. $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=144$ and $|\vec{a}|=4$, then find $|\vec{b}|$.

12. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}+\hat{j}, \vec{c}=\hat{i}$ and $(\vec{a}\times\vec{b})\times\vec{c}=\lambda\vec{a}+\mu\vec{b}$, then find the value of $\lambda+\mu$.

SA I Type Questions (2 marks)

13. If $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$, then find the unit vector in the direction of
(i) $6\vec{b}$ (ii) $2\vec{a}-\vec{b}$

14. Find the angle between the vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$.

OR

If A, B, C and D are the points with position vectors $\hat{i}+\hat{j}-\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$ respectively, then find the projection of $\overrightarrow{AB}$ along $\overrightarrow{CD}$.

15. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B respectively, then find the position vector of a point C in $\overline{BA}$ produced such that $\overrightarrow{BC}=1.5\overrightarrow{BA}$.

16. The position vectors of points A, B, C and D are $\vec{a}$, $\vec{b}$, $2\vec{a}+3\vec{b}$ and $\vec{a}-2\vec{b}$ respectively. Show that $\overrightarrow{DB}=3\vec{b}-\vec{a}$ and $\overrightarrow{AC}=\vec{a}+3\vec{b}$.

SA II Type Questions (3 marks)

17. If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i},\hat{j}$ and $\hat{k}$, $\alpha=3\hat{i}-\hat{j}$, $\beta=2\hat{i}+\hat{j}-3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

18. If $\vec{p}=-3\hat{i}+4\hat{j}-7\hat{k}$ and $\vec{q}=6\hat{i}+2\hat{j}-3\hat{k}$, then find $\vec{p}\times\vec{q}$. Verify that $\vec{p}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other.

19. If the vector $\vec{a}+\vec{b}$ bisects the angle between the non-collinear vectors $\vec{a}$ and $\vec{b}$, then show that $\vec{a}=\vec{b}$.

20. Find the value of the expression $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2$.

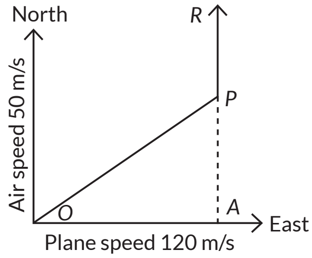
OR

Find the sine of the angle between the vectors $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$.

Case Based Questions (4 marks)

21. A plane started from airport situated at O with a velocity of 120 m/s towards east. Air is blowing at a velocity of 50 m/s towards the north as shown in the figure.

The plane travelled 1 hr in OP direction with the resultant velocity. From P to R the plane travelled 1 hr keeping velocity of 120 m/s and finally landed at R.



Based on the given information, answer the following questions.

(i) What is the resultant velocity from O to P? Also find the direction of travel of plane from O to P with East?

(ii) What is the displacement from O to P?

OR

If $\vec{P}=\hat{i}+2\hat{j}+4\hat{k}$ and $\vec{R}=7\hat{i}+5\hat{j}+2\hat{k}$ then find out the value of $|\overrightarrow{PR}|$.

(iii) What will be the area of ΔPOA where A is the vertical meeting point of plane from $\vec{P}(\hat{i}+2\hat{j}+4\hat{k})$ and $\vec{A}(4\hat{i}-2\hat{j}+\hat{k})$?

LA Type Questions (4/6 marks)

22. Prove that in any ΔABC, $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$, where a, b, and c are the magnitudes of the sides opposite to the vertices A, B and C, respectively.

23. If $\vec{a},\vec{b}$ and $\vec{c}$ determine the vertices of a triangle, show that $\dfrac{1}{2}[\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}]$ gives the vector area of the triangle. Hence, deduce the condition that the three points $\vec{a},\vec{b}$ and $\vec{c}$ are collinear. Also, find the unit vector normal to the plane of the triangle.

24. If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is the angle between them, then show that $\cos\dfrac{\theta}{2}=\dfrac{1}{2}|\hat{a}+\hat{b}|$.

25. Find the value of p for which the vectors $\vec{a}=3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b}=\hat{i}+p\hat{j}+3\hat{j}$ are
(i) perpendicular (ii) parallel

OR

A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3, –6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.

Detailed SOLUTIONS

1. (i) (c) : Clearly, G be the centroid of ΔBCD, therefore coordinates of G are

$$\left(\frac{3+4+2}{3},\frac{0+3+3}{3},\frac{1+6+2}{3}\right)=(3,2,3)$$

(ii) (b): Since, A = (0, 1, 2) and G = (3, 2, 3)

$\therefore\ \overrightarrow{AG}=(3-0)\hat{i}+(2-1)\hat{j}+(3-2)\hat{k}=3\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow |\overrightarrow{AG}|^2 = 3^2 + 1^2 + 1^2 = 9 + 1 + 1 = 11 \Rightarrow |\overrightarrow{AG}|=\sqrt{11}$ units

(iii) (c): Clearly, area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Here, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3-0 & 0-1 & 1-2\\ 4-0 & 3-1 & 6-2\end{vmatrix}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -1 & -1\\ 4 & 2 & 4\end{vmatrix}$

$=\hat{i}(-4+2)-\hat{j}(12+4)+\hat{k}(6+4)=-2\hat{i}-16\hat{j}+10\hat{k}$

$\therefore\ |\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{(-2)^2+(-16)^2+10^2}$

$=\sqrt{4+256+100}=\sqrt{360}=6\sqrt{10}$

Hence, area of $\Delta ABC = \frac{1}{2}\times 6\sqrt{10}=3\sqrt{10}$ sq. units

(iv) (b): Here, $\overrightarrow{AB}=3\hat{i}-\hat{j}-\hat{k}\ \Rightarrow\ |\overrightarrow{AB}|=\sqrt{9+1+1}=\sqrt{11}$

Also, $\overrightarrow{AC}=4\hat{i}+2\hat{j}+4\hat{k}$

$\Rightarrow\ |\overrightarrow{AC}|=\sqrt{16+4+16}=\sqrt{36}=6$

Now, $|\overrightarrow{AB}|+|\overrightarrow{AC}|=\sqrt{11}+6$ = 3.32 + 6 = 9.32 units

(v) (a) : The length of the perpendicular from the vertex *D* on the opposite face

$=|\text{Projection of } \overrightarrow{AD} \text{ on } \overrightarrow{AB}\times\overrightarrow{AC}|$

$=\left|\frac{(2\hat{i}+2\hat{j})\cdot(-2\hat{i}-16\hat{j}+10\hat{k})}{\sqrt{(-2)^2+(-16)^2+10^2}}\right|=\left|\frac{-4-32}{\sqrt{360}}\right|=\frac{36}{6\sqrt{10}}=\frac{6}{\sqrt{10}}$ units

2. (c) : Let $\vec{x}=(\hat{i}-2\hat{j}+2\hat{k})$

Any vector in the direction of a vector $\vec{x}$ is given by $\frac{\vec{x}}{|\vec{x}|}$

$=\frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{1^2+2^2+2^2}}=\frac{\hat{i}-2\hat{j}+2\hat{k}}{3}$

$\therefore$ Vector in the direction of $\vec{x}$ with magnitude 9 is

$9\left(\frac{\hat{i}-2\hat{j}+2\hat{k}}{3}\right)=3(\hat{i}-2\hat{j}+2\hat{k})$

3. (d): Since, two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

$\therefore\ \vec{a}\cdot\vec{b}=0$

$\Rightarrow\ (2\hat{i}+\lambda\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+3\hat{k})=0\ \Rightarrow\ 2+2\lambda+3=0\ \Rightarrow\ \lambda=\frac{-5}{2}$

OR

(c) : Required vector

$=(-3-2)\hat{i}+(7-5)\hat{j}+(4-0)\hat{k}=-5\hat{i}+2\hat{j}+4\hat{k}$

4. (b): Given, $|\vec{a}|=\sqrt{3}, |\vec{b}|=4$ and $\vec{a}\cdot\vec{b}=2\sqrt{3}$

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$,

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow\ 2\sqrt{3}=\sqrt{3}\cdot 4\cdot\cos\theta\ \Rightarrow\ \cos\theta=\frac{1}{2}=\cos\frac{\pi}{3}$

$\therefore\ \theta=\frac{\pi}{3}$

5. (c) : Here, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|^2=4, |\vec{b}|^2=9, |\vec{c}|^2=25$

$\therefore\ (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}$

$\Rightarrow\ |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+|\vec{b}|^2+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+|\vec{c}|^2=\vec{0}$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0\quad [\because\ \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}\ etc]$

$\Rightarrow\ 4+9+25+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-38}{2}=-19$

6. (b): We have, $\vec{w}=x\vec{u}+y\vec{v}$

$\Rightarrow\ 4\hat{i}+3\hat{j}=x(\hat{i}+2\hat{j})+y(-2\hat{i}+\hat{j})$

$\Rightarrow\ (x-2y-4)\hat{i}+(2x+y-3)\hat{j}=\vec{0}$

$\Rightarrow\ x-2y-4=0$ and $2x+y-3=0 \Rightarrow x=2$ and $y=-1$

7. (c) : $|\vec{a}+\vec{b}|<1\Rightarrow|\vec{a}+\vec{b}|^2<1$

$\Rightarrow|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}<1\Rightarrow 1+1+2\vec{a}\cdot\vec{b}<1$

$\Rightarrow\vec{a}\cdot\vec{b}<-\frac{1}{2}\Rightarrow|\vec{a}||\vec{b}|\cos\theta<-\frac{1}{2}$

$\Rightarrow 1\times1\times\cos\theta<-\frac{1}{2}\Rightarrow\cos\theta<-\frac{1}{2}$

$\Rightarrow -1\le\cos\theta<-\frac{1}{2}\Rightarrow\pi\ge\theta>\frac{2\pi}{3}$

8. The unit vector in the direction of a vector $\vec{a}$ is given by $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$. Now, $|\vec{a}|=\sqrt{(2)^2+(3)^2+(4)^2}=\sqrt{29}$

Therefore, $\hat{a}=\frac{2\hat{i}+3\hat{j}+4\hat{k}}{\sqrt{29}}=\frac{2}{\sqrt{29}}\hat{i}+\frac{3}{\sqrt{29}}\hat{j}+\frac{4}{\sqrt{29}}\hat{k}$

9. We are given, $|\vec{a}|=|\vec{b}|, \vec{a}\cdot\vec{b}=8, \theta=60°$

Now $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\Rightarrow\cos 60°=\frac{8}{|\vec{a}|^2}\ \Rightarrow\frac{1}{2}=\frac{8}{|\vec{a}|^2}$

$\Rightarrow\ |\vec{a}|^2=16\ \Rightarrow\ |\vec{a}|=4\ \therefore |\vec{a}|=|\vec{b}|=4$

10. (a) : Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}|2\hat{i}-4\hat{j}+4\hat{k}|$

$=\frac{1}{2}\left[\sqrt{(2)^2+(-4)^2+(4)^2}\right]$ = 3 sq. units

OR

Since, $\vec{r}$ is a non-zero vector and $\vec{r}\cdot\vec{a}=\vec{r}\cdot\vec{b}=\vec{r}\cdot\vec{c}=0$. So, we can say that $\vec{a}, \vec{b}$ and $\vec{c}$ are in a same plane or coplanar.

$\therefore\ \vec{a}\cdot(\vec{b}\times\vec{c})=0$

11. Given, $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=144$ and $|\vec{a}|=4$

$\because\ |\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2$

$\Rightarrow\ 144=16\cdot|\vec{b}|^2\ \Rightarrow |\vec{b}|^2=9\ \Rightarrow\ |\vec{b}|=3$

12. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix}=-\hat{i}+\hat{j}$

Now, $(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 0\\ 1 & 0 & 0\end{vmatrix}$

$\Rightarrow\ \lambda\vec{a}+\mu\vec{b}=\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(0-1)$

$\Rightarrow\ (\lambda+\mu)\hat{i}+(\lambda+\mu)\hat{j}+(\lambda)\hat{k}=-\hat{k}$

On comparing, we get $\lambda=-1$ and $\lambda+\mu=0\ \Rightarrow\ \mu=1$

13. Given that, $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$

(i) Now, $6\vec{b}=6(2\hat{i}+\hat{j}-2\hat{k})=12\hat{i}+6\hat{j}-12\hat{k}$

$\therefore$ Unit vector in the direction of $6\vec{b}=\frac{6\vec{b}}{|6\vec{b}|}$

$$=\frac{12\hat{i}+6\hat{j}-12\hat{k}}{\sqrt{12^2+6^2+12^2}}=\frac{6(2\hat{i}+\hat{j}-2\hat{k})}{\sqrt{324}}=\frac{6(2\hat{i}+\hat{j}-2\hat{k})}{18}=\frac{2\hat{i}+\hat{j}-2\hat{k}}{3}$$

(ii) Since, $2\vec{a}-\vec{b}=2(\hat{i}+\hat{j}+2\hat{k})-(2\hat{i}+\hat{j}-2\hat{k})$

$=2\hat{i}+2\hat{j}+4\hat{k}-2\hat{i}-\hat{j}+2\hat{k}=\hat{j}+6\hat{k}$

Therefore, unit vector in the direction of

$$2\vec{a}-\vec{b}=\frac{2\vec{a}-\vec{b}}{|2\vec{a}-\vec{b}|}=\frac{\hat{j}+6\hat{k}}{\sqrt{1+36}}=\frac{1}{\sqrt{37}}(\hat{j}+6\hat{k})$$

14. Let $\vec{x}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{y}=3\hat{i}+4\hat{j}-\hat{k}$ then, $\cos\theta=\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}$

$$=\frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}+4\hat{j}-\hat{k})}{\sqrt{4+1+1}\sqrt{9+16+1}}=\frac{6-4-1}{\sqrt{6}\sqrt{26}}=\frac{1}{2\sqrt{39}}$$

$\Rightarrow\quad\therefore\ \theta=\cos^{-1}\left(\frac{1}{2\sqrt{39}}\right)$

OR

Given, $\overrightarrow{OA}=\hat{i}+\hat{j}-\hat{k},\ \overrightarrow{OB}=2\hat{i}-\hat{j}+3\hat{k},\ \overrightarrow{OC}=2\hat{i}-3\hat{k}$

and $\overrightarrow{OD}=3\hat{i}-2\hat{j}+\hat{k}$

$\therefore\ \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(2-1)\hat{i}+(-1-1)\hat{j}+(3+1)\hat{k}=\hat{i}-2\hat{j}+4\hat{k}$

and $\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=(3-2)\hat{i}+(-2-0)\hat{j}+(1+3)\hat{k}=\hat{i}-2\hat{j}+4\hat{k}$

Therefore, the projection of $\overrightarrow{AB}$ along $\overrightarrow{CD}=\overrightarrow{AB}\cdot\frac{\overrightarrow{CD}}{|\overrightarrow{CD}|}$

$$=\frac{(\hat{i}-2\hat{j}+4\hat{k})\cdot(\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{1^2+2^2+4^2}}=\frac{1+4+16}{\sqrt{21}}=\frac{21}{\sqrt{21}}=\sqrt{21}\text{ units}$$

15. Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$

Then $\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}=\vec{a}-\vec{b},\ \overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$

Now, $\overrightarrow{BC}=1.5\overrightarrow{BA}$

$\Rightarrow\ \overrightarrow{OC}-\overrightarrow{OB}=1.5(\vec{a}-\vec{b})=1.5\vec{a}-1.5\vec{b}$

$\Rightarrow\ \overrightarrow{OC}=1.5\vec{a}-1.5\vec{b}+\vec{b}=\frac{3\vec{a}-\vec{b}}{2}$

16. We have

$\overrightarrow{DB}$ = Position vector of B – Position vector of D

$=\vec{b}-(\vec{a}-2\vec{b})=\vec{b}-\vec{a}+2\vec{b}=3\vec{b}-\vec{a}$

and $\overrightarrow{AC}$ = Position vector of C – Position vector of A

$=(2\vec{a}+3\vec{b})-(\vec{a})=\vec{a}+3\vec{b}$

17. Let $\vec{\beta}_1=\lambda\vec{\alpha}$, λ is a scalar, *i.e.*, $\vec{\beta}_1=3\lambda\hat{i}-\lambda\hat{j}$

Now, $\vec{\beta}_2=\vec{\beta}-\vec{\beta}_1=(2-3\lambda)\hat{i}+(1+\lambda)\hat{j}-3\hat{k}$

Now, since $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha}\cdot\vec{\beta}_2=0$ *i.e.*, $3(2-3\lambda)-(1+\lambda)=0$ or $\lambda=\frac{1}{2}$

Therefore, $\vec{\beta}_1=\frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}$ and $\vec{\beta}_2=\frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$

18. $\vec{p}=-3\hat{i}+4\hat{j}-7\hat{k}$ and $\vec{q}=6\hat{i}+2\hat{j}-3\hat{k}$

$$\therefore\ \vec{p}\times\vec{q}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-3&4&-7\\6&2&-3\end{vmatrix}$$

$=(-12+14)\hat{i}-(9+42)\hat{j}+(-6-24)\hat{k}$

$\Rightarrow\ \vec{p}\times\vec{q}=2\hat{i}-51\hat{j}-30\hat{k}$

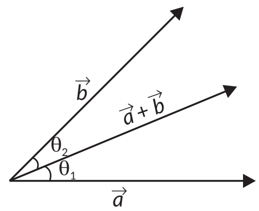
Now, $\vec{p}\cdot(\vec{p}\times\vec{q})=(-3\hat{i}+4\hat{j}-7\hat{k})\cdot(2\hat{i}-51\hat{j}-30\hat{k})=0$

Hence, $\vec{p}$ and $\vec{p}\times\vec{q}$ are perpendicular.

19. Let us consider two non-collinear vectors $\vec{a}$ and $\vec{b}$.

Let $\vec{a}+\vec{b}$ be the vector that bisects the angle between two vectors.

Hence $\theta_1=\theta_2$



$\cos\theta_1=\frac{\vec{a}\cdot(\vec{a}+\vec{b})}{|\vec{a}|\,|\vec{a}+\vec{b}|}$

$\cos\theta_2=\frac{\vec{b}\cdot(\vec{a}+\vec{b})}{|\vec{b}|\,|\vec{a}+\vec{b}|}$

since $\theta_1=\theta_2\Rightarrow\cos\theta_1=\cos\theta_2$

$\Rightarrow\quad\vec{a}=\vec{b}$

So, $\vec{a}$ and $\vec{b}$ are equal vectors.

Hence, $\vec{a}$ and $\vec{b}$ are equal vectors.

20. $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2$

$=|\vec{a}|^2|\vec{b}|^2\sin^2\theta+(\vec{a}\cdot\vec{b})^2$

$=|\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)+(\vec{a}\cdot\vec{b})^2\qquad[\because\sin^2\theta+\cos^2\theta=1]$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{a}|^2|\vec{b}|^2\cos^2\theta+(\vec{a}\cdot\vec{b})^2$

$=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2+(\vec{a}\cdot\vec{b})^2\qquad[\because\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta]$

Hence, $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

OR

Given that, $\vec{a}=3\hat{i}+\hat{j}+2\hat{k},\ \vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$

If θ be the angle between the vectors

then $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=\frac{(3\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{(3)^2+(1)^2+(2)^2}\sqrt{(2)^2+(-2)^2+4^2}}$

$$\cos\theta=\frac{(3\times2)+(1\times(-2))+(2\times4)}{\sqrt{9+1+4}\sqrt{4+4+16}}=\frac{\sqrt{21}}{7}$$

$\therefore\quad\sin\theta=\sqrt{1-\cos^2\theta}=\frac{2}{\sqrt{7}}$

21. **(i)** Resultant velocity from to O to P

$=\sqrt{120^2+50^2}=\sqrt{14400+2500}=\sqrt{16900}=130$ m/s

Direction of travel of plane from O to P with east is $\tan^{-1}\left(\frac{5}{12}\right)$.

(ii) Resultant velocity from O to P = 130 m/s

$=\left(\frac{130\times3600}{1000}\right)$ km/h

Time = 1 hr

$\therefore\quad$ Displacement $=\frac{130\times3600}{1000}=468$ km

OR

We have $\vec{P}=\hat{i}+2\hat{j}+4\hat{k}$

$\vec{R}=-7\hat{i}+5\hat{j}+2\hat{k}$

$\overrightarrow{PR}$ = Position vector of $\vec{R}$ – Position vector of $\vec{P}$

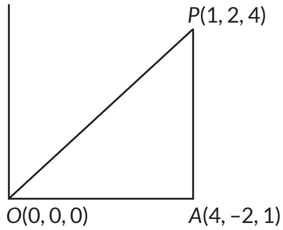
$\Rightarrow \quad \overrightarrow{PR} = 7\hat{i}+5\hat{j}+2\hat{k}-\hat{i}-2\hat{j}-4\hat{k}$

$\Rightarrow \quad \overrightarrow{PR} = 6\hat{i}+3\hat{j}-2\hat{k}$

$\therefore \ |\overrightarrow{PR}| = \sqrt{6^2+3^2+(-2)^2} \ = \sqrt{36+9+4} = \sqrt{49}$

$\Rightarrow |\overrightarrow{PR}| = 7$

(iii) For finding the area of ΔPOA, we must have



$\overrightarrow{OP} = \hat{i}+2\hat{j}+4\hat{k}$

$\overrightarrow{OA} = 4\hat{i}-2\hat{j}+\hat{k}$

Now area of ΔPOA is given by,

$\Delta = \frac{1}{2}|\overrightarrow{OP}\times\overrightarrow{OA}|$

$\therefore \ \overrightarrow{OP}\times\overrightarrow{OA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 4 & -2 & 1 \end{vmatrix}$

$= \hat{i}(2+8)-\hat{j}(1-16)+\hat{k}(-2-8) \ = 10\hat{i}+15\hat{j}-10\hat{k}$

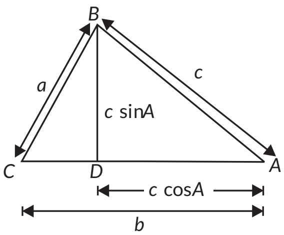
$\Rightarrow \quad \overrightarrow{OP}\times\overrightarrow{OA} = 5(2\hat{i}+3\hat{j}-2\hat{k})$

$\therefore \ |\overrightarrow{OP}\times\overrightarrow{OA}| = \sqrt{(10)^2+15^2+(-10)^2} = \sqrt{425}$

Area of $\Delta POA \ = \frac{1}{2}|\overrightarrow{OP}\times\overrightarrow{OA}|$

ar $(\Delta POA) = \frac{5}{2}\sqrt{17}$ sq.units

22. Since, components of c are $c \cos A$ and $c \sin A$ (as shown in figure)



Also, $|\overrightarrow{CD}| = b - c\cos A$

In ΔBDC,

$a^2 = (b - c\cos A)^2 + (c\sin A)^2$

$\Rightarrow \quad a^2 = b^2 + c^2\cos^2 A - 2bc\cos A + c^2\sin^2 A$

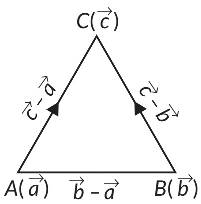
$\Rightarrow \quad 2bc\cos A = b^2 - a^2 + c^2(\cos^2 A + \sin^2 A)$

$\Rightarrow \quad \cos A = \frac{b^2+c^2-a^2}{2bc}$

23. Given, $\vec{a}, \vec{b}$ and $\vec{c}$ are the vertices of a ΔABC.

$\therefore$ area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Since, $\overrightarrow{AB} = \vec{b}-\vec{a}$ and $\overrightarrow{AC} = \vec{c}-\vec{a}$



$\therefore$ Area of $\Delta ABC = \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|$

$= \frac{1}{2}|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}|$

$= \frac{1}{2}|\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{0}|$

$= \frac{1}{2}|\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|$

For collinearity of three points, area of the ΔABC should be equal to zero.

$\Rightarrow \ \frac{1}{2}|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}| = 0 \ \Rightarrow \ \vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b} = 0$

Hence, this is the required condition for three points to be collinear.

Let $\hat{n}$ be the unit vector normal to the plane of the ΔABC.

$\therefore \ \hat{n} = \frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|} = \frac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$

24. Since $\vec{a}$ and $\vec{b}$ are unit vectors $\therefore \ |\vec{a}| = |\vec{b}| = 1$

We have, $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$

$\Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2 \qquad [\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 2+2\cos\theta \qquad [\because |\vec{a}| = |\vec{b}| = 1]$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 2(1+\cos\theta) = 2\left(2\cos^2\frac{\theta}{2}\right) \quad \left[\because \ 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$

$\Rightarrow \ 4\cos^2\frac{\theta}{2} = |\vec{a}+\vec{b}|^2 \ \Rightarrow \ \cos^2\frac{\theta}{2} = \frac{1}{4}|\vec{a}+\vec{b}|^2$

$\Rightarrow \ \cos\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$

25. (i) If vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}\cdot\vec{b} = 0$.

$\Rightarrow \ (3\hat{i}+2\hat{j}+9\hat{k})\cdot(\hat{i}+p\hat{j}+3\hat{k}) = 0$

$\Rightarrow \quad 3 + 2p + 27 = 0 \Rightarrow p = -15$

(ii) We know that, the vectors $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are parallel iff $\vec{a} = \lambda\vec{b}$

$\Leftrightarrow \ (a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) = \lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$

$\Leftrightarrow \quad a_1 = \lambda b_1, a_2 = \lambda b_2, a_3 = \lambda b_3$

$\Leftrightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \lambda$

So, given vectors $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b} = \hat{i}+p\hat{j}+3\hat{k}$ will be parallel if

$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow p = \frac{2}{3}$

OR

Given that, $|\vec{r}| = 14, a = 2k, b = 3k$ and $c = -6k$

$\therefore \quad$ Direction cosines, l, m and n are

$l = \frac{a}{|\vec{r}|} = \frac{2k}{14} = \frac{k}{7}, m = \frac{b}{|\vec{r}|} = \frac{3k}{14}$ and $n = \frac{c}{|\vec{r}|} = \frac{-6k}{14} = \frac{-3k}{7}$

As, we know $l^2 + m^2 + n^2 = 1$

$\therefore \ \frac{k^2}{49} + \frac{9k^2}{196} + \frac{9k^2}{49} = 1$

$\Rightarrow \ \frac{4k^2+9k^2+36k^2}{196} = 1 \ \Rightarrow \ k^2 = \frac{196}{49} = 4$

$\Rightarrow \quad k = \pm 2$

Therefore, the direction cosines (l, m, n) are $\frac{2}{7}, \frac{3}{7}$ and $\frac{-6}{7}$.

[$\because \vec{r}$ makes an acute angle with x-axis $\therefore$ we take $k = 2$]

$\because \ \vec{r} = \hat{r}\cdot|\vec{r}|$

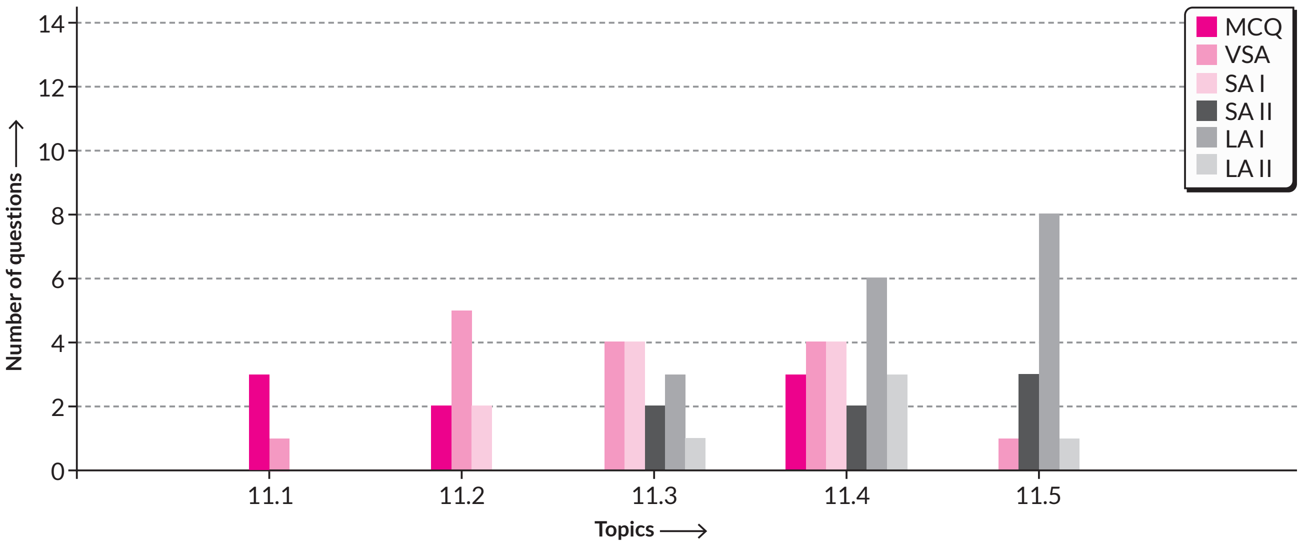
$\therefore \ \vec{r} = (l\hat{i}+m\hat{j}+n\hat{k})|\vec{r}| \ = \left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)\cdot 14 = 4\hat{i}+6\hat{j}-12\hat{k}$

CHAPTER
11

Three Dimensional Geometry

TOPICS

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 11.4 is highly scoring topic.
- Maximum weightage is of Topic 11.4 *Angle between Two Lines.*
- Maximum LA I type questions were asked from Topic 11.5 *Shortest Distance between Two lines.*
- Maximum VSA type questions were asked from Topic 11.2 *Direction Cosines and Directions Ratios of a Line.*

QUICK RECAP

Direction Cosines and Direction Ratios of a Line

- If α, β, γ are the angles which a vector $\overrightarrow{OP}$ makes with the positive directions of the co-ordinate axes OX, OY and OZ respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are known as the direction cosines of $\overrightarrow{OP}$ and are generally denoted by l, m and n respectively *i.e.* $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

  Let l, m, n be the direction cosines of a vector $\vec{r}$ and a, b, c be three numbers such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

  Then a, b, c are known as direction ratios or direction numbers of vector $\vec{r}$.

Note :

(i) $l^2 + m^2 + n^2 = 1$ *i.e.*, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(ii) Direction ratios of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and direction cosines are

$$\pm\frac{x_2-x_1}{|\overrightarrow{PQ}|}, \pm\frac{y_2-y_1}{|\overrightarrow{PQ}|}, \pm\frac{z_2-z_1}{|\overrightarrow{PQ}|}$$

Equation of a Line in Space

- Vector equation of a line passing through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, $\lambda \in R$.
- Cartesian equation of a line passing through a point $A(x_1, y_1, z_1)$ having position vector $\vec{a}$ and in the direction of a vector having a_1, b_1, c_1 as direction ratios, is $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$

Equation of a Line Passing Through Two Given Points

- Let $\vec{a}$ and $\vec{b}$ be position vectors of two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ respectively. Let $\vec{r}$ be the position vector of $P(x, y, z)$. Then, equation of line in
  - **Vector form:** $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \in R$
  - **Cartesian form :** $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

Angle between Two Lines

- **Vector form :** If θ is the angle between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, then

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

- **Cartesian form :** If θ is the angle between the lines

$$l_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and}$$

$$l_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

then $\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

Note : The lines are perpendicular to each other if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Shortest Distance between Two Lines

- **Distance between skew lines**
  - **Vector form :** Let l_1 and l_2 be two lines whose equations are $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ respectively. Let $\overrightarrow{PQ}$ be the shortest distance vector between l_1 and l_2. Then,

$$PQ = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

  - **Cartesian form :** Let two skew lines be

$$l_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and}$$

$$l_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}.$$

The shortest distance between l_1 and l_2 is given by

$$\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-a_1c_2)^2 + (a_1b_2-a_2b_1)^2}}\right|.$$

Note : If the shortest distance between two lines is zero, then the lines are intersecting.

- **Distance between parallel lines**
  - The shortest distance between two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|.$$

BRAIN MAP

THREE DIMENSIONAL GEOMETRY

Equation of Line in Different Forms

Equation of a Line in Space

Vector form : $\vec{r} = \vec{a} + \lambda\vec{b},\ \lambda \in R$

Cartesian form :

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1},$$

where (x_1, y_1, z_1) is passing point and a_1, b_1, c_1 are d.r.s.

Line Passing Through Two Points

Vector form : $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}),\ \lambda \in R$

Cartesian form :

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

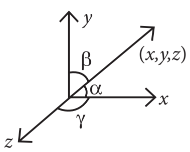
Direction Cosines and Direction Ratios

Direction cosines : If α, β, γ $(0 \le \alpha, \beta, \gamma \le \pi)$ are the angles made by a line with the axes, then $(l, m, n) = (\cos\alpha, \cos\beta, \cos\gamma)$ are the d.c.s. and $l^2 + m^2 + n^2 = 1$

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}$$



D.C.s of axes:

x-axis : (1,0,0)

y-axis : (0,1,0)

z-axis : (0,0,1)

Direction ratios :

The ratio $l : m : n = a : b : c$; i.e., (a,b,c) are called d.r.s.

Shortest Distance Between Two Lines

- **Vector form :** Let l_1 and l_2 be two skew lines whose equations are $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ respectively. Let $\overrightarrow{PQ}$ be the shortest distance vector between l_1 and l_2. Then, $PQ = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$
- **Cartesian form :** Let two skew lines be

  $l_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

  $l_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$.

  The shortest distance between l_1 and l_2 is given by

$$\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-a_1c_2)^2+(a_1b_2-a_2b_1)^2}}\right|.$$

- If lines are parallel, then $d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

Angle Between Two Lines

Vector form : If $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ then $\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

Cartesian form : If $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ then $\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

Perpendicular lines : $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Parallel lines : $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Previous Years' CBSE Board Questions

11.1 Introduction

MCQ

1. Distance of the point (p, q, r) from y-axis is
   (a) q (b) $|q|$
   (c) $|q| + |r|$ (d) $\sqrt{p^2 + r^2}$ *(2023)* U

2. The length of the perpendicular drawn from the point (4, –7, 3) on the y-axis is
   (a) 3 units (b) 4 units
   (c) 5 units (d) 7 units *(2020)* U

3. The vector equation of XY-plane is
   (a) $\vec{r}\cdot\hat{k}=0$ (b) $\vec{r}\cdot\hat{j}=0$
   (c) $\vec{r}\cdot\hat{i}=0$ (d) $\vec{r}\cdot\vec{n}=1$ *(2020)* Ap

VSA (1 mark)

4. Write the distance of a point $P(a, b, c)$ from x-axis. *(2020C, Delhi 2014C)* Ev

11.2 Direction Cosines and Direction Ratios of a Line

MCQ

5. If the direction cosines of a line are $\left(\frac{1}{a},\frac{1}{a},\frac{1}{a}\right)$, then
   (a) $0 < a < 1$ (b) $a > 2$
   (c) $a > 0$ (d) $a = \pm\sqrt{3}$ *(2023)*

6. If a line makes angles of 90°, 135° and 45° with the x, y and z axes respectively, then its direction cosines are
   (a) $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (b) $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$
   (c) $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$ (d) $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ *(2023)*

VSA (1 mark)

7. Find the direction cosines of a line which makes equal angles with the coordinate axes. *(2019)* Ev

8. If a line has the direction ratios – 18, 12, – 4, then what are its direction cosines ? *(2019)* Ev

9. If a line makes angles 90°, 135°, 45° with the x, y and z axes respectively, find its direction cosines. *(NCERT, Delhi 2019)* An

10. If a line makes angles 90° and 60° respectively with the positive directions of x and y axes, find the angle which it makes with the positive direction of z-axis. *(Delhi 2017)* Ev

    OR

    If a line makes angles 90°, 60° and θ with x, y and z-axis respectively, where θ is acute, then find θ. *(Delhi 2015)* Ev

11. If a line makes angles α, β, γ with the positive direction of coordinate axes, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$. *(Delhi 2015C)* U

SA I (2 marks)

12. If a line makes an angle α, β, γ with the coordinate axes, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$. *(Term II, 2021-22)* U

13. Find all the possible vectors of magnitude $5\sqrt{3}$ which are equally inclined to the coordinate axes. *(Term II, 2021-22)* U

11.3 Equation of a Line in Space

VSA (1 mark)

14. The vector equation of a line which passes through the points (3, 4, –7) and (1, –1, 6) is ________. *(2020)* An

15. A line passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction of the vector $\hat{i}+\hat{j}-2\hat{k}$. Find the equation of the line in cartesian form. *(2019)* Ev

16. The equation of a line are $5x – 3 = 15y + 7 = 3 –10z$. Write the direction cosines of the line. *(AI 2015)* Ev

17. If the cartesian equation of a line is $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$, write the vector equation for the line. *(AI 2014)* Ev

SA I (2 marks)

18. The equations of a line are $5x – 3 = 15y + 7 = 3 – 10z$. Write the direction cosines of the line and find the coordinates of a point through which it passes. *(2023)*

19. Write the cartesian equation of the line PQ passing through points $P(2, 2, 1)$ and $Q(5, 1, –2)$. Hence, find the y-coordinate of the point on the line PQ whose z-coordinate is –2. *(Term II, 2021-22)* Ev

20. The Cartesian equation of a line AB is :
    $$\frac{2x-1}{12}=\frac{y+2}{2}=\frac{z-3}{3}$$
    Find the direction cosines of a line parallel to line AB. *(Term II, 2021-22)* Ev

21. The x-coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, –2)$ is 4. Find its z-coordinate. *(AI 2017)* Ap

SA II (3 marks)

22. Find the coordinates of the point where the line through the points (1, 1, 8) and (5, 2, 10) crosses the ZX-plane. *(Term II, 2021-22C)* Ap

23. If a line makes 60° and 45° angles with the positive directions of x-axis and z-axis respectively, then find the angle that it makes with the positive direction of y-axis. Hence, write the direction cosines of the line. *(Term II, 2021-22)* Ev

LA I **(4 marks)**

24. Prove that the line through $A(0, -1, -1)$ and $B(4, 5, 1)$ intersects the line through $C(3, 9, 4)$ and $D(-4, 4, 4)$. *(Foreign 2016)*

25. Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Also find their point of intersection. *(Delhi 2014)* Ev

26. Show that lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ and $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ intersect. Also, find their point of intersection. *(Delhi 2014)* Ev

LA II **(5/6 marks)**

27. A line with direction ratios < 2, 2, 1 > intersects the lines $\frac{x-7}{3}=\frac{y-5}{2}=\frac{z-3}{1}$ and $\frac{x-1}{2}=\frac{y+1}{4}=\frac{z+1}{3}$ at the points P and Q respectively. Find the length and the equation of the intercept PQ. *(2019C)*

11.4 Angle between Two Lines

MCQ

Q. no. 28 is Assertion and Reason based question carrying 1 mark. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (a), (b), (c) and (d) as given below.

28. **Assertion (A) :** The lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are perpendicular, when $\vec{b}_1\cdot\vec{b}_2=0$.
Reason (R) : The angle θ between the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is given by $\cos\theta=\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$.
(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true. *(2023)*

29. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is
(a) 0° (b) 30° (c) 45° (d) 90° *(2023)*

30. If the two lines $L_1: x=5, \frac{y}{3-\alpha}=\frac{z}{-2}$ $L_2: x=2, \frac{y}{-1}=\frac{z}{2-\alpha}$ are perpendicular, then the value of α is
(a) $\frac{2}{3}$ (b) 3 (c) 4 (d) $\frac{7}{3}$ *(2020C)* Ap

VSA **(1 mark)**

31. Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$. *(Delhi 2019)* Ev

32. Find the cartesian equation of the line which passes through the point (–2, 4, –5) and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$. *(Delhi 2019)* Ev

33. Find the angle between the lines
$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and
$\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$. *(Foreign 2014)* Ap

34. Write the equation of the straight line through the point (α, β, γ) and parallel to z-axis. *(AI 2014C)* Ap

SA I **(2 marks)**

35. Find the vector equation of the line passing through the point (2, 1, 3) and perpendicular to both the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$; $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$. *(2023)*

36. Find the vector and the cartesian equations of a line that passes through the point $A(1, 2, -1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$. *(2023)*

37. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y + 1 = -z + 1$ are perpendicular to each other. *(2020)* Ev

38. Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$. *(Delhi 2017)* Ap

SA II **(3 marks)**

39. Find the coordinates of the foot of the perpendicular drawn from point (5, 7, 3) to the line $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. *(2023)*

40. Find the coordinates of the foot of the perpendicular drawn from the point $P(0, 2, 3)$ to the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$. *(2023)*

LA I **(4 marks)**

41. Find the value of λ, so that the lines $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. *(Delhi 2019)* Ev

42. Find the vector and cartesian equations of the line through the point (1, 2, –4) and perpendicular to the two lines
$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$ and
$\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$.
(Delhi 2016, AI 2015) Cr

43. Find the vector and cartesian equations of the line passing through the point (2, 1, 3) and perpendicular

to the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$.

(AI 2014) Ev

44. Find the value of p, so that the lines $l_1: \frac{1-x}{3}=\frac{7y-14}{p}=\frac{z-3}{2}$ and $l_2: \frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$ are perpendicular to each other. Also find the equation of a line passing through a point (3, 2, –4) and parallel to line l_1. *(AI 2014)* Cr

45. A line passes through (2, –1, 3) and is perpendicular to the lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$. Obtain its equation in vector and cartesian form. *(AI 2014)* Ev

46. Find the direction cosines of the line $\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$. Also, find the vector equation of the line through the point $A(-1, 2, 3)$ and parallel to the given line. *(Delhi 2014C)* Ap

LA II (5/6 marks)

47. Show that the following lines do not intersect each other :

$$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}; \frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$$ *(2023)*

48. Find the angle between the lines
$2x = 3y = -z$ and $6x = -y = -4z$. *(2023)*

49. Find the vector and cartesian equations of a line passing through (1, 2, –4) and perpendicular to the two lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. *(Delhi 2017)* Cr

11.5 Shortest Distance between Two Lines

VSA (1 mark)

50. The line of shortest distance between two skew lines is ________ to both the lines. *(2020)* R

SA II (3 marks)

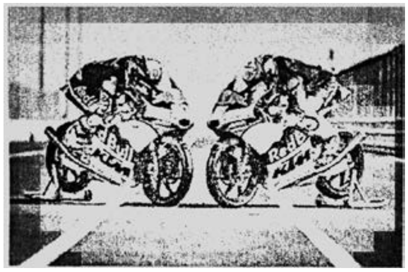
51. Find the distance between the lines $x=\frac{y-1}{2}=\frac{z-2}{3}$ and $x+1=\frac{y+2}{2}=\frac{z-1}{3}$. *(Term II, 2021-22)* Ev

52. Find the distance between the following parallel lines :
$\vec{r}=(2\hat{i}+\hat{j}-\hat{k})+\lambda(\hat{i}+\hat{j}-\hat{k})$
$\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\mu(\hat{i}+\hat{j}-\hat{k})$ *(Term II, 2021-22)* Ev

53. Check whether the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ are skew or not? *(Term II, 2021-22)* Ev

LA I (4 marks)

54. Two motorcycles A and B are running at the speed more than the allowed speed on the roads represented by the lines $\vec{r}=\lambda(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{r}=(3\hat{i}+3\hat{j})+\mu(2\hat{i}+\hat{j}+\hat{k})$ respectively.



Based on the above information, answer the following questions.
(i) Find the shortest distance between the given lines.
(ii) Find the point at which the motorcycles may collide. *(Term II, 2021-22)* Ev

55. Find the shortest distance between the lines
$\vec{r}=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k})$ and
$\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k})$. *(2018)* Ev

56. Find the shortest distance between the lines
$\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and
$\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\mu(2\hat{i}+\hat{j}+2\hat{k})$ *(Delhi 2015C)* Ap

57. Find the shortest distance between the following lines:
$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and
$\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$ *(AI 2015C)* Ev

58. Find the shortest distance between the following lines:
$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}, \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ *(Foreign 2014)* Cr

59. Find the shortest distance between the lines whose vector equations are
$\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}-\hat{j}+\hat{k})$
and $\vec{r}=2\hat{i}+\hat{j}-\hat{k}+\mu(3\hat{i}-5\hat{j}+2\hat{k})$. *(Foreign 2014)* Ev

60. Find the distance between the lines l_1 and l_2 given by
$l_1: \vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$
$l_2: \vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu(4\hat{i}+6\hat{j}+12\hat{k})$. *(Foreign 2014)* Ap

61. Find the shortest distance between the two lines whose vector equations are
$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and
$\vec{r}=(4\hat{i}+5\hat{j}+6\hat{k})+\mu(2\hat{i}+3\hat{j}+\hat{k})$. *(Delhi 2014C)* Ev

LA II (5 / 6 marks)

62. Find the vector equation of a line passing through the point (2, 3, 2) and parallel to the line $\vec{r}=(-2\hat{i}+3\hat{j})+\lambda(2\hat{i}-3\hat{j}+6\hat{k})$. Also, find the distance between these two lines. *(AI 2019)* Cr

CBSE Sample Questions

11.3 Equation of a Line in Space

MCQ

1. P is a point on the line joining the points $A(0, 5, -2)$ and $B(3, -1, 2)$. If the x-coordinate of P is 6, then its z-coordinate is
(a) 10 (b) 6 (c) −6 (d) −10
(2022-23) Ap

SA I (2 marks)

2. Find the direction ratio and direction cosines of a line parallel to the line whose equations are
$6x - 12 = 3y + 9 = 2z - 2.$ *(2022-23)* Ev

3. Find the direction cosines of the following line:
$\frac{3-x}{-1}=\frac{2y-1}{2}=\frac{z}{4}$ *(Term II, 2021-22)* Ap

11.4 Angle between Two Lines

MCQ

4. **Assertion (A) :** The acute angle between the line $\vec{r}=\hat{i}+\hat{j}+2\hat{k}+\lambda(\hat{i}-\hat{j})$ and the x-axis is $\frac{\pi}{4}$.
Reason (R) : The acute angle θ between the lines $\vec{r}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}+\lambda(a_1\hat{i}+b_1\hat{j}+c_1\hat{k})$ and $\vec{r}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}+\mu(a_2\hat{i}+b_2\hat{j}+c_2\hat{k})$ is given by
$$\cos\theta=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$
(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true. *(2022-23)* Ap

11.5 Shortest Distance between Two Lines

SA II (3 marks)

5. Find the shortest distance between the following lines:
$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+s(2\hat{i}+\hat{j}+\hat{k}),\ \vec{r}=(\hat{i}+\hat{j}+2\hat{k})+t(4\hat{i}+2\hat{j}+2\hat{k})$
(Term II, 2021-22) Ev

LA II (5/6 marks)

6. An insect is crawling along the line $\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda(\hat{i}-2\hat{j}+2\hat{k})$ and another insect is crawling along the line $\vec{r}=-4\hat{i}-\hat{k}+\mu(3\hat{i}-2\hat{j}-2\hat{k})$. At what points on the lines should they reach so that the distance between them is shortest? Find the shortest possible distance between them.
(2022-23)

7. The equation of motion of a rocket are : $x = 2t$, $y = -4t$, $z = 4t$, where the time t is given in seconds, and the coordinates of a moving point in km. What is the path of the rocket? At what distances will the rocket be from the starting point $O(0, 0, 0)$ and from the following line in 10 seconds?
$\vec{r}=20\hat{i}-10\hat{j}+40\hat{k}+\mu(10\hat{i}-20\hat{j}+10\hat{k})$ *(2022-23)* U

8. Find the shortest distance between the lines
$\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$
and $\vec{r}=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$.
If the lines intersect, then find their point of intersection. *(2020-21)* Ap

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. **(d):** Given point is (p, q, r)
The foot of perpendicular drawn from point (p, q, r) on the y-axis is $(0, q, 0)$.
Now, distance between these two points is
$\sqrt{(p-0)^2+(q-q)^2+(r-0)^2}=\sqrt{p^2+r^2}$

2. **(c) :** Let $P(4, -7, 3)$ be the given point and A be a point on y-axis s.t. $PA \perp$ to y-axis.
$\therefore\ A \equiv (0, -7, 0)$
Now, $PA = \sqrt{(4-0)^2+(-7-(-7))^2+(3-0)^2}$
$= \sqrt{4^2+3^2}=\sqrt{16+9}=\sqrt{25}=5$ units

Answer Tips

➲ Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

3. **(a) :** Vector equation of XY-plane is $\vec{r}\cdot\hat{k}=0$.

4. We have equation of x-axis is $y = 0, z = 0$
$\therefore$ Distance of $P(a, b, c)$ from x-axis
$=\sqrt{(a-a)^2+b^2+c^2}=\sqrt{b^2+c^2}$ units.

5. **(d):** Given that the direction cosines of a line are $\left(\frac{1}{a},\frac{1}{a},\frac{1}{a}\right)$

We know that the sum of squares of the direction cosines is 1.

$\Rightarrow \frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2}=1 \Rightarrow \frac{3}{a^2}=1 \Rightarrow a^2 = 3$

$\Rightarrow a=\pm\sqrt{3}$

6. **(a):** Direction cosines are $\langle \cos 90°, \cos 135°, \cos 45° \rangle$

$=\left\langle 0, \cos(90°+45°), \frac{1}{\sqrt{2}} \right\rangle = \left\langle 0, -\sin 45°, \frac{1}{\sqrt{2}} \right\rangle$

$=\left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

7. If a line makes α, β, γ with positive direction of x, y, z axis respectively, then direction cosines of line will be $\cos\alpha, \cos\beta, \cos\gamma$ or $-\cos\alpha, -\cos\beta, -\cos\gamma$.
and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
Since, $\alpha = \beta = \gamma$

$\therefore \quad \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Therefore, direction cosines are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

8. Since, D.R'.s are –18, 12, –4

$\therefore$ D.C'.s are $\frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$

$\Rightarrow \frac{-18}{22}, \frac{12}{22}, \frac{-4}{22} \Rightarrow \frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

9. Since the line makes angles 90°, 135° and 45° with x, y and z axes respectively.

$\therefore \quad l = \cos 90° = 0,\ m = \cos 135° = -\frac{1}{\sqrt{2}}$ and $n = \cos 45° = \frac{1}{\sqrt{2}}$

Hence, direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Concept Applied

- If a line makes angles α, β and γ with x, y and z-axes respectively, then $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ are direction cosines of line.

10. Let the line makes an angle α, β, γ with the positive direction of x, y, z axes respectively.

$\therefore \quad \alpha = 90°, \beta = 60°$ and $\gamma = \theta$ (say)

Since, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad \cos^2 90° + \cos^2 60° + \cos^2\theta = 1$

$\Rightarrow \frac{1}{4}+\cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$

11. Here, the direction cosines of the given line are $\cos\alpha, \cos\beta, \cos\gamma$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$[\because \sin^2\alpha + \cos^2\alpha = 1]$

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$

12. Here, the direction cosines of the given line are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

By using $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

$\cos^2\beta = \frac{1+\cos 2\beta}{2}$ and so on.

$\Rightarrow \quad \frac{1}{2}[\cos 2\alpha + 1 + \cos 2\beta + 1 + \cos 2\gamma + 1] = 1$

$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

13. $\pm 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)$ are two possible vectors of magnitude $5\sqrt{3}$, which are equally inclined to the coordinate axes.

14. Vector equation of a line passes through the points (3, 4, –7) and (1, –1, 6) is given by

$\vec{r} = (3\hat{i}+4\hat{j}-7\hat{k}) + \lambda[(\hat{i}-\hat{j}+6\hat{k})-(3\hat{i}+4\hat{j}-7\hat{k})]$

$\therefore \quad \vec{r} = 3\hat{i}+4\hat{j}-7\hat{k}+\lambda(-2\hat{i}-5\hat{j}+13\hat{k})$

15. The equation of line in vector from $\vec{r} = \vec{a}+\lambda\vec{b}$.

Here, $\vec{a} = 2\hat{i}-\hat{j}+4\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}-2\hat{k}$

$\therefore \quad (a_1, a_2, a_3) \equiv (2, -1, 4)$

D.R'.s. b_1, b_2, b_3 are 1, 1, –2

The equation of line in cartesian form is given by

$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} \Rightarrow \frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$

16. The given line is $5x - 3 = 15y + 7 = 3 - 10z$

$\Rightarrow \frac{x-\frac{3}{5}}{\frac{1}{5}} = \frac{y+\frac{7}{15}}{\frac{1}{15}} = \frac{z-\frac{3}{10}}{-\frac{1}{10}}$

Its direction ratios are $\frac{1}{5}, \frac{1}{15}, -\frac{1}{10}$

i.e., Its direction ratios are proportional to 6, 2, –3.

Now, $\sqrt{6^2+2^2+(-3)^2} = 7$

$\therefore \quad$ Its direction cosines are $\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}$.

17. The cartesian equation of a line is

$\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$ ...(i)

$\Rightarrow \frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2} = \lambda$ (say)

$\Rightarrow \quad x = 3 - 5\lambda, y = -4 + 7\lambda, z = 3 + 2\lambda$

Take $\vec{a} = 3\hat{i}-4\hat{j}+3\hat{k}$ and $\vec{b} = -5\hat{i}+7\hat{j}+2\hat{k}$.

$\therefore$ The vector equation of the line (i) is $\vec{r} = \vec{a}+\lambda\vec{b}$

$\Rightarrow \vec{r} = (3\hat{i}-4\hat{j}+3\hat{k})+\lambda(-5\hat{i}+7\hat{j}+2\hat{k})$

Answer Tips

- If cartesian equation of a line is $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$, then its vector equation is $\vec{r} = (x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$

18. The given line is $5x - 3 = 15y + 7 = 3 - 10z$

$$\Rightarrow \frac{x-\frac{3}{5}}{\frac{1}{5}}=\frac{y+\frac{7}{15}}{\frac{1}{15}}=\frac{z-\frac{3}{10}}{-\frac{1}{10}}$$

Its direction ratios are $\frac{1}{5}, \frac{1}{15}, -\frac{1}{10}$

i.e., Its direction ratios are proportional to 6, 2, −3.

Now, $\sqrt{6^2+2^2+(-3)^2}=7$

$\therefore$ Its direction cosines are $\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}$.

19. We have $P(2, 2, 1)$ and $Q(5, 1, -2)$, then the equation of line PQ is

$$\frac{x-2}{5-2}=\frac{y-2}{1-2}=\frac{z-1}{-2-1} \text{ or } \frac{x-2}{3}=\frac{y-2}{-1}=\frac{z-1}{-3} \quad \text{...(i)}$$

Given, $z = -2$, then from (i), we have $\frac{y-2}{-1}=\frac{-2-1}{-3}$

$\Rightarrow$ $y = 1$

20. The cartesian equation of line AB is

$$\frac{2x-1}{12}=\frac{y+2}{2}=\frac{z-3}{3}$$

can be rearranged as $\frac{x-\frac{1}{2}}{6}=\frac{y+2}{2}=\frac{z-3}{3}$

So, $a = 6, b = 2, c = 3$

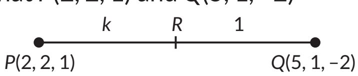
$\Rightarrow$ $\sqrt{a^2+b^2+c^2}=\sqrt{6^2+2^2+3^2}=7$

$\therefore$ Required direction cosines are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$.

Answer Tips

- Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

21. Given that $P(2, 2, 1)$ and $Q(5, 1, -2)$



Let the point R on the line PQ, divides the line in the ratio $k : 1$. And x-coordinate of point R on the line is 4.

So, by section formula $4=\frac{5k+2}{k+1} \Rightarrow k=2$

Now, z-coordinate of point R, $z=\frac{-2k+1}{k+1}=\frac{-2\times 2+1}{2+1}=-1$

$\Rightarrow$ z-coordinate of point $R = -1$

22. We have the points $P(1, 1, 8)$ and $Q(5, 2, 10)$, then the equations of line is $\frac{x+1}{5+1}=\frac{y-1}{-2-1}=\frac{z-(-8)}{10-(-8)}$

or $\frac{x+1}{6}=\frac{y-1}{-3}=\frac{z+8}{18}$ ...(i)

If the line crosses the zx-plane, then $y = 0$, so from (1)

$\frac{x+1}{6}=\frac{1}{3}$ and $\frac{z+8}{18}=\frac{1}{3}$

$x = 1$ and $z = -2$

$\therefore$ The coordinates of the required point is $(1, 0, -2)$.

23. Since, α and β be the angle made by x-axis and z-axis and $\alpha = 60°$ and $\beta = 45°$ (given)

Let θ be the angle made by the line with y-axis.

Then, $\cos^2 60° + \cos^2 45° + \cos^2\theta = 1$

[$\because$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$]

$\Rightarrow$ $\frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$

$\Rightarrow$ $\cos^2\theta=1-\frac{1}{2}-\frac{1}{4}=\frac{1}{4} \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta = 60°$

$\therefore$ Direction cosines of the line are

$\langle\cos 60°, \cos 45°, \cos 60°\rangle$ *i.e.*, $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$.

24. The equation of line AB is given by

$$\frac{x-0}{4-0}=\frac{y+1}{5+1}=\frac{z+1}{1+1}=\lambda \text{ (say)}$$

$\Rightarrow$ $x = 4\lambda, y = 6\lambda - 1, z = 2\lambda - 1$

The coordinates of a general point on AB are $(4\lambda, 6\lambda - 1, 2\lambda - 1)$

The equation of line CD is given by

$$\frac{x-3}{3+4}=\frac{y-9}{9-4}=\frac{z-4}{4-4}=\mu \text{ (say)}$$

$\Rightarrow$ $x = 7\mu + 3,\ y = 5\mu + 9, z = 4$

The coordinates of a general point on CD are $(7\mu + 3, 5\mu + 9, 4)$

If the line AB and CD intersect then they have a common point. So, for some values of λ and μ, we must have

$4\lambda = 7\mu + 3, 6\lambda - 1 = 5\mu + 9, 2\lambda - 1 = 4$

$\Rightarrow$ $4\lambda - 7\mu = 3$ ...(i)

$6\lambda - 5\mu = 10$ ...(ii)

and $\lambda=\frac{5}{2}$ ...(iii)

Substituting $\lambda=\frac{5}{2}$ in (ii), we get $\mu = 1$

Since $\lambda=\frac{5}{2}$ and $\mu = 1$ satisfy (i), so the given lines AB and CD intersect.

Key Points

- Equation of a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by
  $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

25. Any point on the line

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=r \text{ (say)} \quad \text{...(i)}$$

is $(3r - 1, 5r - 3, 7r - 5)$.

Any point on the line

$$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=k \text{ (say)} \quad \text{...(ii)}$$

is $(k + 2, 3k + 4, 5k + 6)$

For lines (i) and (ii) to intersect, we must have

$3r - 1 = k + 2, 5r - 3 = 3k + 4, 7r - 5 = 5k + 6$

On solving these, we get $r=\frac{1}{2}, k=-\frac{3}{2}$

$\therefore$ Lines (i) and (ii) intersect and their point of intersection is $\left(\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\right)$

26. The given lines are

$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})=(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k}$ ...(i)

and $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})=(2\mu+4)\hat{i}+0\cdot\hat{j}+(3\mu-1)\hat{k}$ ...(ii)

If the lines (i) & (ii) intersect, then they have a common point. So, we must have

$(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k}=(2\mu+4)\hat{i}+0\cdot\hat{j}+(3\mu-1)\hat{k}$

$\Rightarrow$ $3\lambda+1=2\mu+4$, $1-\lambda=0$ and $-1=3\mu-1$

On solving last two equations, we get $\lambda = 1$ and $\mu = 0$. These values of λ and μ satisfy the first equation.

So, the given lines intersect.

Putting $\lambda = 1$ in (i), we get the position vector of the point of intersection.

Thus, the coordinates of the point of intersection are $(4, 0, -1)$.

27. Let $\frac{x-7}{3}=\frac{y-5}{2}=\frac{z-3}{1}=\alpha$ (say)

$\Rightarrow$ Any point $P(3\alpha+7, 2\alpha+5, \alpha+3)$ lie on this line.

Let $\frac{x-1}{2}=\frac{y+1}{4}=\frac{z+1}{3}=\beta$ (say)

$\Rightarrow$ Any point $Q(2\beta+1, 4\beta-1, 3\beta-1)$ lie on this line.

D.R's. of line PQ are 2, 2, 1, then

$$\frac{2\beta+1-(3\alpha+7)}{2}=\frac{4\beta-1-(2\alpha+5)}{2}=\frac{3\beta-1-(\alpha+3)}{1}$$

$\Rightarrow$ $\alpha=\frac{-2}{3}$ and $\beta=\frac{1}{3}$

$\Rightarrow$ Point are $P\left(5,\frac{11}{3},\frac{5}{3}\right)$ and $Q\left(\frac{5}{3},\frac{1}{3},0\right)$

Length of $PQ=\sqrt{\left(\frac{5}{3}-5\right)^2+\left(\frac{1}{3}-\frac{11}{3}\right)^2+\left(0-\frac{5}{3}\right)^2}$

$=\sqrt{\frac{225}{9}}=\frac{15}{3}=5$ units

The equation of intercept PQ is

$$\frac{x-5}{\frac{5}{3}-5}=\frac{y-\frac{11}{3}}{\frac{1}{3}-\frac{11}{3}}=\frac{z-\frac{5}{3}}{0-\frac{5}{3}}$$

$$\Rightarrow \frac{x-5}{-\frac{10}{3}}=\frac{y-\frac{11}{3}}{-\frac{10}{3}}=\frac{z-\frac{5}{3}}{-\frac{5}{3}}$$

28. **(a):** If lines are perpendicular, then $\theta=\frac{\pi}{2}$

$\therefore$ $\cos\frac{\pi}{2}=\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \Rightarrow \cos\frac{\pi}{2}=\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$

$\Rightarrow$ $\vec{b}_1\cdot\vec{b}_2=0$

$\therefore$ Both A and R are true and R is the correct explanation of A.

29. **(d):** The given equation of lines can be rewritten as

$\frac{x-0}{1/2}=\frac{y-0}{1/3}=\frac{z-0}{-1}$ and $\frac{x-0}{1/6}=\frac{y-0}{-1}=\frac{z-0}{-1/4}$

$\therefore$ $a_1=\frac{1}{2}, b_1=\frac{1}{3}, c_1=-1$

and $a_2=\frac{1}{6}, b_2=-1, c_2=\frac{-1}{4}$

Now, $\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$$=\frac{\frac{1}{2}\cdot\frac{1}{6}+\frac{1}{3}\cdot(-1)+(-1)\cdot\left(\frac{-1}{4}\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+(-1)^2}\sqrt{\left(\frac{1}{6}\right)^2+(-1)^2+\left(\frac{-1}{4}\right)^2}}=0$$

$\Rightarrow$ $\cos\theta=0 \Rightarrow \theta=90°$

30. **(d):** The given lines are perpendicular, if

$a_1a_2+b_1b_2+c_1c_2=0$ ...(i)

Here, $L_1:\frac{x-5}{0}=\frac{y-0}{3-\alpha}=\frac{z-0}{-2}$

$L_2:\frac{x-2}{0}=\frac{y-0}{-1}=\frac{z-0}{2-\alpha}$

Here, a_1, b_1, c_1 are $0, 3-\alpha, -2$, and a_2, b_2, c_2 are $0, -1, 2-\alpha$ respectively.

$\therefore$ $0\times 0-(3-\alpha)-2(2-\alpha)=0$

$\Rightarrow$ $\alpha=\frac{7}{3}$ [from (i)]

31. We know that vector equation of a line passing through point $\vec{a}$ and parallel to vector $\vec{b}$ is given by $\vec{r}=\vec{a}+\lambda\vec{b}$

Here $\vec{a}=3\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{b}=2\hat{i}+2\hat{j}-3\hat{k}$

$\therefore$ Required equation is

$\vec{r}=(3\hat{i}+4\hat{j}+5\hat{k})+\lambda(2\hat{i}+2\hat{j}-3\hat{k})$

32. Equation of the line can be written as

$\frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$

Direction ratios of this line are $3, -5, 6$.

The required line passes through $(-2, 4, -5)$ and its direction ratios are proportional to $3, -5, 6$. So, its equation is $\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$.

33. For $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$, we have

$\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$

Also for

$\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$, we have $\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$

Let θ be the angle between the lines.

So, $\theta=\cos^{-1}\left|\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

$\Rightarrow \theta=\cos^{-1}\left|\frac{(3\hat{i}+2\hat{j}+6\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{3^2+2^2+6^2}\sqrt{1^2+2^2+2^2}}\right|$

$\Rightarrow \theta=\cos^{-1}\left|\frac{3+4+12}{7\times 3}\right| \Rightarrow \theta=\cos^{-1}\left(\frac{19}{21}\right)$

34. Any line parallel to z-axis has direction ratios proportional to 0, 0, 1.

$\therefore$ The equation of a line through (α, β, γ) and parallel to z-axis is $\frac{x-\alpha}{0}=\frac{y-\beta}{0}=\frac{z-\gamma}{1}$

35. Let the equation of line passing through (2, 1, 3) and perpendicular to the lines

$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$ be

$\frac{x-2}{l}=\frac{y-1}{m}=\frac{z-3}{n}$ ...(i)

$\therefore$ $l\cdot 1+m\cdot 2+n\cdot 3=0$ and $l\cdot(-3)+m\cdot 2+n\cdot 5=0$

$\Rightarrow \frac{l}{10-6}=\frac{m}{-9-5}=\frac{n}{2+6} \Rightarrow \frac{l}{2}=\frac{m}{-7}=\frac{n}{4}.$

$\therefore$ The equation of the required line is

$\frac{x-2}{2}=\frac{y-1}{-7}=\frac{z-3}{4}.$

Also its vector equation is

$\vec{r}=(2\hat{i}+\hat{j}+3\hat{k})+\lambda(2\hat{i}-7\hat{j}+4\hat{k}).$

36. Let the equation of line passing through $A(1, 2, -1)$ be

$\frac{x-1}{a}=\frac{y-2}{b}=\frac{z+1}{c}$ ...(i)

Now, given equation of line is,

$5x - 25 = 14 - 7y = 35z$

$\Rightarrow \frac{x-5}{1/5}=\frac{y-2}{-1/7}=\frac{z-0}{1/35}$

$\Rightarrow \frac{x-5}{7}=\frac{y-2}{-5}=\frac{z-0}{1}$ ...(ii)

Since (i) and (ii) are parallel lines.

$\therefore \frac{a}{7}=\frac{b}{-5}=\frac{c}{1}$

From (i), we get

$\frac{x-1}{7}=\frac{y-2}{-5}=\frac{z+1}{1}$ is the cartesian equation of line.

Also, the vector equation of line is $(\hat{i}+2\hat{j}-\hat{k})+\lambda(7\hat{i}-5\hat{j}+\hat{k})$.

37. The given lines are perpendicular, if

$a_1a_2+b_1b_2+c_1c_2=0$

The given lines are as

$l_1: \frac{x-0}{1}=\frac{y-0}{-1}=\frac{z-0}{\frac{1}{k}}$; $l_2: \frac{x-2}{1}=\frac{y+\frac{1}{2}}{\frac{1}{2}}=\frac{z-1}{-1}$

$\because$ l_1 is perpendicular to l_2

here, a, b, c, are 1, –1, 1/k and a_2, b_2, c_2 are 1, 1/2, –1 respectively

$\therefore \quad 1(1)+(-1)\left(\frac{1}{2}\right)+\left(\frac{1}{k}\right)(-1)=0$

$\Rightarrow \quad 1-\frac{1}{2}-\frac{1}{k}=0 \Rightarrow \frac{1}{2}=\frac{1}{k} \Rightarrow k=2$

Commonly Made Mistake

- Remember two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are perpendicular to each other if $a_1a_2+b_1b_2+c_1c_2=0$ and parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

38. Vector equation of the line passing through (1, 2, –1) and parallel to the line

$5x - 25 = 14 - 7y = 35z$

i.e., $\frac{x-5}{1/5}=\frac{y-2}{-1/7}=\frac{z}{1/35}$ or $\frac{x-5}{7}=\frac{y-2}{-5}=\frac{z-0}{1}$

is $\vec{r}=(\hat{i}+2\hat{j}-\hat{k})+\lambda(7\hat{i}-5\hat{j}+\hat{k})$

39. We have point $P(5, 7, 3)$ and equation of line as

$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}=k$ (say)

Any point on this line is given by

$Q(3k + 15, 8k + 29, -5k + 5)$

Direction ratio of line PQ are

$\langle 3k + 15 - 5, 8k + 29 - 7, -5k + 5 - 3\rangle$

i.e., $\langle 3k + 10, 8k + 22, -5k + 2\rangle$

As, PQ is perpendicular to given line

$\therefore$ $3(3k + 10) + 8(8k + 22) - 5(-5k + 2) = 0$

$\Rightarrow$ $98k + 196 = 0 \Rightarrow k = -2$

$\therefore$ Foot of perpendicular drawn from given point $P(5, 7, 3)$ on the given line is

$(-6 + 15, -16 + 29, 10 + 5)$ *i.e.*, (9, 13, 15)

40. Let M be the foot of the perpendicular drawn from point $P(0, 2, 3)$ to the line

$\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}=\lambda$ (say) ... (i)

$\therefore$ Any point on line (i) is $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$

So, coordinates of $M \equiv (5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$ ... (ii)

Now, direction ratios of PM are $\langle 5\lambda - 3 - 0, 2\lambda + 1 - 2, 3\lambda - 4 - 3\rangle$

i.e., $\langle 5\lambda - 3, 2\lambda - 1, 3\lambda - 7\rangle$

Since, PM is perpendicular to line (i).

$\therefore$ $5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$

$\Rightarrow$ $25\lambda - 15 + 4\lambda - 2 + 9\lambda - 21 = 0$

$\Rightarrow$ $38\lambda - 38 = 0 \Rightarrow \lambda = 1$

So, coordinates of M are (2, 3, –1).

41. The given lines are

$l_1: \frac{x-1}{-3}=\frac{y-2}{\lambda/7}=\frac{z-3}{2}$ and $l_2: \frac{x-1}{-3\lambda/7}=\frac{y-5}{1}=\frac{z-6}{-5}$

Now, $l_1 \perp l_2$ [Given]

$\therefore (-3)\left(-\frac{3\lambda}{7}\right)+\frac{\lambda}{7}-10=0$

$\Rightarrow \frac{9\lambda}{7}+\frac{\lambda}{7}-10=0 \Rightarrow \frac{10\lambda}{7}=10 \Rightarrow \lambda=7$

Since for $\lambda = 7$, given lines are at right angle.

$\therefore$ Lines are intersecting.

42. The given lines are

$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

Equation of any line through (1, 2, –4) with d.r's l, m, n is

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + p(l\hat{i} + m\hat{j} + n\hat{k})$ ...(i)

Since, the required line is perpendicular to both the given lines.

$\therefore$ $3l - 16m + 7n = 0$ and $3l + 8m - 5n = 0$

$\Rightarrow \dfrac{l}{80-56} = \dfrac{m}{21+15} = \dfrac{n}{24+48} \Rightarrow \dfrac{l}{2} = \dfrac{m}{3} = \dfrac{n}{6}$

$\therefore$ From (i), the required line is

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + p(2\hat{i} + 3\hat{j} + 6\hat{k})$

Here, the position vector of passing point is $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$ and parallel vector is $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

$\therefore$ Cartesian equation of line is given by

$\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z+4}{6}$

43. Let the equation of line passing through (2, 1, 3) and perpendicular to the lines

$\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$ and $\dfrac{x}{-3} = \dfrac{y}{2} = \dfrac{z}{5}$ be

$\dfrac{x-2}{l} = \dfrac{y-1}{m} = \dfrac{z-3}{n}$ ...(i)

$\therefore$ $l \cdot 1 + m \cdot 2 + n \cdot 3 = 0$ and $l \cdot (-3) + m \cdot 2 + n \cdot 5 = 0$

$\Rightarrow \dfrac{l}{10-6} = \dfrac{m}{-9-5} = \dfrac{n}{2+6} \Rightarrow \dfrac{l}{2} = \dfrac{m}{-7} = \dfrac{n}{4}.$

$\therefore$ The equation of the required line is

$\dfrac{x-2}{2} = \dfrac{y-1}{-7} = \dfrac{z-3}{4}.$

Also its vector equation is

$\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k}).$

44. The given lines are

$l_1: \dfrac{x-1}{-3} = \dfrac{y-2}{p/7} = \dfrac{z-3}{2}$

$l_2: \dfrac{x-1}{-3p/7} = \dfrac{y-5}{1} = \dfrac{z-6}{-5}$

$\because$ l_1 is perpendicular to l_2.

$\therefore$ $(-3)\left(\dfrac{-3p}{7}\right) + \dfrac{p}{7} \cdot 1 + 2(-5) = 0$

$\Rightarrow \dfrac{9p}{7} + \dfrac{p}{7} = 10 \Rightarrow \dfrac{10p}{7} = 10 \Rightarrow p = 7$

Now, equation of the line passing through (3, 2, –4) and parallel to l_1 is

$\dfrac{x-3}{-3} = \dfrac{y-2}{1} = \dfrac{z+4}{2}.$

45. The given lines are

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

Equation of any line through (2, –1, 3) with d.r's l, m, n is

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + p(l\hat{i} + m\hat{j} + n\hat{k})$ ...(i)

Since, the required line is perpendicular to both the given lines.

$\therefore$ $2l - 2m + n = 0$ and $l + 2m + 2n = 0$

$\Rightarrow \dfrac{l}{-4-2} = \dfrac{m}{1-4} = \dfrac{n}{4+2} \Rightarrow \dfrac{l}{2} = \dfrac{m}{1} = \dfrac{n}{-2}$

$\therefore$ From (i), the required line is

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + p(2\hat{i} + \hat{j} - 2\hat{k}).$

46. The given line is $\dfrac{x+2}{2} = \dfrac{2y-7}{6} = \dfrac{5-z}{6}$

$\Rightarrow \dfrac{x+2}{2} = \dfrac{y-\frac{7}{2}}{3} = \dfrac{z-5}{-6}$ ...(i)

Its d.r's are 2, 3, –6

$\because$ $\sqrt{2^2 + 3^2 + (-6)^2} = 7$

$\therefore$ Its d.c's are $\dfrac{2}{7}, \dfrac{3}{7}, -\dfrac{6}{7}$

Equation of a line through (–1, 2, 3) and parallel to (i) is

$\dfrac{x+1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{-6} = \lambda$ (say)

$\therefore$ Vector equation of a line passing through (–1, 2, 3) and parallel to (i) is given by

$\vec{r} = (-\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 6\hat{k})$

Answer Tips

- If a, b, c are d.r.'s of a line, then d.c.'s of the line are $\dfrac{a}{\sqrt{a^2+b^2+c^2}}, \dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}$.

47. The given lines are

$\dfrac{x-1}{3} = \dfrac{y+1}{2} = \dfrac{z-1}{5} = \lambda$ ...(i) $\dfrac{x+2}{4} = \dfrac{y-1}{3} = \dfrac{z+1}{-2} = \mu$ ...(ii)

Let P be the general point on line (i), then

$x = 3\lambda + 1,\ y = 2\lambda - 1$ and $z = 5\lambda + 1$

$\therefore$ $P \equiv (3\lambda + 1, 2\lambda - 1, 5\lambda + 1)$

Let Q be the general point on line (ii), then

$x = 4\mu - 2,\ y = 3\mu + 1$ and $z = -2\mu - 1$

$\therefore$ $Q \equiv (4\mu - 2, 3\mu + 1, -2\mu - 1)$

Let the given lines intersects.

So P and Q coincide for some particular values of λ and μ.

$\therefore$ $3\lambda + 1 = 4\mu - 2 \Rightarrow 3\lambda - 4\mu = -3$ ...(iii)

$2\lambda - 1 = 3\mu + 1 \Rightarrow 2\lambda - 3\mu = 2$ ...(iv)

and $5\lambda + 1 = -2\mu - 1 \Rightarrow 5\lambda + 2\mu = -2$ ...(v)

Solving equation (iii) and (iv), we get

$\lambda = -17$ and $\mu = -12$

But $\lambda = -17$ and $\mu = -12$ do not satisfy the equation (v).

It means our assumption is wrong hence the given lines do not intersect.

48. Given line, $2x = 3y = -z$ can be written as

$\dfrac{x}{1/2} = \dfrac{y}{1/3} = \dfrac{z}{-1}$ ...(i)

The direction ratios of line (i) are $<\frac{1}{2},\frac{1}{3},-1>$

and the line, $6x = -y = -4z$ can be written as

$$\frac{x}{1/6}=\frac{y}{-1}=\frac{z}{-1/4} \quad \text{...(ii)}$$

The direction ratios of line (ii) are $<\frac{1}{6},-1,-\frac{1}{4}>$

It is known that if two lines are perpendicular then the dot product of the direction ratios of the two lines is equal to 0.

Product of direction ratios $=\frac{1}{2}\times\frac{1}{6}+\frac{1}{3}\times(-1)+(-1)\times\left(-\frac{1}{4}\right)$

$=\frac{1}{12}-\frac{1}{3}+\frac{1}{4}=0$

So, angle between the lines is 90°.

49. Let the equation of line passing through (1, 2, –4) and perpendicular to the lines

$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

be $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z+4}{n}$ ...(i)

$\therefore$ $l(3)+m(-16)+n(7)=0$ and $l(3)+m(8)+n(-5)=0$

$\Rightarrow \frac{l}{80-56}=\frac{m}{21+15}=\frac{n}{24+48}$

$\Rightarrow \frac{l}{24}=\frac{m}{36}=\frac{n}{72} \Rightarrow \frac{l}{2}=\frac{m}{3}=\frac{n}{6}$

$\therefore$ The equation of the required line is

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$

and its vector equation is

$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$

50. The line of shortest distance between two skew lines is perpendicular to both the lines.

51. For the given lines, $\frac{l_1}{l_2}=\frac{1}{1}=1$; $\frac{m_1}{m_2}=\frac{2}{2}=1$; $\frac{n_1}{n_2}=\frac{3}{3}=1$

Since, $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$, therefore the given lines are parallel.

Let $x=\frac{y-1}{2}=\frac{z-2}{3}=\lambda$ (say)

Any point on this line is $P(\lambda, 2\lambda+1, 3\lambda+2)$

If it is the foot of the perpendicular on the line, then

$1(\lambda)+2(2\lambda+1)+3(3\lambda+2)=0$

$\Rightarrow \lambda=-\frac{4}{7}$

$\therefore$ $P(\lambda,2\lambda+1,3\lambda+2)\equiv P\left(-\frac{4}{7},\frac{3}{7},\frac{2}{7}\right)$

Similarly $x+1=\frac{y+2}{2}=\frac{z-1}{3}=\mu$ (say)

Any point on this line is $Q(\mu-1, 2\mu-2, 3\mu+1)$.

$\Rightarrow$ $1(\mu-1)+2(2\mu-2)+3(3\mu+1)=0$

$\Rightarrow \mu=\frac{1}{7}$

$\therefore$ $Q(\mu-1, 2\mu-2, 3\mu+1)\equiv\left(-\frac{6}{7},-\frac{12}{7},\frac{10}{7}\right)$

$$PQ=\sqrt{\left(\frac{-6}{7}+\frac{4}{7}\right)^2+\left(\frac{-12}{7}-\frac{3}{7}\right)^2+\left(\frac{10}{7}-\frac{2}{7}\right)^2}=\frac{\sqrt{293}}{7}\text{ units}$$

52. Comparing the gives lines with $\vec{r}=\vec{a}+s\vec{b}, \vec{r}=\vec{c}+t\vec{b}$

$\vec{a}_1=2\hat{i}+\hat{j}-\hat{k}$; $\vec{b}=\hat{i}+\hat{j}-\hat{k}$; $\vec{a}_2=\hat{i}-2\hat{j}+\hat{k}$

Distance between two given lines $=\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

$$=\left|\frac{(\hat{i}+\hat{j}-\hat{k})\times(-\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1+1+1}}\right|=\frac{1}{\sqrt{3}}\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&1&-1\\-1&-3&2\end{vmatrix}$$

$$=\frac{1}{\sqrt{3}}\left|\hat{i}(2-3)+\hat{j}(-2+1)+\hat{k}(-3+1)\right|=\frac{1}{\sqrt{3}}|-\hat{i}-\hat{j}-2\hat{k}|$$

$$=\frac{\sqrt{1+1+4}}{\sqrt{3}}=\sqrt{2}\text{ units}$$

53. For the given lines,

$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$

$$\Delta=\left|\frac{\begin{vmatrix}4-1&1-2&0-3\\2&3&4\\5&2&1\end{vmatrix}}{\sqrt{470}}\right|=\left|\frac{\begin{vmatrix}3&-1&-3\\2&3&4\\5&2&1\end{vmatrix}}{\sqrt{470}}\right|$$

$\left[\because \sqrt{(-5)^2+(18)^2+(11)^2}=\sqrt{470}\right]$

$$\Delta=\left|\frac{3(3-8)+1(2-20))-3(4-15)}{\sqrt{470}}\right|$$

$$\Delta=\left|\frac{-15-18+33}{\sqrt{470}}\right|$$

$\Delta=0$

Since, $\Delta=0$, therefore, the gives lines are not skew lines.

Concept Applied

- Shortest distance between two lines

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is gives by

$$\Delta=\left|\frac{\begin{vmatrix}x_2-x_1&y_2-y_1&z_2-z_1\\a_1&b_1&c_1\\a_2&b_2&c_2\end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

54. (i) We have, $\vec{r}=\lambda(\hat{i}+2\hat{j}-\hat{k})$ ...(i)

and $\vec{r}=(3\hat{i}+3\hat{j})+\mu(2\hat{i}+\hat{j}+\hat{k})$ ...(ii)

Here, $\vec{a}_1=0\hat{i}+0\hat{j}+0\hat{k}$, $\vec{a}_2=3\hat{i}+3\hat{j}$;

$\vec{b}_1=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}_2=2\hat{i}+\hat{j}+\hat{k}$

$\vec{a}_2-\vec{a}_1=3\hat{i}+3\hat{j}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ 2 & 1 & 1\end{vmatrix}=\hat{i}(2+1)-\hat{j}(1+2)+\hat{k}(1-4)=3\hat{i}-3\hat{j}-3\hat{k}$$

Now, $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=(3\hat{i}+3\hat{j})\cdot(3\hat{i}-3\hat{j}-3\hat{k}) = 9-9=0$

Hence, shortest distance between the given lines is 0.

(ii) Equation of line (i) : $x=\lambda, y=2\lambda, z=-\lambda$

Equation of line (ii) : $x=3+2\mu, y=3+\mu, z=\mu$

So, $\lambda = 3+2\mu$ ...(iii)

$2\lambda = 3+\mu$ ...(iv)

$-\lambda=\mu$ ...(v)

Substitute $\mu=-\lambda$ is (iii), we get

$\lambda=3-2\lambda$

$\Rightarrow\quad 3\lambda=3$

$\Rightarrow\quad \lambda=1$

$\Rightarrow\quad \mu=-1$

So, the two lines intersect at point (1, 2, – 1).

Since, the point (1, 2, –1) satisfies both the equation of lines, therefore point of intersection of given lines is (1, 2, –1).

So, the motorcycles may collide at point (1, 2, –1).

55. We have, $\vec{r}=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k})$ ...(i)

$\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k})$ ...(ii)

Comparing with lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$, we get

$\vec{a}_1=(4\hat{i}-\hat{j})$; $\vec{b}_1=\hat{i}+2\hat{j}-3\hat{k}$

$\vec{a}_2=\hat{i}-\hat{j}+2\hat{k}$; $\vec{b}_2=2\hat{i}+4\hat{j}-5\hat{k}$

Now, $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}=\hat{i}(-10+12)-\hat{j}(-5+6)+\hat{k}(4-4)=2\hat{i}-\hat{j}$

and $|\vec{b}_1\times\vec{b}_2|=\sqrt{(2)^2+(1)^2}=\sqrt{5}$

$\vec{a}_2-\vec{a}_1=-3\hat{i}+2\hat{k}$

$\therefore$ Shortest distance, $d=\left|\dfrac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$$=\left|\frac{(2\hat{i}-\hat{j})\cdot(-3\hat{i}+2\hat{k})}{\sqrt{5}}\right|=\left|\frac{-6+0+0}{\sqrt{5}}\right|=\frac{6}{\sqrt{5}}$$

56. The given lines are

$\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and

$\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\mu(2\hat{i}+\hat{j}+2\hat{k})$

On comparing, we get

$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k}, \vec{b}_1=\hat{i}-\hat{j}+\hat{k}$; $\vec{a}_2=2\hat{i}-\hat{j}-\hat{k}, \vec{b}_2=2\hat{i}+\hat{j}+2\hat{k}$

$\therefore\ \vec{a}_2-\vec{a}_1=\hat{i}-3\hat{j}-2\hat{k}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}$$

$$=\hat{i}(-2-1)-\hat{j}(2-2)+k(1+2)=-3\hat{i}+0\cdot\hat{j}+3\hat{k}$$

$\Rightarrow\ |\vec{b}_1\times\vec{b}_2|=\sqrt{(-3)^2+3^2}=3\sqrt{2}$

$\Rightarrow\ (\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2) = 1(-3)-3(0)-2(3)=-9.$

$\therefore\ d=\left|\dfrac{-9}{3\sqrt{2}}\right|=\dfrac{3}{\sqrt{2}}=\dfrac{3}{2}\sqrt{2}$ units

Concept Applied

➲ S.D. between the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$, and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is given by $d=\left|\dfrac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

57. The given lines are

$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and

$\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$

S.D. between the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is given by

$$d=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

On comparing, we get

$\vec{a}_1=2\hat{i}-5\hat{j}+\hat{k}, \vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$; $\vec{a}_2=7\hat{i}-6\hat{k}, \vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$

$\therefore\ \vec{a}_2-\vec{a}_1=5\hat{i}+5\hat{j}-7\hat{k}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & 6\\ 1 & 2 & 2\end{vmatrix}=\hat{i}(4-12)-\hat{j}(6-6)+\hat{k}(6-2)=-8\hat{i}+4\hat{k}$$

$\therefore\ |\vec{b}_1\times\vec{b}_2|=\sqrt{(-8)^2+4^2}=4\sqrt{5}$

Hence, $d=\left|\dfrac{(5\hat{i}+5\hat{j}-7\hat{k})\cdot(-8\hat{i}+4\hat{k})}{4\sqrt{5}}\right|$

$$=\frac{|5(-8)-7(4)|}{4\sqrt{5}}=\frac{68}{4\sqrt{5}}=\frac{17\sqrt{5}}{5}\text{units}$$

58. Let $l_1:\dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}$

$\Rightarrow\ \dfrac{x-(-1)}{7}=\dfrac{y-(-1)}{-6}=\dfrac{z-(-1)}{1}$ and $l_2:\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}$

$\therefore$ Vector equation of lines are

$\vec{r}=-\hat{i}-\hat{j}-\hat{k}+\lambda(7\hat{i}-6\hat{j}+\hat{k})$ and $\vec{r}=3\hat{i}+5\hat{j}+7\hat{k}+\mu(\hat{i}-2\hat{j}+\hat{k})$

We get $\vec{a}_1=-\hat{i}-\hat{j}-\hat{k}, \vec{b}_1=7\hat{i}-6\hat{j}+\hat{k}$

and $\vec{a}_2=3\hat{i}+5\hat{j}+7\hat{k}, \vec{b}_2=\hat{i}-2\hat{j}+\hat{k}$

So, $\vec{a}_2-\vec{a}_1=(3\hat{i}+5\hat{j}+7\hat{k})-(-\hat{i}-\hat{j}-\hat{k})=4\hat{i}+6\hat{j}+8\hat{k}$

And, $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 7 & -6 & 1\\ 1 & -2 & 1\end{vmatrix}=-4\hat{i}-6\hat{j}-8\hat{k}$

Shortest distance between two skew lines is,

$$d=\left|\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$\Rightarrow\ d=\left|\dfrac{(-4\hat{i}-6\hat{j}-8\hat{k}).(4\hat{i}+6\hat{j}+8\hat{k})}{\sqrt{(-4)^2+(-6)^2+(-8)^2}}\right|$

$\Rightarrow\ d=\left|\dfrac{-16-36-64}{\sqrt{116}}\right| \Rightarrow d=2\sqrt{29}$ units

59. We have, $\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}-\hat{j}+\hat{k})$

$\therefore\ \vec{a}_1=\hat{i}+\hat{j}, \vec{b}_1=2\hat{i}-\hat{j}+\hat{k}$

Also, $\vec{r}=2\hat{i}+\hat{j}-\hat{k}+\mu(3\hat{i}-5\hat{j}+2\hat{k})$

$\therefore\ \vec{a}_2=2\hat{i}+\hat{j}-\hat{k}, \vec{b}_2=3\hat{i}-5\hat{j}+2\hat{k}$

So, $\vec{a}_2-\vec{a}_1=\hat{i}-\hat{k}$

And, $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 3 & -5 & 2\end{vmatrix}=\hat{i}(-2+5)-\hat{j}(4-3)+\hat{k}(-10+3)=3\hat{i}-\hat{j}-7\hat{k}$

Shortest distance between two skew lines is,

$$d=\left|\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$$\Rightarrow\ d=\left|\frac{(3\hat{i}-\hat{j}-7\hat{k})\cdot(\hat{i}-\hat{k})}{\sqrt{3^2+(-1)^2+(-7)^2}}\right|\ \Rightarrow\ d=\left|\frac{3+7}{\sqrt{59}}\right|=\frac{10}{\sqrt{59}}\text{ units.}$$

60. Given lines are

$l_1:\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k});$

$l_2:\vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu(4\hat{i}+6\hat{j}+12\hat{k})$

$\therefore$ We have $\vec{a}_1=\hat{i}+2\hat{j}-4\hat{k}, \vec{b}_1=2\hat{i}+3\hat{j}+6\hat{k}$

and $\vec{a}_2=3\hat{i}+3\hat{j}-5\hat{k}, \vec{b}_2=4\hat{i}+6\hat{j}+12\hat{k}$

So, $\vec{a}_2-\vec{a}_1=2\hat{i}+\hat{j}-\hat{k}$

Also, $\vec{b}_2=4\hat{i}+6\hat{j}+12\hat{k}=2\vec{b}_1\ \Rightarrow\ \vec{b}_1\,||\,\vec{b}_2$

Hence l_1 and l_2 are parallel lines.

Shortest distance between two parallel lines is,

$$d=\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$$

$$\Rightarrow\ d=\left|\frac{(2\hat{i}+3\hat{j}+6\hat{k})\times(2\hat{i}+\hat{j}-\hat{k})}{\sqrt{2^2+3^2+6^2}}\right|\ \Rightarrow\ d=\left|\frac{-9\hat{i}+14\hat{j}-4\hat{k}}{7}\right|$$

$$\Rightarrow\ d=\frac{\sqrt{(-9)^2+14^2+(-4)^2}}{7}=\frac{\sqrt{293}}{7}\text{ units.}$$

61. Here, the lines are

$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and

$\vec{r}=(4\hat{i}+5\hat{j}+6\hat{k})+\mu(2\hat{i}+3\hat{j}+\hat{k})$

Here, $\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k}, \vec{b}_1=\hat{i}-3\hat{j}+2\hat{k}$ and

$\vec{a}_2=4\hat{i}+5\hat{j}+6\hat{k}, \vec{b}_2=2\hat{i}+3\hat{j}+\hat{k}$

The shortest distance between the lines is given by

$$d=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$\vec{a}_2-\vec{a}_1=3\hat{i}+3\hat{j}+3\hat{k}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ 2 & 3 & 1\end{vmatrix}=\hat{i}(-3-6)-\hat{j}(1-4)+\hat{k}(3+6)=-9\hat{i}+3\hat{j}+9\hat{k}$$

$\Rightarrow\ |\vec{b}_1\times\vec{b}_2|=\sqrt{(-9)^2+3^2+9^2}=3\sqrt{19}$

Also, $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=(3\hat{i}+3\hat{j}+3\hat{k})\cdot(-9\hat{i}+3\hat{j}+9\hat{k})$

$= 3\times(-9)+3\times 3+3\times 9=9$

$$\therefore\ d=\left|\frac{9}{3\sqrt{19}}\right|=\frac{3}{\sqrt{19}}\text{ unit.}$$

62. Vector equation of a line passing through (2, 3, 2) and parallel to the line

$\vec{r}=(-2\hat{i}+3\hat{j})+\lambda(2\hat{i}-3\hat{j}+6\hat{k})$ is given by

$\vec{r}=(2\hat{i}+3\hat{j}+2\hat{k})+\mu(2\hat{i}-3\hat{j}+6\hat{k})$

Now, $\vec{a}_1=-2\hat{i}+3\hat{j},\ \vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$

and $\vec{a}_2=2\hat{i}+3\hat{j}+2\hat{k}$

Distance between given parallel lines

$$=\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|=\left|\frac{(2\hat{i}-3\hat{j}+6\hat{k})\times(4\hat{i}+0\hat{j}+2\hat{k})}{|\sqrt{4+9+36}|}\right|$$

$$=\frac{1}{7}|\hat{i}(-6)+20\hat{j}+12\hat{k}|$$

$$=\frac{1}{7}\sqrt{(-6)+(20)^2+(12)^2}=\frac{\sqrt{580}}{7}\text{ units}$$

CBSE Sample Questions

1. **(b) :** The line through the points (0, 5, –2) and (3, –1, 2) is

$$\frac{x}{3-0}=\frac{y-5}{-1-5}=\frac{z+2}{2+2}$$

or $\dfrac{x}{3}=\dfrac{y-5}{-6}=\dfrac{z+2}{4}$

Any point on the line is $P(3k, -6k+5, 4k-2)$, where k is an arbitrary scalar.

$\because\ 3k=6$

$\Rightarrow\ k=2$

The z-coordinate of the point P will be $4\times 2-2=6$. **(1)**

2. The equations of the line are $6x-12=3y+9=2z-2$, which, when written in standard symmetric form, will be

$\dfrac{x-2}{1/6}=\dfrac{y-(-3)}{1/3}=\dfrac{z-1}{1/2}$ **(1/2)**

Since, lines are parallel, we have $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$ **(1/2)**

Hence, the required direction ratios are

$\left(\dfrac{1}{6},\dfrac{1}{3},\dfrac{1}{2}\right)$ or $(1,2,3)$ **(1/2)**

and the required direction cosines are $\left(\dfrac{1}{\sqrt{14}},\dfrac{2}{\sqrt{14}},\dfrac{3}{\sqrt{14}}\right)$ **(1/2)**

3. The given line can be written as $\dfrac{x-3}{1}=\dfrac{y-\frac{1}{2}}{1}=\dfrac{z}{4}$ **(1)**

$\therefore$ Direction ratios of the given line are $<1, 1, 4>$. **(1/2)**

Thus, direction cosines are $\left\langle\dfrac{1}{3\sqrt{2}},\dfrac{1}{3\sqrt{2}},\dfrac{4}{3\sqrt{2}}\right\rangle$. **(1/2)**

4. **(a):** The equation of the x-axis may be written as $\vec{r}=t\hat{i}$. Hence, the acute angle θ between the given line and the x-axis is given by

$$\cos\theta=\frac{|1\times1+(-1)\times0+0\times0|}{\sqrt{1^2+(-1)^2+0^2}\times\sqrt{1^2+0^2+0^2}}=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4}$$

Hence, both A and R are true and R is the correct explanation of A. **(1)**

5. Let $\vec{a}_1=\hat{i}+\hat{j}-\hat{k}, \vec{a}_2=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}+\hat{k}$

Here, the lines are parallel.

$\therefore$ Shortest distance between lines $=\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{b}|}$

$=\frac{|(3\hat{k})\times(2\hat{i}+\hat{j}+\hat{k})|}{\sqrt{4+1+1}}$ **(1½)**

Now, $(3\hat{k})\times(2\hat{i}+\hat{j}+\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 3\\ 2 & 1 & 1\end{vmatrix}=-3\hat{i}+6\hat{j}$ **(1)**

$\Rightarrow |(3\hat{k})\times(2\hat{i}+\hat{j}+\hat{k})|=\sqrt{9+36}=3\sqrt{5}$

Hence, the required shortest distance $=\frac{3\sqrt{5}}{\sqrt{6}}$ units. **(1/2)**

6. The given lines are non-parallel lines. There is a unique line-segment AB. A lying on one and B on the other, which is at right angles to both the lines, AB is the shortest distance between the lines. Hence, the shortest possible distance between the insects = AB

The position vector of A lying on the line

$\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda(\hat{i}-2\hat{j}+2\hat{k})$

is $(6+\lambda)\hat{i}+(2-2\lambda)\hat{j}+(2+2\lambda)\hat{k}$ for some λ. **(1)**

The position vector of B lying on the line

$\vec{r}=-4\hat{i}-\hat{k}+\mu(3\hat{i}-2\hat{j}-2\hat{k})$

is $(-4+3\mu)\hat{i}+(-2\mu)\hat{j}+(-1-2\mu)\hat{k}$ for some μ. **(1)**

$AB=(-10+3\mu-\lambda)\hat{i}+(-2\mu-2+2\lambda)\hat{j}+(-3-2\mu-2\lambda)\hat{k}$

Since, AB is perpendicular to both the lines

$(-10+3\mu-\lambda)+(-2\mu-2+2\lambda)(-2)+(-3-2\mu-2\lambda)2=0,$ **(1)**

i.e., $\mu-3\lambda=4$ ...(i)

and $(-10+3\mu-\lambda)3+(-2\mu-2+2\lambda)(-2)+(-3-2\mu-2\lambda)(-2)=0$

i.e., $17\mu-3\lambda=20$ ...(ii)

Solving (i) and (ii) for λ and μ, we get $\mu=1, \lambda=-1$. **(1)**

The position vector of the points, at which they should be so that the distance between them is the shortest, are

$5\hat{i}+4\hat{j}$ and $-\hat{i}-2\hat{j}-3\hat{k}$

$AB=-6\hat{i}-6\hat{j}-3\hat{k}$

$\therefore$ The shortest distance $=|AB|=\sqrt{6^2+6^2+3^2}=9$ **(1)**

7. Eliminating t between the equations, we obtain the equations of the path $\frac{x}{2}=\frac{y}{-4}=\frac{z}{4}$, which are the equations of the line passing through the origin having direction ratios $<2,-4,4>$. This line is the path of the rocket. **(1)**

When t = 10 seconds, the rocket will be at the point (20, −40, 40). **(1)**

Hence, the required distance from the origin at 10 seconds $=\sqrt{20^2+40^2+40^2}$ km $=20\times3$ km $=60$ km **(1)**

The distance of the point (20, −40, 40) from the given line

$=\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{b}|}=\frac{|-30\hat{j}\times(10\hat{i}-20\hat{j}+10\hat{k})|}{|10\hat{i}-20\hat{j}+10\hat{k}|}$ km **(1)**

$=\frac{|-300\hat{i}+300\hat{k}|}{|10\hat{i}-20\hat{j}+10\hat{k}|}$ km $=\frac{300\sqrt{2}}{10\sqrt{6}}$ km $=10\sqrt{3}$ km **(1)**

8. We have, $\vec{a}_1=3\hat{i}+2\hat{j}-4\hat{k}$, $\vec{b}_1=\hat{i}+2\hat{j}+2\hat{k}$

$\vec{a}_2=5\hat{i}-2\hat{j}$, $\vec{b}_2=3\hat{i}+2\hat{j}+6\hat{k}$

$\vec{a}_2-\vec{a}_1=2\hat{i}-4\hat{j}+4\hat{k}$ **(1)**

$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 2\\ 3 & 2 & 6\end{vmatrix}=\hat{i}(12-4)-\hat{j}(6-6)+\hat{k}(2-6)$ **(1)**

$=8\hat{i}-4\hat{k}$

And $(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)=16-16=0$ **(1)**

$\therefore$ The lines are intersecting and the shortest distance between the lines is 0.

Now for point of intersection, consider

$3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$

$\Rightarrow$ $3+\lambda=5+3\mu$ ... (i)

$2+2\lambda=-2+2\mu$ ... (ii)

$-4+2\lambda=6\mu$ ... (iii) **(1)**

Solving (i) and (ii) we get $\mu=-2$ and $\lambda=-4$.

These values satisfy equation (iii) also.

Now, substituting the value of μ in equation of line, we get

$\vec{r}=5\hat{i}-2\hat{j}+(-2)(3\hat{i}+2\hat{j}+6\hat{k})=-\hat{i}-6\hat{j}-12\hat{k}$

$\therefore$ Point of intersection is (−1, −6, −12). **(1)**

Self Assessment

Case Based Objective Questions (4 marks)

1. Two motorcycles A and B are running at the speed more than allowed speed on the road along the lines $\vec{r}=\lambda(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{r}=3\hat{i}+3\hat{j}+\mu(2\hat{i}+\hat{j}+\hat{k})$, respectively.

Based on the above information, attempt any 4 out of 5 subparts.

(i) The cartesian equation of the line along which motorcycle A is running, is

(a) $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z-1}{-1}$ (b) $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$

(c) $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ (d) None of these

(ii) The direction cosines of line along which motorcycle A is running, are

(a) $< 1, -2, 1 >$ (b) $< 1, 2, -1 >$

(c) $<\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}>$ (d) $<\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{-1}{\sqrt{6}}>$

(iii) The direction ratios of line along which motorcycle B is running, are

(a) $< 1, 0, 2 >$ (b) $< 2, 1, 0 >$

(c) $< 1, 1, 2 >$ (d) $< 2, 1, 1 >$

(iv) The shortest distance between the gives lines is

(a) 4 units (b) $2\sqrt{3}$ units

(c) $3\sqrt{2}$ units (d) 0 unit

(v) The motorcycles will meet with an accident at the point

(a) $(-1, 1, 2)$ (b) $(2, 1, -1)$

(c) $(1, 2, -1)$ (d) does not exist

Multiple Choice Questions (1 mark)

2. Distance of the point (α, β, γ) from y-axis is

(a) β (b) $|\beta|$

(c) $|\beta| + |\gamma|$ (d) $\sqrt{\alpha^2+\gamma^2}$

3. If the direction cosines of a line are k, k, k, then

(a) $k > 0$ (b) $0 < k < 1$

(c) $k = 1$ (d) $k=\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

4. A line makes angles α, β and γ with the co-ordinate axes. If $\alpha + \beta = 90°$, then γ is equal to

(a) 0 (b) 90°

(c) 180° (d) None of these

OR

The equation of a line passing through the point $(-3, 2, -4)$ and equally inclined to the axes are

(a) $x - 3 = y + 2 = z - 4$ (b) $x + 3 = y - 2 = z + 4$

(c) $\frac{x+3}{1}=\frac{y-2}{2}=\frac{z+4}{3}$ (d) None of these

5. The vector equation of the symmetrical form of equation of straight line $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ is

(a) $\vec{r}=(3\hat{i}+7\hat{j}+2\hat{k})+\mu(5\hat{i}+4\hat{j}-6\hat{k})$

(b) $\vec{r}=(5\hat{i}+4\hat{j}-6\hat{k})+\mu(3\hat{i}+7\hat{j}+2\hat{k})$

(c) $\vec{r}=(5\hat{i}-4\hat{j}-6\hat{k})+\mu(3\hat{i}-7\hat{j}-2\hat{k})$

(d) $\vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\mu(3\hat{i}+7\hat{j}+2\hat{k})$

6. The angle between the lines $x = 1$, $y = 2$ and $y = -1$, $z = 0$ is

(a) 90° (b) 30° (c) 60° (d) 0°

7. The angle between the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

VSA Type Questions (1 mark)

8. Find the direction cosines of the vector $(2\hat{i}+2\hat{j}-\hat{k})$.

9. Write the given equation of the line $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ in vector form.

10. If α, β, γ are the direction angles of a vector and $\cos\alpha = \frac{14}{15}$, $\cos\beta = \frac{1}{3}$, then find $\cos\gamma$.

OR

If the vectors $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$, where $A \equiv (3, 4, -2)$, $B \equiv (1, -1, 2)$, $C \equiv (k, 3, 2)$ and $D \equiv (3, 5, 6)$, then find the value of k.

11. If $\left(\frac{1}{2},\frac{1}{3},n\right)$ are the direction cosines of a line, then what is the value of n?

12. Find the equation of a line, which is parallel to $2\hat{i}+\hat{j}+3\hat{k}$ and which passes through the point $(5, -2, 4)$.

SA I Type Questions (2 marks)

13. Write the vector equation of the line through the points $(3, 4, -7)$ and $(1, -1, 6)$.

14. Find the angle between the straight lines $\frac{x+1}{2}=\frac{y-2}{5}=\frac{z+3}{4}$ and $\frac{x-1}{1}=\frac{y+2}{2}=\frac{z-3}{-3}$.

15. Find the shortest distance between the lines $x = y = z$ and $x=1-y=\frac{z}{0}$.

OR

Find the angle between the lines passing through the points $(4, 7, 8)$, $(2, 3, 4)$ and $(-1, -2, 1)$, $(1, 2, 5)$.

16. Find the shortest distance between the lines $\frac{x-5}{4}=\frac{y-7}{-5}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-7}{1}=\frac{z-5}{-3}$.

SA II Type Questions (3 marks)

17. $A(0, 6, -9)$, $B(-3, -6, 3)$ and $C(7, 4, -1)$ are three points. Find the equation of line AB. If D is the foot of the perpendicular drawn from the point C to the line AB, then find the coordinates of the point D.

18. Prove that the line through $A(0, -1, -1)$ and $B(4, 5, 1)$ intersects the line through $C(3, 9, 4)$ and $D(-4, 4, 4)$.

19. Find the angle between the pairs of lines given by

$\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and

$\vec{r}=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$.

OR

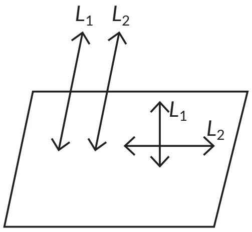
Find the foot of perpendicular from $(0, 2, 7)$ on the line $\frac{x+2}{-1}=\frac{y-1}{3}=\frac{z-3}{-2}$.

20. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec{r}=(3\hat{i}+8\hat{j}+3\hat{k})+\lambda(3\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}=(-3\hat{i}-7\hat{j}+6\hat{k})+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$.

21. Find the angle between the lines whose direction cosines are given by the equations $l+m+n=0$ and $l^2+m^2-n^2=0$

Case Based Questions (4 marks)

22. If a_1, b_1, c_1 and a_2, b_2, c_2 are direction ratios of two lines say L_1 and L_2 respectively. Then $L_1 \parallel L_2$ iff $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ and $L_1 \perp L_2$ iff $a_1a_2+b_1b_2+c_1c_2=0$.



Based on the above information, answer the following questions.

(i) Find the coordinates of the foot of the perpendicular drawn from the point $A(1, 2, 1)$ to the line joining $B(1, 4, 6)$ and $C(5, 4, 4)$.

(ii) Find the direction ratios of the line which is perpendicular to the lines with direction ratios proportional to $(1, -2, -2)$ and $(0, 2, 1)$.

LA Type Questions (4 /6 marks)

23. Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.

24. Find the shortest distance between the lines given by $\vec{r}=(8+3\lambda)\hat{i}-(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k}$ and $\vec{r}=15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k})$.

OR

Show that the straight lines whose direction cosines are given by $2l+2m-n=0$ and $mn+nl+lm=0$ are at right angles.

25. Find the equation of the perpendicular drawn from the point $P(2, 4, -1)$ to the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$. Also, write down the coordinates of the foot of the perpendicular.

Detailed SOLUTIONS

1. **(i) (b) :** The line along which motorcycle A is running, is $\vec{r}=\lambda(\hat{i}+2\hat{j}-\hat{k})$, which can be rewritten as

$(x\hat{i}+y\hat{j}+z\hat{k})=\lambda\hat{i}+2\lambda\hat{j}-\lambda\hat{k}$

$\Rightarrow x=\lambda, y=2\lambda, z=-\lambda \Rightarrow \frac{x}{1}=\lambda, \frac{y}{2}=\lambda, \frac{z}{-1}=\lambda$

Thus, the required cartesian equation is $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$

(ii) (d): Clearly, D.R.'s of the required line are $< 1, 2, -1 >$.

$\therefore$ D.C.'s are

$$<\frac{1}{\sqrt{1^2+2^2+(-1)^2}}, \frac{2}{\sqrt{1^2+2^2+(-1)^2}}, \frac{-1}{\sqrt{1^2+2^2+(-1)^2}}>$$

i.e., $<\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}>$

(iii) (d): The line along which motorcycle B is running, is $\vec{r}=(3\hat{i}+3\hat{j})+\mu(2\hat{i}+\hat{j}+\hat{k})$, which is parallel to the vector $2\hat{i}+\hat{j}+\hat{k}$.

$\therefore$ D.R.'s of the required line are <2, 1, 1>.

(iv) (d): Here, $\vec{a}_1=0\hat{i}+0\hat{j}+0\hat{k}, \vec{a}_2=3\hat{i}+3\hat{j}, \vec{b}_1=\hat{i}+2\hat{j}-\hat{k}$,

$\vec{b}_2=2\hat{i}+\hat{j}+\hat{k}$

$\therefore\ \vec{a}_2-\vec{a}_1=3\hat{i}+3\hat{j}$

and $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ 2 & 1 & 1\end{vmatrix}=3\hat{i}-3\hat{j}-3\hat{k}$

Now, $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=(3\hat{i}+3\hat{j})\cdot(3\hat{i}-3\hat{j}-3\hat{k})$
$= 9 - 9 = 0$

Hence, shortest distance between the given lines is 0.

(v) (c): Since, the point (1, 2, –1) satisfies both the equations of lines, therefore point of intersection of given lines is (1, 2, –1).

So, the motorcycles will meet with an accident at the point (1, 2, –1).

2. (d): Distance of point (α, β, γ) from y-axis or any point $(0, \beta, 0)$ $=\sqrt{(\alpha-0)^2+(\beta-\beta)^2+(\gamma-0)^2}=\sqrt{\alpha^2+\gamma^2}$

3. (d): If direction cosines of a line are k, k and k, then $l = k$, $m = k$ and $n = k$

Since, $l^2 + m^2 + n^2 = 1$

$\therefore\ k^2 + k^2 + k^2 = 1 \Rightarrow k^2=\frac{1}{3} \Rightarrow k=\pm\frac{1}{\sqrt{3}}$

4. (b): We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \cos^2(90° - \alpha) + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \sin^2\alpha + \cos^2\gamma = 1 \Rightarrow 1 + \cos^2\gamma = 1 \Rightarrow \cos^2\gamma = 0$

$\Rightarrow \cos\gamma = 0 \Rightarrow \gamma = \frac{\pi}{2} = 90°$.

OR

(b) : Since, the line is equally inclined to the axes.

$\therefore\ l = m = n$ ...(i)

The required equation of the line is

$$\frac{x+3}{l}=\frac{y-2}{l}=\frac{z+4}{l} \quad \text{[Using (i)]}$$

$\Rightarrow \frac{x+3}{1}=\frac{y-2}{1}=\frac{z+4}{1} \Rightarrow x + 3 = y - 2 = z + 4$

5. (d): $\because$ The vector form of $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ is

$\vec{r}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$

$\therefore$ Required equation in vector form is

$\Rightarrow \vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\mu(3\hat{i}+7\hat{j}+2\hat{k})$

6. (a): The first line is parallel to z-axis and the second line parallel to x-axis, therefore the angle between the required line is 90°.

7. (b): Direction ratios of the given lines are respectively 7, –5, 1 and 1, 2, 3.

Here, $a_1a_2 + b_1b_2 + c_1c_2 = (7)(1) + (-5)(2) + (1)(3)$
$= 7 - 10 + 3 = 0$

$\therefore$ Angle between the given lines is $\frac{\pi}{2}$.

8. Direction cosines of $(2\hat{i}+2\hat{j}-\hat{k})$ are

$$\frac{2}{\sqrt{(2)^2+(2)^2+(-1)^2}}, \frac{2}{\sqrt{(2)^2+(2)^2+(-1)^2}}, \frac{-1}{\sqrt{(2)^2+(2)^2+(-1)^2}}$$

$$=\frac{2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{-1}{\sqrt{4+4+1}}$$

i.e., $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$

9. We have, $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$ and $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

$\therefore$ Vector equation of line is $\vec{r}=\vec{a}+\lambda\vec{b}$ or

$\vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$

10. We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \frac{196}{225}+\frac{1}{9}+\cos^2\gamma=1$

$\Rightarrow \cos^2\gamma=1-\frac{221}{225}=\frac{4}{225}\Rightarrow\cos\gamma=\pm\frac{2}{15}$

OR

The direction ratios of the line along the vector $\overrightarrow{AB}$ are $a_1 = 1 - 3 = -2$, $b_1 = -1 - 4 = -5$, $c_1 = 2 - (-2) = 4$

The direction ratios of the line along the vector $\overrightarrow{CD}$ are $a_2 = 3 - k$, $b_2 = 5 - 3 = 2$, $c_2 = 6 - 2 = 4$

Given $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$.

$\therefore\ a_1a_2 + b_1b_2 + c_1c_2 = 0 \Rightarrow (-2)(3 - k) + (-5)(2) + 4(4) = 0$

$\Rightarrow -6 + 2k - 10 + 16 = 0 \Rightarrow k = 0$

11. Given, $\left(\frac{1}{2},\frac{1}{3},n\right)$ are the direction cosines of a line

$\therefore \left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+n^2=1 \Rightarrow n^2=\frac{23}{36} \Rightarrow n=\frac{\pm\sqrt{23}}{6}$

12. We have, $x_1 = 5$, $y_1 = -2$, $z_1 = 4$

and $a = 2$, $b = 1$, $c = 3$

$\therefore$ Required equation of the line is $\frac{x-5}{2}=\frac{y+2}{1}=\frac{z-4}{3}$

13. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}, \vec{a}=3\hat{i}+4\hat{j}-7\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+6\hat{k}$

Now, $\vec{b}-\vec{a}=-2\hat{i}-5\hat{j}+13\hat{k}$

Since, vector equation of a line passes through two points is represented by $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$.

Therefore, the required equation is

$$x\hat{i}+y\hat{j}+z\hat{k}=3\hat{i}+4\hat{j}-7\hat{k}+\lambda(-2\hat{i}-5\hat{j}+13\hat{k})$$

$\Rightarrow (x-3)\hat{i}+(y-4)\hat{j}+(z+7)\hat{k}=\lambda(-2\hat{i}-5\hat{j}+13\hat{k})$

14. The direction ratios of lines are

$a_1 = 2$, $b_1 = 5$, $c_1 = 4$ and $a_2 = 1$, $b_2 = 2$, $c_2 = -3$

$\therefore \cos\theta=\frac{2\cdot1+5\cdot2+4(-3)}{\sqrt{2^2+5^2+4^2}\sqrt{1^2+2^2+(-3)^2}}$

$=\frac{(2+10-12)}{\sqrt{4+25+16}\sqrt{1+4+9}}=0 \Rightarrow \theta = 90°$

15. The lines are $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y-1}{-1}=\frac{z}{0}$.

Here, $\vec{a}_1=0, \vec{b}_1=\hat{i}+\hat{j}+\hat{k}, \vec{a}_2=\hat{j}$ and $\vec{b}_2=\hat{i}-\hat{j}$

$\therefore \quad \vec{b}_1\times\vec{b}_2=(\hat{i}+\hat{j}+\hat{k})\times(\hat{i}-\hat{j})=\hat{i}+\hat{j}-2\hat{k}$

Shortest distance between the lines is

$$\frac{|(\vec{a}_2-\vec{a}_1)(\vec{b}_1\times\vec{b}_2)|}{|\vec{b}_1\times\vec{b}_2|}$$

$$\therefore \quad \frac{|\hat{j}\cdot(\hat{i}+\hat{j}-2\hat{k})|}{|\hat{i}+\hat{j}-2\hat{k}|}=\frac{1}{\sqrt{6}}.$$

OR

Let the given points are $A(4, 7, 8)$, $B(2, 3, 4)$, $C(-1, -2, 1)$ and $D(1, 2, 5)$.

Direction ratios of AB (a_1, b_1, c_1) are $(-2, -4, -4)$

Direction ratios of CD (a_2, b_2, c_2) are $(2, 4, 4)$

We have, $\frac{a_1}{a_2}=\frac{-2}{2}=-1, \frac{b_1}{b_2}=\frac{-4}{4}=-1$ and $\frac{c_1}{c_2}=\frac{-4}{4}=-1$

$$\therefore \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Thus, $AB||CD$.

Hence, angle between the lines is 0.

16. Here, $x_1 = 5, y_1 = 7, z_1 = -3; a_1 = 4, b_1 = -5, c_1 = -5; x_2 = 8, y_2 = 7, z_2 = 5$ and $a_2 = 7, b_2 = 1, c_2 = -3$

$$\therefore \text{ Shortest distance} = \left|\frac{\begin{vmatrix} 3 & 0 & 8 \\ 4 & -5 & -5 \\ 7 & 1 & -3 \end{vmatrix}}{\sqrt{(15+5)^2+(-35+12)^2+(4+35)^2}}\right|$$

$$=\left|\frac{3(15+5)+8(4+35)}{\sqrt{400+529+1521}}\right|=\left|\frac{60+312}{\sqrt{2450}}\right|=\frac{372}{35\sqrt{2}}$$

$$=\frac{372\sqrt{2}}{70}=\frac{186\sqrt{2}}{35} \text{ units}$$

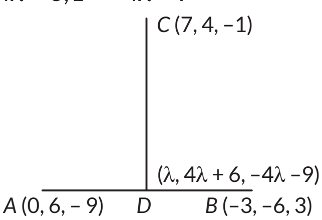
17. We have, $A(0, 6, -9)$, $B(-3, -6, 3)$ and $C(7, 4, -1)$.

The equation of line passing through the points $A(0, 6, -9)$ and $B(-3, -6, 3)$ is

$$\frac{x-0}{-3-0}=\frac{y-6}{-6-6}=\frac{z+9}{3+9} \Rightarrow \frac{x}{-3}=\frac{y-6}{-12}=\frac{z+9}{12}$$

$$\Rightarrow \quad \frac{x}{1}=\frac{y-6}{4}=\frac{z+9}{-4}=\lambda \text{ (say)}$$

$\therefore \quad x = \lambda, y = 4\lambda + 6, z = -4\lambda - 9$



Let the coordinates of D be $(\lambda, 4\lambda + 6, - 4\lambda - 9)$. ...(i)

Direction ratios of CD be $(\lambda - 7, 4\lambda + 2, - 4\lambda - 8)$ and direction ratios of AB are $(1, 4, - 4)$.

Now, CD is perpendicular to AB.

So, $1(\lambda - 7) + 4(4\lambda + 2) - 4(-4\lambda - 8) = 0$

$\Rightarrow \quad \lambda - 7 + 16\lambda + 8 + 16\lambda + 32 = 0 \Rightarrow \lambda = -1$

Substituting the value of λ in (i), we get $(-1, 2, -5)$

Hence, the coordinates of D are $(-1, 2, -5)$.

18. As we know, the cartesian equation of line passes through $A(x_1, y_1, z_1) = (0, -1, -1)$ and $B(x_2, y_2, z_2) = (4, 5, 1)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\therefore \quad \frac{x-0}{4-0}=\frac{y+1}{5+1}=\frac{z+1}{1+1}$$

$$\Rightarrow \quad \frac{x}{4}=\frac{y+1}{6}=\frac{z+1}{2} \quad \text{...(i)}$$

and cartesian equation of the line passes through $C(3, 9, 4)$ and $D(-4, 4, 4)$ is

$$\frac{x-3}{-4-3}=\frac{y-9}{4-9}=\frac{z-4}{4-4}$$

$$\Rightarrow \quad \frac{x-3}{-7}=\frac{y-9}{-5}=\frac{z-4}{0} \quad \text{...(ii)}$$

Now, shortest distance between the lines

$$=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

$$=\frac{\begin{vmatrix} 3-0 & 9+1 & 4+1 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{(0+10)^2+(-14-0)^2+(-20+42)^2}}$$

$$=\frac{\begin{vmatrix} 3 & 10 & 5 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{100+196+484}}$$

$$=\frac{3(0+10)-10(14)+5(-20+42)}{\sqrt{780}}=\frac{30-140+110}{\sqrt{780}}=0$$

Hence, the given lines intersect.

[$\because$ If shortest distance between lines is zero then the lines intersect]

19. We have $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$

$\therefore \quad \vec{a}_1=3\hat{i}+2\hat{j}-4\hat{k}$ and $\vec{b}_1=\hat{i}+2\hat{j}+2\hat{k}$

and $\vec{r}=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$

$\therefore \quad \vec{a}_2=5\hat{i}-2\hat{j}$ and $\vec{b}_2=3\hat{i}+2\hat{j}+6\hat{k}$

Here, $\vec{b}_1=\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}_2=3\hat{i}+2\hat{j}+6\hat{k}$

Let θ be the angle between the given lines, then

$$\cos\theta=\left|\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|=\frac{1\times3+2\times2+2\times6}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+2^2+6^2}}$$

$= \frac{3+4+12}{\sqrt{9}\sqrt{49}} = \frac{19}{3\times 7} = \frac{19}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$

OR

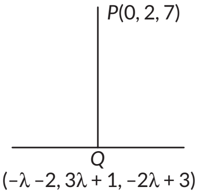
We have, equation of line is $\frac{x+2}{-1} = \frac{y-1}{3} = \frac{z-3}{-2}$.

$\Rightarrow \quad \frac{x+2}{-1} = \frac{y-1}{3} = \frac{z-3}{-2} = \lambda$ (say) ...(i)

$\Rightarrow \quad x = -\lambda - 2, y = 3\lambda + 1, z = -2\lambda + 3$

Any point on the line (i) be $(-\lambda - 2, 3\lambda + 1, -2\lambda + 3)$.

Let Q be the foot of the perpendicular drawn from the point $P(0, 2, 7)$ to the given line.



Let the coordinates of Q be $(-\lambda - 2, 3\lambda + 1, -2\lambda + 3)$ ...(ii)

Direction ratios of PQ be $-\lambda - 2, 3\lambda - 1, -2\lambda - 4$

PQ is the perpendicular to the given line.

$\therefore \quad -1(-\lambda - 2) + 3(3\lambda - 1) - 2(-2\lambda - 4) = 0$

$\Rightarrow \quad \lambda + 2 + 9\lambda - 3 + 4\lambda + 8 = 0 \Rightarrow \lambda = -\frac{1}{2}$

Substituting the value of λ in (ii), we get $\left(\frac{-3}{2}, \frac{-1}{2}, 4\right)$

Hence, the coordinates of Q are $\left(-\frac{3}{2}, -\frac{1}{2}, 4\right)$.

20. Given lines are $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$ and

$\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$

In cartesian forms, we have

$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda$ ...(i)

and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$ ...(ii)

Any point on (i) has coordinates $P(3\lambda + 3, -\lambda + 8, \lambda + 3)$ and on (ii) $Q(-3\mu - 3, 2\mu - 7, 4\mu + 6)$

Direction ratios of PQ are

$a = -3\mu - 3 - 3\lambda - 3 = -3\mu - 3\lambda - 6$

$b = 2\mu - 7 + \lambda - 8 = 2\mu + \lambda - 15$

$c = 4\mu + 6 - \lambda - 3 = 4\mu - \lambda + 3$

As PQ is perpendicular to (i) and (ii).

$\therefore \quad -9\mu - 9\lambda - 18 - 2\mu - \lambda + 15 + 4\mu - \lambda + 3 = 0$

$\Rightarrow \quad -7\mu - 11\lambda = 0 \Rightarrow 7\mu + 11\lambda = 0$ ...(iii)

and $9\mu + 9\lambda + 18 + 4\mu + 2\lambda - 30 + 16\mu - 4\lambda + 12 = 0$

$\Rightarrow \quad 29\mu + 7\lambda = 0$ ...(iv)

Solving (iii) and (iv), we get $\mu = 0, \lambda = 0$.

$\therefore$ Coordinates of $P(3, 8, 3)$, $Q(-3, -7, 6)$ and direction ratios of PQ are $<-6, -15, 3>$ or $<2, 5, -1>$

So, shortest distance $PQ = \sqrt{6^2 + 15^2 + (-3)^2}$

$= \sqrt{270} = 3\sqrt{30}$ units.

and vector equation of PQ is

$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 5\hat{j} - \hat{k})$

21. We have, $l + m + n = 0$ ...(i)

or $\quad n = -(l + m)$

and $l^2 + m^2 - n^2 = 0$ ...(ii)

Substituting $n = -(l + m)$ in (ii), we get

$l^2 + m^2 - (l + m)^2 = 0$

$\Rightarrow \quad l^2 + m^2 - l^2 - m^2 - 2ml = 0 \Rightarrow 2lm = 0$

$\Rightarrow \quad lm = 0 \Rightarrow (-m - n)m = 0 \quad [\because \; l = -m - n]$

$\Rightarrow \quad (m + n)m = 0$

$\Rightarrow \quad m = -n$ or $m = 0$

$\Rightarrow \quad l = 0, l = -n$ (Using $l = -m - n$)

Hence, dr's of two lines are proportional to $0, -n, n$ and $-n, 0, n$ *i.e.*, $0\ -1, 1$ and $-1, 0, 1$.

Therefore, the vector parallel to these given lines are

$\vec{a} = -\hat{j} + \hat{k}$ and $\vec{b} = -\hat{i} + \hat{k}$

Now, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$

$\Rightarrow \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore \quad \theta = \frac{\pi}{3}$

22. (i) Equation of line joining $B(1, 4, 6)$ and $C(5, 4, 4)$ is

$\frac{x-1}{4} = \frac{y-4}{0} = \frac{z-6}{-2}$

Let coordinates of foot of perpendicular be $D(x, y, z)$ drawn from the point $A(1, 2, 1)$

$\therefore \quad$ D.R.'s of AD are $< x - 1, y - 2, z - 1 >$.

Now, $4(x - 1) + 0(y - 2) + (-2)(z - 1) = 0 \Rightarrow 4x - 2z = 2$

Also, (x, y, z) will satisfy equation of line BC.

Here, $(3, 4, 5)$ satisfy both the equations.

$\therefore \quad$ Required coordinates are $(3, 4, 5)$.

(ii) Let a, b, c be the direction ratios of the required line. Since it is perpendicular to the lines whose direction ratios are proportional to $(1, -2, -2)$ and $(0, 2, 1)$ respectively.

$\therefore \quad a - 2b - 2c = 0$ ...(i)

$0\cdot a + 2b + c = 0$ ...(ii)

On solving (i) and (ii) by cross-multiplication, we get

$\frac{a}{-2+4} = \frac{b}{0-1} = \frac{c}{2} \Rightarrow \frac{a}{2} = \frac{b}{-1} = \frac{c}{2}$

Thus, the direction ratios of the required line are $< 2, -1, 2 >$.

23. We have, equation of line as

$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$

$\Rightarrow \quad \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = \lambda$ (say)

and Dr's are $-2, 6, -3$.

$\Rightarrow \quad x = -2\lambda + 4, y = 6\lambda$ and $z = -3\lambda + 1$

If the coordinates of L be $(4 - 2\lambda, 6\lambda, 1 - 3\lambda)$ and P be $(2, 3, -8)$ then direction ratios of PL are proportional to $(4 - 2\lambda - 2, 6\lambda - 3, 1 - 3\lambda + 8)$ *i.e.*, $(2 - 2\lambda, 6\lambda - 3, 9 - 3\lambda)$.

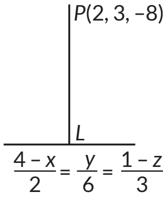
Since, PL is perpendicular to give line

$\therefore \quad -2(2 - 2\lambda) + 6(6\lambda - 3) - 3(9 - 3\lambda) = 0$

$\Rightarrow \quad -4 + 4\lambda + 36\lambda - 18 - 27 + 9\lambda = 0$

$\Rightarrow \quad 49\lambda = 49 \Rightarrow \lambda = 1$

Hence, the coordinates of L are
$(4 - 2\lambda, 6\lambda, 1 - 3\lambda)$ *i.e.*, $(2, 6, -2)$



Now, $PL = \sqrt{(2-2)^2 + (6-3)^2 + (-2+8)^2}$

$= \sqrt{0+9+36} = 3\sqrt{5}$ units

24. Given, $\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$

$= 8\hat{i} - 9\hat{j} + 10\hat{k} + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$

Compare with $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$, we get

$\vec{a}_1 = 8\hat{i} - 9\hat{j} + 10\hat{k}$ and $\vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}$ ...(i)

Also, $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

$\Rightarrow$ $\vec{a}_2 = 15\hat{i} + 29\hat{j} + 5\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}$ ...(ii)

Now, shortest distance between two lines is given by

$$\left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

Since, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$

$= \hat{i}(80-56) - \hat{j}(-15-21) + \hat{k}(24+48) = 24\hat{i} + 36\hat{j} + 72\hat{k}$

$\Rightarrow$ $|\vec{b}_1 \times \vec{b}_2| = \sqrt{(24)^2 + (36)^2 + (72)^2}$

$= 12\sqrt{2^2 + 3^2 + 6^2} = 84$

and $(\vec{a}_2 - \vec{a}_1) = (15-8)\hat{i} + (29+9)\hat{j} + (5-10)\hat{k} = 7\hat{i} + 38\hat{j} - 5\hat{k}$

Hence, shortest distance

$$= \left|\frac{(24\hat{i} + 36\hat{j} + 72\hat{k})\cdot(7\hat{i} + 38\hat{j} - 5\hat{k})}{84}\right|$$

$$= \left|\frac{168 + 1368 - 360}{84}\right| = \left|\frac{1176}{84}\right| = 14 \text{ units}$$

OR

We have, $2l + 2m - n = 0$ ...(i)

$\Rightarrow$ $m = \dfrac{n - 2l}{2}$

and $mn + nl + lm = 0$ ...(ii)

Substituting $m = \dfrac{n-2l}{2}$ in (ii), we get

$$\left(\frac{n-2l}{2}\right)n + nl + l\left(\frac{n-2l}{2}\right) = 0$$

$\Rightarrow$ $\dfrac{n^2 - 2nl + 2nl + nl - 2l^2}{2} = 0$

$\Rightarrow$ $n^2 + nl - 2l^2 = 0$

$\Rightarrow$ $(n + 2l)(n - l) = 0$ $\Rightarrow$ $n = -2l$ and $n = l$

$\therefore$ $m = \dfrac{-2l - 2l}{2}$, $m = \dfrac{l - 2l}{2}$ $\Rightarrow$ $m = -2l$, $m = \dfrac{-l}{2}$

Hence, the direction ratios of two lines are proportional to $l, -2l, -2l$ and $l, \dfrac{-l}{2}, l$.

$\Rightarrow$ $1, -2, -2$ and $1, \dfrac{-1}{2}, 1$

$\Rightarrow$ $1, -2, -2$ and $2, -1, 2$

Also, the vectors parallel to these lines are

$\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively.

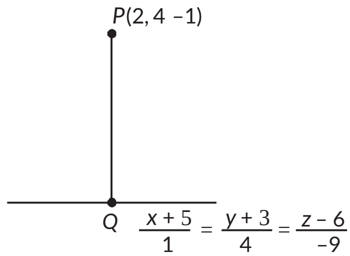
$\therefore$ $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{(\hat{i} - 2\hat{j} - 2\hat{k})\cdot(2\hat{i} - \hat{j} + 2\hat{k})}{3\cdot 3} = \dfrac{2+2-4}{9} = 0$

$\Rightarrow$ $\theta = \dfrac{\pi}{2}$ Hence proved.

25. Let Q be the foot of the perpendicular drawn from the point $P(2, 4, -1)$ to the given line. The coordinates of a general point on $\dfrac{x+5}{1} = \dfrac{y+3}{4} = \dfrac{z-6}{-9} = \lambda$ (say)

are $x = \lambda - 5$, $y = 4\lambda - 3$, $z = -9\lambda + 6$

Let the coordinates of Q are $(\lambda - 5, 4\lambda - 3, -9\lambda + 6)$ ...(i)



$\therefore$ Direction ratios of PQ are

$\lambda - 5 - 2, 4\lambda - 3 - 4, -9\lambda + 6 + 1$

i.e., $\lambda - 7, 4\lambda - 7, -9\lambda + 7$

Direction ratios of the given line are proportional to $1, 4, -9$

Since, PQ is perpendicular to the given line

$\therefore$ $1(\lambda - 7) + 4(4\lambda - 7) - 9(-9\lambda + 7) = 0$

$\Rightarrow$ $\lambda - 7 + 16\lambda - 28 + 81\lambda - 63 = 0$

$\Rightarrow$ $98\lambda = 98$

$\Rightarrow$ $\lambda = 1$

Putting $\lambda = 1$ in (i), we get the coordinates of Q as $(-4, 1, -3)$.

$\therefore$ The equation of line is $\dfrac{x-2}{-4-2} = \dfrac{y-4}{1-4} = \dfrac{z+1}{-3+1}$

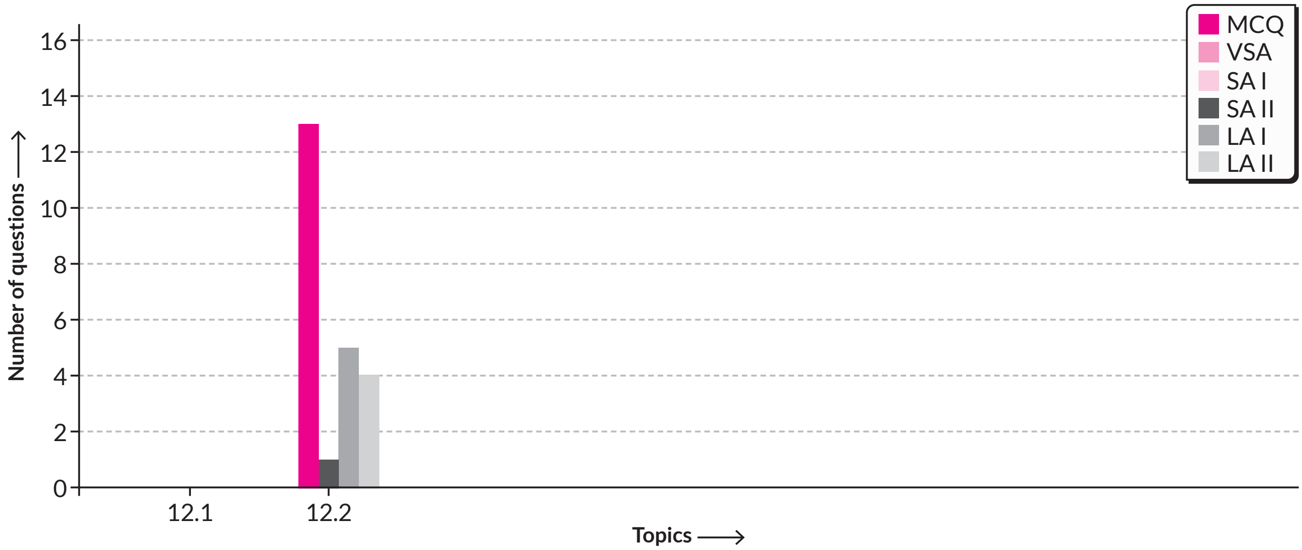
$\Rightarrow$ $\dfrac{x-2}{-6} = \dfrac{y-4}{-3} = \dfrac{z+1}{-2}$ $\Rightarrow$ $\dfrac{x-2}{6} = \dfrac{y-4}{3} = \dfrac{z+1}{2}$

CHAPTER 12

Linear Programming

TOPICS

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Maximum weightage is of Topic 12.2 *Linear Programming Problem and its Mathematical Formulation*.
- Maximum MCQ, LA II type questions were asked from Topic 12.2 *Linear Programming Problem and its Mathematical Formulation*.

QUICK RECAP

Linear Programming

- Linear programming (LP) is an optimisation technique in which a linear function is optimised (*i.e.*, minimised or maximised) subject to certain restrictions which are in the form of linear inequalities.

Linear Programming Problem

- A linear programming problem (LPP) is a problem that is concerned with finding the optimal value of a linear function subject to given constraints.

Optimisation problem

- A problem which maximise or minimise a linear function subject to the given constraints.
  - **Feasible Region :** The common region determined by all the constraints of an LPP is called the feasible region. The feasible region may be either bounded or unbounded.

(i) **Bounded feasible region :** If the feasible region is enclosed within a circle, then it is called bounded feasible region.

(ii) **Unbounded feasible region :** If the feasible region is not bounded, then it is called unbounded feasible region.

- **Feasible Solution :** The set of points, within or on the boundary of the feasible region is said to be the feasible solution.

Note :

(i) The region other than feasible region is called infeasible region.

(ii) Any point outside the feasible region is called an infeasible solution.

- **Optimal Value :** The maximum or minimum value of the objective function is called optimal value.

- **Optimal Solution :** Any point in the feasible region which gives the optimal value is called optimal solution.

- **Corner Point :** The intersection point of two boundary lines of the feasible region.

Some Important Theorems of LPP

- **Theorem 1 :** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point of the feasible region.

- **Theorem 2 :** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both maximum and minimum value on R and each of these occurs at a corner point of R.

  Remark : If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R.

Steps to solve LPP

- Here are the following steps to solve an LPP.

  Step 1 : Convert inequations into equations.

  Step 2 : Find the point of intersection.

  Step 3 : Draw the graph of inequations.

  Step 4 : Find the value of the objective function corresponding to each corner point.

BRAIN MAP

LINEAR PROGRAMMING

Theorems

- Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear ineqa lities, this optimal value must occur at a corner point (vertex) of the feasible region.

  Remark: If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R.
- Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

Definition

Finding the optimal values (maximum or minimum) of quantities subject to the constraints when relationship is expressed as linear inequations or inequalities.

General Form of LPP

Maximize (or minimize} $Z = c_1x_1 + c_2x_2 + ...+ c_nx_n$ (Objective function) Subject to

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n(\leq, = , \geq)b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n(\leq, = , \geq)b_2$

$\vdots \qquad \vdots \qquad \vdots$ (constraints)

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... +a_{mn}x_n (\leq, = , \geq)b_m$

$x_1, x_2, x_3, ..., x_n \geq 0$ (non-negative restrictions)

where all a_{ij}'s, b_i's and c_j's are constants and x_j's are variables.

Graphical Method of Solving LPP :

Step - 1 : Feasible Solution

Find set of points, whose co-ordinates satisfy the constraints and non negativity of a linear programming problem.

Step - 2 : Feasible Region

Mark common region determined by all the constraints or points satisfying the constraints of an LPP.

Step - 3 : Optimal Feasible Solution

Any point which maximises or minimises the objective function gives the solution of objective function.

Formulation of LPP

The three steps in the mathematical formulation of an L.P.P. are as follows :

- Identify the objective function as a linear combination of variables (x and y) and construct all constraints *ie*. linear equations and inequations involving these variables. An L.P.P. stated mathematically as

  Maximise (or minimise) $Z = ax + by$

  Subject to the constraints:

  $a_ix + b_iy \leq$ (or $\geq$ or $=$ or $>$ or $<$) c_i, where $i = 1,..., n$,

  $x \geq 0, y \geq 0$ (non-negative constraints).
- Find the solutions (feasible region) of these equations and inequations by some mathematical method.
- Find an optimal solution *ie*. select particular values of the variables x and y that give the desired value (maximum/minimum) of the objective function.

Previous Years' CBSE Board Questions

12.2 Linear Programming Problem and its Mathematical Formulation

MCQ

1. Which of the following points satisfies both the inequations $2x + y \le 10$ and $x + 2y \ge 8$?
(a) (–2, 4) (b) (3, 2) (c) (–5, 6) (d) (4, 2)
(2023) U

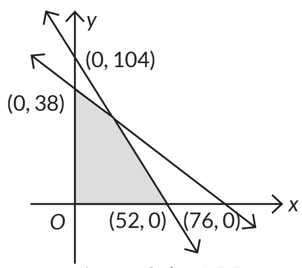
2. The solution set of the inequation $3x + 5y < 7$ is
(a) whole *xy*-plane except the points lying on the line $3x + 5y = 7$.
(b) whole *xy*-plane along with the points lying on the line $3x + 5y = 7$.
(c) open half plane containing the origin except the points of line $3x + 5y = 7$.
(d) open half plane not containing the origin.
(2023) U

3. If the corner points of the feasible region of an LPP are (0, 3), (3, 2) and (0, 5), then the minimum value of $Z = 11x + 7y$ is
(a) 21 (b) 33 (c) 14 (d) 35
(Term I, 2021-22) Ev

4. The number of solutions of the system of inequations $x + 2y \le 3$, $3x + 4y \ge 12$, $x \ge 0$, $y \ge 1$ is
(a) 0 (b) 2 (c) finite (d) infinite
(Term I, 2021-22) U

5. The maximum value of $Z = 3x + 4y$ subject to the constraints $x \ge 0$, $y \ge 0$ and $x + y \le 1$ is
(a) 7 (b) 4 (c) 3 (d) 10
(Term I, 2021-22) Ev

6. The feasible region of an LPP is given in the following figure



Then, the constraints of the LPP are $x \ge 0$, $y \ge 0$ and
(a) $2x + y \le 52$ and $x + 2y \le 76$
(b) $2x + y \le 104$ and $x + 2y \le 76$
(c) $x + 2y \le 104$ and $2x + y \le 76$
(d) $x + 2y \le 104$ and $2x + y \le 38$
(Term I, 2021-22) Ap

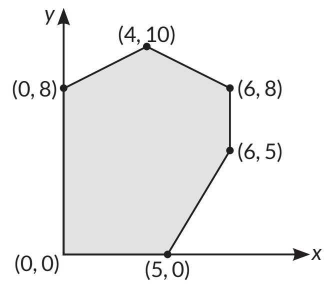
7. If the minimum value of an objective function $Z = ax + by$ occurs at two points (3, 4) and (4, 3) then
(a) $a + b = 0$ (b) $a = b$
(c) $3a = b$ (d) $a = 3b$
(Term I, 2021-22) U

8. For the following LPP, maximise $Z = 3x + 4y$ subject to constraints $x - y \ge -1$, $x \le 3$, $x \ge 0$, $y \ge 0$ the maximum value is
(a) 0 (b) 4 (c) 25 (d) 30
(Term I, 2021-22) Ap

9. The corner points of the feasible region determined by the system of linear inequalities are (0, 0), (4, 0), (2, 4) and (0, 5). If the maximum value of $z = ax + by$, where $a, b > 0$ occurs at both (2, 4) and (4, 0), then
(a) $a = 2b$ (b) $2a = b$
(c) $a = b$ (d) $3a = b$ *(2020)* U

10. In an LPP, if the objective function $z = ax + by$ has the same maximum value on two corner points of the feasible region, then the number of points at which z_{max} occurs is
(a) 0 (b) 2 (c) finite (d) infinite
(2020) U

11. The feasible region for an LPP is shown below :
Let $z = 3x - 4y$ be the objective function. Minimum of z occurs at



(a) (0, 0) (b) (0, 8)
(c) (5, 0) (d) (4, 10)
(NCERT Exemplar, 2020) Ap

12. The graph of the inequality $2x + 3y > 6$ is
(a) half plane that contains the origin
(b) half plane that neither contains the origin nor the points of the line $2x + 3y = 6$.
(c) whole *XOY*-plane excluding the points on the line $2x + 3y = 6$.
(d) entire *XOY*-plane. *(2020)* U

13. The objective function of an LPP is
(a) a constant
(b) a linear function to be optimised
(c) an inequality
(d) a quadratic expression *(2020)* R

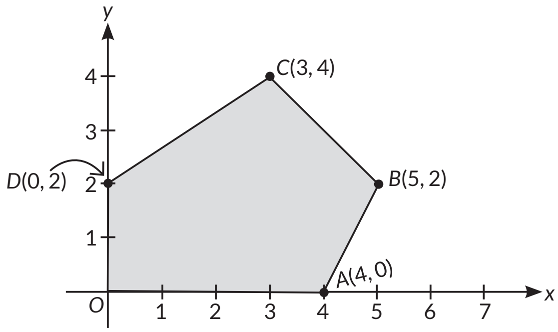
SA II (3 marks)

14. Solve the following linear programming problem graphically :
Maximise $z = -3x - 5y$
Subject to the constraints
$-2x + y \le 4$
$x + y \ge 3$
$x - 2y \le 2$,
$x \ge 0, y \ge 0$. *(2023)* EV

LA I (4 marks)

15. Solve the following linear programming problem graphically :
Maximize $z = 3x + 9y$
Subject to constraints
$x + 3y \leq 60$
$x + y \geq 10$
$x \leq y$
$x, y \geq 0$ *(2021)* Ev

16. The corner points of the feasible region determined by the system of linear inequations are as shown below:



Answer each of the following :
(i) Let $z = 13x - 15y$ be the objective function. Find the maximum and minimum values of z and also the corresponding points at which the maximum and minimum values occur.
(ii) Let $z = kx + y$ be the objective function. Find k, if the value of z at A is same as the value of z at B. *(2021)*

17. Solve the following LPP graphically :
Minimize $z = 5x + 7y$
Subject to the constraints
$2x + y \geq 8, x + 2y \geq 10, \ x, y \geq 0$ *(2020)* Ev

18. Solve the following LPP graphically :
Minimise $Z = 5x + 10y$
Subject to constraints $x + 2y \leq 120, x + y \geq 60,$
$x - 2y \geq 0$ and $x, y \geq 0$
(NCERT Exemplar, Delhi 2017) Ev

19. Maximise $Z = x + 2y$
Subject to the constraints:
$x + 2y \geq 100, 2x - y < 0, \ 2x + y \leq 200, \ x, y \geq 0$
Solve the above LPP graphically.
(NCERT, AI 2017) Ev

LA II (5 / 6 marks)

20. Solve the following linear programming problem graphically.
Maximize : $P = 70x + 40y$
Subject to : $3x + 2y \leq 9, 3x + y \leq 9, x \geq 0, y \geq 0.$
(2023) Ev

21. Solve the following linear programming problem graphically.
Minimize : $Z = 60x + 80y$
Subject to constraints:
$3x + 4y \geq 8$
$5x + 2y \geq 11$
$x, y \geq 0$ *(2023)* Cr

22. Find graphically, the maximum value of $z = 2x + 5y$, subject to constraints given below:
$2x + 4y \leq 8, 3x + y \leq 6, x + y \leq 4; x \geq 0, y \geq 0$
(Delhi 2015) Ev

23. Maximise $z = 8x + 9y$ subject to the constraints given below :
$2x + 3y \leq 6, 3x - 2y \leq 6, y \leq 1; x, y \geq 0$
(Foreign 2015) Ev

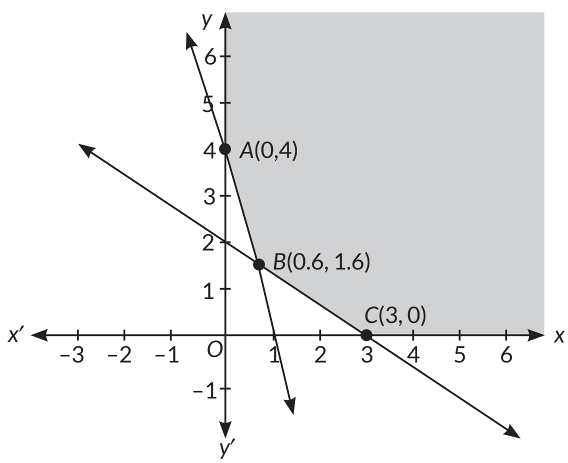
CBSE Sample Questions

12.2 Linear Programming Problem and its Mathematical Formulation

MCQ

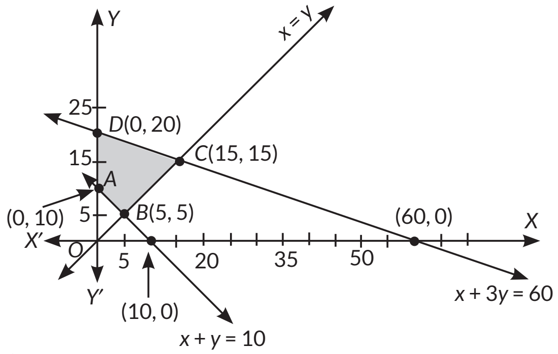
1. The solution set of the inequality $3x + 5y < 4$ is
(a) an open half-plane not containing the origin.
(b) an open half-plane containing the origin.
(c) the whole XY-plane not containing the line $3x + 5y = 4$.
(d) a closed half plane containing the origin.
(2022-23) Ev

2. The corner points of the shaded unbounded feasible region of an LPP are (0, 4), (0.6, 1.6) and (3, 0) as shown in the figure. The minimum value of the objective function $Z = 4x + 6y$ occurs at



(a) (0.6, 1.6) only
(b) (3, 0) only
(c) (0.6, 1.6) and (3, 0) only
(d) at every point of the line-segment joining the points (0.6, 1.6) and (3, 0) *(2022-23)* Ev

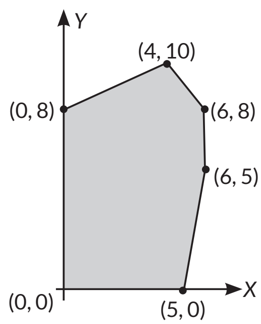
3. Based on the given shaded region as the feasible region in the graph, at which point(s) is the objective function $Z = 3x + 9y$ maximum?



(a) Point B
(b) Point C
(c) Point D
(d) every point on the line segment CD

(Term I, 2021-22) U

4. In the given graph, the feasible region for a LPP is shaded. The objective function $Z = 2x - 3y$, will be minimum at



(a) (4, 10) (b) (6, 8)
(c) (0, 8) (d) (6, 5)

(Term I, 2021-22) Ap

5. A linear programming problem is as follows :
Minimize $Z = 30x + 50y$
Subject to the constraints,
$3x + 5y \geq 15$
$2x + 3y \leq 18$
$x \geq 0, y \geq 0$
In the feasible region, the minimum value of Z occurs at
(a) a unique point
(b) no point
(c) infinitely many points
(d) two points only *(Term I, 2021-22)* U

6. For an objective function $Z = ax + by$, where $a, b > 0$; the corner points of the feasible region determined by a set of constraints (linear inequalities) are (0, 20), (10, 10), (30, 30) and (0, 40). The condition on a and b such that the maximum Z occurs at both the points (30, 30) and (0, 40) is
(a) $b - 3a = 0$ (b) $a = 3b$
(c) $a + 2b = 0$ (d) $2a - b = 0$

(Term I, 2021-22)

7. In a linear programming problem, the constraints on the decision variables x and y are $x - 3y \geq 0$, $y \geq 0$, $0 \leq x \leq 3$. The feasible region
(a) is not in the first quadrant
(b) is bounded in the first quadrant
(c) is unbounded in the first quadrant
(d) does not exist *(Term I, 2021-22)* U

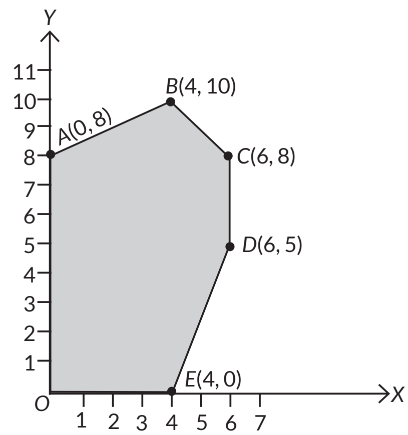
SA II (3 marks)

8. Solve the following Linear Programming Problem graphically:
Maximize $Z = 400x + 300y$ subject to $x + y \leq 200$, $x \leq 40$, $x \geq 20$, $y \geq 0$ *(2022-23)* Ev

LA II (5 / 6 marks)

9. Solve the following linear programming problem (L.P.P) graphically.
Maximize $Z = 3x + y$
Subject to constraints;
$x + 2y \geq 100$
$2x - y \leq 0$
$2x + y \leq 200$
$x, y \geq 0$ *(2020-21)* Ap

10. The corner points of the feasible region determined by the system of linear constraints are as shown below:



Answer each of the following:
(i) Let $Z = 3x - 4y$ be the objective function. Find the maximum and minimum value of Z and also the corresponding points at which the maximum and minimum value occurs.
(ii) Let $Z = px + qy$,where $p, q > 0$ be the objective function. Find the condition on p and q so that the maximum value of Z occurs at $B(4,10)$ and $C(6, 8)$. Also mention the number of optimal solutions in this case. *(2020-21)* Ev

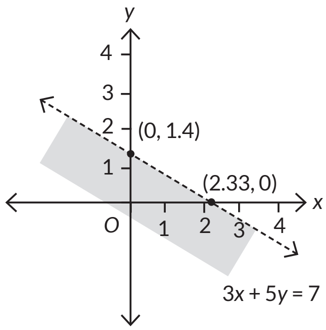
Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. **(d):** We have, $2x + y \leq 10$ and $x + 2y \geq 8$
Let us check which of the given points satisfy the given inequation one by one.
(a) $(-2, 4)$
$2 \times (-2) + 4 = -4 + 4 = 0 \leq 10$
and $-2 + 2 \times 4 = -2 + 8 = 6 \ngeq 8$
(b) $(3, 2)$
$2 \times 3 + 2 = 6 + 2 = 8 \leq 10$
$3 + 2 \times 2 = 3 + 4 = 7 \ngeq 8$
(c) $(-5, 6)$
$2 \times (-5) + 6 = -10 + 6 = -4 \leq 10$
$-5 + 2 \times 6 = -5 + 12 = 7 \ngeq 8$
(d) $(4, 2)$
$2 \times 4 + 2 = 10 \leq 10\,;\, 4 + 2 \times 2 = 8 \geq 8$
$\therefore$ (4, 2) satisfy both the inequations.

2. **(c):** Given inequation is $3x + 5y < 7$
Let us draw the graph of $3x + 5y = 7$

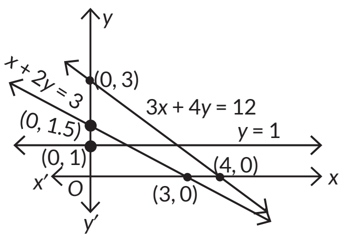
x	0	2.33
y	1.4	0



Substitute, $x = 0$ and $y = 0$ in the inequation, we get
$3(0) + 5(0) < 7$
i.e., $0 < 7$ which is true.
$\therefore$ The solution set of the inequality is an open half plane containing the origin except the points on line $3x + 5y = 7$.

3. **(a):** Given, $Z = 11x + 7y$
At (0, 3), $Z = 11 \times 0 + 7 \times 3 = 21$
At (3, 2), $Z = 11 \times 3 + 7 \times 2 = 47$
At (0, 5), $Z = 11 \times 0 + 7 \times 5 = 35$
Thus, Z is minimum at (0, 3) and minimum value of Z is 21.

4. **(a):** Given,
$x + 2y \leq 3$, $3x + 4y \geq 12$, $x \geq 0$, $y \geq 1$
The graph of given constraints is shown here.

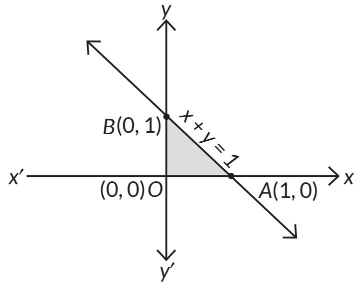


Since, there is no common region, so, no solution exists.

Key Points

- A feasible region is an area defined by a set of coordinates that satisfy a system of inequalities.

5. **(b):** We have to maximise $Z = 3x + 4y$
Subject to constraints, $x \geq 0$, $y \geq 0$ and $x + y \leq 1$



The shaded portion OAB is the feasible region, where $O(0, 0)$, $A(1, 0)$ and $B(0, 1)$ are the corner points.
At $O(0, 0)$, $Z = 3 \times 0 + 4 \times 0 = 0$
At $A(1, 0)$, $Z = 3 \times 1 + 4 \times 0 = 3$
At $B(0, 1)$, $Z = 3 \times 0 + 4 \times 1 = 4$
$\therefore$ Maximum value of Z is 4, which occurs at $B(0, 1)$.

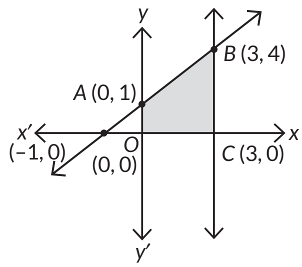
Concept Applied

- Any point in the feasible region of a linear programming problem that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

6. **(b):** Clearly, the pair of points given in graph, and (0, 104); (52, 0) and (0, 38); (76, 0) satisfy the corresponding equations given in option(b) *i.e.*, $2x + y \leq 104$ and $x + 2y \leq 76$.

7. **(b):** Since, minimum value of $Z = ax + by$ occurs at two points (3, 4) and (4, 3).
$\therefore$ $3a + 4b = 4a + 3b \Rightarrow a = b$

8. **(c):** Given, $Z = 3x + 4y$
Subject to constraints, $x - y \geq -1$, $x \leq 3$; $x \geq 0$, $y \geq 0$



The shaded region $OABC$ is the feasible region, where corner points are $O(0, 0)$, $A(0, 1)$, $B(3, 4)$ and $C(3, 0)$
At O (0, 0), $Z = 3(0) + 4(0) = 0$
At A (0, 1), $Z = 3(0) + 4(1) = 4$
At B (3, 4), $Z = 3(3) + 4(4) = 25$
At C (3, 0), $Z = 3(3) + 4(0) = 9$
$\therefore$ Maximum value of Z is 25, which occurs at $B(3, 4)$.

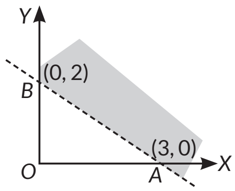
9. **(a):** Since, maximum value of $z = ax + by$ occurs at both (2, 4) and (4, 0).
$\therefore$ $2a + 4b = 4a + 0 \Rightarrow 4b = 2a \Rightarrow 2b = a$

10. **(d):** In an LPP, if the objective function $z = ax + by$ has the same maximum value on two corner points of the feasible region, then the number of points at which z_{max} occurs is infinite.

11. (b) : We know that minimum of objective function occurs at corner points.

Corner points	Value of $z = 3x - 4y$
(0, 0)	0
(5, 0)	15
(6, 5)	–2
(6, 8)	–14
(4, 10)	–28
(0, 8)	–32 ← Minimum

12. (b) : From the graph of inequality $2x + 3y > 6$. It is clear that it does not contain the origin nor the points of the line $2x + 3y = 6$.



13. (b) : A linear function to be optimized is called an objective function.

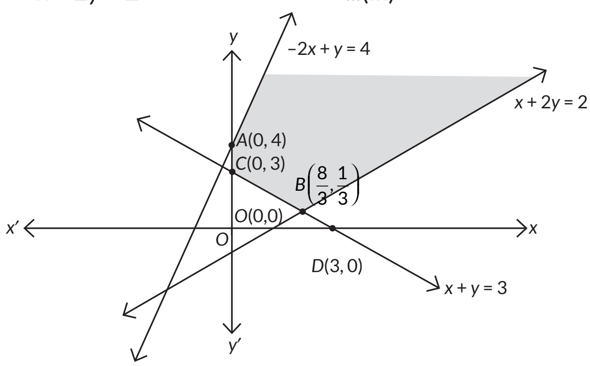
14. We have, maximise $z = -3x - 5y$

Converting the given inequations into equations, we get

$-2x + y = 4$ ...(i)

$x + y = 3$ ...(ii)

$x - 2y = 2$ ...(iii)



We draw the graph of these lines.

As, $x \geq 0, y \geq 0$ so the solution lies in first quadrant.

From graph, corner point of feasible region are $A(0, 4)$, $B(8/3, 1/3)$ and $C(0, 3)$

The value of z at these corner points are shown as :

Corner points	$z = -3x - 5y$	
$A(0, 4)$	–20	
$B(8/3, 1/3)$	–29/3	← Maximum
$C(0, 3)$	–15	

Hence maximum value of $z = \dfrac{-29}{3}$.

15. We have, maximize $z = 3x + 9y$

Subject to constraints, $x + 3y \leq 60$, $x + y \geq 10$, $x \leq y$, $x, y \geq 0$

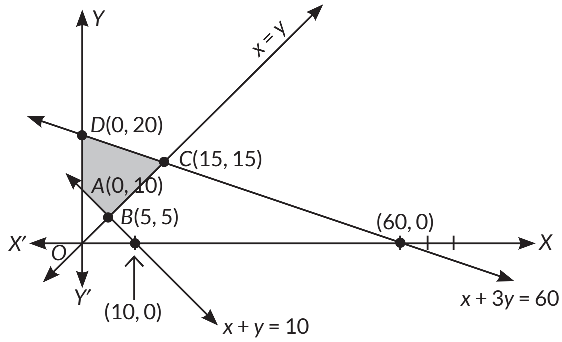
To solve L.P.P. graphically, we convert inequations into equations.

$l_1 : x + 3y = 60$, $l_2 : x + y = 10$, $l_3 : x = y$, $x = 0$ and $y = 0$

l_2 and l_3 intersect at (5, 5). l_1 and l_3 intersect at (15, 15).

The shaded region $ABCD$ is the feasible region and is bounded. The corner points of the feasible region are

$A(0, 10)$, $B(5, 5)$, $C(15, 15)$ and $D(0, 20)$



Corner Points	Value of $z = 3x + 9y$	
$A(0, 10)$	90	
$B(5, 5)$	60	
$C(15, 15)$	180	Maximum (Multiple optimal solutions)
$D(0, 20)$	180	

The maximum value of Z on the feasible region occurs at the two corner points $C(15, 15)$ and $D(0, 20)$ and it is 180 in each case.

16. (i)

Corner Points	$z = 13x - 15y$
$O(0, 0)$	0
$A(4, 0)$	52 (Maximum)
$B(5, 2)$	35
$C(3, 4)$	–21
$D(0, 2)$	–30 (Minimum)

Thus, maximum value of Z is 52 at $A(4, 0)$ and minimum value of Z is –30 at $D(0, 2)$

(ii) Since value of $z = kx + y$ at $A(4, 0)$ is same as the value of Z at $B(5, 2)$.

$\therefore \quad k \cdot 4 + 0 = k \cdot 5 + 2 \Rightarrow 4k = 5k + 2 \Rightarrow k = -2$

17. We have, minimize $z = 5x + 7y$,

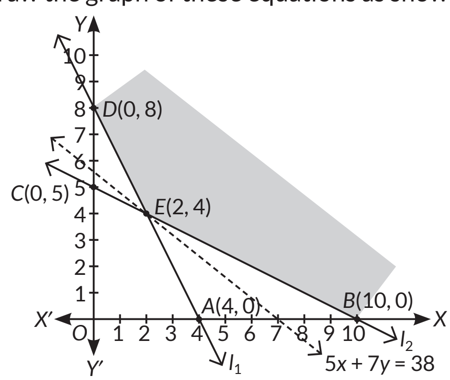
Subject to constraints, $2x + y \geq 8$, $x + 2y \geq 10$, $x, y \geq 0$

To solve LPP graphically, we convert inequations into equations.

Now, $l_1 : 2x + y = 8$, $l_2 : x + 2y = 10$ and $x = 0, y = 0$

l_1 and l_2 intersect at $E(2, 4)$.

Let us draw the graph of these equations as shown below.



The corner points of the feasible region are $D(0, 8)$, $B(10, 0)$ and $E(2, 4)$.

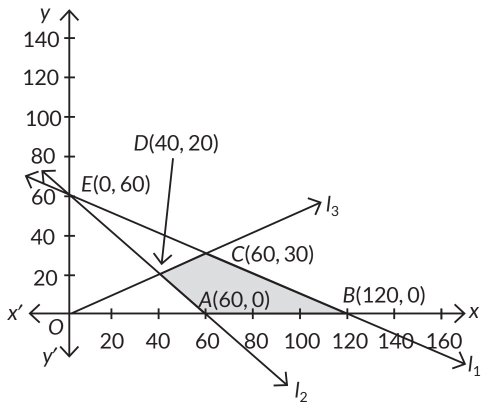
Corner points	Value of $z = 5x + 7y$
D (0, 8)	56
B (10, 0)	50
E (2, 4)	38 (Minimum)

From the table, we find that 38 is the minimum value of z at $E(2, 4)$. Since the region is unbounded, so we draw the graph of inequality $5x + 7y < 38$ to check whether the resulting open half plane has any point common with the feasible region. Since it has no point in common. So, the minimum value of z is obtained at $E(2, 4)$ and the minimum value of $z = 38$.

Answer Tips

- If the region is unbounded, then a maximum or minimum value of the objective function may not exist. If it exists, it must occur at a corner point of region.

18. We have, Minimise $Z = 5x + 10y$,
Subject to constraints :
$x + 2y \le 120$
$x + y \ge 60$
$x - 2y \ge 0$ and $x, y \ge 0$
To solve L.P.P graphically, we convert inequations into equations.
$l_1 : x + 2y = 120$, $l_2 : x + y = 60$, $l_3 : x - 2y = 0$ and $x = 0, y = 0$
l_1 and l_2 intersect at $E(0, 60)$, l_1 and l_3 intersect at $C(60, 30)$, l_2 and l_3 intersect at $D(40, 20)$.
The shaded region $ABCD$ is the feasible region and is bounded. The corner points of the feasible region are $A(60, 0)$, $B(120, 0)$, $C(60, 30)$ and $D(40, 20)$.



Corner points	Value of $Z = 5x + 10y$
$A(60, 0)$	300 ← (Minimum)
$B(120, 0)$	600
$C(60, 30)$	600
$D(40, 20)$	400

Hence, Z is minimum at $A(60, 0)$ *i.e.*, 300.

Commonly Made Mistake

- Remember to convert inequations into equations.

19. Maximise $Z = x + 2y$, Subject to constraints :
$x + 2y \ge 100$, $2x - y < 0$, $2x + y \le 200$ and $x, y \ge 0$.
Converting the inequations into equations, we obtain the lines

$l_1 : x + 2y = 100$ ...(i)
$l_2 : 2x - y = 0$ ...(ii)
$l_3 : 2x + y = 200$ ...(iii)
$l_4 : x = 0$ ...(iv)
and $l_5 : y = 0$ ...(v)

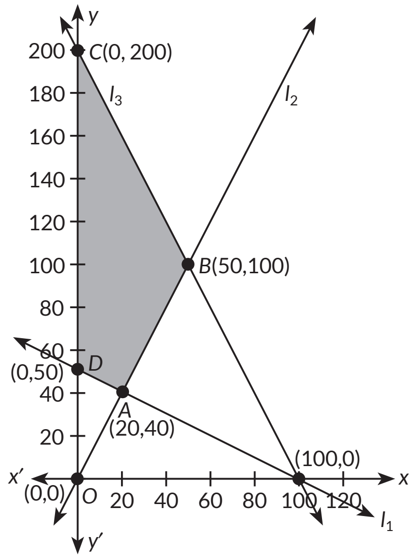
By intercept form, we get

$l_1 : \frac{x}{100} + \frac{y}{50} = 1$

$\Rightarrow$ The line l_1 meets the coordinate axes at (100, 0) and (0, 50).
$l_2 : 2x = y$
$\Rightarrow$ The line l_2 passes through origin and cuts l_1 and l_3 at (20, 40) and (50, 100) respectively.

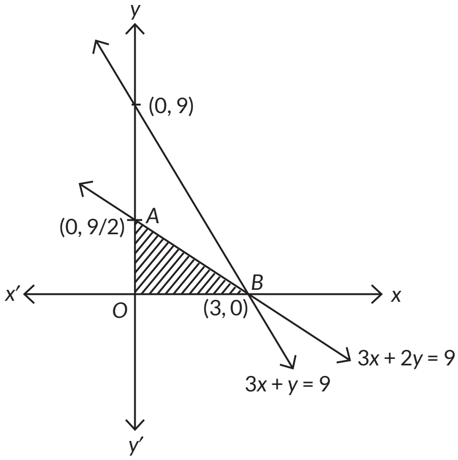
$l_3 : \frac{x}{100} + \frac{y}{200} = 1$

$\Rightarrow$ The line l_3 meets the coordinates axes at (100, 0) and (0, 200).
$l_4 : x = 0$ is the y-axis, $l_5 : y = 0$ is the x-axis



Now, plotting the above points on the graph, we get the feasible region of the LPP as shaded region $ABCD$. The coordinates of the corner points of the feasible region $ABCD$ are $A(20, 40)$, $B(50, 100)$, $C(0, 200)$, $D(0, 50)$.
Now, $Z_A = 20 + 2 \times 40 = 100$
$Z_B = 50 + 2 \times 100 = 250$, $Z_C = 0 + 2 \times 200 = 400$
$Z_D = 0 + 2 \times 50 = 100$
$\therefore$ Z is maximum at $C(0, 200)$ and having value 400.

20. We have, maximize $P = 70x + 40y$
Subject to : $3x + 2y \le 9$
$3x + y \le 9$
$x \ge 0, y \ge 0$
Convert all inequations into equation, we get
$3x + 2y = 9$ ... (i)
$3x + y = 9$ ... (ii)
$x = 0$ and $y = 0$
Solving (i) and (ii), we get
$x = 3, y = 0$
So, point of intersection of equation (i) and (ii) are (3, 0).



The given shaded region is the feasible region.

The corner points of the feasible region are $O(0, 0)$, $A(0, 9/2)$ and $B(3, 0)$.

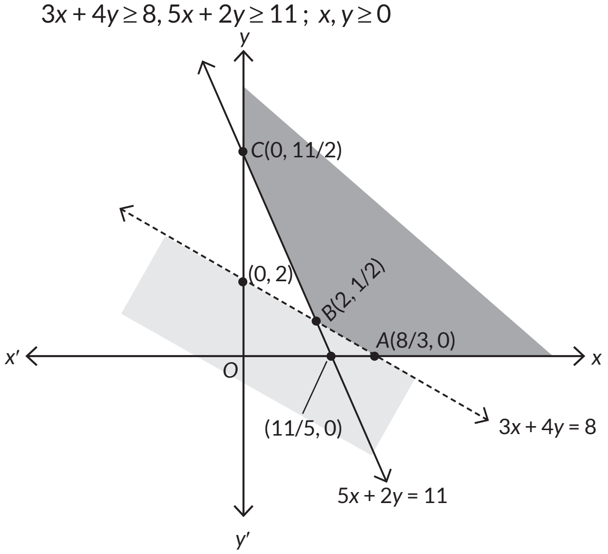
Corner points	Value of $p = 70x + 40y$
$O(0, 0)$	$70 \times 0 + 40 \times 0 = 0$
$A(0, 9/2)$	$70 \times 0 + 40 \times \frac{9}{2} = 180$
$B(3, 0)$	$70 \times 3 + 40 \times 0 = 210$ (maximum)

So, P is maximum at point $B(3, 0)$.

21. We have, min$z = 60x + 80y$;

Subject to constraints ;

$3x + 4y \geq 8,\ 5x + 2y \geq 11;\ x, y \geq 0$



From graph, it is clear that feasible region is unbounded.

The corner point of the feasible region are $A(8/3, 0)$, $B(2, 1/2)$ and $C(0, 11/2)$.

The value of Z at these corner points are as follows :

Corner Points	$Z = 60x + 80y$
$A(8/3, 0)$	160 (Minimum)
$B(2, 1/2)$	160 (Minimum)
$C(0, 11/2)$	440

As the feasible region is unbounded,

$\therefore$ 160 may or may not be the minimum value of Z.

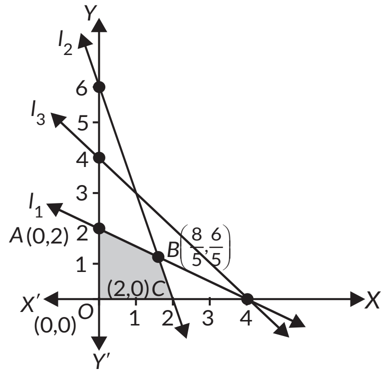
So, we graph the inequality $60x + 80y < 160$ *i.e.*, $3x + 4y < 8$ and check whether the resulting half plane has points in common with the feasible region or not.

From graph, it can be seen that feasible region has no common point with $3x + 4y < 8$

$\therefore$ Minimum value of Z is 160 at the line joining the points (8/3, 0) and (2, 1/2).

22. Let $l_1 : 2x + 4y = 8, l_2 : 3x + y = 6, l_3 : x + y = 4; x = 0, y = 0$

Solving l_1 and l_2 we get $B\left(\frac{8}{5}, \frac{6}{5}\right)$



Shaded portion $OABC$ is the feasible region, where coordinates of the corner points are $O(0, 0)$, $A(0, 2)$, $B\left(\frac{8}{5}, \frac{6}{5}\right)$, $C(2, 0)$

The value of objective function at these points are :

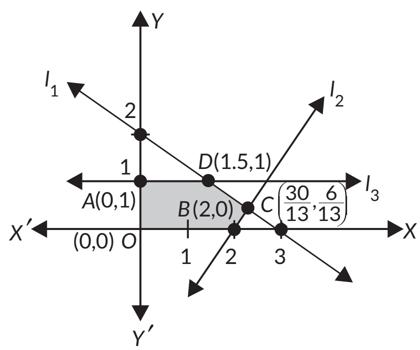
Corner points	Value of the objective function $z = 2x + 5y$
$O(0, 0)$	$2 \times 0 + 5 \times 0 = 0$
$A(0, 2)$	$2 \times 0 + 5 \times 2 = 10$ (Maximum)
$B\left(\frac{8}{5}, \frac{6}{5}\right)$	$2\times\frac{8}{5}+5\times\frac{6}{5}=9.2$
$C(2, 0)$	$2 \times 2 + 5 \times 0 = 4$

$\therefore$ The maximum value of z is 10, which is at $A(0, 2)$.

Concept Applied

- If the region is bounded then the objective function Z has both maximum and minimum value of region.

23. Let $l_1 : 2x + 3y = 6, l_2 : 3x - 2y = 6, l_3 : y = 1; x = 0, y = 0$



Solving l_1 and l_3, we get D (1.5, 1)

Solving l_1 and l_2, we get $C\left(\frac{30}{13}, \frac{6}{13}\right)$

Shaded portion $OADCB$ is the feasible region, where coordinates of the corner points are $O(0, 0)$, $A(0, 1)$, $D(1.5, 1)$, $C\left(\frac{30}{13}, \frac{6}{13}\right)$, $B(2, 0)$.

The value of the objective function at these points are :

Corner points	Value of the objective function $z = 8x + 9y$
$O(0, 0)$	$8 \times 0 + 9 \times 0 = 0$
$A(0, 1)$	$8 \times 0 + 9 \times 1 = 9$
$D(1.5, 1)$	$8 \times 1.5 + 9 \times 1 = 21$
$C\left(\frac{30}{13}, \frac{6}{13}\right)$	$8 \times \frac{30}{13} + 9 \times \frac{6}{13} = 22.6$ (Maximum)
$B(2, 0)$	$8 \times 2 + 9 \times 0 = 16$

The maximum value of z is 22.6, which is at $C\left(\frac{30}{13}, \frac{6}{13}\right)$.

Commonly Made Mistake

- Remember the difference between feasible solutions and infeasible solutions.

CBSE Sample Questions

1. **(b):** The strict inequality represents an open half plane and it contains the origin, as (0, 0) satisfies it. **(1)**

2. **(d):** The minimum value of the objective function occurs at two adjacent corner points (0.6, 1.6) and (3, 0) and there is no point in the half plane $4x + 6y < 12$ in common with the feasible region.
So, the minimum value occurs at every point of the line-segment joining the two points. **(1)**

3. **(d):** We have,

Corner points	Value of $Z = 3x + 9y$
$A(0, 10)$	$3 \times 0 + 9 \times 10 = 90$
$B(5, 5)$	$3 \times 5 + 9 \times 5 = 60$
$C(15, 15)$	$3 \times 15 + 9 \times 15 = 180$ (Maximum)
$D(0, 20)$	$3 \times 0 + 9 \times 20 = 180$ (Maximum)

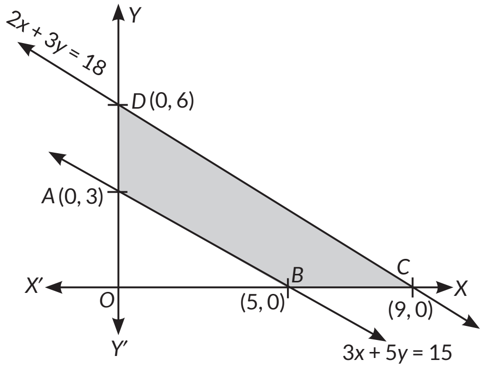
$\because$ Z is maximum at $C(15, 15)$ and $D(0, 20)$.
$\therefore$ Z is maximum at every point on the line joining CD. **(1)**

4. **(c):** We have,

Corner points	Value of $Z = 2x - 3y$
(0, 0)	$2 \times 0 - 3 \times 0 = 0$
(0, 8)	$2 \times 0 - 3 \times 8 = -24$ (Minimum)
(4, 10)	$2 \times 4 - 3 \times 10 = -22$
(6, 8)	$2 \times 6 - 3 \times 8 = -12$
(6, 5)	$2 \times 6 - 3 \times 5 = -3$
(5, 0)	$2 \times 5 - 3 \times 0 = 10$

$\therefore$ Value of Z is minimum at (0, 8). **(1)**

5. **(c):** Here, the feasible region is shaded.

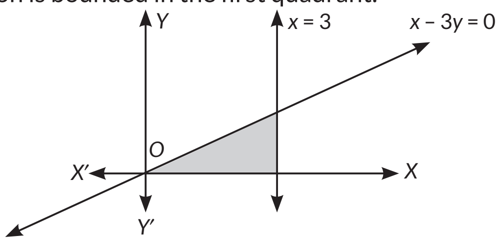


Corner points	Value of $Z = 30x + 50y$
$A(0, 3)$	$30 \times 0 + 50 \times 3 = 150$ (Minimum)
$B(5, 0)$	$30 \times 5 + 50 \times 0 = 150$ (Minimum)
$C(9, 0)$	$30 \times 9 + 50 \times 0 = 270$
$D(0, 6)$	$30 \times 0 + 50 \times 6 = 300$

Since, minimum value of Z occurs at both A and B. So, Z is minimum at every point on the line joining AB. So, minimum value of Z occurs at infinitely many points. **(1)**

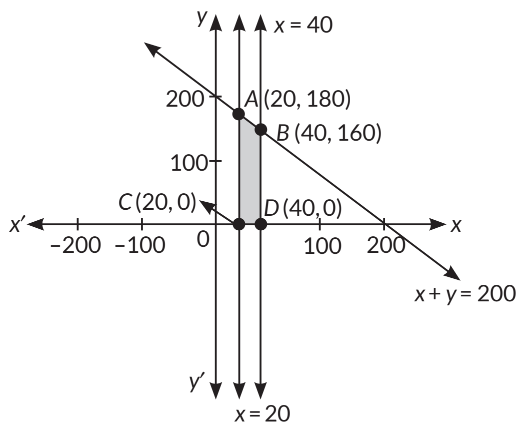
6. **(a):** As, Z is maximum at (30, 30) and (0, 40).
$\Rightarrow$ $30a + 30b = 40b \Rightarrow b - 3a = 0$ **(1)**

7. **(b):** From the graph, we can say that the feasible region is bounded in the first quadrant.



(1)

8. We have $Z = 400x + 300y$ subject to
$x + y \le 200, x \le 40, x \ge 20, y \ge 0$
The corner points of the feasible region are $C(20, 0)$, $D(40, 0)$, $B(40, 1600)$, $A(20, 180)$



(1)

Corner points	$Z = 400x + 300y$
$C(20, 0)$	8000
$D(40, 0)$	16000
$B(40, 160)$	64000
$A(20, 180)$	62000

(1)

Maximum profit occurs at $x = 40$, $y = 160$

and the maximum profit = ₹ 64,000 **(1)**

9. Maximize $Z = 3x + y$

Subject to constraints

$x + 2y \geq 100$

$2x - y \leq 0$

$2x + y \leq 200$

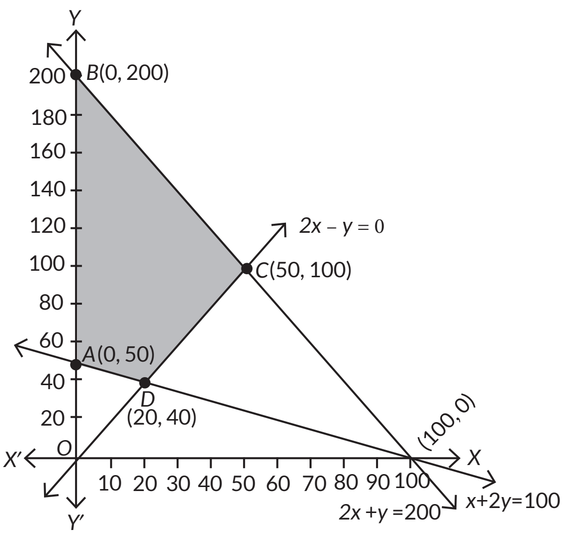
$x \geq 0, y \geq 0$

Converting the given inequations into equations, we get

$x + 2y = 100$ ... (i) $2x - y = 0$ ... (ii)

$2x + y = 200$ ... (iii)

Now, draw the graphs of (i), (ii) and (iii).



(3)

The feasible region is shaded region and corner points are $A(0, 50)$, $B(0, 200)$, $C(50, 100)$ and $D(20, 40)$. **(1)**

The values of Z at corner points are shown in the following table:

Corner points	$Z = 3x + y$
A (0, 50)	50
B (0, 200)	200
C (50, 100)	250 (Maximum)
D (20, 40)	100

Thus, maximum value of Z is 250 at $x = 50$, $y = 100$. **(1)**

10. (i)

Corner points	$Z = 3x - 4y$
O (0, 0)	0
A (0, 8)	–32 (Minimum)
B (4, 10)	–28
C (6, 8)	–14
D (6, 5)	–2
E (4, 0)	12 (Maximum)

(1½)

Thus, maximum value of Z is 12 at $E(4, 0)$.

and minimum value of Z is –32 at $A(0, 8)$. **(1)**

(ii) Since maximum value of Z occurs at $B(4, 10)$ and $C(6, 8)$.

$\therefore \quad 4p + 10q = 6p + 8q \Rightarrow 2q = 2p \Rightarrow p = q$ **(2)**

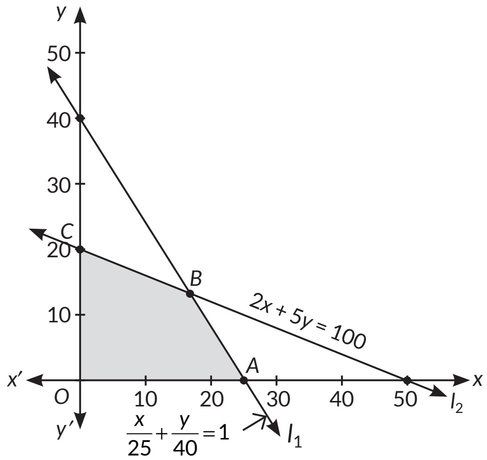
Number of optional solutions are infinite.

[$\because$ Every point on the line segment BC joining the two corner points B and C also give the same maximum value] **(1/2)**

Self Assessment

Case Based Objective Questions (4 marks)

1. Deepa rides her car at 25 km/hr. She has to spend ₹ 2 per km on diesel and if she rides it at a faster speed of 40 km/hr, the diesel cost increases to ₹ 5 per km. She has ₹ 100 to spend on diesel. Let she travels x kms with speed 25 km/hr and y kms with speed 40 km/hr. The feasible region for the LPP is shown below :



Based on above information, attempt any 4 out of 5 subparts.

(i) What is the point of intersection of line l_1 and l_2?

(a) $\left(\frac{40}{3},\frac{50}{3}\right)$ (b) $\left(\frac{50}{3},\frac{40}{3}\right)$

(c) $\left(\frac{-50}{3},\frac{40}{3}\right)$ (d) $\left(\frac{-50}{3},\frac{-40}{3}\right)$

(ii) The corner points of the feasible region shown in above graph are

(a) (0, 25), (20, 0), $\left(\frac{40}{3},\frac{50}{3}\right)$

(b) (0, 0), (25, 0), (0, 20)

(c) (0, 0), $\left(\frac{40}{3},\frac{50}{3}\right)$, (0, 20)

(d) (0, 0), (25, 0), $\left(\frac{50}{3},\frac{40}{3}\right)$, (0, 20)

(iii) If $Z = x + y$ be the objective function and max $Z = 30$. The maximum value occurs at point

(a) $\left(\frac{50}{3},\frac{40}{3}\right)$ (b) (0, 0)

(c) (25, 0) (d) (0, 20)

(iv) If $Z = 6x - 9y$ be the objective function, then maximum value of Z is

(a) –20 (b) 150

(c) 180 (d) 20

(v) If $Z = 6x + 3y$ be the objective function, then what is the minimum value of Z?

(a) 120 (b) 130 (c) 0 (d) 150

Multiple Choice Questions (1 mark)

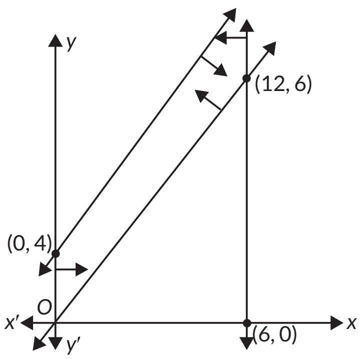
2. The optimal value of the objective function is attained at the points
(a) on X-axis
(b) on Y-axis
(c) which are corner points of the feasible region
(d) none of these

OR

Region represented by $x \geq 0, y \geq 0$ is
(a) first quadrant (b) second quadrant
(c) third quadrant (d) fourth quadrant

3. Which of the following statement is false?
(a) The feasible region is always a concave region.
(b) The maximum (or minimum) solution of the objective function occurs at the vertex of the feasible region.
(c) If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.
(d) None of these

4. The feasible region for an LPP is shown in the following figure. Let $F = 3x - 4y$ be the objective function. Maximum value of F is



(a) 0 (b) 8
(c) 12 (d) –18

5. Corner points of the feasible region determined by the system of linear constraints are (0, 3), (1, 1) and (3, 0). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q, so that the minimum of Z occurs at (3, 0) and (1, 1) is

(a) $p = 2q$ (b) $p=\frac{q}{2}$

(c) $p = 3q$ (d) $p = q$

6. Which of the following sets is convex?
   (a) $\{(x, y) : x^2 + y^2 \geq 1\}$ (b) $\{(x, y) : y^2 \geq x\}$
   (c) $\{(x, y) : 3x^2 + 4y^2 \geq 5\}$ (d) $\{(x, y) : y \geq 2, y \leq 4\}$

VSA Type Questions (1 mark)

7. If the feasible region for a LPP is ______, then the optimal value of the objective function $Z = ax + by$ may or may not exist.

8. The value of objective function is maximum under linear constraints at ______ of feasible region.

9. In a LPP, if the objective function $Z = ax + by$ has the same maximum value on two corner points of the feasible region, then every point on the line segment joining these two points give the same ______ value.

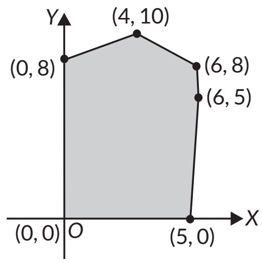
10. The feasible region for a LPP is always a ______ polygon.

OR

Corner points of the feasible region for an LPP are (0, 2), (3, 0), (6, 0), (6, 8) and (0, 5). Let $F = 4x + 6y$ be the objective function. Then minimum value of F occurs at which point?

Case Based Questions (4 marks)

11. **Case Study -** Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.
    Here are the following steps to solve an LPP.
    Step 1 : Convert inequations into equations.
    Step 2 : Find the point of intersection.
    Step 3 : Draw the graph of inequations.
    Step 4 : Find the value of the objective function corresponding to each corner point.
    Based on the given information, answer the following questions.
    (i) In solving the LPP : "minimize $f = 6x + 10y$ subject to constraints $x \geq 6, y \geq 2, 2x + y \geq 10, x \geq 0, y \geq 0$", what are the redundant constraints ?
    (ii) The feasible region for a LPP is shown shaded in the figure.



Let $Z = 3x - 4y$ be the objective function. What is the minimum value of Z?

LA Type Questions (4 / 6 marks)

12. Minimise $Z = 30x + 20y$ subject to
    $x + y \leq 8, x + 4y \geq 12, 5x + 8y \geq 20, x, y \geq 0$

13. Let $z = x + y$, then what is the maximum of z subject to constraints $y \geq |x| - 1, y \leq 1 - |x|, x \geq 0, y \geq 0$?

14. What is the maximum value of $Z = 4x + y$ subject to the constraints, $x + y \leq 50, 3x + y \leq 90, x \geq 0, y \geq 0$?

OR

Given the LPP
Minimise $f = 2x_1 - x_2$
$x_1 \geq 0, x_2 \geq 0;$
$x_1 + x_2 \geq 5;$
$-x_1 + x_2 \leq 1;$
$5x_1 + 4x_2 \leq 40$
Then, find the minimum value of the given LPP.

15. Maximise the function $Z = 11x + 7y$, subject to the constraints: $x \leq 3, y \leq 2, x \geq 0$ and $y \geq 0$.

Detailed SOLUTIONS

1. **(i) (b) :** Let $B(x, y)$ be the point of intersection of the given lines

$2x + 5y = 100$ ...(i)

and $\frac{x}{25} + \frac{y}{40} = 1 \Rightarrow 8x + 5y = 200$ ...(ii)

On solving (i) and (ii), we get

$x = \frac{50}{3}, y = \frac{40}{3}$

$\therefore$ The point of intersection $B(x, y) = \left(\frac{50}{3}, \frac{40}{3}\right)$.

(ii) (d): The corner points of the feasible region shown in the given graph are

$O(0, 0), A(25, 0), B\left(\frac{50}{3}, \frac{40}{3}\right)$ and $C(0, 20)$.

(iii) (a): Here, $Z = x + y$

Corner points	Value of $Z = x + y$
(0, 0)	0
(25, 0)	25
$\left(\frac{50}{3}, \frac{40}{3}\right)$	30 ← Maximum
(0, 20)	20

Thus, max $Z = 30$ occurs at point $\left(\frac{50}{3}, \frac{40}{3}\right)$.

(iv) (b) :

Corner points	Value of $Z = 6x - 9y$
(0, 0)	0
(25, 0)	150 ← Maximum
$\left(\frac{50}{3}, \frac{40}{3}\right)$	–20
(0, 20)	–180

(v) (c) :

Corner points	Value of $Z = 6x + 3y$
(0, 0)	0 ← Minimum
(25, 0)	150
$\left(\frac{50}{3}, \frac{40}{3}\right)$	140
(0, 20)	60

2. **(c) :** When we solve an L.P.P. graphically, the optimal (or optimum) value of the objective function is attained at corner points of the feasible region.

OR

(a) Region represented by $x \geq 0, y \geq 0$ is first quadrant.

3. **(a) :** The feasible region is always a concave region.

4. **(c) :** The feasible region as shown in the figure, has objective function $F = 3x - 4y$.

Corner points	$F = 3x - 4y$
(0, 0)	$3 \times 0 - 4 \times 0 = 0$
(12, 6)	$3 \times 12 - 4 \times 6 = 12$ ← Maximum
(0, 4)	$3 \times 0 - 4 \times 4 = -16$ ← Minimum

Hence, the maximum value of F is 12.

5. **(b) :**

Corner points	$Z = px + qy; p, q > 0$
(0, 3)	$p \times 0 + q \times 3 = 3q$
(1, 1)	$p \times 1 + q \times 1 = p + q$
(3, 0)	$p \times 3 + q \times 0 = 3p$

The minimum of Z occurs at (3, 0) and (1, 1).

$\therefore \quad p + q = 3p \Rightarrow 2p = q$

$\Rightarrow p = \frac{q}{2}$

6. **(d)**

7. If the feasible region for a LPP is unbounded, then optimal value of the objective function $Z = ax + by$ may or may not exist.

8. The value of objective function is maximum under linear constraints at any vertex of feasible region.

9. In a LPP, if the objective function $Z = ax + by$ has the same maximum value on two corner points of the feasible region, then every point on the line segment joining these two points give the same maximum value.

10. The feasible region for a LPP is always a convex polygon.

OR

Corner points	$F = 4x + 6y$
(0, 2)	$4 \times 0 + 6 \times 2 = 12$ ← Minimum
(3, 0)	$4 \times 3 + 6 \times 0 = 12$ ← Minimum
(6, 0)	$4 \times 6 + 6 \times 0 = 24$
(6, 8)	$4 \times 6 + 6 \times 8 = 72$ ← Maximum
(0, 5)	$4 \times 0 + 6 \times 5 = 30$

Hence, minimum value of F occurs at (0, 2) and (3, 0). So it will occur at any point of the line segment joining the points (0, 2) and (3, 0).

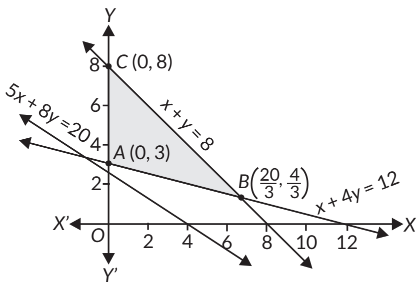
11. **(i)** When $x \geq 6$ and $y \geq 2$, then
$2x + y \geq 2 \times 6 + 2$, *i.e.*, $2x + y \geq 14$
Hence, $x \geq 0, y \geq 0$ and $2x + y \geq 10$ are automatically satisfied by every point of the region
$\{(x, y) : x \geq 6\} \cap \{(x, y) : y \geq 2\}$

(ii) Construct the following table of values of the objective function :

Corner points	Value of $Z = 3x - 4y$
(0, 0)	$3 \times 0 - 4 \times 0 = 0$
(5, 0)	$3 \times 5 - 4 \times 0 = 15$
(6, 5)	$3 \times 6 - 4 \times 5 = -2$
(6, 8)	$3 \times 6 - 4 \times 8 = -14$
(4,10)	$3 \times 4 - 4 \times 10 = -28$
(0, 8)	$3 \times 0 - 4 \times 8 = -32$ ← Minimum

$\therefore$ Minimum of Z is – 32 which occurs at (0, 8).

12. Converting the given inequation into equation, we get $x + y = 8$, $x + 4y = 12$, $5x + 8y = 20$
Let us draw the graph of these equations as shown below



The point of intersection of the lines $x + 4y = 12$ and $x + y = 8$ is $B = \left(\frac{20}{3}, \frac{4}{3}\right)$.

We have, corner points $A(0, 3)$, $B\left(\frac{20}{3}, \frac{4}{3}\right)$ and $C(0, 8)$.

Now, $Z = 30x + 20y$

$\therefore \quad Z(0, 3) = 30(0) + 20(3) = 60$

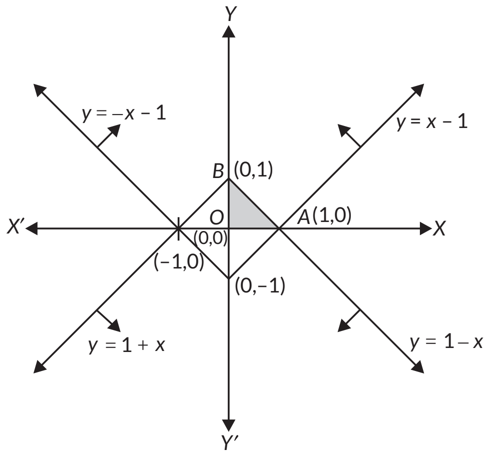
$Z\left(\frac{20}{3}, \frac{4}{3}\right) = 30\left(\frac{20}{3}\right) + 20\left(\frac{4}{3}\right) = 226.6$

$Z(0, 8) = 30(0) + 20(8) = 160$

$\therefore$ Minimum value of Z is 60 which is attained at point $A(0, 3)$.

13. Max $z = x + y$

subject to $y \geq \begin{cases} x - 1, & x \geq 0 \\ -x - 1, & x < 0 \end{cases}$, $y \leq \begin{cases} 1 - x, & x \geq 0 \\ 1 + x, & x < 0 \end{cases}$ and $x, y \geq 0$



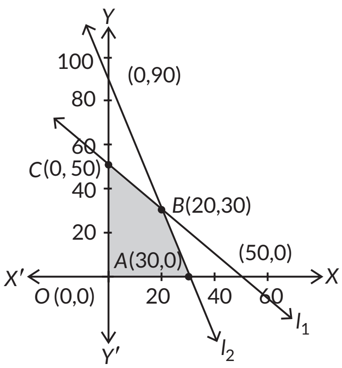
The common region *OAB* is showing with shades.

Corner points	Value of $z = x + y$
O (0, 0)	$z = 0 + 0 = 0$
A (1, 0)	$z = 1 + 0 = 1$ (Maximum)
B (0, 1)	$z = 0 + 1 = 1$ (Maximum)

From table, maximum value of $z = 1$.

14. Let $l_1 : x + y = 50$ ... (i)

$l_2 : 3x + y = 90$ ... (ii)



From graph the corner points are $O(0, 0)$, $C(0, 50)$, $B(20, 30)$ and $A(30, 0)$.

The value of objective function at these corner points are as follows :

Corner points	Value of $Z = 4x + y$
O (0, 0)	$Z = 0$
A (30, 0)	$Z = 120$ (Maximum)
B (20, 30)	$Z = 110$
C (0,50)	$Z = 50$

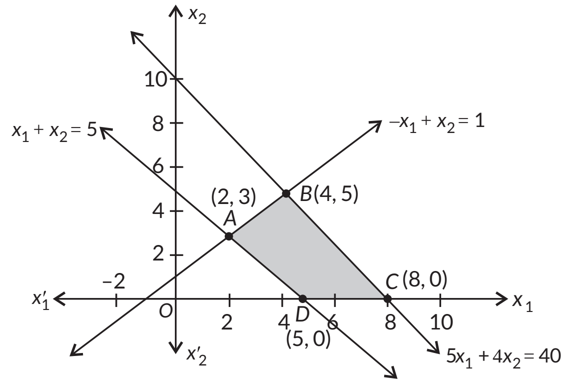
$\therefore$ Maximum value of $Z = 120$ which is attained at point (30, 0).

OR

Converting inequations into equations, we get

$x_1 + x_2 = 5$, $-x_1 + x_2 = 1$, $5x_1 + 4x_2 = 40$, $x_1 = 0$, $x_2 = 0$

We draw the graph of these equations as shown below:



Here, *ABCD* is the feasible region

$\therefore$ The values of f at the points A, B, C and D are

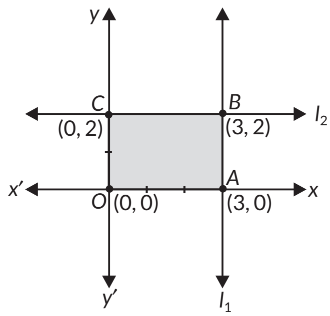
Corner points	Value of $F = 2x_1 - x_2$
$A(2, 3)$	$2(2) - 3 = 1$ (Minimum)
$B(4, 5)$	$2(4) - 5 = 3$
$C(8, 0)$	$2(8) - 0 = 16$
$D(5, 0)$	$2(5) - 0 = 10$

$\therefore$ F is minimum at $A(2, 3)$ and minimum value of F is 1.

15. We have, maximise $Z = 11x + 7y$, subject to the constraints $x \le 3$, $y \le 2$, $x \ge 0$, $y \ge 0$.

Let $l_1 : x = 3$

$l_2 : y = 2$



The feasible (shaded) region as shown in the figure as *OABC* is bounded. Let us evaluate Z at corner points $O(0, 0)$, $A(3, 0)$, $B(3, 2)$ and $C(0, 2)$.

Corner points	$Z = 11x + 7y$
(0, 0)	$11 \times 0 + 7 \times 0 = 0$
(3, 0)	$11 \times 3 + 7 \times 0 = 33$
(3, 2)	$11 \times 3 + 7 \times 2 = 47$ $\leftarrow$ Maximum
(0, 2)	$11 \times 0 + 7 \times 2 = 14$

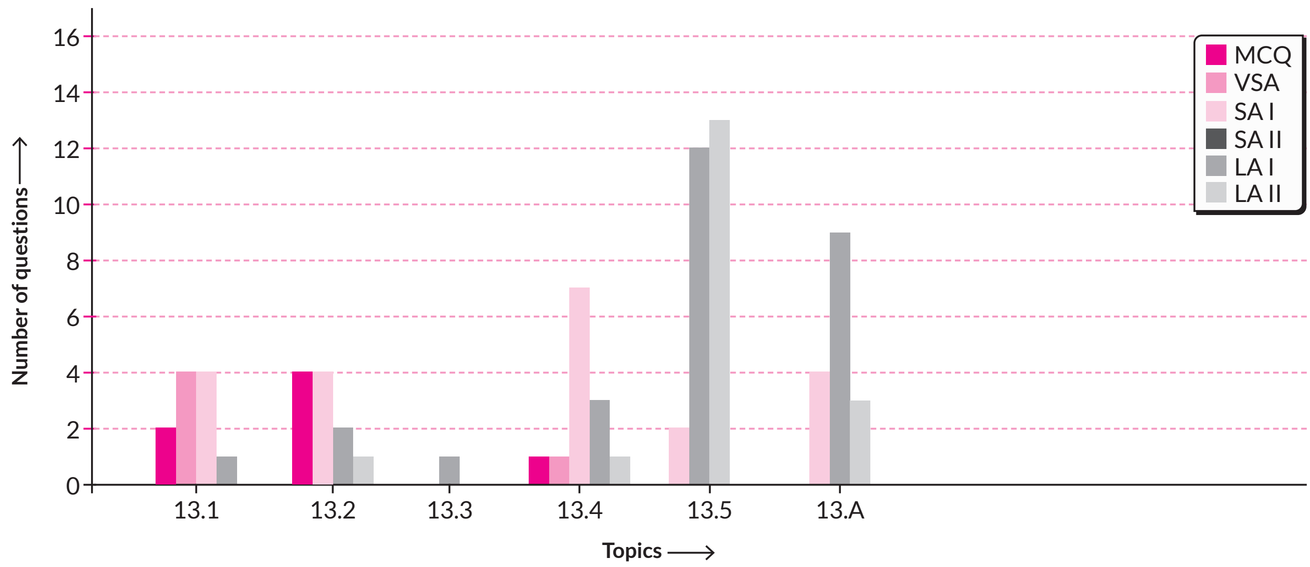
Hence, Z is maximum at (3, 2) and its maximum value is 47.

CHAPTER 13

Probability

TOPICS

Analysis of Last 10 Years' CBSE Board Questions (2023-2014)



Weightage Xtract

- Topic 13.5 is highly scoring topic.
- Maximum weightage is of Topic 13.5 *Bayes' Theorem*.
- Maximum SA I type questions were asked from Topic 13.4 *Independent Events*.
- Maximum LA I and LA II type questions were asked from Topic 13.5 *Bayes' Theorem*

QUICK RECAP

Conditional Probability

- Let E and F be two events of a random experiment, then the conditional probability of occurrence of E under the condition that F has already occurred, *i.e.*, $P(E/F)$ is given by

$$P(E/F)=\frac{P(E\cap F)}{P(F)}, \text{ where } P(F)\neq 0$$

This topic is not in the NCERT Textbook but given in current CBSE Syllabus.

- **Properties of Conditional Probability**
  (i) For an event A and sample space S,
  $P(S/A) = P(A/A) = 1$
  (ii) If A and B are any two events of a sample space S and F is an event of S such that $P(F) \neq 0$, then
  $P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)$
  In particular, if A and B are disjoint events, then
  $P((A \cup B)/F) = P(A/F) + P(B/F)$
  (iii) $P(A'/B) = 1 - P(A/B)$
  (iv) $0 \leq P(A/B) \leq 1$

Multiplication Theorem on Probability

- Let A and B are two events associated with a sample space S. Then $A \cap B$ denotes the event that both A and B have occurred. The event $A \cap B$ is also written as AB.
  Also, $P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$, provided $P(A) \neq 0$, $P(B) \neq 0$
  Note : If A, B and C are three events associated with a random experiment, then
  $P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/(A \cap B))$
  $= P(A) \cdot P(B/A) \cdot P(C/AB)$
  Similarly, multiplication rule of probability can be extended to four or more events.

Independent Events

- Events are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence or non-occurrence of the other.
- Two events A and B associated with a random experiment are said to be independent if
  (i) $P(A/B) = P(A)$ provided $P(B) \neq 0$
  (ii) $P(B/A) = P(B)$ provided $P(A) \neq 0$
  Note :
  (i) Two events A and B associated with a random experiment are independent if
  $P(A \cap B) = P(A) \cdot P(B)$
  (ii) Two events A and B are said to be dependent if they are not independent, *i.e.*, if
  $P(A \cap B) \neq P(A) \cdot P(B)$
  (iii) Events $A_1, A_2, ..., A_n$ are mutually independent if the probability of the simultaneous occurrence of (any) finite number of events is equal to the product of their separate probabilities. While these events are pairwise independent if $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.
  (iv) If A and B are independent events, then so are the events A and B'; A' and B; A' and B'.

Theorem of Total Probability

- If $E_1, E_2, E_3, ..., E_n$ are mutually exclusive and exhaustive events associated with a sample space S of a random experiment and A is any event associated with S, then

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + ... + P(E_n)\,P(A/E_n)$$

Bayes' Theorem

- If $E_1, E_2, E_3, ..., E_n$ are mutually exclusive and exhaustive events associated with a random experiment and A is any event associated with the experiment, then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_i P(E_i)P(A|E_i)}, \quad \text{where } i = 1, 2, 3, ..., n$$

Random Variable

- A random variable is a real valued function, whose domain is the sample space of a random experiment. Generally, it is denoted by X.

Probability Distributions

- Let real numbers $x_1, x_2, ..., x_n$ are the possible values of a random variable X and $p_1, p_2, ..., p_n$ are the corresponding probabilities to each value of the random variable X. Then the probability distribution is

X :	x_1	x_2	...	x_n
$P(X)$:	p_1	p_2	...	p_n.

  Note :
  (i) $1 > p_i > 0$
  (ii) Sum of probabilities $(p_1 + p_2 + ... + p_n) = 1$

- **Mean :** If X is a random variable, then mean $(\bar{X})$ of X is defined as

$$\bar{X} = \mu = p_1x_1 + p_2x_2 + ... + p_nx_n = \sum_{i=1}^{n} p_ix_i$$

  Mean of a random variable is also called the expectation of X, denoted by $E(X)$.

BRAIN MAP

PROBABILITY

Conditional Probability

- Probability of occurrence of an event A, given that B has already occurred *i.e.*, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Properties

- $0 \le P(A|B) \le 1$
- $P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$ where $P(F) \neq 0$
  $= P(A|F) + P(B|F)$ (if A & B are disjoint)
- $P(A'|B) = 1 - P(A|B)$

Events

Independent

- Occurrence or non-occurrence of one does not affect the occurrence of the other *i.e.*,
  $P(A \mid B) = P(A)$ provided $P(B) \neq 0$,
  $P(B \mid A) = P(B)$ provided $P(A) \neq 0$
- In general, if n events $A_1, A_2, ..., A_n$ associated with a random experiment are independent, then $P(A_1 \cap A_2 \cap ..... \cap A_n) = P(A_1) \cdot P(A_2) ..... P(A_n)$

Mutually Independent

- If $P(A \cap B) = P(A)\ P(B)$, $P(A \cap C) = P(A)\ P(C)$, $P(B \cap C) = P(B)\ P(C)$, and
  $P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$

Mutually Exclusive

- If and only if $A \cap B = \phi$.

Dependent

- If $P(A \cap B) \neq P(A) \cdot P(B)$

Theorems

Total Probability

- $P(A) = P(E_1)\ P(A \mid E_1) + P(E_2)\ P(A \mid E_2) + ... + P(E_n)\ P(A \mid E_n)$ where, $E_1, E_2, E_3, ..., En$ are mutually exclusive and exhaustive events.

Bayes'

$P(E_i|A) = \frac{P(E_i)\ P(A|E_i)}{\Sigma P(E_i)\ P(A|E_i)}$, where $i = 1, 2, 3, ..., n$,
where, $E_1, E_2, E_3, ..., E_n$ are mutually exclusive and exhaustive events.

Multiplication

- $P(A \cap B) = P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B)$, provided $P(A) \neq 0$, $P(B) \neq 0$ *ie*., $P(AB) = P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B)$, provided $P(A) \neq 0$, $P(B) \neq 0$

Extension

- If A, B, C are three events associated with a random experiment, then $P(ABC)$ or $P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot P(C \mid (A \cap B)) = P(A)\ P(B \mid A)\ P(C \mid AB)$
- If $A_1, A_2, ..., A_n$ are n events associated with a random experiment, then $P(A_1 \cap A_2 ...... \cap A_n) = P(A_1)\ P(A_2 \mid A_1)\ P(A_3 \mid (A_1 \cap A_2)) ...... P(A_n \mid (A_1 \cap A_2 ...... \cap A_{n-1}))$.

Partition of a Sample Space

A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of sample space S if

- $E_i \cap E_j = \phi$, $i \neq j$; $i, j = 1, 2, 3, ..., n$
- $E_1 \cup E_2 \cup .... \cup E_n = S$ and $P(E_i) > 0$ for all $i = 1, 2, ..., n$

In other words, the events $E_1, E_2, ..., E_n$ represent a partition of the sample space S if they are pairwise disjoint, exhaustive and have non-zero probabilities.

Random Variables and its Probability Distributions

- A real valued function, whose domain is the sample space of a random experiment. Generally, it is denoted by X.
- Let real numbers $x_1, x_2, ..., x_n$ are the possible values of a random variable X and $p_1, p_2, ..., p_n$ are the corresponding probabilities to each value of the random variable X. Then the probability distribution is
  $X \ : x_1 \quad x_2 \quad ... \quad x_n$
  $P(X) \ : p_1 \quad p_2 \quad ... \quad p_n$.
- $P(x_k)$ *ie* . $P(X = x_k)$ lies between 0 and 1 for $k = 1, 2, ..., n$
- $\sum_{i=1}^{n} p_i = 1$
- $P(X \le x_i) = p_1 + p_2 + ... + p_i$,
  $P(X \ge x_i) = p_i + p_{i+1} + ... + p_n$,
  $P(X \le x_i) = P(X < x_i) + P(X = x_i)$ etc.

Mean

- $\bar{X} = E(X) = \mu = p_1x_1 + p_2x_2 + ... + p_nx_n = \sum_{i=1}^{n} p_i x_i$

Previous Years' CBSE Board Questions

13.1 Introduction

MCQ

1. Five fair coins are tossed simultaneously. The probability of the events that atleast one head comes up is
   (a) $\frac{27}{32}$ (b) $\frac{5}{32}$ (c) $\frac{31}{32}$ (d) $\frac{1}{32}$
   (2023)

2. A die is thrown once. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is
   (a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) 0 (d) 1 *(2020)*

VSA (1 mark)

3. A coin is tossed once. If head comes up, a die is thrown, but if tail comes up, the coin is tossed again. Find the probability of obtaining head and number 6. *(2021 C)*

4. Two cards are drawn at random and one-by-one without replacement from a well-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black. *(2020)*

5. From a pack of 52 cards, 3 cards are drawn at random (without replacement). The probability that they are two red cards and one black card, is ________. *(2020C)*

6. A bag contains 3 black, 4 red and 2 green balls. If three balls are drawn simultaneously at random, then the probability that the balls are of different colours is ________. *(2020)* Ap

SA I (2 marks)

7. A box B_1 contains 1 white ball and 3 red balls. Another box B_2 contains 2 white balls and 3 red balls. If one ball is drawn at random from each of the boxes B_1 and B_2, then find the probability that the two balls drawn are of the same colour. *(Term II, 2021-22)*

8. A bag contains cards numbered 1 to 25. Two cards are drawn at random, one after the other, without replacement. Find the probability that the number on each card is a multiple of 7. *(Term II, 2021-22 C)*

9. If A and B are two events such that $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.6$, then find $P(B' \cap A)$. *(2020)* An

10. Out of 8 outstanding students of a school, in which there are 3 boys and 5 girls, a team of 4 students is to be selected for a quiz competition. Find the probability that 2 boys and 2 girls are selected. *(AI 2019)*

LA I (4 marks)

11. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black. *(Delhi 2015)* Cr

13.2 Conditional Probability

MCQ

12. For two events A and B, if $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$, then $P(A \cup B)$ is
    (a) 0.24 (b) 0.3 (c) 0.48 (d) 0.96
    (2023)

13. If for any two events A and B,
    $P(A)=\frac{4}{5}$ and $P(A\cap B)=\frac{7}{10}$, then $P(B/A)$ is
    (a) $\frac{1}{10}$ (b) $\frac{1}{8}$ (c) $\frac{7}{8}$ (d) $\frac{17}{20}$
    (2023)

14. **In the following questions, a statements are Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices:**
    Assertion (A) : Two coins are tossed simultaneously. The probability of getting two heads, if it is known that at least one head comes up, is $\frac{1}{3}$.
    Reason (R) : Let E and F be two events with a random experiment, then $P(F/E)=\frac{P(E\cap F)}{P(E)}$.
    (a) Both (A) and (R) are true and (R) is the correct explanation of (A)
    (b) Both (A) and (R) are true, but (R) is not the correct explanation of the (A)
    (c) (A) is true, and (R) is False.
    (d) (A) is false, but (R) is true. *(2023)*

15. A card is picked at random from a pack of 52 playing cards. Given that the picked card is a queen, the probability of this card to be a card of spade is
    (a) $\frac{1}{3}$ (b) $\frac{4}{13}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$
    (2020) Ap

SA I (2 marks)

16. Find $[P(B/A) + P(A/B)]$, if $P(A) = \frac{3}{10}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{3}{5}$. *(2020)*

17. 12 cards numbered 1 to 12 (one number on one card), are placed in a box and mixed up thoroughly. Then a card is drawn at random from the box. If it is known that the number on the drawn card is greater than 5, find the probability that the card bears an odd number. *(AI 2019)* An

18. Mother, father and son line up at random for a family photo. If A and B are two events given by A = Son on one end, B = Father in the middle, find $P(B/A)$. *(2019)*

19. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4. *(2018)*

LA I (4 marks)

20. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that
   (i) the youngest is a girl.
   (ii) atleast one is a girl. *(Delhi 2014)* Ap

21. A couple has 2 children. Find the probability that both are boys, if it is known that
   (i) one of them is a boy,
   (ii) the older child is a boy. *(Delhi 2014C)*

LA II (5/6 marks)

22. Consider the experiment of tossing a coin. If the coin shows head, toss it again, but if it shows tail, then throw a die. Find the conditional probability of the event 'the die shows a number greater than 4' given that 'there is at least one tail'. *(Delhi 2014C)*

13.3 Multiplication Theorem on Probability

LA I (4 marks)

23. A bag contains 3 red and 7 black balls. Two balls are selected at random one-by-one without replacement. If the second selected ball happens to be red, what is the probability that the first selected ball is also red? *(Delhi 2014C)* Ev

13.4 Independent Events

MCQ

24. If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B'|A)$ is equal to
   (a) $\frac{1}{4}$ (b) $\frac{1}{3}$
   (c) $\frac{3}{4}$ (d) 1 *(2020)* R

VSA (1 mark)

25. A problem is given to three students whose probabilities of solving it are $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{6}$ respectively. If the events of solving the problem are independent, find the probability that at least one of them solves it. *(2020)* U

SA I (2 marks)

26. The probability that A hits the target is $\frac{1}{3}$ and the probability that B hits it, is $\frac{2}{5}$. If both try to hit the target independently, find the probability that the target is hit. *(Term II, 2021-22)*

27. Events A and B are such that
$$P(A)=\frac{1}{2},\ P(B)=\frac{7}{12} \text{ and } P(\bar{A}\cup\bar{B})=\frac{1}{4}$$
Find whether the events A and B are independent or not. *(Term II, 2021-22)* Ap

28. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three? *(2020)*

29. Given two independent events A and B such that $P(A) = 0.3$ and $P(B) = 0.6$, find $P(A' \cap B')$. *(2020)*

30. The probability of two students A and B coming to school on time are $\frac{2}{7}$ and $\frac{4}{7}$, respectively. Assuming that the events 'A coming on time' and 'B coming on time' are independent, find the probability of only one of them coming to school on time. *(2019)*

31. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event "number is even" and B be the event "number is marked red". Find whether the events A and B are independent or not. *(Delhi 2019)* An

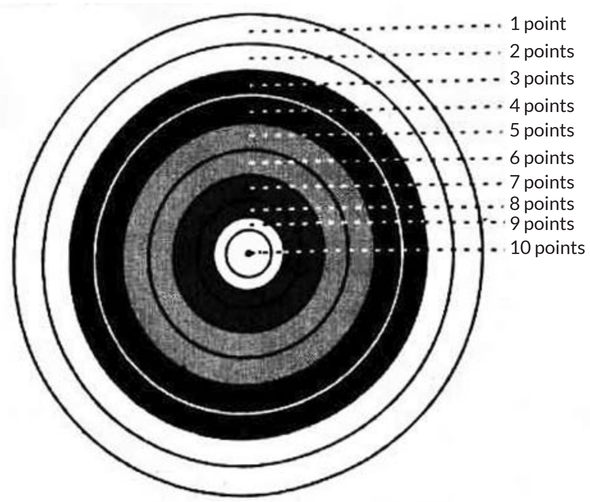
OR

A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events. *(AI 2017)*

32. Prove that if E and F are independent events, then the events E' and F' are also independent. *(Delhi 2017)*

LA I (4 marks)

33. In a game of Archery, each ring of the Archery target is valued. The centremost ring is worth 10 points and rest of the rings are allotted points 9 to 1 in sequential order moving outwards.
Archer A is likely to earn 10 points with a probability of 0.8 and Archer B is likely to earn 10 points with a probability of 0.9.



Based on the above information, answer the following questions :
If both of them hit the Archery target, then find the probability that

(a) exactly one of them earns 10 points.
(b) both of them earn 10 points.

(Term II, 2021-22, 2020)

34. A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins. *(Delhi 2016)*

35. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(i) the problem is solved
(ii) exactly one of them solved the problem.

(Delhi 2015C) Ap

LA II (5/6 marks)

36. If A and B are two independent events such that $P(\bar{A}\cap B)=\frac{2}{15}$ and $P(A\cap\bar{B})=\frac{1}{6}$, then find $P(A)$ and $P(B)$. *(Delhi 2015)*

13.5 Bayes' Theorem

SA I (2 marks)

37. There are two bags. Bag I contains 1 red and 3 white balls, and Bag II contains 3 red and 5 white balls. A bag is selected at random and a ball is drawn from it. Find the probability that the ball so drawn is red in colour. *(Term II, 2021-22)*

38. A purse contains 3 silver and 6 copper coins and a second purse contains 4 silver and 3 copper coins. If a coin is drawn at random from one of the two purses, find the probability that it is a silver coin. *(2020)* Ev

LA I (4 marks)

39. **Case study :** A building contractor undertakes a job to construct 4 flats on a plot along with parking area. Due to strike the probability of many construction workers not being present for the job is 0.65. The probability that many are not present and still the work gets completed on time is 0.35. The probability that work will be completed on time when all workers are present is 0.80.
Let E_1 : represent the event when many workers were not present for the job;
E_2 : represent the event when all workers were present; and
E : represent completing the construction work on time.
Based on the above information, answer the following questions :

(i) What is the probability that all the workers are present for the job?
(ii) What is the probability that construction will be completed on time?
(iii) What is the probability that many workers are not present given that the construction work is completed on time?

OR

What is the probability that all workers were present given that the construction job was completed on time? *(2023)*

40. There are two boxes, namely box-I and box-II. Box-I contains 3 red and 6 black balls. Box-II contains 5 red and 5 black balls. One of the two boxes, is selected at random and a ball is drawn at random. The ball drawn is found to be red. Find the probability that this red ball comes out from box-II. *(Term II, 2021-22)*

41. There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black. *(2020)* Ev

42. A bag contains 5 red and 4 black balls, a second bag contains 3 red and 6 black balls. One of the two bags is selected at random and two balls are drawn at random (without replacement), both of which are found to be red. Find the probability that these two balls are drawn from the second bag. *(2020 C)*

43. A bag contains two coins, one biased and the other unbiased. When tossed, the biased coin has a 60% chance of showing heads. One of the coins is selected at random and on tossing it shows tails. What is the probability it was an unbiased coin? *(2020)*

44. In a shop X, 30 tins of ghee of type A and 40 tins of ghee of type B which look alike, are kept for sale. While in shop Y, similar 50 tins of ghee of type A and 60 tins of ghee of type B are there. One tin of ghee is purchased from one of the randomly selected shop and is found to be of type B. Find the probability that it is purchased from shop Y. *(2020)* Ev

45. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die? *(2018)* An

46. Often it is taken that a truthful person commands, more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society? *(Delhi 2017)*

47. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance ? Is regularity required only in school? Justify your answer. *(AI 2017)*

48. Three persons *A*, *B* and *C* apply for a job of Manager in a Private Company. Chances of their selection (*A*, *B* and *C*) are in the ratio 1 : 2 : 4. The probabilities that *A*, *B* and *C* can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the change does not take place, find the probability that it is due to the appointment of *C*. *(Delhi 2016)* An

49. A bag contains 4 balls. Two balls are drawn at random (without replacement) and are found to be white. What is the probability that all balls in the bag are white? *(AI 2016)*

50. Three machines E_1, E_2 and E_3 in a certain factory producing electric bulbs, produce 50%, 25% and 25% respectively, of the total daily output of electric bulbs. It is known that 4% of the bulbs produced by each of machines E_1 and E_2 are defective and that 5% of those produced by machine E_3 are defective. If one bulb is picked up at random from a day's production, calculate the probability that it is defective. *(Foreign 2015)* An

LA II (5/6 marks)

51. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{3}$. What is the probability that the student knows the answer, given that he answered it correctly? *(2023)*

52. A manufacturer has three machine operators *A*, *B* and *C*. The first operator *A* produces 1% of defective items, whereas the other two operators *B* and *C* produces 5% and 7% defective items respectively. *A* is on the job for 50% of the time. *B* on the job 30% of the time and *C* on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by *A*? *(Delhi 2019)* Ap

53. An insurance company insured 3000 cyclists, 6000 scooter drivers and 9000 car drivers. The probability of an accident involving a cyclist, a scooter driver and a car driver are 0.3, 0.05 and 0.02 respectively. One of the insured persons meets with an accident. What is the probability that he is a cyclist? *(AI 2019)* Ev

54. There are two boxes I and II. Box I contains 3 red and 6 black balls. Box II contains 5 red and '*n*' black balls. One of the two boxes, box I and box II is selected at random and a ball is drawn at random. The ball drawn is found to be red. If the probability that this red ball comes out from box II is $\frac{3}{5}$, find the value of '*n*'. *(2019)*

55. Bag *A* contains 3 red and 5 black balls, while bag *B* contains 4 red and 4 black balls. Two balls are transferred at random from bag *A* to bag *B* and then a ball is drawn from bag *B* at random. If the ball drawn from bag *B* is found to be red find the probability that two red balls were transferred from *A* to *B*. *(Foreign 2016)*

56. In a factory which manufactures bolts, machines *A*, *B* and *C* manufacture respectively 30%, 50% and 20% of the bolts. Of their outputs 3, 4 and 1 percent respectively are defective bolts. *A* bolt is drawn at random from the product and is found to be defective. Find the probability that this is not manufactured by machine *B*. *(AI 2015)*

57. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and two balls are drawn at random without replacement from the bag and are found to be both red. Find the probability that the balls are drawn from the first bag. *(Delhi 2015C)* An

58. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses the answer will be correct with probability $\frac{1}{3}$, what is the probability that the student knows the answer given that he answered it correctly? *(AI 2015C)*

59. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade. *(Delhi 2014)* An

60. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows head. What is the probability that it was the two-headed coin? *(AI 2014)*

61. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident for them are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver or a car driver? *(Foreign 2014)* Ev

62. A man is known to speak the truth 3 out of 5 times. He throws a die and reports that it is '1'. Find the probability that it is actually 1. *(Delhi 2014C)* Ap

63. An urn contains 4 balls. Two balls are drawn at random from the urn (without replacement) and are found to be white. What is the probability that all the four balls in the urn are white? *(AI 2014C)*

Random Variables and its Probability Distributions

SA I (2 marks)

64. Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$. Find the probability distribution of X. *(Term II, 2021-22)*

65. A coin is tossed twice. The following table shows the probability distribution of number of tails :

X	0	1	2
$P(X)$	K	$6K$	$9K$

(a) Find the value of K.
(b) Is the coin tossed biased or unbiased? Justify your answer. *(Term II, 2021-22)*

66. Two balls are drawn at random from a bag containing 2 red balls and 3 blue balls, without replacement. Let the variable X denotes the number of red balls. Find the probability distribution of X. *(Term II, 2021-22)*

67. The random variable X has a probability distribution $P(X)$ of the following form, where 'k' is some number,

$$P(X=x)=\begin{cases} k, & \text{if } x=0 \\ 2k, & \text{if } x=1 \\ 3k, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of 'k'. *(Delhi 2019)* An

LA I (4 marks)

68. The probability distribution of a random variable X, where k is a constant is given below :

$$P(X = x) = \begin{cases} 0.1, & \text{if } x=0 \\ kx^2, & \text{if } x=1 \\ kx, & \text{if } x=2 \text{ or } 3 \\ 0, & \text{otherwise} \end{cases}$$

Determine
(a) the value of k
(b) $P(x \le 2)$
(c) Mean of the variable X *(2020)* An

69. Three rotten apples are mixed with seven fresh apples. Find the probability distribution of the number of rotten apples, if three apples are drawn one by one with replacement. Find the mean of the number of rotten apples. *(2020)*

70. The random variable X can take only the values 0, 1, 2, 3. Given that $P(X = 0) = P(X = 1) = p$ and $P(X = 2) = P(X = 3)$ such that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p. *(Delhi 2017)* An

71. In a game, a man wins ₹ 5 for getting a number greater than 4 and loses ₹ 1 otherwise, when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a number greater than 4. Find the expected value of the amount he wins/loses. *(AI 2016)*

72. Let X denote the number of colleges where you will apply after your results and $P(X = x)$ denotes your probability of getting admission in x number of colleges. It is given that

$$P(X=x)=\begin{cases} kx & , \text{if } x=0 \text{ or } 1 \\ 2kx & , \text{if } x=2 \\ k(5-x) & , \text{if } x=3 \text{ or } 4 \\ 0 & , \text{if } x>4 \end{cases}$$

where k is a positive constant. Find the value of k. Also find the probability that you will get admission in
(i) exactly one college
(ii) atmost 2 colleges
(iii) atleast 2 colleges. *(Foreign 2016)* An

73. Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spades. Hence find the mean of the distribution. *(AI 2015)* Ev

74. From a lot of 15 bulbs which include 5 defectives, a sample of 2 bulbs is drawn at random (without replacement). Find the probability distribution of the number of defective bulbs. *(Delhi 2015C)*

75. Three cards are drawn at random (without replacement) from a well shuffled pack of 52 playing cards. Find the probability distribution of number of red cards. Hence find the mean of the distribution. *(Foreign 2014)* Ev

76. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find the mean of X. *(AI 2014C)*

LA II (5/6 marks)

77. A box contains 10 tickets, 2 of which carry a prize of ₹ 8 each, 5 of which carry a prize of ₹ 4 each, and remaining 3 carry a prize of ₹ 2 each. If one ticket is drawn at random, find the mean value of the prize. *(2023)*

78. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution. *(AI 2014)*

79. In a game, a man wins rupees five for a six and loses rupee one for any other number, when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/loses. *(AI 2014C)* An

CBSE Sample Questions

13.1 Introduction

SA II (3 marks)

1. Three friends go for coffee. They decide who will pay the bill, by each tossing a coin and then letting the "odd person" pay. There is no odd person if all three tosses produce the same result. If there is no odd person in the first round, they make a second round of tosses and they continue to do so until there is an odd person. What is the probability that exactly three rounds of tosses are made? *(2022-23)*

13.2 Conditional Probability

SA I (2 marks)

2. Given that E and F are events such that $P(E) = 0.8$, $P(F) = 0.7$, $P(E \cap F) = 0.6$. Find $P(\bar{E} | \bar{F})$. *(2020-21)* Ap

13.4 Independent Events

VSA (1 mark)

3. The probabilities of A and B solving a problem independently are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. If both of them try to solve the problem independently, what is the probability that the problem is solved? *(2020-21)*

13.5 Bayes' Theorem

MCQ

Case study-based questions are compulsory. Attempt any 4 sub parts from each question. Each sub-part carries 1 mark.

4. In an office three employees Vinay, Sonia and Iqbal process incoming copies of a certain form. Vinay process 50% of the forms, Sonia processes 20% and Iqbal the remaining 30% of the forms. Vinay has an error rate of 0.06, Sonia has an error rate of 0.04 and Iqbal has an error rate of 0.03.

Based on the above information answer the following:

(i) The conditional probability that an error is committed in processing given that Sonia processed the form is

(a) 0.0210 (b) 0.04 (c) 0.47 (d) 0.06

(ii) The probability that Sonia processed the form and committed an error is

(a) 0.005 (b) 0.006 (c) 0.008 (d) 0.68

(iii) The total probability of committing an error in processing the form is

(a) 0 (b) 0.047 (c) 0.234 (d) 1

(iv) The manager of the company wants to do a quality check. During inspection he selects a form at random from the days output of processed forms. If the form selected at random has an error, the probability that the form is not processed by Vinay is

(a) 1 (b) $\frac{30}{47}$ (c) $\frac{20}{47}$ (d) $\frac{17}{47}$

(v) Let A be the event of committing an error in processing the form and let E_1, E_2 and E_3 be the events that Vinay, Sonia and Iqbal processed the form. The value of $\sum_{i=1}^{3} P(E_i | A)$ is

(a) 0 (b) 0.03 (c) 0.06 (d) 1

(2020-21)

LA I (4 marks)

5. There are two antiaircraft guns, named as A and B. The probabilities that the shell fired from them hits an airplane are 0.3 and 0.2 respectively. Both of them fired one shell at an airplane at the same time.

(i) What is the probability that the shell fired from exactly one of them hit the plane?

(ii) If it is known that the shell fired from exactly one of them hit the plane, then what is the probability that it was fired from B? *(2022-23)*

6. An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at sometime within a fixed one-year period with probability 0.6, whereas this probability is 0.2 for a person who is not accident prone. The company knows that 20 percent of the population is accident prone.

Based on the given information, answer the following questions.

(i) What is the probability that a new policyholder will have an accident within a year of purchasing a policy?

(ii) Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone? *(Term II, 2021-22)*

Random Variables and its Probability Distributions

SA I (2 marks)

7. A bag contains 1 red and 3 white balls. Find the probability distribution of the number of red balls if 2 balls are drawn at random from the bag one-by-one without replacement. *(Term II, 2021-22)*

8. A refrigerator box contains 2 milk chocolates and 4 dark chocolates. Two chocolates are drawn at random. Find the probability distribution of the number of milk chocolates. What is the most likely outcome? *(2020-21)*

SA II (3 marks)

9. Find the mean number of defective items in a sample of two items drawn one-by-one without replacement from an urn containing 6 items, which include 2 defective items. Assume that the items are identical in shape and size. *(2022-23)*

Detailed SOLUTIONS

1. **(c) :** Since each coin turns up on either a head or tail.

$\therefore$ Total possible outcomes = $2^5 = 32$

Let A be the event that all tails comes up.

$\therefore$ $n(A) = 1$ {i.e., (T, T, T, T, T)}

So, required probability $= 1 - P(A) = 1 - \frac{1}{32} = \frac{31}{32}$

2. **(d) :** Here, $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4\}$

$A \cap B = \{4\}$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{6} + \frac{4}{6} - \frac{1}{6} = 1$

3. We have the sample space associated with the given random experiment as follows :

$\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, H), (T, T)\}$

So, the total number of elementary events = 8 = $n(S)$

There is only one way in which head and number 6 occurring *i.e.*, $(H, 6)$

$\therefore$ $n(E) = 1$

So, the required probability $= \frac{n(E)}{n(S)} = \frac{1}{8}$

4. Required probability $= \frac{26}{52} \times \frac{26}{51} + \frac{26}{52} \times \frac{26}{51}$

$= 2 \times \frac{26}{52} \times \frac{26}{51} = \frac{26}{51}$

5. We have, $n(s) = 52$

Probability that first drawn card is red *i.e.*, $P(R_1) = \frac{26}{52}$

Probability that second drawn card is black *i.e.*, $P(B) = \frac{26}{51}$

Probability that third drawn card is red *i.e.*, $P(R_2) = \frac{25}{50}$

So, required probability $= P(R_1) \times P(B) \times P(R_2)$

$= \frac{26}{52} \times \frac{26}{51} \times \frac{25}{50} = \frac{13}{102}$

6. Required probability $= \frac{{}^4C_1 \times {}^3C_1 \times {}^2C_1}{{}^9C_3}$

$= \frac{4 \times 3 \times 2}{\frac{9 \times 7 \times 8}{3 \times 2}} = \frac{2}{7}$

7. B_1 contains 1 white ball and 3 red balls.
B_2 contains 2 white balls and 3 red balls.
P(two ball drawn of same colour)
= P(white ball of B_1 and white ball of B_2) or P(red ball of B_1 and red ball of B_2)

$$=\frac{1}{4}\times\frac{2}{5}+\frac{3}{4}\times\frac{3}{5}=\frac{2}{20}+\frac{9}{20}=\frac{11}{20}$$

8. Since the bag contains cards numbered 1 to 25.
So, the numbers which are multiple of 7 are {7, 14, 21}.

Required probability $=\frac{3}{25}\times\frac{2}{24}=\frac{1}{100}$

9. We have, $P(A) = 0.4$, $P(B) = 0.3$ and $P(A\cup B) = 0.6$
9, $P(A\cap B) = P(A) + P(B) - P(A\cup B) = 0.4 + 0.3 - 0.6 = 0.1$
Now, $P(B'\cap A) = P(A-B) = P(A) - P(A\cap B) = 0.4 - 0.1 = 0.3$

10. Total number of students = 8
The number of ways to select 4 students out of 8 students

$$={}^8C_4=\frac{8!}{4!4!}=70$$

The number of ways to select 2 boys and 2 girls

$$={}^3C_2\times{}^5C_2=\frac{3!}{2!1!}\times\frac{5!}{2!3!}=3\times10=30$$

$\therefore$ Required probability $=\frac{30}{70}=\frac{3}{7}$.

11. Probability of choosing bag A = $P(A)=\frac{2}{6}=\frac{1}{3}$

Probability of choosing bag B = $P(B)=\frac{4}{6}=\frac{2}{3}$

Let E_1 and E_2 be the events of drawing a red and a black ball from bag A and B respectively.

$$\therefore\quad P(E_1)=\frac{{}^6C_1\times{}^4C_1}{{}^{10}C_2}\text{ and }P(E_2)=\frac{{}^7C_1\times{}^3C_1}{{}^{10}C_2}$$

$\therefore$ Required probability = $P(A)\times P(E_1)+P(B)\times P(E_2)$

$$=\frac{1}{3}\times\frac{{}^6C_1\times{}^4C_1}{{}^{10}C_2}+\frac{2}{3}\times\frac{{}^7C_1\times{}^3C_1}{{}^{10}C_2}=\frac{8}{45}+\frac{14}{45}=\frac{22}{45}$$

12. **(d):** We have, $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$

We know that $P(B/A)=\frac{P(A\cap B)}{P(A)}$

$$\Rightarrow\quad 0.6=\frac{P(A\cap B)}{0.4}$$

$\Rightarrow\quad P(A\cap B) = 0.24$
Now, $P(A\cup B) = P(A) + P(B) - P(A\cap B) = 0.4 + 0.8 - 0.24 = 0.96$
Hence, $P(A\cup B) = 0.96$

13. **(c):** We know that, $P(B/A)=\frac{P(B\cap A)}{P(A)}=\frac{7/10}{4/5}=\frac{7}{8}$

14. **(a):** Sample space = {HH, HT, TH, TT}
Let A be the event of coming up two heads

$\therefore\quad A=\{HH\}\Rightarrow P(A)=\frac{1}{4}$

and B be the event of coming up atleast one head

$\therefore\quad B=\{HH, HT, TH\}\Rightarrow P(B)=\frac{3}{4}$

Also, $A\cap B=\{HH\}\Rightarrow P(A\cap B)=\frac{1}{4}$

So, required probability $=P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$

So, assertion is true.
Also, reason is true and it is the correct explanation of assertion.

15. **(c) :** Let A be the event that the card is a spade and B be the event that the picked card is a queen.
We have a total of 13 spades and 4 queen cards.
Also only one queen is from spade.

$$\therefore\quad P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{52}}{\frac{4}{52}}=\frac{1}{4}$$

Key Points

- A standard 52–card deck comprises 13 cards in each of the four suits : clubs, diamonds, hearts and spades.

16. We have, $P(A)=\frac{3}{10}, P(B)=\frac{2}{5}$ and $P(A\cup B)=\frac{3}{10}$
Now,

$$P(A\cap B)=P(A)+P(B)-P(A\cup B)=\frac{3}{10}+\frac{2}{5}-\frac{3}{5}=\frac{1+4-6}{10}=\frac{1}{10}$$

$$\therefore\quad [P(B/A)+P(A/B)]=\frac{P(A\cap B)}{P(A)}+\frac{P(A\cap B)}{P(B)}$$

$$=\frac{\frac{1}{10}}{3/10}+\frac{\frac{1}{10}}{2/5}=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$$

17. The sample space, S is given by
$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
Let A be the event that number on the drawn card is odd, and B be the event that number on the drawn card is greater than 5.
$\therefore\quad A = \{1, 3, 5, 7, 9, 11\}$
$B = \{6, 7, 8, 9, 10, 11, 12\}$
and, $A\cap B=\{7, 9, 11\}$

Now, $P(A)=\frac{n(A)}{n(S)}=\frac{6}{12}$,

$P(B)=\frac{n(B)}{n(S)}=\frac{7}{12}$

$P(A\cap B)=\frac{n(A\cap B)}{n(S)}=\frac{3}{12}$

Now, $P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{3/12}{7/12}=\frac{3}{7}$

Hence, required probability is $\frac{3}{7}$.

18. Let M, F and S denote mother, father and son respectively.

Sample space $S = \{MFS, MSF, FMS, FSM, SMF, SFM\}$

Given, A = Son on one end *i.e.*, $\{MFS, FMS, SMF, SFM\}$

and B = Father in the middle *i.e.*, $B = \{MFS, SFM\}$

$A \cap B = \{MFS, SFM\}$

$$P(A)=\frac{4}{6}=\frac{2}{3}, P(B)=\frac{2}{6}=\frac{1}{3} \text{ and } P(A\cap B)=\frac{2}{6}=\frac{1}{3}$$

$$\therefore \quad P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{1/3}{2/3}=\frac{1}{2}$$

19. E : Sum 8' and F : 'red die resulted in a number less than 4'

i.e., $E = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

i.e., $F = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (6, 1), (6, 2), (6, 3)\}$

Hence, $E \cap F = \{(5, 3), (6, 2)\}$, $P(E) = 5/36$,

$P(F) = 18/36$, $P(E \cap F) = 2/36$

$\therefore$ Required probability $= P(E/F)$

$$=\frac{P(E\cap F)}{P(F)}=\frac{2/36}{18/36}=\frac{2}{18}=\frac{1}{9}.$$

20. Let G_i $(i = 1, 2)$ and B_i $(i = 1, 2)$ denote the i^{th} child is a girl or a boy respectively.

Then sample space is,

$S = \{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$

Let A be the event that both children are girls, B be the event that the youngest child is a girl and C be the event that at least one of the children is a girl.

Then $A = \{G_1G_2\}$, $B = \{G_1G_2, B_1G_2\}$

and $C = \{B_1G_2, G_1G_2, G_1B_2\}$

$\Rightarrow \quad A \cap B = \{G_1G_2\}$ and $A \cap C = \{G_1G_2\}$

(i) Required probability $= P(A/B) = \dfrac{P(A\cap B)}{P(B)}=\dfrac{1/4}{2/4}=\dfrac{1}{2}$

(ii) Required probability $= P(A/C) = \dfrac{P(A\cap C)}{P(C)}=\dfrac{1/4}{3/4}=\dfrac{1}{3}$

21. Let $B_i(i = 1, 2)$ and $G_i(i = 1, 2)$ denote the i^{th} child is a boy or a girl respectively.

Then sample space is, $S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$

Let A be the event that both are boys, B be the event that one of them is a boy and C be the event that the older child is a boy.

$A = \{B_1B_2\}$, $B = \{G_1B_2, B_1G_2, B_1B_2\}$

$C = \{B_1B_2, B_1G_2\} \Rightarrow A\cap B = \{B_1B_2\}$ and $A\cap C = \{B_1B_2\}$

(i) Required probability $= P(A/B) = \dfrac{P(A\cap B)}{P(B)}=\dfrac{1/4}{3/4}=\dfrac{1}{3}$

(ii) Required probability $= P(A/C) = \dfrac{P(A\cap C)}{P(C)}=\dfrac{1/4}{2/4}=\dfrac{1}{2}$

22. The sample space S of the given random experiment is

$S = \{(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

Let A be the event that the die shows a number greater than 4 and B be the event that there is at least one tail.

$\therefore \quad A = \{(T, 5), (T, 6)\}$

and $B = \{(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6), (H, T)\}$

$A\cap B = \{(T, 5), (T, 6)\}$

$\therefore \quad P(B) = P\{(T, 1)\} + P\{(T, 2)\} + P\{(T, 3)\} + P\{(T, 4)\} + P\{(T, 5)\} + P\{(T, 6)\} + P\{(H, T)\}$

$$=\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{4}=\frac{3}{4}$$

$$P(A\cap B)=P\{(T,5)\}+P\{(T,6)\}=\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$$

$\therefore$ Required probability $=P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{1/6}{3/4}=\dfrac{2}{9}$

23. Let A be the event of drawing a red ball in first draw and B be the event of drawing a red ball in second draw.

$$\therefore \quad P(A)=\frac{{}^3C_1}{{}^{10}C_1}=\frac{3}{10}$$

Now, $P(B/A)$ = Probability of drawing a red ball in the second draw, when a red ball already has been drawn in the first draw $=\dfrac{{}^2C_1}{{}^9C_1}=\dfrac{2}{9}$

$\therefore$ The required probability $= P(A \cap B)$

$$= P(A)\cdot P(B/A) = \frac{3}{10}\times\frac{2}{9}=\frac{1}{15}$$

Answer Tips

- Conditional probability is calculated by multiplied the probability of the preceding event by the renewed probability of the succeeding event.

24. **(c) :** Given, A and B are independent events.

Also, $P(A) = \dfrac{1}{3}$ and $P(B) = \dfrac{1}{4}$

Now, $P(B'|A) = \dfrac{P(B'\cap A)}{P(A)}$

$=\dfrac{P(B')P(A)}{P(A)}$ [$\because$ A, B are independent events]

$= P(B') = 1 - P(B) = 1-\dfrac{1}{4}=\dfrac{3}{4}$

25. Let A, B, C be respectively the events of solving problem by three students and $P(A)$, $P(B)$, $P(C)$ be their probability of solving the problem respectively.

$\therefore \quad P(A) = \dfrac{1}{3}$, $P(B) = \dfrac{1}{4}$ and $P(C) = \dfrac{1}{6}$

Required probability $= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

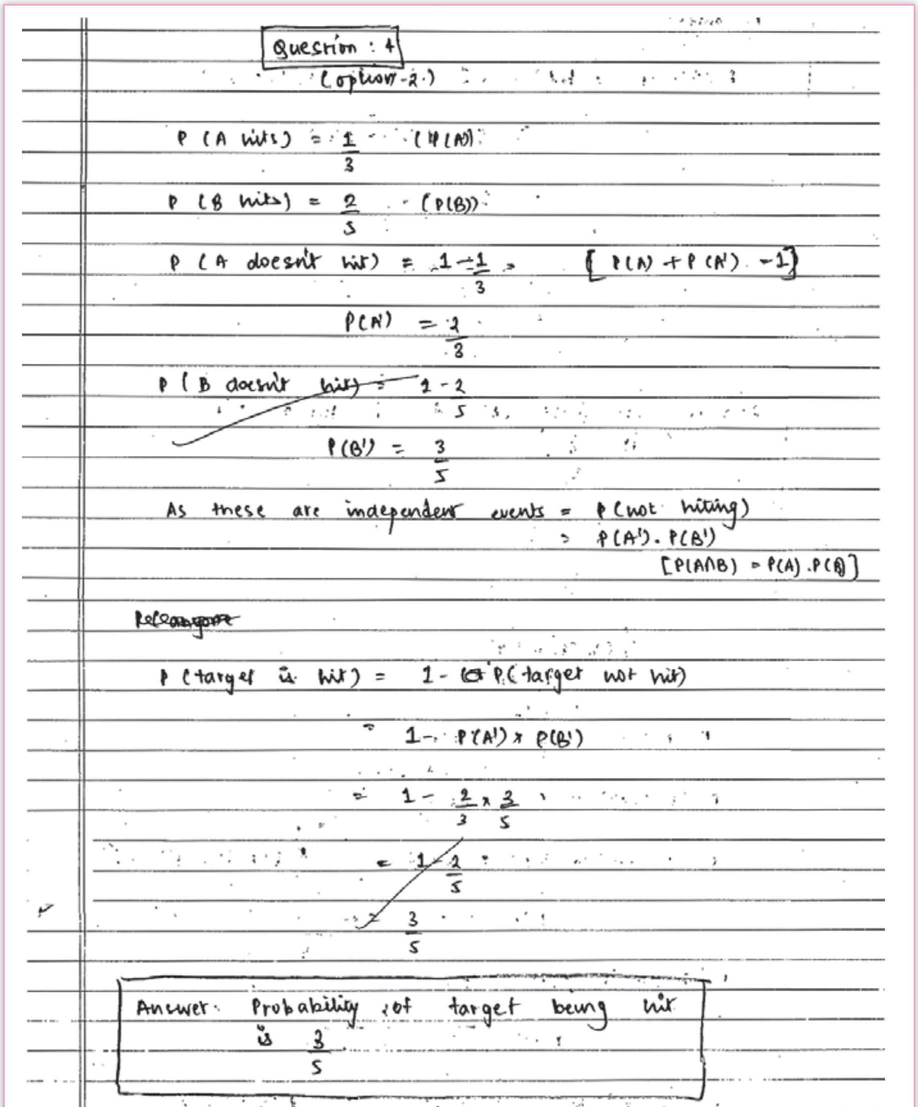
$= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})$

($\because A, B, C$ are independent $\therefore \bar{A}, \bar{B}, \bar{C}$ are also independent)

$= 1 - [1 - P(A)]\, [1 - P(B)]\, [1 - P(C)]$

$$=1-\frac{2}{3}\times\frac{3}{4}\times\frac{5}{6}=1-\frac{5}{12}=\frac{7}{12}$$

26.



Question : 4
(option-2)

P (A hits) = $\frac{1}{3}$ (P(A))

P (B hits) = $\frac{2}{5}$ (P(B))

P (A doesn't hit) = $1 - \frac{1}{3}$ [P(A) + P(A') = 1]

P(A') = $\frac{2}{3}$

P (B doesn't hit) = $1 - \frac{2}{5}$

P(B') = $\frac{3}{5}$

As these are independent events = P (not hiting)
= P(A') . P(B')
[P(A∩B) = P(A) . P(B)]

P (target is hit) = 1 − P (target not hit)

= 1 − P(A') × P(B')

= $1 - \frac{2}{3} \times \frac{3}{5}$

= $1 - \frac{2}{5}$

= $\frac{3}{5}$

Answer : Probability of target being hit is $\frac{3}{5}$

[Topper's Answer, 2022]

27. Given : $P(A)=\frac{1}{2}, P(B)=\frac{7}{12}, \; P(\bar{A}\cup\bar{B})=\frac{1}{4}$

To find whether A and B are independent or not.

Two events are independent if $P(A\cap B)=P(A)\cdot P(B)$

$P(\bar{A}\cup\bar{B})=1-P(A\cap B)$

$\Rightarrow \frac{1}{4}=1-P(A\cap B) \Rightarrow P(A\cap B) = 1-\frac{1}{4}=\frac{3}{4}$

and $P(A)\cdot P(B) = \frac{1}{2}\times\frac{7}{12}=\frac{7}{24}$

Since $P(A\cap B) \neq P(A)\cdot P(B)$

$\therefore$ A and B are not independent.

28. Let G be the event of a green signal.

Required probability = $P(GGG') + P(G'GG)$

$$= \left(\frac{3}{10}\right)^2\frac{7}{10}+\frac{7}{10}\cdot\left(\frac{3}{10}\right)^2 = \frac{9}{100}\cdot\frac{7}{10}+\frac{7}{10}\cdot\frac{9}{100}$$

$$= \frac{63}{1000}+\frac{63}{1000}=\frac{126}{1000}=\frac{63}{500}$$

29. Given, A and B are independent events. So, A' and B' are also independent events.

Now, $P(A'\cap B') = P(A') \times P(B')$

$= [1 - P(A)][1 - P(B)] = [1 - 0.3][1 - 0.6]$

[Given, $P(A) = 0.3$ and $P(B) = 0.6$]

$= 0.7 \times 0.4 = 0.28$

30. Let A denotes the student A coming school on time and B denotes the student B coming school on time.

$\therefore \; P(A)=\frac{2}{7}$ and $P(B)=\frac{4}{7}$

So, we have, $P(\bar{A})=1-P(A)=1-\frac{2}{7}=\frac{5}{7}$

and $P(\bar{B})=1-P(B)=1-\frac{4}{7}=\frac{3}{7}$

$\therefore$ Probability of only one of them coming to school on time $=P(\bar{A}\cap B)+P(A\cap\bar{B})$

$=P(\bar{A})\times P(B)+P(A)\times P(\bar{B})$

$=\frac{5}{7}\times\frac{4}{7}+\frac{2}{7}\times\frac{3}{7} = \frac{20}{49}+\frac{6}{49}=\frac{26}{49}$

31. We have, $S = \{1, 2, 3, 4, 5, 6\}$ and A be the event that number is even = {2, 4, 6}

$\Rightarrow P(A)=\frac{3}{6}=\frac{1}{2}$

B be the event that number is red = {1, 2, 3}

$\Rightarrow P(B)=\frac{3}{6}=\frac{1}{2}$

and $A \cap B = \{2\}$

$\Rightarrow P(A\cap B)=\frac{1}{6}$ ...(i)

Also, $P(A)\cdot P(B)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ ...(ii)

From (i) and (ii),

$P(A) \cdot P(B) \neq P(A \cap B)$

So, A and B are not independent.

32. Since, E and F are independent events.

$\therefore$ $P(E \cap F) = P(E)\, P(F)$ ...(i)

Now, $P(E' \cap F') = 1 - P(E \cup F)$ $[\because P(E' \cap F') = P((E \cup F)')]$

$= 1 - [P(E) + P(F) - P(E \cap F)]$

$= 1 - P(E) - P(F) + P(E)\, P(F)$ [Using (i)]

$= (1 - P(E))\,(1 - P(F)) = P(E')\, P(F')$

Hence, E' and F' are also independent events.

33. (a) We have, $P(A) = 0.8$, $P(B) = 0.9$

P(exactly one of them earn 10 points) $= P(A \cup B) - P(A \cap B)$

$= P(A) + P(B) - 2P(A \cap B)$

$= P(A) + P(B) - 2P(A)\, P(B)$ ($\because$ A & B are independent)

$= 0.8 + 0.9 - 0.8 \times 0.9 \times 2$

$= 0.26$

(b) P(both of them earn 10 points) $= P(A \cap B)$

$= P(A)\, P(B) = 0.8 \times 0.9 = 0.72$

34. Total outcomes = 36

Favourable outcomes for A to win

= {(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}

$\therefore$ Probability of A to win, $P(A) = \frac{6}{36}=\frac{1}{6}$

Probability of A to lose, $P(\bar{A})=1-\frac{1}{6}=\frac{5}{6}$

Favourable outcomes for B to win = {(4, 6), (6, 4), (5, 5)}

$\therefore$ Probability of B to win, $P(B) = \frac{3}{36}=\frac{1}{12}$

Probability of B to lose, $P(\bar{B})=1-\frac{1}{12}=\frac{11}{12}$

$\therefore$ Required probability

$=P(\bar{A})P(B)+P(\bar{A})P(\bar{B})P(\bar{A})P(B)$

$+P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(\bar{A})P(B)+....$

$=\frac{5}{6}\times\frac{1}{12}+\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{1}{12}+\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{1}{12}+....$

$=\frac{5/72}{1-\frac{5}{6}\times\frac{11}{12}}=\frac{5}{17}$

35. Let X and Y denote the respective events of solving the given specific problem by A and B,

then $P(X) = \frac{1}{2}$ and $P(Y)=\frac{1}{3}$

(i) P (problem is solved)

$=P(X\cup Y)=1-P(\bar{X})P(\bar{Y})=1-\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=1-\frac{1}{2}\times\frac{2}{3}=\frac{2}{3}.$

(ii) P (Exactly one of A and B solves the problem)

$P(X)\cdot P(\bar{Y})+P(\bar{X})\cdot P(Y)$

$=\frac{1}{2}\left(1-\frac{1}{3}\right)+\left(1-\frac{1}{2}\right)\cdot\frac{1}{3}=\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{2}\left(\frac{2}{3}+\frac{1}{3}\right)=\frac{1}{2}$

Concept Applied

- $P(\text{not } A) = 1 - P(A)$

36. It is given that A and B are independent events and

$P(\bar{A}\cap B)=\frac{2}{15}$

$\Rightarrow$ $P(\bar{A})P(B)=\frac{2}{15}$ ...(i)

Also, $P(A\cap\bar{B})=\frac{1}{6} \Rightarrow P(A)P(\bar{B})=\frac{1}{6}$ ...(ii)

Let $p = P(A) \Rightarrow P(\bar{A}) = 1 - P(A) = 1 - p$

and $q = P(B) \Rightarrow P(\bar{B}) = 1 - P(B) = 1 - q$

Now, from (i) and (ii), we get

$(1 - p)q = \frac{2}{15}$ ...(iii)

and $p(1 - q) = \frac{1}{6}$ ...(iv)

Subtracting (iii) from (iv), we get

$p - q = \frac{1}{6}-\frac{2}{15}=\frac{1}{30}\Rightarrow p=q+\frac{1}{30}$

Putting this value of p in (iii), we get

$\left(1-q-\frac{1}{30}\right)q=\frac{2}{15} \Rightarrow \frac{29}{30}q-q^2=\frac{2}{15}$

$\Rightarrow$ $30q^2 - 29q + 4 = 0 \Rightarrow 30q^2 - 24q - 5q + 4 = 0$

$\Rightarrow$ $6q(5q - 4) - 1(5q - 4) = 0 \Rightarrow (5q - 4)(6q - 1) = 0$

$\Rightarrow$ $q=\frac{4}{5}$ or $\frac{1}{6}$

For $q=\frac{4}{5}$, from (iv), we have

$p\left(1-\frac{4}{5}\right)=\frac{1}{6}\Rightarrow p\left(\frac{1}{5}\right)=\frac{1}{6}\Rightarrow p=\frac{5}{6}$

For $q=\frac{1}{6}$, from (iv), we have

$p\left(1-\frac{1}{6}\right)=\frac{1}{6}\Rightarrow p\left(\frac{5}{6}\right)=\frac{1}{6}\Rightarrow p=\frac{1}{5}$

$\therefore$ $P(A)=\frac{5}{6}, P(B)=\frac{4}{5}$ or $P(A)=\frac{1}{5}, P(B)=\frac{1}{6}$

Commonly Made Mistake

- Remember the difference between exclusive and exhaustive events.

37. Let E_1 be the event that bag I is chosen,
E_2 be the event that bag II is chosen and A be the event that red ball is drawn.
Clearly, E_1 and E_2 are mutually exclusive and exhaustive events.
Since, one of the bag is chosen at random

$\therefore \quad P(E_1)=\frac{1}{2}$ and $P(E_2)=\frac{1}{2}\quad P(A|E_1)=\frac{1}{4}$ and $P(A|E_2)=\frac{3}{8}$

By using law of total probability, we get
$P(A) = P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2)]$

$$=\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{3}{8}=\frac{1}{8}+\frac{3}{16}=\frac{5}{16}$$

38. Let E_1, E_2 and A denote the events defined as follow :
E_1 = selecting a purse 1
E_2 = selecting a purse 2
A = drawing a silver coin
Since one of two purses is selected randomly

$\therefore\ P(E_1)=\frac{1}{2}$ and $P(E_2)=\frac{1}{2}$

Now, $P(A/E_1)=\frac{3}{9}=\frac{1}{3}$ and $P(A/E_2)=\frac{4}{7}$

Using the total law of probability, we have,
Required probability, $P(A) = P(E_1)\cdot P(A/E_1) + P(E_2)\cdot(A/E_2)$

$$\Rightarrow\ P(A)=\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\times\frac{4}{7}=\frac{1}{6}+\frac{2}{7}=\frac{19}{42}$$

39. Given, E_1 : represent the event when many workers were not present for the job.
$P(E_1) = 0.65$
E_2 : represent the event when all workers were present.
$P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35$
E = represent completing the construction work on time.
(i) Required probability $=P(E_2) = 0.35$
(ii) Given, $P(E/E_1)=0.35$ and $P(E/E_2)=0.80$

$P(E)=P(E_1)P(E/E_1)+P(E_2)P(E/E_2)$ ($\because$ Law of total probability)

$= 0.65 \times 0.35 + 0.35 \times 0.80 = 0.2275 + 0.28 = 0.5075$

(iii) (a) We have to find $P(E_1|E)$
By using Bayes' theorem,

$$P(E_1|E)=\frac{P(E_1)\cdot P(E|E_1)}{P(E_1)\cdot P(E|E_1)+P(E_2)\cdot P(E|E_2)}$$

$$=\frac{P(E_1)\cdot P(E|E_1)}{P(E)}=\frac{0.65\times0.35}{0.5075}\approx0.448$$

OR

(b) We have to find $P(E_2|E)$
By using Bayes' theorem

$$P(E_2|E)=\frac{P(E_2)\cdot P(E|E_2)}{P(E_1)\cdot P(E|E_1)+P(E_2)\cdot P(E|E_2)}=\frac{P(E_2)\cdot P\left(\frac{E}{E_2}\right)}{P(E)}$$

$$=\frac{0.35\times0.80}{0.5075}=\frac{0.28}{0.5075}\approx0.551$$

40. Let E_1, E_2 and A denote the events defined as follows :
E_1 = Selecting box I
E_2 = Selecting box II
A = Getting a red ball

$P(E_1)=\frac{1}{2}, P(E_2)=\frac{1}{2}$

$P(A|E_1) = \frac{3}{9}=\frac{1}{3}, P(A|E_2)=\frac{5}{10}=\frac{1}{2}$

Using Bayes' Theorem

$$P(E_2|A)=\frac{P(E_2)\cdot P(A|E_2)}{P(E_1)P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{2}}=\frac{\frac{1}{4}}{\frac{1}{6}+\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{2+3}{12}}=\frac{1}{4}\times\frac{12}{5}=\frac{3}{5}$$

$\therefore$ The probability that a red ball comes out from box II is $\frac{3}{5}$.

41. Let E_1 be the event that ball transferred from bag I to bag II is red, E_2 be the event that ball transferred from bag I to bag II is black and B be the event that ball drawn from bag II is black.

So, $P(E_1)=\frac{3}{8}$, $P(E_2)=\frac{5}{8}$

$P(B|E_1)=\frac{3}{8}$, $P(B|E_2)=\frac{4}{8}$

So, required probability = $P(E_2 \mid B)$

$$=\frac{P(E_2)\times P(B|E_2)}{P(E_1)\times P(B|E_1)+P(E_2)\times P(B|E_2)}=\frac{\frac{5}{8}\times\frac{4}{8}}{\frac{3}{8}\times\frac{3}{8}+\frac{5}{8}\times\frac{4}{8}}=\frac{20}{9+20}=\frac{20}{29}$$

42. Let E_1, E_2 and A denote the events defined as follows :
E_1 = First bag is chosen
E_2 = Second bag is chosen
and A = two balls drawn at random are red
Since, one of the bag is chosen at random

$\therefore\ P(E_1)=P(E_2)=\frac{1}{2}$

If E_1 has already occurred, *i.e.*, first bag is chosen.
Therefore, the probability of drawing two red balls in this case $=P(A|E_1)=\frac{{}^5C_2}{{}^9C_2}=\frac{10}{36}$

Similarly, $P(A|E_2)=\frac{{}^3C_2}{{}^9C_2}=\frac{3}{36}$

By Bayes' theorem,

Required probability, $P(E_2|A)=\frac{P(E_2)\cdot P(A|E_2)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}$

$$=\frac{\frac{1}{2}\times\frac{3}{36}}{\frac{1}{2}\times\frac{10}{36}+\frac{1}{2}\times\frac{3}{36}}=\frac{\frac{3}{72}}{\frac{10}{72}+\frac{3}{72}}=\frac{\frac{3}{72}}{\frac{13}{72}}=\frac{3}{13}$$

43. Let E_1 be the event of choosing a biased coin and E_2 be the event of choosing an unbiased coin.

$\Rightarrow \quad P(E_1) = P(E_2) = \frac{1}{2}$

Given, probability of biased coin has the chance of showing heads is 60%

$\therefore$ Probability of biased coin has the chance of showing tail is 40%

Let A be the event of showing tail.

$\therefore \quad P(A|E_1)=\frac{40}{100}=\frac{2}{5} \quad P(A|E_2)=\frac{1}{2}$

Using Bayes' theorem, we get

$$P(E_2|A)=\frac{P(E_2)P(A|E_2)}{P(E_1)(A|E_1)+P(E_2)(A|E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{1}{2}}=\frac{\frac{1}{4}}{\frac{1}{5}+\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{9}{20}}=\frac{5}{9}$$

44. Let E_1 be the event of getting ghee from shop X, E_2 be the event of getting ghee from shop Y and A be the event of getting ghee of type B.

$\therefore \quad P(E_1)=\frac{1}{2},\ P(E_2)=\frac{1}{2},\ P(A|E_1)=\frac{40}{70}=\frac{4}{7},$

$P(A|E_2)=\frac{60}{110}=\frac{6}{11}$

Using Bayes' Theorem, we have

$$P(E_2|A)=\frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)}$$

$$\Rightarrow \quad P(E_2|A)=\frac{\frac{1}{2}\times\frac{6}{11}}{\frac{1}{2}\times\frac{4}{7}+\frac{1}{2}\times\frac{6}{11}}=\frac{\frac{6}{11}}{\frac{4}{7}+\frac{6}{11}}=\frac{42}{44+42}=\frac{42}{86}=\frac{21}{43}$$

45. Let E_1 be the event that the outcome on the die is 1 or 2, E_2 be the event that the outcome on the die is 3, 4, 5, 6.

$\therefore\ P(E_1)=\frac{2}{6}=\frac{1}{3}$ and $P(E_2)=\frac{4}{6}=\frac{2}{3}$

Let A be the event of getting exactly one tail.

Now, $P(A|E_1)$ be the probability of getting exactly one tail by tossing the coin three times if she gets 1 or 2 = $\frac{3}{8}$ and $P(A|E_2)$ be the probability of getting exactly one tail in a single throw of coin if she gets 3, 4, 5, 6 = $\frac{1}{2}$

The probability that the girl threw 3, 4, 5, 6 with the die, if she obtained exactly one tail is given by $P(E_2|A)$.

$$\therefore \quad P(E_2|A)=\frac{P(E_2)\cdot P(A|E_2)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

$$=\frac{\frac{2}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{3}{8}+\frac{2}{3}\cdot\frac{1}{2}}=\frac{8}{11}$$

Concept Applied

- If $E_1, E_2, E_3, ..., E_n$ are mutually exclusive and exhaustive events associated with a random experiment and A is any event associated with the experiment, then

$$P(E_i|A)=\frac{P(E_i)P(A|E_i)}{\sum_i P(E_i)P(A|E_i)}, \quad \text{where } i = 1, 2, 3, ..., n$$

46. Let E_1 be the event that '6' occurs, E_2 be the event that '6' does not occur and A be the event that the man reports that it is '6'.

$\therefore \quad P(E_1)=\frac{1}{6},\ P(E_2)=\frac{5}{6}$

Now, $P(A|E_1)$ be the probability that the man reports that there is '6' on the die and '6' actually occurs

= Probability that the man speaks the truth $=\frac{4}{5}$

And $P(A|E_2)$ be the probability that the man reports that there is '6' when actually '6' does not occurs

= Probability that man does not speaks the truth

$=1-\frac{4}{5}=\frac{1}{5}$

$\therefore \quad$ Required probability = $P(E_1|A)$

$$=\frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

$$=\frac{\frac{1}{6}\times\frac{4}{5}}{\frac{1}{6}\times\frac{4}{5}+\frac{5}{6}\times\frac{1}{5}}=\frac{4}{4+5}=\frac{4}{9}$$

Yes, we are agree that the value of truthfulness leads to more respect in the society.

47. Let E_1 be event of students which have 100% attendance, E_2 be the event of students which are irregular and A be the event of students which have an A grade.

Then, $P(E_1)$ = 0.3, $P(E_2)$ = 0.7, $P(A|E_1)$ = 0.7 and $P(A|E_2)$ = 0.1

So, P(Probability that student has 100% attendance given that he has A grade)

$$=P(E_1|A)=\frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

[Using Bayes' theorem]

$$=\frac{0.3\times0.7}{0.3\times0.7+0.7\times0.1}$$

$$=\frac{0.3\times0.7}{0.7(0.3+0.1)}=\frac{0.3}{0.4}=\frac{3}{4}=0.75$$

As per answer, the probability of regular students having grade A is more than 50%. So, the regularity is required. No, regularity is required everywhere as it maintains our respect in society.

48. Let I be the event that changes take place to improve profits.

Probability of selection of A, $P(A)=\frac{1}{7}$

Probability of selection of B, $P(B)=\frac{2}{7}$

Probability of selection of C, $P(C)=\frac{4}{7}$

Probability that A does not introduce changes,

$P(\bar{I}|A)=1-0.8=0.2$

Probability that B does not introduce changes,

$P(\bar{I}|B)=1-0.5=0.5$

Probability that C does not introduce changes,

$P(\bar{I}|C)=1-0.3=0.7$

So, required probability $=P(C|\bar{I})$

$$=\frac{P(C)P(\bar{I}|C)}{P(A)P(\bar{I}|A)+P(B)P(\bar{I}|B)+P(C)P(\bar{I}|C)}$$

$$=\frac{\frac{4}{7}\times 0.7}{\frac{1}{7}\times 0.2+\frac{2}{7}\times 0.5+\frac{4}{7}\times 0.7}=0.7$$

49. Consider the following events.
E : Two balls drawn are white
A : There are 2 white balls in the bag
B : There are 3 white balls in the bag
C : There are 4 white balls in the bag

$P(A) = P(B) = P(C) = \frac{1}{3}$

$$P(E|A)=\frac{^2C_2}{^4C_2}=\frac{1}{6},\ P(E|B)=\frac{^3C_2}{^4C_2}=\frac{3}{6}=\frac{1}{2}$$

$$P(E|C)=\frac{^4C_2}{^4C_2}=1$$

$$\therefore\quad P(C|E)=\frac{P(C)\cdot P(E|C)}{P(A)\cdot P(E|A)+P(B)\cdot P(E|B)+P(C)\cdot P(E|C)}$$

$$=\frac{\frac{1}{3}\times 1}{\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 1}=\frac{3}{5}$$

50. Let A be the event that the bulb is defective.

$$\therefore\quad P(E_1)=\frac{50}{100},\ P(E_2)=\frac{25}{100},\ P(E_3)=\frac{25}{100}$$

$$P(A|E_1)=\frac{4}{100},\ P(A|E_2)=\frac{4}{100},\ P(A|E_3)=\frac{5}{100}$$

$\therefore$ Required probability, $P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)$

$$=\frac{50}{100}\times\frac{4}{100}+\frac{25}{100}\times\frac{4}{100}+\frac{25}{100}\times\frac{5}{100}$$

$$=\frac{200+100+125}{10000}=\frac{425}{10000}=\frac{17}{400}$$

51. Let E_1, E_2 and A be the events defined as follows :
E_1 : The student knows the answer
E_2 : The student guesses the answer
A : The student answers correctly

We have, $P(E_1)=\frac{3}{5}, P(E_2)=\frac{2}{5}$

Also, $P(A|E_2)=\frac{1}{3}$ and $P(A|E_1) = 1$

$\therefore$ Required probability

$$=P(E_1|A)=\frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

$$=\frac{\frac{3}{5}\cdot 1}{\frac{3}{5}\cdot 1+\frac{2}{5}\cdot\frac{1}{3}}=\frac{3\times 3}{3\times 3+2}=\frac{9}{11}.$$

52. Let E_1, E_2, E_3 and E be the events defined as follows:
E_1 : The item is manufactured by operator A
E_2 : The item is manufactured by operator B
E_3 : The item is manufactured by operator C
E : The item is defective.

$$\therefore\ P(E_1)=\frac{50}{100}=\frac{5}{10},\ P(E_2)=\frac{30}{100}=\frac{3}{10},\ P(E_3)=\frac{20}{100}=\frac{2}{10}$$

$$P(E|E_1)=\frac{1}{100};P(E|E_2)=\frac{5}{100};P(E|E_3)=\frac{7}{100}$$

Now, we have, to find $P(E_1/E)$ (*i.e.*, item is defective and it is produced by operator A)

$$\therefore\quad P(E_1/E)=\frac{P(E_1)P(E|E_1)}{P(E_1)P(E|E_1)+P(E_2)P(E|E_2)+P(E_3)P(E|E_3)}$$

$$=\frac{\frac{5}{10}\times\frac{1}{100}}{\frac{5}{10}\times\frac{1}{100}+\frac{3}{10}\times\frac{5}{100}+\frac{2}{10}\times\frac{7}{100}}=\frac{5}{5+15+14}=\frac{5}{34}$$

53. Total number of persons insured
= 3000 + 6000 + 9000 = 18000

Let E_1, E_2 and E_3 be the event that the person is a cyclist, a scooter driver and a car driver respectively.

$$\therefore\ P(E_1)=\frac{3000}{18000}=\frac{1}{6},\ P(E_2)=\frac{6000}{18000}=\frac{1}{3}$$

and $P(E_3)=\frac{9000}{18000}=\frac{1}{2}$

Let E be the event that insured person meets with an accident.

$\therefore\quad P(E|E_1) = 0.3, P(E|E_2) = 0.05, P(E|E_3) = 0.02$

By Bayes' theorem,

$\therefore$ Required probability

$$=\frac{P(E|E_1)\cdot P(E_1)}{P(E|E_1)\cdot P(E_1)+P(E|E_2)\cdot P(E_2)+P(E|E_3)P(E_3)}$$

$$=\frac{0.3\times\frac{1}{6}}{0.3\times\frac{1}{6}+0.05\times\frac{1}{3}+0.02\times\frac{1}{2}}=\frac{\frac{0.3}{6}}{\frac{0.3+0.1+0.06}{6}}=\frac{0.3}{0.46}=\frac{15}{23}.$$

54. Let us consider the following events
E_1 = bag I is selected
E_2 = bas II is selected
A = getting a red ball

Here $P(E_1)=P(E_2)=\frac{1}{2}$ $P(A|E_1)=\frac{3}{9}=\frac{1}{3}$ and $P(A|E_2)=\frac{5}{5+n}$

By Baye's theorem, we have

$$P(E_2|A)=\frac{P(E_2)\cdot P(A|E_2)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

$$\Rightarrow\ \frac{3}{5}=\frac{\frac{1}{2}\times\frac{5}{5+n}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{5}{5+n}}\ \Rightarrow\ \frac{3}{5}=\frac{\frac{5}{5+n}}{\frac{1}{3}+\frac{5}{5+n}}$$

$$\Rightarrow\ \frac{3}{5}=\frac{\frac{5}{5+n}}{(5+n+15)/[3(5+n)]}=\frac{5}{5+n}\times\frac{3(n+5)}{n+20}$$

$$\Rightarrow\ \frac{3}{5}=\frac{15}{n+20}\Rightarrow 3n+60=75\ \Rightarrow\ 3n=15\Rightarrow n=5$$

Hence, the value of n is 5.

55. Let E_1, E_2, E_3 and C be the events as defined below :
E_1 : Two red balls are transferred from bag A to bag B.
E_2 : One red ball and one black ball is transferred from bag A to bag B.
E_3 : Two black balls are transferred from bag A to bag B.
C : Ball drawn from bag B is red.

So, $P(E_1) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$, $P(E_2) = \frac{{}^3C_1 \times {}^5C_1}{{}^8C_2} = \frac{15}{28}$

$P(E_3) = \frac{{}^5C_2}{{}^8C_2} = \frac{10}{28}$

Also, $P(C|E_1) = \frac{6}{10}$, $P(C|E_2) = \frac{5}{10}$, $P(C|E_3) = \frac{4}{10}$

$\therefore$ Required probability, $P(E_1|C)$

$$= \frac{P(E_1)P(C|E_1)}{P(E_1)P(C|E_1)+P(E_2)P(C|E_2)+P(E_3)P(C|E_3)}$$

$$= \frac{\frac{3}{28}\times\frac{6}{10}}{\frac{3}{28}\times\frac{6}{10}+\frac{15}{28}\times\frac{5}{10}+\frac{10}{28}\times\frac{4}{10}} = \frac{18}{18+75+40} = \frac{18}{133}$$

56. Let E_1, E_2, E_3 and E be the events as follows:

E_1 : The bolt is manufactured by the machine A

E_2 : The bolt is manufactured by the machine B

E_3 : The bolt is manufactured by the machine C

E : The bolt is defective.

$\therefore \quad P(E_1) = \frac{30}{100} = \frac{3}{10}$; $P(E_2) = \frac{50}{100} = \frac{5}{10}$;

$P(E_3) = \frac{20}{100} = \frac{2}{10}$

$P(E|E_1) = \frac{3}{100}$; $P(E|E_2) = \frac{4}{100}$; $P(E|E_3) = \frac{1}{100}$

Now, $P(E_2|E) = \frac{P(E_2)\cdot P(E|E_2)}{\sum_{i=1}^{3} P(E_i)\cdot P(E|E_i)}$

$$= \frac{\frac{5}{10}\cdot\frac{4}{100}}{\frac{3}{10}\cdot\frac{3}{100}+\frac{5}{10}\cdot\frac{4}{100}+\frac{2}{10}\cdot\frac{1}{100}} = \frac{20}{9+20+2} = \frac{20}{31}$$

$\therefore$ Required probability = The probability that bolt is defective and not manufactured by machine B.

$$= 1 - P(E_2|E) = 1 - \frac{20}{31} = \frac{11}{31}.$$

57. Let E_1 and E_2 denote the events of selection of first bag and second bag respectively. Let A be the event that 2 balls drawn are both red.

$\therefore \quad P(E_1) = \frac{1}{2} = P(E_2)$

Now, $P(A|E_1) = \frac{{}^4C_2}{{}^8C_2} = \frac{4\cdot 3}{8\cdot 7} = \frac{3}{14}$ $\quad P(A|E_2) = \frac{{}^2C_2}{{}^8C_2} = \frac{1\times 2}{8\cdot 7} = \frac{1}{28}$

The required probability = $P(E_1|A)$

$$= \frac{P(E_1)\cdot P(A|E_1)}{P(E_1).P(A|E_1)+P(E_2).P(A|E_2)} = \frac{\frac{1}{2}\cdot\frac{3}{14}}{\frac{1}{2}\cdot\frac{3}{14}+\frac{1}{2}\cdot\frac{1}{28}} = \frac{3\times 2}{3\times 2+1} = \frac{6}{7}.$$

58. Let E_1, E_2 and A be the events defined as follows :

E_1 : The student knows the answer

E_2 : The student guesses the answer

A : The student answers correctly

We have, $P(E_1) = \frac{3}{5}$, $P(E_2) = \frac{2}{5}$

Also, $P(A|E_2) = \frac{1}{3}$ and $P(A|E_1) = 1$

$\therefore$ Required probability

$$= P(E_1|A) = \frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)}$$

$$= \frac{\frac{3}{5}\cdot 1}{\frac{3}{5}\cdot 1+\frac{2}{5}\cdot\frac{1}{3}} = \frac{3\times 3}{3\times 3+2} = \frac{9}{11}.$$

59. Let E_1, E_2, E_3, E_4 and A be the events defined as below :

E_1 : Missing card is a card of heart.

E_2 : Missing card is a card of spade.

E_3 : Missing card is a card of club.

E_4 : Missing card is a card of diamond.

A : Drawing three spade cards from the remaining cards.

Now, $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{13}{52} = \frac{1}{4}$

$P(A|E_2) = \frac{{}^{12}C_3}{{}^{51}C_3}$

$P(A|E_1) = P(A|E_3) = P(A|E_4) = \frac{{}^{13}C_3}{{}^{51}C_3}$

$\therefore$ Required probability = $P(E_2|A)$

$$= \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)}$$

$$= \frac{\frac{1}{4}\times\frac{{}^{12}C_3}{{}^{51}C_3}}{\frac{1}{4}\times\frac{{}^{13}C_3}{{}^{51}C_3}+\frac{1}{4}\times\frac{{}^{12}C_3}{{}^{51}C_3}+\frac{1}{4}\times\frac{{}^{13}C_3}{{}^{51}C_3}+\frac{1}{4}\times\frac{{}^{13}C_3}{{}^{51}C_3}}$$

$$= \frac{220}{286+220+286+286} = \frac{220}{1078} = \frac{10}{49}$$

60. Let A be the two-headed coin, B be the biased coin showing up heads 75% of the times and C be the biased coin showing up tails 40% (*i.e.*, showing up heads 60%) of the times.

Let E_1, E_2 and E_3 be the events of choosing coins of the type A, B and C respectively. Let S be the event of getting a head. Then

$P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{3}$, $P(E_3) = \frac{1}{3}$

$P(S|E_1) = 1$, $P(S|E_2) = \frac{75}{100} = \frac{3}{4}$,

$P(S|E_3) = \frac{60}{100} = \frac{3}{5}$

$\therefore$ Required probability $= P(E_1|S) = \frac{P(E_1)\cdot P(S|E_1)}{\sum_{i=1}^{3} P(E_i)\cdot P(S|E_i)}$

$$= \frac{\frac{1}{3}\cdot 1}{\frac{1}{3}\cdot 1+\frac{1}{3}\cdot\frac{3}{4}+\frac{1}{3}\cdot\frac{3}{5}} = \frac{20}{20+15+12} = \frac{20}{47}$$

61. Let the events are defined as

E_1 : Person is a scooter driver

E_2 : Person is a car driver

E_3 : Person is a truck driver

A : Person meets with an accident.

Then, $P(E_1)=\frac{2000}{12000}=\frac{1}{6}, P(E_2)=\frac{4000}{12000}=\frac{2}{6},$

$P(E_3)=\frac{6000}{12000}=\frac{3}{6}.$

Also, $P(A|E_1)=0.01=\frac{1}{100}, P(A|E_2)=0.03=\frac{3}{100},$

$P(A|E_3)=0.15=\frac{15}{100}.$

$\therefore$ Required probability = 1 – P(the person who meets with accident is a truck driver)

i.e., Required probability = 1 – $P(E_3|A)$

$$=1-\frac{P(A|E_3)P(E_3)}{P(A|E_1)P(E_1)+P(A|E_2)P(E_2)+P(A|E_3)P(E_3)}$$

$$=1-\frac{\frac{15}{100}\times\frac{3}{6}}{\frac{1}{100}\times\frac{1}{6}+\frac{3}{100}\times\frac{2}{6}+\frac{15}{100}\times\frac{3}{6}}=1-\frac{45}{1+6+45}$$

$$=1-\frac{45}{52}=\frac{7}{52}.$$

62. Let E_1 be the event that '1' occurs, E_2 be the event that '1' does not occur and A be the event that the man reports that it is '1'.

$\therefore \quad P(E_1)=\frac{1}{6},\ P(E_2)=\frac{5}{6}$

Now, $P\left(\frac{A}{E_1}\right)$ be the probability that the man reports that there is '1' on the die given that '1' actually occurs.

So, $P\left(\frac{A}{E_1}\right)$ = Probability that the man speaks the truth

$=\frac{3}{5}$

And $P\left(\frac{A}{E_2}\right)$ be the probability that the man reports that there is '1' when actually '1' does not occur.

So, $P\left(\frac{A}{E_2}\right)$ = Probability that man does not speak the truth

$=1-\frac{3}{5}=\frac{2}{5}.$

$\therefore$ Required probability = $P\left(\frac{E_1}{A}\right)$

$$=\frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)}=\frac{\frac{1}{6}\times\frac{3}{5}}{\frac{1}{6}\times\frac{3}{5}+\frac{5}{6}\times\frac{2}{5}}=\frac{3}{13}.$$

63. Consider the following events.

E : Two balls drawn are white

A : There are 2 white balls in the urn

B : There are 3 white balls in the urn

C : There are 4 white balls in the urn

$P(A) = P(B) = P(C) = \frac{1}{3}$

$P(E|A)=\frac{{}^2C_2}{{}^4C_2}=\frac{1}{6},\ P(E|B)=\frac{{}^3C_2}{{}^4C_2}=\frac{3}{6}=\frac{1}{2}$

$P(E|C)=\frac{{}^4C_2}{{}^4C_2}=1$

$$\therefore \quad P(C|E)=\frac{P(C)\cdot P(E|C)}{P(A)\cdot P(E|A)+P(B)\cdot P(E|B)+P(C)\cdot P(E|C)}$$

$$=\frac{\frac{1}{3}\times 1}{\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 1}=\frac{3}{5}$$

64. $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$

Let $P(X = x_3) = k$.

So $P(X=x_1)=\frac{k}{2}$; $P(X=x_2)=\frac{k}{3}$; $P(X=x_4)=\frac{k}{5}$

We know that sum of all probabilities in probability distribution is 1.

So, $P(X=x_1)+P(X=x_2)+P(X=x_3)+P(X=x_4)=1$

$\Rightarrow \quad \frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1$

$\Rightarrow \quad \frac{15k+10k+30k+6k}{30}=1 \Rightarrow 61k = 30$

$\Rightarrow \quad k=\frac{30}{61}$

So, probability distribution of X :

$P(X=x_1)=\frac{30}{61\times 2}=\frac{15}{61}$; $P(X=x_2)=\frac{30}{61\times 3}=\frac{10}{61}$

$P(X=x_3)=\frac{30}{61}$; $P(X=x_4)=\frac{30}{61\times 5}=\frac{6}{61}$

65. (a) : We know $\Sigma p(x_i) = 1$

$\Rightarrow \quad K + 6K + 9K = 1 \Rightarrow 16K = 1 \Rightarrow K=\frac{1}{16}$

(b) $\quad P$(getting 2 heads) $=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ (if the coin was unbiased)

But from given p.d. table, P(getting 2 heads) $=\frac{1}{16}\neq\frac{1}{4}$

$\therefore \quad$ Coin tossed is biased

66. Given, the number of red balls in a bag is = 2

The number of blue balls in a bag is = 3

So, total number of balls in a bag is = 2 + 3 = 5

Since, two balls are drawn at random without replacement and X denotes the number of red balls. So X can be 0, 1 and 2.

Case I : When no red ball is drawn, X = 0

$P(X = 0) = P(BB) = P(B) \cdot P(B)$

$=\frac{3}{5}\times\frac{2}{4}=\frac{6}{20}=\frac{3}{10}$

Case II : When one red ball is drawn, X = 1

$P(X=1)=P(RB)+P(BR)=P(R)P(B)+P(B)P(R)$

$=\frac{2}{5}\times\frac{3}{4}+\frac{3}{5}\times\frac{2}{4}=\frac{6}{20}+\frac{6}{20}=\frac{12}{20}=\frac{3}{5}$

Case III : When two red ball are drawn, $X = 2$

$P(X = 2) = P(RR) = P(R) \cdot P(R)$

$= \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$

Hence, the required probability distribution is given by

X	0	1	2
$P(X)$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$

67. We have, $P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$

Since, $\Sigma P(x_i) = 1 \Rightarrow k + 2k + 3k = 1$

$\Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$

68. The probability distribution of x is

$X = x$	0	1	2	3
$P(X = x)$	0.1	k	$2k$	$3k$

(a) $\because \sum P(X) = 1 \Rightarrow 0.1 + k + 2k + 3k = 1$

$\Rightarrow 6k = 1 - 0.1 \Rightarrow 6k = 0.9 \Rightarrow k = \frac{0.9}{6} = 0.15$

(b) $P(x \le 2) = P(0) + P(1) + P(2) = 0.1 + 0.15 + 0.3 = 0.55$

(c) Mean, $\bar{X} = \sum X \cdot P(X)$

$= 0.15 \times 1 + 2 \times 0.3 + 3 \times 0.45 = 2.1$

69. We have, number of rotten apples = 3 and number of good apples = 7

$\therefore$ Total number of apples = 10

Let X be number of rotten apples.

So, X can take values 0, 1, 2, 3

Let E be the event of getting a rotten apple.

$P(E) = \frac{3}{10}, P(\bar{E}) = \frac{7}{10}$

Now, $P(X = 0) = {}^3C_0 \cdot \frac{7}{10} \cdot \frac{7}{10} \cdot \frac{7}{10} = \frac{343}{1000}$

$P(X = 1) = {}^3C_1 \cdot \frac{3}{10} \cdot \frac{7}{10} \cdot \frac{7}{10} = \frac{441}{1000}$

$P(X = 2) = {}^3C_2 \cdot \frac{3}{10} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{189}{1000}$

$P(X = 3) = {}^3C_3 \cdot \frac{3}{10} \cdot \frac{3}{10} \cdot \frac{3}{10} = \frac{27}{1000}$

So, probability distribution table is given by

X	0	1	2	3
$P(X)$	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

Now, mean $(\bar{X}) = \sum X \cdot P(X)$

$= 0 \times \frac{343}{1000} + 1 \times \frac{441}{1000} + 2 \times \frac{189}{1000} + 3 \times \frac{27}{1000}$

$= \frac{441}{1000} + \frac{378}{1000} + \frac{81}{1000} = \frac{900}{1000} = \frac{9}{10}$

70. We have, $P(X = 0) = P(X = 1) = p$

Let $P(X = 2) = P(X = 3) = k$

Since, X is a random variable taking values 0, 1, 2, 3

$\therefore P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$

$\Rightarrow p + p + k + k = 1 \Rightarrow 2p + 2k = 1 \Rightarrow p + k = \frac{1}{2}$ ...(i)

Now, $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$

$\Rightarrow p(0) + p(1) + k(4) + k(9) = 2[p(0) + p(1) + k(2) + k(3)]$

$\Rightarrow p + 13k = 2p + 10k$

$\Rightarrow p - 3k = 0$ ...(ii)

Subtracting (ii) from (i), we get $4k = \frac{1}{2} \Rightarrow k = \frac{1}{8}$

$\therefore$ From (i), we get $p = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

71. Let X be the amount he wins/loses.

Then, X can take values $-3, 3, 4, 5$.

$P(X = 5) = P$(Getting a number greater than 4 in the first throw) $= \frac{2}{6} = \frac{1}{3}$

$P(X = 4) = P$(Getting a number less than or equal to 4 in the first throw and a number greater than 4 in the second throw) $= \frac{4}{6} \times \frac{2}{6} = \frac{2}{9}$

$P(X = 3) = P$(Getting a number less than or equal to 4 in the first two throws and a number greater than 4 in the third throw) $= \frac{4}{6} \times \frac{4}{6} \times \frac{2}{6} = \frac{4}{27}$

$P(X = -3) = P$(Getting a number less than or equal to 4 in all three throws) $= \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} = \frac{8}{27}$

$\therefore$ The probability distribution is

X	5	4	3	-3
$P(X)$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{4}{27}$	$\frac{8}{27}$

$\therefore E(X) = \sum XP(X) = 5\left(\frac{1}{3}\right) + 4\left(\frac{2}{9}\right) + 3\left(\frac{4}{27}\right) - 3\left(\frac{8}{27}\right)$

$= \frac{57}{27} = \frac{19}{9}$

Hence, expected value of the amount he wins/loses is $\frac{19}{9}$.

72. The probability distribution of X is

X	0	1	2	3	4
$P(X)$	0	k	$4k$	$2k$	k

The given distribution is a probability distribution.

$\therefore \sum_{i=0}^{4} p_i = 1$

$\Rightarrow 0 + k + 4k + 2k + k = 1 \Rightarrow 8k = 1 \Rightarrow k = \frac{1}{8} = 0.125$

(i) P (getting admission in exactly one college)

$= P(X = 1) = k = 0.125$

(ii) P (getting admission in atmost 2 colleges)
$= P(X \le 2) = 0 + k + 4k = 5k = 0.625$

(iii) P (getting admission in atleast 2 colleges)
$= P(X \ge 2) = 4k + 2k + k = 7k = 0.875$

73. Let X denote the number of spade cards in a sample of 3 cards drawn from a well-shuffled pack of 52 cards.
Since there are 13 spade cards in the pack, so in a sample of 3 cards drawn, either there is no spade card or one spade card or two spade cards or 3 spade cards. Thus $X = 0, 1, 2$ and 3.
Now, $P(X = 0)$ = Probability of getting no spade card

$$= \frac{39}{52}\cdot\frac{39}{52}\cdot\frac{39}{52} = \frac{27}{64}$$

$P(X = 1)$ = Probability of getting one spade card

$$= \frac{13}{52}\cdot\frac{39}{52}\cdot\frac{39}{52} + \frac{39}{52}\cdot\frac{13}{52}\cdot\frac{39}{52} + \frac{39}{52}\cdot\frac{39}{52}\cdot\frac{13}{52} = \frac{27}{64}$$

$P(X = 2)$ = Probability of getting 2 spade cards

$$= \frac{13}{52}\cdot\frac{13}{52}\cdot\frac{39}{52} + \frac{13}{52}\cdot\frac{39}{52}\cdot\frac{13}{52} + \frac{39}{52}\cdot\frac{13}{52}\cdot\frac{13}{52} = \frac{9}{64}$$

$P(X = 3)$ = Probability of getting 3 spade cards

$$= \frac{13}{52}\cdot\frac{13}{52}\cdot\frac{13}{52} = \frac{1}{64}$$

Hence, the probability distribution of X is

X	0	1	2	3
$P(X)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

Now, mean of this distribution is given by

$$\bar{X} = 0\times\frac{27}{64} + 1\times\frac{27}{64} + 2\times\frac{9}{64} + 3\times\frac{1}{64} = \frac{48}{64} = \frac{3}{4}$$

74. Let X denote the number of defective bulbs in a sample of 2 bulbs which are to be drawn.
Here, number of defective bulbs = 5
Number of non-defective bulbs = 15 – 5 = 10
∴ X can take values 0, 1, 2.
Now, $P(X = 0)$ = Probability of getting no defective bulb
= Probability of getting 2 non-defective bulbs.

$$= \frac{^{10}C_2}{^{15}C_2} = \frac{10\times 9}{15\times 14} = \frac{3}{7} = \frac{9}{21}$$

$P(X = 1)$ = Probability of getting 1 defective bulb

$$= \frac{^{5}C_1 \times {}^{10}C_1}{^{15}C_2} = \frac{5\times 10\times 2}{15\times 14} = \frac{10}{21}$$

$P(X = 2)$ = Probability of getting 2 defective bulbs

$$= \frac{^{5}C_2}{^{15}C_2} = \frac{5\times 4}{15\times 14} = \frac{2}{21}$$

Thus the probability distribution of X is given by

X	0	1	2
$P(X)$	$\frac{9}{21}$	$\frac{10}{21}$	$\frac{2}{21}$

Concept Applied

- A combination determines the number of possible selection in a collection of items where the order of the selection does not matter.

75. Let X denote the number of red cards. So X can take values 0, 1, 2, 3.
Total number of cards = 52
Number of red cards = 26.

Now, $P(X = 0) = \frac{^{26}C_3}{^{52}C_3} = \frac{26\times 25\times 24}{52\times 51\times 50} = \frac{4}{34}$

$P(X = 1) = \frac{^{26}C_1 \times {}^{26}C_2}{^{52}C_3} = \frac{26\times 26\times 25\times 6}{2\times 52\times 51\times 50} = \frac{13}{34}$

$P(X = 2) = \frac{^{26}C_2 \times {}^{26}C_1}{^{52}C_3} = \frac{26\times 25\times 26\times 6}{2\times 52\times 51\times 50} = \frac{13}{34}$

$P(X = 3) = \frac{^{26}C_3}{^{52}C_3} = \frac{26\times 25\times 24}{52\times 51\times 50} = \frac{4}{34}$

∴ Probability distribution of X is given by

X	0	1	2	3
$P(X)$	$\frac{4}{34}$	$\frac{13}{34}$	$\frac{13}{34}$	$\frac{4}{34}$

Mean $(\bar{X}) = \sum XP(X)$

$$= 0\left(\frac{4}{34}\right) + 1\left(\frac{13}{34}\right) + 2\left(\frac{13}{34}\right) + 3\left(\frac{4}{34}\right) = \frac{3}{2}$$

76. Here the ages of the given 15 students are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years.
∴ The required probability distribution of X is given by

X	14	15	16	17	18	19	20	21
$P(X)$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

Mean, $\bar{X} = \sum XP(X)$

$$= 14\times\frac{2}{15} + 15\times\frac{1}{15} + 16\times\frac{2}{15} + 17\times\frac{3}{15} + 18\times\frac{1}{15} + 19\times\frac{2}{15} + 20\times\frac{3}{15} + 21\times\frac{1}{15}$$

$$= \frac{1}{15}(28+15+32+51+18+38+60+21) = \frac{263}{15}$$

77. The probability of drawing a ticket out of $10 = \frac{1}{10}$
The probability of drawing a ticket with prize of ₹ 8 is $2\times\frac{1}{10}$.
The probability of drawing a ticket with prize of ₹ 4 is $5\times\frac{1}{10}$.
The probability of drawing a ticket with prize of ₹ 2 is $3\times\frac{1}{10}$.

We can show this on a table as :

Number of tickets	2	5	3
X	8	4	2
$P(X)$	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{3}{10}$

$\because$ Mean = $\sum X_i\, P(X_i)$

Hence the mean prize

$$8\times\frac{2}{5}+4\times\frac{5}{10}+2\times\frac{3}{10}\ =\frac{8}{5}+2+\frac{3}{5}=₹\frac{21}{5}₹\,4.20$$

78. The first six positive integers are 1, 2, 3, 4, 5 and 6.
Let X be the larger number of two numbers selected the possible outcomes are :
Sample space S is given by
S = {(1, 2) (1, 3), (1, 4) (1, 5) (1, 6), (2, 1), (2, 3) .... (2, 6), (3, 1), (3, 2), (3, 4) .... (3, 6), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6), (5, 1), (5, 2) .... (5, 4), (5, 6), (6, 1), (6, 2) .... (6, 5)}

$\therefore$ X can take values 2, 3, 4, 5, or 6.

Total number of ways = ${}^6C_2$ = 15

Theprobabilitydistributionofarandomvariable$X$isgivenby

X	2	3	4	5	6
$P(X)$	1/15	2/15	3/15	4/15	5/15

$\therefore$ Mean = $\sum XP(X)$

$$= 2\times\frac{1}{15}+3\times\frac{2}{15}+4\times\frac{3}{15}+5\times\frac{4}{15}+6\times\frac{5}{15}$$

$$= \frac{2}{15}+\frac{6}{15}+\frac{12}{15}+\frac{20}{15}+\frac{30}{15}=\frac{70}{15}=\frac{14}{3}.$$

79. When a die is thrown, probability of getting a six = $\frac{1}{6}$

$\therefore$ Probability of not getting a six = $1-\frac{1}{6}=\frac{5}{6}.$

If he gets a six in first throw, then, probability of getting a six = $\frac{1}{6}.$

If he does not get a six in first throw, but he gets a six in the second throw, then

Probability = $\frac{5}{6}\times\frac{1}{6}=\frac{5}{36}.$

Probability that he does not get a six in first two throws and he gets a six in third throw $=\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{216}$

Probability that he does not get a six in any of the three throws $=\left(\frac{5}{6}\right)^3=\frac{125}{216}.$

In first throw he gets a six, will receive ₹ 5.

If he gets a six in second throw, he will receive ₹ (5 – 1) = 4

If he gets a six in third throw, he will receive ₹ (–1 – 1 + 5) = ₹ 3

If he does not get a six in all three throws, he will receive ₹ (–1 – 1 – 1) = ₹ –3.

Let X be the amount he wins/losses.

Then, X can take values –3, 3, 4, 5

$\therefore$ The probability distribution is

X	5	4	3	–3
$P(X)$	1/6	5/36	25/216	25/216

$\therefore$ Expected value $=\frac{1}{6}\times5+\left(\frac{5}{36}\right)\times4+\left(\frac{25}{216}\right)\times3$

$$+\left(\frac{125}{216}\right)\times(-3)$$

$$=\frac{5}{6}+\frac{20}{36}+\frac{75}{216}-\frac{375}{216}=0$$

He neither loses or wins.

CBSE Sample Questions

1. P(not obtaining an odd person in a single round) = P(All three of them throw tails or All three of them throw heads)

$$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$$ **(1½)**

P(obtaining an odd person in a single round)

= 1 – P(not obtaining an odd person in a single round) = 3/4

Required probability = P(In first round there is no odd person' and 'In second round there is no odd person' and 'In third round there is an odd person)

$$=\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{64}$$ **(1½)**

2. $$P(\bar{E}\,|\,\bar{F})=\frac{P(\bar{E}\cap\bar{F})}{P(\bar{F})}=\frac{P(\overline{E\cup F})}{P(\bar{F})}=\frac{1-P(E\cup F)}{1-P(F)}$$ ... (i)

Now, $P(E\cup F) = P(E) + P(F) - P(E\cap F) = 0.8 + 0.7 - 0.6 = 0.9$ **(1)**

Substituting value of $P(E\cup F)$ in (i), we get **(1/2)**

$$P(\bar{E}\,|\,\bar{F})=\frac{1-0.9}{1-0.7}=\frac{0.1}{0.3}=\frac{1}{3}$$ **(1/2)**

3. Given, $P(A)=\frac{1}{3}$ then $P(A)=1-\frac{1}{3}=213$ and

$$P(B)=\frac{1}{4}\text{ then }P(B)=1-\frac{1}{4}=\frac{3}{4}$$

Required probability

= 1 – P(problem is not solved)

$$=1-P(\bar{A})\cdot P(\bar{B})=1-\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}$$ **(1)**

4. Let A be the event of commiting an error and E_1, E_2 and E_3 be the events that Vinay, Sonia and Iqbal processed the form.

(i) **(b) :** Required probability = $P(A|E_2)$

$$=\frac{P(A\cap E_2)}{P(E_2)}=\frac{\left(0.04\times\frac{20}{100}\right)}{\left(\frac{20}{100}\right)}=0.04$$ **(1)**

(ii) **(c) :** Required probability = $P(A\cap E_2)$

$$=0.04\times\frac{20}{100}=0.008$$ **(1)**

(iii) (b) : Total probability is given by

$P(A) = P(E_1)\cdot P(A|E_1) + P(E_2)\cdot P(A|E_2) + P(E_3)\cdot P(A|E_3)$

$$=\frac{50}{100}\times 0.06+\frac{20}{100}\times 0.04+\frac{30}{100}\times 0.03=0.047$$ **(1)**

(iv) (d) : Using Bayes' theorem, we have

$$P(E_1|A)=\frac{P(E_1)\cdot P(A|E_1)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)+P(E_3)\cdot P(A|E_3)}$$

$$=\frac{0.5\times 0.06}{0.5\times 0.06+0.2\times 0.04+0.3\times 0.03}=\frac{30}{47}$$

$\therefore$ Required probability = $P(\bar{E}_1|A)$

$$=1-P(E_1|A)=1-\frac{30}{47}=\frac{17}{47}$$ **(1)**

(v) (d) : $\sum_{i=1}^{3} P(E_i|A) = P(E_1|A) + P(E_2|A) + P(E_3|A)$

$= 1$

[$\because$ Sum of posterior probabilities is 1] **(1)**

5. (i) Let P be the event that the shell fired from A hits the plane. Q be the event that the shell fired from B hits the plane. The following four hypotheses are possible the trial, with the guns operating independently:

$$E_1=PQ, E_2=\bar{P}\bar{Q}, E_3=\bar{P}Q, E_4=P\bar{Q}$$

Let E = The shell fired from exactly one of them hits the plane. **(1/2)**

$P(E_1) = 0.3 \times 0.2 = 0.06$, $P(E_2) = 0.7 \times 0.8 = 0.56$,

$P(E_3) = 0.7 \times 0.2 = 0.14$, $P(E_4) = 0.3 \times 0.8 = 0.24$ **(1/2)**

$$P\left(\frac{E}{E_1}\right)=0, P\left(\frac{E}{E_2}\right)=0, P\left(\frac{E}{E_3}\right)=1, P\left(\frac{E}{E_4}\right)=1$$ **(1/2)**

$$P(E)=P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+P(E_3)\cdot P\left(\frac{E}{E_3}\right)+P(E_4)\cdot P\left(\frac{E}{E_4}\right)$$ **(1/2)**

$= 0.14 + 0.24 = 0.38$

(ii) By Bayes' Theorem, $P\left(\frac{E_3}{E}\right)$ **(1/2)**

$$=\frac{P(E_3)\cdot P\left(\frac{E}{E_3}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+P(E_3)\cdot P\left(\frac{E}{E_3}\right)+P(E_4)\cdot P\left(\frac{E}{E_4}\right)}$$ **(1)**

$$=\frac{0.14}{0.38}=\frac{7}{19}$$ **(1/2)**

6. Let E_1 = The policyholder is accident prone.

E_2 = The policyholder is not accident prone.

E = The new policyholder has an accident withing a year of purchasing a policy.

(i) $P(E) = P(E_1) \times P(E|E_1) + P(E_2) \times P(E|E_2)$

$$=\frac{20}{100}\times\frac{6}{10}+\frac{80}{100}\times\frac{2}{10}=\frac{7}{25}$$ **(2)**

(ii) Using Bayes' Theorem, we have $P(E_1|E)$

$$=\frac{P(E_1)\times P(E|E_1)}{P(E)}=\frac{\frac{20}{100}\times\frac{6}{10}}{\frac{7}{25}}=\frac{3}{7}$$ **(2)**

7. Let X be the random variable defined as the number of red balls.

Then, $X = 0, 1$ **(1/2)**

$$P(X=0)=\frac{3}{4}\times\frac{2}{3}=\frac{6}{12}=\frac{1}{2}$$ **(1/2)**

$$P(X=1)=\frac{1}{4}\times\frac{3}{3}+\frac{3}{4}\times\frac{1}{3}=\frac{6}{12}=\frac{1}{2}$$ **(1/2)**

Probability Distribution Table :

X	0	1
$P(X)$	$\frac{1}{2}$	$\frac{1}{2}$

(1/2)

8. Let X denotes the number of milk chocolates drawn. Then probability distribution table is

X	$P(X)$
0	$\frac{4}{6}\times\frac{3}{5}=\frac{12}{30}$
1	$\left(\frac{2}{6}\times\frac{4}{5}\right)+\left(\frac{4}{6}\times\frac{2}{5}\right)=\frac{16}{30}$
2	$\frac{2}{6}\times\frac{1}{5}=\frac{2}{30}$

(1½)

Most likely outcome is getting one chocolate of each type. **(1/2)**

9. Suppose X denotes the Random Variable defined by the number of defective items.

$$P(X=0)=\frac{4}{6}\times\frac{3}{5}=\frac{2}{5}$$ **(1/2)**

$$P(X=1)=\left(\frac{2}{6}\times\frac{4}{5}\right)+\left(\frac{4}{6}\times\frac{2}{5}\right)=\frac{8}{15}$$ **(1/2)**

$$P(X=2)=\frac{2}{6}\times\frac{1}{5}=\frac{1}{15}$$ **(1/2)**

x_i	0	1	2
p_i	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$
$p_i x_i$	0	$\frac{8}{15}$	$\frac{2}{15}$

$$\therefore \text{Mean}=\sum p_i x_i=\frac{10}{15}=\frac{2}{3}$$ **(1½)**

Self Assessment

Case Based Objective Questions (4 marks)

1. One day, a sangeet mahotsav is to be organised in an open area of Rajasthan. In recent years, it has rained only 6 days each year. Also, it is given that, when it actually rains, the weatherman correctly forecasts rain 80% of the time. When it doesn't rain, he incorrectly forecasts rain 20% of the time. Here, leap year is considered.

Based on the above information, attempt any 4 out of 5 subparts.

(i) The probability that it rains on chosen day is

(a) $\frac{1}{366}$ (b) $\frac{1}{73}$ (c) $\frac{1}{60}$ (d) $\frac{1}{61}$

(ii) The probability that it does not rain on chosen day is

(a) $\frac{1}{366}$ (b) $\frac{5}{366}$
(c) $\frac{360}{366}$ (d) none of these

(iii) The probability that the weatherman predicts correctly is

(a) $\frac{5}{6}$ (b) $\frac{7}{8}$ (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

(iv) The probability that it will rain on the chosen day, if weatherman predicts rain for that day, is

(a) 0.0625 (b) 0.0725
(c) 0.0825 (d) 0.0925

(v) The probability that it will not rain on the chosen day, if weatherman predicts rain for that day, is

(a) 0.94 (b) 0.84
(c) 0.74 (d) 0.64

Multiple Choice Questions (1 mark)

2. If $P(A)=\frac{4}{5}$ and $P(A \cap B)=\frac{7}{10}$, then $P(B|A)$ is equal to

(a) $\frac{1}{10}$ (b) $\frac{1}{8}$ (c) $\frac{7}{8}$ (d) $\frac{17}{20}$

3. A and B are events such that $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.5$. Then $P(B' \cap A)$ equals

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{3}{10}$ (d) $\frac{1}{5}$

OR

If two events are independent, then
(a) they must be mutually exclusive
(b) the sum of their probabilities must be equal to 1
(c) (a) and (b) both are correct
(d) None of the above is correct

4. A die is thrown and a card is selected at random from a deck of 52 playing cards, then the probability of getting an even number on the die and a spade card is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{3}{4}$

5. If the events A and B are independent, then $P(A \cap B)$ is equal to

(a) $P(A) + P(B)$ (b) $P(A) - P(B)$
(c) $P(A) \cdot P(B)$ (d) $P(A) \mid P(B)$

6. Two cards are drawn from a well shuffled deck of 52 playing cards with replacement. The probability, that both cards are queens, is

(a) $\frac{1}{13} \times \frac{1}{13}$ (b) $\frac{1}{13} + \frac{1}{13}$
(c) $\frac{1}{13} \times \frac{1}{17}$ (d) $\frac{1}{13} \times \frac{4}{51}$

7. The probability distribution of a discrete random variable X is given below

X	2	3	4	5
$P(X)$	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

The value of k is

(a) 8 (b) 16 (c) 32 (d) 48

VSA Type Questions (1 mark)

8. Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

9. Let A and B be independent events with $P(A) = 1/4$ and $P(A \cup B) = 2P(B) - P(A)$. Find $P(B)$.

10. A random variable X has the following distribution.

X	1	2	3	4	5	6	7	8
$P(X)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the event E = {X is prime number}, find $P(E)$.

OR

If three events of a sample space are E, F and G, then what is the value of $P(E \cap F \cap G)$?

11. For the following probability distribution :

X	–4	–3	–2	–1	0
$P(X)$	0.1	0.2	0.3	0.2	0.2

what is the value of $E(X)$?

12. If A and B are two events such that $P(A|B) = p$, $P(A) = p$, $P(B)=\frac{1}{3}$ and $P(A\cup B)=\frac{5}{9}$, then find the value of p.

SA I Type Questions (2 marks)

13. One card is drawn from a well shuffled pack of 52 cards. If E is the event "the card drawn is a king or a queen" and F is the event "the card drawn is a queen or an ace" then find the probability of the conditional event $E|F$.

14. Given, $P(A) = 0.4$, $P(B) = 0.7$ and $P(B|A) = 0.6$. Find $P(A\cup B)$.

OR

P has 2 children. He has a son, Jatin. What is the probability that Jatin's sibling is a brother?

15. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement, then find the probability that both drawn balls are black.

16. A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then, another ball is drawn at random. What is the probability of second ball being blue?

SA II Type Questions (3 marks)

17. A bag contains 4 balls. Two balls are drawn at random (without replacement) and are found to be white. What is the probability that all balls in the bag are white?

OR

One-third of the students of a class are boys and the rest are girls. It is known that the probability of a girl getting a first class marks in Board's Exam is 0.6 and a boy getting a first class marks is 0.35. Find the probability that a student chosen at random will get a first class marks in exam.

18. A box contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the box and kept aside. From the remaining balls in the box, another ball is drawn at random and kept beside the first. The process is repeated, find the probability that the balls drawn are in the sequence of 2 black, 4 white and 3 red.

19. Prove that if E and F are independent events, then events E and F' are also independent.

20. The probability that a student A can solve a question is $\frac{6}{7}$ and that another student B can solve a question is $\frac{3}{4}$. Assuming that two events "A can solve the question" and "B can solve the question" are independent, find the probability that only one of them solves the question.

21. If $P(\text{not } A) = 0.7$, $P(B) = 0.7$ and $P(B \mid A) = 0.5$, then find $P(A \mid B)$ and $P(A \cup B)$.

Case Based Questions (4 marks)

22. A student is preparing for the competitive examinations of LIC AAO, SSC CGL and Bank P.O. The probabilities that the student is selected independently in competitive examinations of LIC AAO, SSC CGL and Bank P.O. are a, b and c respectively. Of these examinations, students have 50% chance of selection in at least one, 40% chance of selection in at least two and 30% chance of selection in exactly two examinations.

Based on the above information, answer the following questions.

(i) Find the value of $a + b + c - ab - bc - ca + abc$.

(ii) Find the value of $ab + bc + ac - 2abc$.

LA Type Questions (4/6 marks)

23. A bag contains 6 red and 5 blue balls and another bag contains 5 red and 8 blue balls. A ball is drawn from the first bag and without noticing its colour is put in the second bag. A ball is then drawn from the second bag. Find the probability that the ball drawn is blue in colour.

24. A and B are two events such that

$P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$ and $P(A\cap B)=\frac{1}{4}$. Find

(i) $P(A|B)$ (ii) $P(B|A)$

(iii) $P(A'|B)$ (iv) $P(A'|B')$

25. A shopkeeper sells three types of flower seeds A_1, A_2 and A_3. They are sold as a mixture, where the proportions are 4:4 : 2, respectively. The germination rates of the three types of seeds are 45%, 60% and 35%. Calculate the probability
(i) of a randomly chosen seed to germinate.
(ii) that it will not germinate given that the seed is of type A_3.
(iii) that it is of the type A_2 given that a randomly chosen seed does not germinate.

OR

In a college, 70% students pass in Physics, 75% students pass in Mathematics and 10% students fail in both. One student in chosen at random. What is the probability that
(i) he passes in Physics and Mathematics?
(ii) he passes in Mathematics, given that he passes in Physics?
(iii) he passes in Physics, given that he passes in Mathematics?

Detailed SOLUTIONS

1. (i) (d): Since, it rained only 6 days each year, therefore, probability that it rains on chosen day is $\frac{6}{366}=\frac{1}{61}$

(ii) (c): The probability that it does not rain on chosen day = $1-\frac{1}{61}=\frac{60}{61}=\frac{360}{366}$

(iii) (c): It is given that, when it actually rains, the weatherman correctly forecasts rain 80% of the time.

$\therefore$ Required probability = $\frac{80}{100}=\frac{8}{10}=\frac{4}{5}$

(iv) (a): Let A_1 be the event that it rains on chosen day, A_2 be the event that it does not rain on chosen day and E be the event the weatherman predicts rain.

Then, we have, $P(A_1) = \frac{6}{366}$, $P(A_2) = \frac{360}{366}$,

$P(E\mid A_1) = \frac{8}{10}$ and $P(E\mid A_2) = \frac{2}{10}$

Required probability = $P(A_1 \mid E)$

$$= \frac{P(A_1)\cdot P(E|A_1)}{P(A_1)\cdot P(E|A_1)+P(A_2)\cdot P(E|A_2)}$$

$$= \frac{\frac{6}{366}\times\frac{8}{10}}{\frac{6}{366}\times\frac{8}{10}+\frac{360}{366}\times\frac{2}{10}}=\frac{48}{768} = 0.0625$$

(v) (a): Required probability = $1 - P(A_1 \mid E)$
$= 1 - 0.0625 = 0.9375 \approx 0.94$

2. (c) : Given, $P(A)=\frac{4}{5}, P(A\cap B)=\frac{7}{10}$

Now, $P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{7/10}{4/5}=\frac{7}{8}$

3. (d) : We have, $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.5$

$\because P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow P(A \cap B) = 0.4 + 0.3 - 0.5 = 0.2$

Now, $P(B' \cap A) = P(A) - P(A \cap B)$

$=0.4-0.2=0.2=\frac{1}{5}$

OR

(d) : If two events A and B are independent, then

$P(A \cap B) = P(A)\cdot P(B), P(A) \neq 0, P(B) \neq 0$

$\Rightarrow$ A and B have a common outcome.

Further, mutually exclusive events never have a common outcome.

4. (c) : Let E_1 be the event for getting an even number on the die and E_2 be the event that a spade card is selected.

$\therefore P(E_1)=\frac{3}{6}=\frac{1}{2}$ and $P(E_2)=\frac{13}{52}=\frac{1}{4}$

Now, $P(E_1\cap E_2)=P(E_1)\cdot P(E_2)=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$

5. (c) : For independent events A and B,
$P(A \cap B) = P(A)\cdot P(B)$

6. (a) : As, there is replacement.

$\therefore$ Probability of drawing each card of queens is $\frac{4}{52}$

$\therefore$ Required probability $=\frac{4}{52}\cdot\frac{4}{52}=\frac{1}{13}\times\frac{1}{13}$

7. (c) : Since, sum of all probabilities = 1

$\therefore \Sigma P(X) = 1$

$\Rightarrow \frac{5}{k}+\frac{7}{k}+\frac{9}{k}+\frac{11}{k}=1 \Rightarrow \frac{32}{k}=1$

$\therefore k = 32$

8. Let D_1, D_2 be the events that we find a defective fuse in the first and second test respectively.

Required probability = $P(D_1 \cap D_2)$

$=P(D_1)P(D_2|D_1) =\frac{2}{7}\cdot\frac{1}{6}=\frac{1}{21}$

9. We have, $P(A) = 1/4$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)\,P(B)$ ($\because$ A, B are independent)

$= 1/4 + P(B) - (1/4)\,P(B) = 2P(B) - 1/4$ (Given)

$\Rightarrow P(B) = 2/5.$

10. $P(E) = P(X = 2) + P(X = 3) + P(X = 5) + P(X = 7)$
$= 0.23 + 0.12 + 0.20 + 0.07 = 0.62$

OR

If E, F and G are the three events of a sample space, then we have

$P(E \cap F \cap G) = P(E)\, P(F|E)\, P(G|(E \cap F))$
$= P(E)\, P(F|E)\, P(G|EF)$

11. $E(X) = \Sigma XP(X)$
$= (-4) \times 0.1 + (-3) \times 0.2 + (-2) \times 0.3 + (-1) \times 0.2 + 0 \times 0.2$
$= -0.4 - 0.6 - 0.6 - 0.2 = -1.8$

12. We have, $P(A) = p,\ P(B) = \frac{1}{3}$

and $P(A \cup B) = \frac{5}{9}$

Now, $P(A|B) = \frac{P(A \cap B)}{P(B)} = p \Rightarrow P(A \cap B) = \frac{p}{3}$

Since, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \frac{5}{9} = p + \frac{1}{3} - \frac{p}{3}$ (Using above values)

$\Rightarrow \frac{5-3}{9} = \frac{2p}{3} \Rightarrow p = \frac{2}{9} \times \frac{3}{2} = \frac{1}{3}$

13. There are 4 kings and 4 queens in pack of 52 cards.
Let E = the card drawn is a king or a queen

$\therefore \quad P(E) = \frac{8}{52} = \frac{2}{13}$

There are 4 queens and 4 aces in a pack of 52 cards.
Let F = the card drawn is a queen or an ace

$\therefore \quad P(F) = \frac{8}{52} = \frac{2}{13}$

Then, $E \cap F$ contains drawing a queen.

$\therefore \quad P(E \cap F) = \frac{4}{52} = \frac{1}{13}$

Required probability, $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/13}{2/13} = \frac{1}{2}$

14. $P(A) = 0.4$, $P(B) = 0.7$ and $P(B\,|\,A) = 0.6$

Now, $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$\Rightarrow \quad P(A \cap B) = P(B\,|\,A) \times P(A)$
$\therefore \quad P(A \cap B) = 0.6 \times 0.4 = 0.24$

Now, we have
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\Rightarrow \quad P(A \cup B) = 0.4 + 0.7 - 0.24 \quad \Rightarrow P(A \cup B) = 0.86$

OR

Consider the experiment of selecting a random family having two children.
Then, the sample space is $S = \{BB, BG, GB, GG\}$, where we are assuming boys and girls are equally likely to be born, the 4 elements of S are equally likely.
Let E be the event that the P has a son, then
$E = \{BB, BG, GB\}$.
Let F be the event that the P has two sons, then
$F = \{BB\}$.

Required probability is $P(F|E) = \frac{P(F \cap E)}{P(E)}$

$= \frac{P(\{BB\})}{P(\{BB, BG, GB\})} = \frac{1/4}{3/4} = \frac{1}{3}$

15. Let E and F denote the events that first and second ball drawn is black respectively. We have to find $P(E \cap F)$.

Now, $P(E) = P(\text{black ball in first draw}) = \frac{10}{15}$

and $P(F|E) = \frac{9}{14}$

By multiplication rule of probability, we have

$P(E \cap F) = P(E) \cdot P(F|E) = \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$

16. We have, A box = {5 blue, 4 red}
Let E_1 = Event that first ball drawn is blue
E_2 = Event that first ball drawn is red
and E = Event that second ball drawn is blue.
Then $P(E) = P(E_1) \cdot P(E|E_1) + P(E_2) \cdot P(E|E_2)$

$= \frac{5}{9} \cdot \frac{4}{8} + \frac{4}{9} \cdot \frac{5}{8} = \frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$

17. Consider the following events.
E : Two balls drawn are white
A : There are 2 white balls in the bag
B : There are 3 white balls in the bag
C : There are 4 white balls in the bag
$P(A) = P(B) = P(C) = 1/3$

$P(E/A) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6},\ P(E/B) = \frac{{}^3C_2}{{}^4C_2} = \frac{3}{6} = \frac{1}{2}$

$P(E/C) = \frac{{}^4C_2}{{}^4C_2} = 1$

$\therefore \quad P(C/E) = \frac{P(C) \cdot P(E|C)}{P(A) \cdot P(E|A) + P(B) \cdot P(E|B) + P(C) \cdot P(E|C)}$

$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1} = \frac{3}{5}$

OR

Let E_1, E_2 and A be the events defined as follows:
E_1 : a boy is chosen
E_2 : a girl is chosen
and A : the student gets a first class marks

Then, $P(E_1) = \frac{1}{3}$ and $P(E_2) = \frac{2}{3}$

Note that, E_1 and E_2 are mutually exclusive and exhaustive events.
$P(A|E_1)$ = probability of a boy getting a first class marks

$= 0.35 = \frac{35}{100} = \frac{7}{20}$

$P(A|E_2)$ = probability of a girl getting a first class marks

$= 0.6 = \frac{6}{10} = \frac{3}{5}$

By using law of total probability, we get
$P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2)$

$= \frac{1}{3} \cdot \frac{7}{20} + \frac{2}{3} \cdot \frac{3}{5} = \frac{7}{60} + \frac{2}{5} = \frac{31}{60}$

18. In total, the number of balls = 2 + 4 + 3 = 9 and the balls are being drawn one by one without replacement.

Now, P (first drawing gives a black ball) = $\frac{2}{9}$

P (second drawing gives a black ball) = $\frac{1}{8}$

P (third drawing gives a white ball) = $\frac{4}{7}$, ... and so on.

Hence, the required probability = P (the balls drawn are in sequence of 2 black, 4 white and 3 red)

= P (BBWWWWRRR)

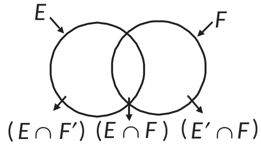
$$= \frac{2}{9}\times\frac{1}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}\times\frac{2}{2}\times\frac{1}{1}=\frac{1}{1260}$$

19. Since, E and F are independent, we have

$P(E \cap F) = P(E) \cdot P(F)$ ...(i)

From venn diagram in fig., it is clear that $E \cap F$ and $E \cap F'$ are mutually exclusive events and also

$E = (E \cap F) \cup (E \cap F')$.



Therefore, $P(E) = P(E \cap F) + P(E \cap F')$

or $P(E \cap F') = P(E) - P(E \cap F)$

$= P(E) - P(E) \cdot P(F)$ (By (i))

$= P(E)\,(1 - P(F)) = P(E) \times P(F')$

Hence, E and F' are also independent.

20. Let E be the event that the problem is solved by A and F be the event that the problem solved by B.

$$P(E)=\frac{6}{7}, P(\bar{E})=1-\frac{6}{7}=\frac{1}{7}$$

$$P(F)=\frac{3}{4}, P(\bar{F})=1-\frac{3}{4}=\frac{1}{4}$$

P(only one of them solves the question)

$$=P(E\cap\bar{F})+P(\bar{E}\cap F)$$

$$=P(E)\times P(\bar{F})+P(\bar{E})\times P(F)$$ [∵ E and F are independent events]

$$=\frac{6}{7}\times\frac{1}{4}+\frac{1}{7}\times\frac{3}{4}=\frac{6+3}{28}=\frac{9}{28}$$

21. We have, $P(\text{not } A) = 0.7$ or $P(\bar{A})=0.7$

$\therefore$ $1 - P(A) = 0.7 \Rightarrow P(A) = 0.3$ $\left[\because P(A)+P(\bar{A})=1\right]$

Now, $P(B|A)=\dfrac{P(A\cap B)}{P(A)}$

$\Rightarrow$ $0.5=\dfrac{P(A\cap B)}{0.3} \Rightarrow P(A \cap B) = 0.15$

$\therefore$ $P(A|B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{0.15}{0.7}=\dfrac{3}{14}$ [∵ $P(B)=0.7$]

and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.7 - 0.15 = 0.85$.

22. Let A be the event that the student is selected for LIC AAO, B be the event that the student is selected for SSC CGL and C be the event that the student is selected for Bank P.O.

Then, $P(A) = a$, $P(B) = b$ and $P(C) = c$

(i) We have, $P(A \cup B \cup C) = 0.5$

$\Rightarrow$ $1-P(\overline{A\cup B\cup C})=0.5$

$\Rightarrow$ $1-P(\bar{A}\cap\bar{B}\cap\bar{C})=0.5$

$\Rightarrow$ $1-P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C})=0.5$

$\Rightarrow$ $1 - (1 - a)(1 - b)(1 - c) = 0.5$

[Since, $P(A)$, $P(B)$ and $P(C)$ are independent events]

$\Rightarrow$ $1 - [(1 - a - b + ab)(1 - c)] = 0.5$

$\Rightarrow$ $1 - [1 - c - a + ac - b + bc + ab - abc] = 0.5$

$\Rightarrow$ $a + b + c - ab - bc - ca + abc = 0.5$

(ii) We have, P(selection in at least two competitive exams) = 0.4

$\Rightarrow$ $P(A\cap B\cap\bar{C})+P(\bar{A}\cap B\cap C)+P(A\cap\bar{B}\cap C)+P(A\cap B\cap C)=0.4$

$\Rightarrow$ $ab(1 - c) + (1 - a)bc + a(1 - b)c + abc = 0.4$

$\Rightarrow$ $ab - abc + bc - abc + ac - abc + abc = 0.4$

$\Rightarrow$ $ab + bc + ac - 2abc = 0.4$

23. A blue colour ball can be drawn from the second bag in the following mutually exclusive ways:

(I) By transferring a blue ball from first bag to the second bag and then drawing a blue ball from the second bag.

(II) By transferring a red ball from first bag to the second bag and then drawing a blue ball from the second bag.

Let E_1, E_2 and A be the events defined as follows:

E_1 = ball drawn from first bag is blue

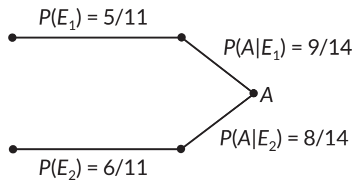
E_2 = ball drawn from first bag is red

and A = ball drawn from the second bag is blue

Since, first bag contains 6 red and 5 blue balls, we have

$$P(E_1)=\frac{5}{11} \text{ and } P(E_2)=\frac{6}{11}$$

If E_1 has already occurred, that is, if a blue ball is transferred from the first bag to the second bag, then the second bag contains 5 red and 9 blue balls, therefore the probability of drawing a blue ball from the second bag is $\frac{9}{14}$.



$\therefore$ $P(A|E_1)=\dfrac{9}{14}$

Similarly, we have $P(A|E_2)=\dfrac{8}{14}$

By the law of total probability, we have

$$P(A)=P(E_1)P(A|E_1)+P(E_2)P(A|E_2)$$

$$=\frac{5}{11}\times\frac{9}{14}+\frac{6}{11}\times\frac{8}{14}=\frac{93}{154}$$

24. (i) $P(A|B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{1/4}{1/3}=\dfrac{3}{4}$

(ii) $P(B|A)=\dfrac{P(A\cap B)}{P(A)}=\dfrac{1/4}{1/2}=\dfrac{1}{2}$

(iii) $P(A'|B)=\dfrac{P(A'\cap B)}{P(B)}=\dfrac{P(B)-P(A\cap B)}{P(B)}$

$$=\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{3}}=\frac{\frac{1}{12}}{\frac{1}{3}}=\frac{1}{4}$$

(iv) $P(A'|B')=\dfrac{P(A'\cap B')}{P(B')}=\dfrac{P((A\cup B)')}{P(B')}$

$$=\frac{1-P(A\cup B)}{1-P(B)}=\frac{1-[P(A)+P(B)-P(A\cap B)]}{1-P(B)}$$

$$=\frac{1-\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right]}{1-\frac{1}{3}}=\frac{1-(7/12)}{2/3}=\frac{5}{8}$$

25. We have, $A_1 : A_2 : A_3 = 4 : 4 : 2$

$\therefore\quad P(A_1)=\dfrac{4}{10}, P(A_2)=\dfrac{4}{10}$ and $P(A_3)=\dfrac{2}{10}$

where A_1, A_2 and A_3 denote the three types of flower seeds.

If E be the event that a seed germinates and $\bar{E}$ be the event that a seed does not germinate.

Then $P(E|A_1)=\dfrac{45}{100}, P(E|A_2)=\dfrac{60}{100}$

and $P(E|A_3)=\dfrac{35}{100}$

and $P(\bar{E}|A_1)=1-P(E|A_1)=\dfrac{55}{100}$,

$P(\bar{E}|A_2)=1-P(E|A_2)=\dfrac{40}{100}$

and $P(\bar{E}|A_3)=1-P(E|A_3)=\dfrac{65}{100}$

(i) $P(E) = P(A_1)\cdot P(E|A_1) + P(A_2)\cdot P(E|A_2) + P(A_3)\cdot P(E|A_3)$

$$=\frac{4}{10}\cdot\frac{45}{100}+\frac{4}{10}\cdot\frac{60}{100}+\frac{2}{10}\cdot\frac{35}{100}$$

[Substituting above values]

$$=\frac{180}{1000}+\frac{240}{1000}+\frac{70}{1000}=\frac{490}{1000}=0.49$$

(ii) $P(\bar{E}|A_3)=1-P(E|A_3)=1-\dfrac{35}{100}=\dfrac{65}{100}$

(iii) $P(A_2|\bar{E})=\dfrac{P(A_2)\cdot P(\bar{E}|A_2)}{P(A_1)\cdot P(\bar{E}|A_1)+P(A_2)\cdot P(\bar{E}|A_2)+P(A_3)\cdot P(\bar{E}|A_3)}$

$$=\frac{\frac{4}{10}\cdot\frac{40}{100}}{\frac{4}{10}\cdot\frac{55}{100}+\frac{4}{10}\cdot\frac{40}{100}+\frac{2}{10}\cdot\frac{65}{100}}$$

$$=\frac{\frac{160}{1000}}{\frac{220}{1000}+\frac{160}{1000}+\frac{130}{1000}}=\frac{160/1000}{510/1000}$$

$$=\frac{16}{51}=0.314$$

OR

$P(A)$ = Probability of students passed in Physics = 70/100

$P(B)$ = Probability of students passed in Mathematics = 75/100

$P(A'\cap B')$ = Probability of students fail in both = $\dfrac{10}{100}$

$P(A'\cup B') = 1 - P(A'\cap B') = 1 - \dfrac{10}{100}=\dfrac{90}{100}$

(i) The student passes in Physics and Mathematics)

$= P(A\cap B) = P(A) + P(B) - P(A\cup B)$

$$=\frac{70}{100}+\frac{75}{100}-\frac{90}{100}=\frac{55}{100}=\frac{11}{20}$$

(ii) The student passes in Mathematics given that he passes in Physics)

$$=P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{\frac{55}{100}}{\frac{70}{100}}=\frac{55}{70}=\frac{11}{14}$$

(iii) The student passes in Physics given that he passes in Mathematics)

$$=P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{55}{100}}{\frac{75}{100}}=\frac{55}{75}=\frac{11}{15}$$

PRACTICE PAPER 1

General Instructions :

1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub parts.

Time Allowed : 3 hours **Maximum Marks :** 80

Section A

(Multiple Choice Questions)
Each question carries 1 mark

1. If A is a square matrix such that $A^2 = A$, then find $(I - A)^3 + A$.
(a) A (b) A^2 (c) I (d) A^3

2. Find the value of $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A \mid B) = \frac{2}{5}$.
(a) 11/26 (b) 11/23
(c) 11/25 (d) 11/24

3. If $\begin{pmatrix} 2x+y & 3y \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 6 & 4 \end{pmatrix}'$, then find the value of x.
(a) 1 (b) 2 (c) 3 (d) 4

4. Find the solution of the differential equation $\frac{dy}{dx} = x^3 e^{-2y}$.
(a) $e^{2y} = x^4 + C$ (b) $e^y = x^4 + C$
(c) $2e^{2y} = x^4 + C$ (d) None of these

5. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
(a) πab sq. units (b) $\frac{\pi ab}{2}$ sq. units
(c) $\sqrt{\pi ab}$ sq. units (d) $\frac{\pi ab}{3}$ sq. units

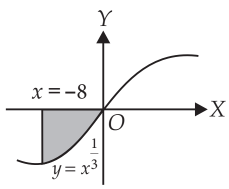
6. What is the degree of the differential equation $5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$?
(a) 2 (b) 1
(c) 3 (d) None of these

7. Two dice are thrown together. What is the probability that the sum of the numbers on the two faces is neither 9 nor 11?
(a) 1/6 (b) 5/6 (c) 3/5 (d) 1/2

8. Evaluate : $\int_{-\pi}^{\pi} x^{10} \sin^7 x \, dx$
(a) 0 (b) 2 (c) 1 (d) 4

9. The value of $\frac{d}{dx}\left(\cot^{-1} a - \cot^{-1} x\right)$ at $x = 1$, is
(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

10. Compute the shaded area shown in the given figure.



(a) 36 sq. units (b) 12 sq. units
(c) 18 sq. units (d) 15 sq. units

11. The function $f(x) = x^2 - 2x$ is strictly decreasing in the interval
(a) $(-\infty, 1)$ (b) $(1, \infty)$
(c) R (d) None of these

12. If $y = e^{\frac{1}{2}\log(1+\tan^2 x)}$, then $\frac{dy}{dx}$ is equal to
(a) $\frac{1}{2}\sec^2 x$ (b) $\sec^2 x$
(c) $\sec x \tan x$ (d) $e^{\frac{1}{2}\log(1+\tan^2 x)}$

13. If $P(A) = 0.4$, $P(B) = 0.8$ and $P(B \mid A) = 0.6$, then find $P(A \cup B)$.
(a) 0.92 (b) 0.94 (c) 0.96 (d) 0.98

14. If $ax^2 + 2hxy + by^2 = 1$, then $\frac{dy}{dx}$ equals
(a) $\frac{hx+by}{ax+hy}$ (b) $\frac{ax+hy}{hx+by}$
(c) $\frac{ax+hx}{hy+by}$ (d) $\frac{-(ax+hy)}{hx+by}$

15. The length of the longest interval, in which the function $3\sin x - 4\sin^3 x$ is increasing, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

16. $2\int_0^3 \sqrt{4-y}\, dy$ is equal to

(a) $\frac{10}{3}$ (b) $\frac{28}{3}$ (c) 14 (d) 21

17. Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

(a) $e^y = e^x + \frac{x^3}{3} + C$ (b) $e^y = e^x - \frac{x^3}{3} + C$

(c) $e^y = e^x + \frac{x^2}{2} + C$ (d) $e^y = e^x - \frac{x^2}{2} + C$

18. Solve the differential equation $\cos^2(x-2y) = 1 - 2\frac{dy}{dx}$.

(a) $y = \tan(x - 2y) + C$ (b) $x = \tan(x - 2y) + C$

(c) $y = \tan(2y - x) + C$ (d) None of these

ASSERTION-REASON BASED QUESTIONS

In the following questions, a statement of assertion (A) is followed by a statement of reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true.

19. **Assertion (A) :** If $2(\sin^{-1}x)^2 - 5(\sin^{-1}x) + 2 = 0$, then x has 2 solutions.

Reason (R) : $\sin^{-1}(\sin x) = x$, if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

20. Consider the system of equations,

$x + y + z = 1$, $2x + 2y + 2z = 2$, $4x + 4y + 4z = 3$

Assertion (A) : The above system has infinitely many solutions.

Reason (R) : For the above system det $A = 0$ and $(\text{adj } A)B = O$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Section B

This section comprises of very short answer type questions (VSA) of 2 marks each

21. If $\cot^{-1}\left(\frac{-1}{5}\right) = x$, then find the values of $\sin x$ and $\cos x$.

22. Evaluate : $\int_1^3 \frac{1}{x(1+\log x)^2} dx$

OR

Evaluate : $\int_0^{2\pi} e^x\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$

23. A couple has 2 children. Find the probability that both are boys, if it is known that

(i) atleast one of them is a boy,

(ii) the elder child is a boy.

24. Check whether the relation R in the set $\mathbb{R}$ of real numbers defined as $R = \{(a, b) : a < b\}$ is

(i) symmetric, (ii) transitive.

OR

Find the range of the function

$f : [0, 1] \to R$, $f(x) = x^3 - x^2 + 4x + 2\sin^{-1}x$.

25. If A, B, C have position vectors (2, 0, 0), (0, 1, 0), (0, 0, 2), show that ΔABC is isosceles.

Section C

This section comprises of short answer type questions (SA) of 3 marks each

26. Show that the function $f : (-\infty, 0) \to (-1, 0)$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in (-\infty, 0)$ is one-one and onto.

27. If $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{for } x \ne 0 \\ 1, & \text{for } x = 0 \end{cases}$, then show that the function is discontinuous at $x = 0$.

OR

If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$, then find $\frac{dy}{dx}$.

28. Sketch the graph $y = |x + 1|$. Evaluate $\int_{-4}^{2} |x+1|\, dx$.

29. Find the matrix X for which $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$.

OR

Find the adjoint of the matrix

$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence show that $A(\text{adj } A) = |A|\, I_3$.

30. Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if A and B are

(i) mutually exclusive (ii) independent.

31. Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular, if $pp' + rr' + 1 = 0$

OR

Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$

Section D

This section comprises of long answer type questions (LA) of 5 marks each

32. Evaluate : $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2 \sin^2 x}$

33. Solve the following differential equations.

(i) $(x+y)^2\dfrac{dy}{dx}=1$ (ii) $\dfrac{dy}{dx}=\sec(x+y)$

OR

Solve the following differential equations.

(i) $\dfrac{dy}{dx}=(x+y+1)^2$ (ii) $\dfrac{dy}{dx}=\dfrac{(x-y)+3}{2(x-y)+5}$

34. Find the vector and cartesian equation of the line through the point $\hat{i}+\hat{j}-3\hat{k}$ and perpendicular to the lines $\vec{r}=2\hat{i}-3\hat{j}+\lambda(2\hat{i}+\hat{j}-3\hat{k})$ and $\vec{r}=3\hat{i}-5\hat{j}+\mu(\hat{i}+\hat{j}+\hat{k})$.

35. Solve the following LPP graphically :

Maximize $Z = 600x + 400y$

subject to the constraints : $x + 2y \le 12$, $2x + y \le 12$; $x + \dfrac{5}{4}y \ge 5$ and $x, y \ge 0$.

OR

Find the number of points at which the objective function $z = 3x + 2y$ can be maximized subject to the constraints : $3x + 5y \le 15$, $5x + 2y \le 20$, $x \ge 0$, $y \ge 0$.

Section E

This section comprises of 3 case-study/passage-based questions of 4 marks each. First two case study questions have three sub-parts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two sub-parts of 2 marks each.

36. **Case-Study 1 :** Read the following passage and answer the questions given below.

To promote plantation of trees in parks, an organisation tried to spread awareness through (i) Phone calls (ii) Posters (iii) Announcements. The cost of each mode per attempt is given below :

(i) ₹ 30 (ii) ₹ 50 (iii) ₹ 40

The number of attempts made in the city P, Q and R by above three modes are given below :

City	(i)	(ii)	(iii)
P	100	50	30
Q	200	40	70
R	300	80	25

Also, the chances of trees plantation corresponding to one attempt of given mode is

(i) 3% (ii) 10% (iii) 20%

(i) Find the cost incurred by organisation on city P.

(ii) What cost is incurred by organisation on city R than city Q?

(iii) What is the total number of tree plantation that can be expected after the promotion in city P.

OR

How many trees are planted in all the three cities that can be expected after the promotion?

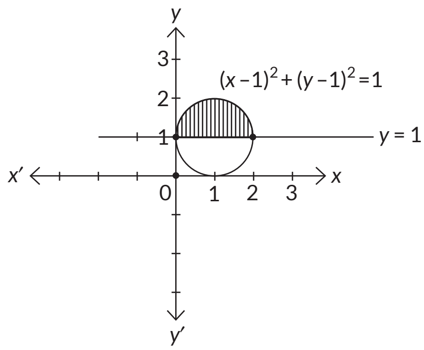
37. **Case-Study 2 :** Read the following passage and answer the questions given below.

Komal participated in the drawing competition organised in her school. She wanted to make a scenery in a circle. So, she drew a circle represented by the equation $(x - 1)^2 + (y - 1)^2 = 1$ and a straight line $y = 1$ as shown in the figure.

(i) Find the points of intersection of the given curve and the line.

(ii) What is the area of whole scenery?

(iii) Find the area of shaded region.



OR

Find : $\displaystyle\int_0^4 \sqrt{4-(x-2)^2}\,dx$.

38. **Case-Study 3:** Read the following passage and answer the questions given below.

The helicopters H_1 and H_2 move in 3D space along the lines $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{3}=t$ and $\dfrac{x}{3}=\dfrac{y}{2}=\dfrac{z}{1}=t$ respectively, where t is the time in seconds.

(i) Find the angle between the path of the two helicopters.

(ii) Find the direction ratios and direction of the line joining the position cosines vectors of H_1 and H_2 after 3 seconds.

Detailed SOLUTIONS

1. (c) : We have, $A^2 = A$
Now, L.H.S. $= (I - A)^3 + A = (I - A)(I - A)(I - A) + A$
$= (I \cdot I - I \cdot A - A \cdot I + A \cdot A)(I - A) + A$
$= (I - A - A + A)(I - A) + A$ $[\because I \cdot A = A \cdot I = A \text{ and } A^2 = A]$
$= (I - A)(I - A) + A = (I \cdot I - I \cdot A - A \cdot I + A \cdot A) + A$
$= (I - A - A + A) + A = (I - A) + A = I$
Hence, $(I - A)^3 + A = I$

2. (a) : Given, $P(A|B) = \frac{2}{5} \Rightarrow \frac{P(A\cap B)}{P(B)} = \frac{2}{5}$

$\Rightarrow P(A\cap B) = \frac{2}{5}P(B) = \frac{2}{5}\times\frac{5}{13} = \frac{2}{13}$

Hence, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2}\times\frac{5}{13} + \frac{5}{13} - \frac{2}{13}$ $\left(\because 2P(A) = \frac{5}{13}, \therefore P(A) = \frac{1}{2}\times\frac{5}{13}\right)$

$= \frac{5+10-4}{26} = \frac{11}{26}.$

3. (b) : We have, $\begin{pmatrix} 2x+y & 3y \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 6 & 4 \end{pmatrix}'$

$\Rightarrow \begin{pmatrix} 2x+y & 3y \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 0 & 4 \end{pmatrix}$

By equality of two matrices, we have
$2x + y = 6$ and $3y = 6 \Rightarrow y = 2.$
Putting the value of y, we get
$2x + 2 = 6 \Rightarrow 2x = 4 \Rightarrow x = 2.$

4. (c) : We have, $\frac{dy}{dx} = x^3 e^{-2y} \Rightarrow e^{2y}\,dy = x^3 dx$

On integrating, we get $\frac{e^{2y}}{2} = \frac{x^4}{4} + C'$

$\Rightarrow 2e^{2y} = x^4 + C$, where $C = 4C'$

5. (a) : We know that, the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq. units.

6. (b) : Since greatest power of highest order derivative is 1, therefore degree of the given differential equation is 1.

7. (b) : If two dice are thrown, then total number of cases = 36
Cases for total of 9 or 11 are {(3, 6), (4, 5), (6, 3), (5, 4), (6, 5), (5, 6)}, *i.e.*, 6 in number.

$\therefore \quad P(\text{total 9 or 11}) = \frac{6}{36} = \frac{1}{6}$

P(sum is neither 9 nor 11) = 1 – P(sum is 9 or 11)

$= 1 - \frac{1}{6} = \frac{5}{6}$

8. (a) : Let $I = \int_{-\pi}^{\pi} x^{10}\sin^7 x\,dx$

Also, let $f(x) = x^{10}\sin^7 x$

Then, $f(-x) = (-x)^{10}[\sin(-x)]^7 = -x^{10}\sin^7 x = -f(x)$

$\Rightarrow f(x)$ is an odd function. $\therefore \quad I = \int_{-\pi}^{\pi} x^{10}\sin^7 x\,dx = 0$

9. (a) : $\frac{d}{dx}[\cot^{-1}a - \cot^{-1}x] = 0 - \left(\frac{-1}{1+x^2}\right) = \frac{1}{1+x^2}$

At $x = 1$, $\frac{d}{dx}[\cot^{-1}a - \cot^{-1}x] = \frac{1}{1+1} = \frac{1}{2}$

10. (b) : Required area

$= \left|\int_{-8}^{0} x^{1/3}\,dx\right| = \left|\left[\frac{x^{4/3}}{4/3}\right]_{-8}^{0}\right| = \left|\frac{3}{4}[0-(-8)^{4/3}]\right|$

$= \left|\frac{3}{4}[-(-2)^4]\right| = \frac{3}{4}\times 16 = 12 \text{ sq. units}$

11. (a) : $f'(x) = 2x - 2 = 2(x - 1) < 0$ if $x < 1$ *i.e.*, $x \in (-\infty, 1)$.
Hence, f is strictly decreasing in $(-\infty, 1)$.

12. (c) : $y = e^{\frac{1}{2}\log(1+\tan^2 x)} = (\sec^2 x)^{1/2} = \sec x$

$\therefore \quad \frac{dy}{dx} = \sec x \tan x$

13. (c) : Given, $P(A) = 0.4$, $P(B) = 0.8$ and $P(B|A) = 0.6$
Clearly, $P(A \cap B) = P(B|A)\cdot P(A) = 0.6 \times 0.4 = 0.24$
Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.8 - 0.24 = 0.96$

14. (d) : Given, $ax^2 + 2hxy + by^2 = 1$
On differentiating w.r.t. x, we get

$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{ax+hy}{hx+by}\right).$

15. (a) : Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad -\frac{\pi}{2} \le 3x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{6} \le x \le \frac{\pi}{6}$

Thus, length of interval $= \left|\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{\pi}{3}$

16. (b) : We have, $2\int_0^3 \sqrt{4-y}\,dy = 2\left[\frac{2}{3}\left(\frac{(4-y)^{3/2}}{(-1)}\right)\right]_0^3$

$= -\frac{4}{3}[(1)^{3/2} - (4)^{3/2}] = -\frac{4}{3}[1-8] = \frac{28}{3}$

17. (a) : We have, $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\Rightarrow dy = (e^{x-y} + x^2 e^{-y})dx \Rightarrow e^y\,dy = (e^x + x^2)dx$

$\Rightarrow \int e^y\,dy = \int (e^x + x^2)\,dx$ [Integrating both sides]

$\Rightarrow e^y = e^x + \frac{x^3}{3} + C$, which is the required solution.

18. (b) : Given, $\cos^2(x - 2y) = 1 - 2\frac{dy}{dx}$ ...(i)

Put $x - 2y = u \Rightarrow 1 - \frac{2dy}{dx} = \frac{du}{dx}$

$\therefore$ Equation (i) becomes $\cos^2 u = \frac{du}{dx}$

$\Rightarrow \int dx = \int \sec^2 u \, du$

$\Rightarrow x = \tan u + C \Rightarrow x = \tan(x - 2y) + C$

19. (d): $2(\sin^{-1}x)^2 - 5(\sin^{-1}x) + 2 = 0$

$\Rightarrow \sin^{-1}x = \frac{5 \pm \sqrt{25-16}}{4} \Rightarrow \sin^{-1}x = \frac{1}{2}, \sin^{-1}x = 2$

$\Rightarrow x = \sin\left(\frac{1}{2}\right)$ is only solution

$[\because \frac{-\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}, \sin^{-1}x = 2$ is not possible$]$

20. (d): The given system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e. $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Here, $\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{vmatrix} = (2 \times 4)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Also, $(\text{adj } A)B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = O$

$\therefore$ Reason (R) is true.

However, Assertion (A) is not true as the given system is inconsistent. Here, the third equation contradicts the first and second which are identical.

So, the given system has no solution.

21. Given $\cot^{-1}\left(\frac{-1}{5}\right) = x \Rightarrow \cot x = \frac{-1}{5}, x \in (0, \pi)$

Now, $\operatorname{cosec} x = \sqrt{1 + \cot^2 x} = \sqrt{1 + \left(\frac{-1}{5}\right)^2} = \sqrt{\frac{26}{25}} = \frac{\sqrt{26}}{5}$

[$\operatorname{cosec} x$ is +ve since $0 < x < \pi$]

$\Rightarrow \sin x = \frac{5}{\sqrt{26}}$

Also, $\cos x = \frac{\cos x}{\sin x} \cdot \sin x = \cot x \cdot \sin x = \left(\frac{-1}{5}\right)\frac{5}{\sqrt{26}} = \frac{-1}{\sqrt{26}}$

22. $I = \int_1^3 \frac{1}{x(1+\log x)^2} dx$

Put $1 + \log x = t \Rightarrow \frac{dx}{x} = dt$

When $x = 1, t = 1$ and when $x = 3, t = 1 + \log 3$

$\therefore I = \int_1^{1+\log 3} \frac{dt}{t^2}$

$= \left[\frac{t^{-1}}{-1}\right]_1^{1+\log 3} = -\left[\frac{1}{1+\log 3} - 1\right] = \frac{\log 3}{1+\log 3}$

OR

$\int_0^{2\pi} e^x\left(\frac{x}{2} + \frac{\pi}{4}\right)dx = \int_0^{2\pi}\left(\frac{x}{2}e^x + \frac{\pi}{4}e^x\right)dx$

$= \left[\left(\frac{x}{2}\right)(e^x) - \left(\frac{1}{2}\right)(e^x) + \frac{\pi}{4}e^x\right]_0^{2\pi}$

$= \pi e^{2\pi} - \frac{1}{2}e^{2\pi} + \frac{\pi}{4}e^{2\pi} + \frac{1}{2} - \frac{\pi}{4} = e^{2\pi}\left(\frac{5\pi}{4} - \frac{1}{2}\right) + \frac{1}{2} - \frac{\pi}{4}$

23. Let $B_i(i = 1, 2)$ denote the i^{th} child is a boy and $G_i(i = 1, 2)$ denote the i^{th} child is a girl.

Then sample space is,

$S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$

Let A be the event that both are boys, B be the event that one of them is a boy and C be the event that the elder child is a boy.

$A = \{B_1B_2\}, B = \{G_1B_2, B_1G_2, B_1B_2\}$

$C = \{B_1B_2, B_1G_2\} \Rightarrow A \cap B = \{B_1B_2\}$ and $A \cap C = \{B_1B_2\}$

(i) Required probability = $P(A|B)$

$= \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$

(ii) Required probability = $P(A|C)$

$= \frac{P(A \cap C)}{P(C)} = \frac{1/4}{2/4} = \frac{1}{2}$

24. We have, $R = \{(a, b) : a < b\}$, where $a, b \in \mathbb{R}$

(i) Symmetric : Let $(x, y) \in R$, *i.e.*, $x R y \Rightarrow x < y$

Then, $y \nless x$, so $(x, y) \in R \Rightarrow (y, x) \notin R$

Thus, R is not symmetric.

(ii) Transitive : Let $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow x < y$ and $y < z \Rightarrow x < z$

$\Rightarrow (x, z) \in R$. Thus, R is transitive.

OR

We have, $f(x) = x^3 - x^2 + 4x + 2\sin^{-1}x$

For $x = 0$, $f(0) = 0 + 2\sin^{-1}(0) = 0$

For $x = 1$, $f(1) = 1 - 1 + 4 + 2\sin^{-1}(1) = 4 + 2 \times \frac{\pi}{2} = 4 + \pi$

Since, $f(x)$ is an increasing function.

$\therefore$ Its range is $[0, 4 + \pi]$.

25. We have, $\overrightarrow{AB}$ = P.V. of B – P.V. of A

$\Rightarrow \overrightarrow{AB} = (0\hat{i} + \hat{j} + 0\hat{k}) - (2\hat{i} + 0\hat{j} + 0\hat{k}) = -2\hat{i} + \hat{j} + 0\hat{k}$

$\therefore AB = |\overrightarrow{AB}| = \sqrt{(-2)^2 + 1^2 + 0^2} = \sqrt{5}$

$\overrightarrow{BC}$ = P.V. of C – P.V. of B

$\Rightarrow \overrightarrow{BC} = (0\hat{i} + 0\hat{j} + 2\hat{k}) - (0\hat{i} + \hat{j} + 0\hat{k}) = 0\hat{i} - \hat{j} + 2\hat{k}$

$\therefore BC = |\overrightarrow{BC}| = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$

$\Rightarrow AB = BC$, therefore, ΔABC is isosceles.

26. Given, $f(x) = \frac{x}{1+|x|}, x \in (-\infty, 0)$

$= \frac{x}{1-x}$ $\quad (\because x \in (-\infty, 0), |x| = -x)$

For one-one : Let $f(x_1) = f(x_2)$, $x_1x_2 \in (-\infty, 0)$

$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1(1 - x_2) = x_2(1 - x_1)$

$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2 \Rightarrow x_1 = x_2$

Hence, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

$\therefore$ f is one-one

For onto : Let $f(x) = y$

$\Rightarrow \quad y = \dfrac{x}{1-x} \Rightarrow y(1-x) = x \Rightarrow y - xy = x$

$\Rightarrow \quad x + xy = y \Rightarrow x(1+y) = y \Rightarrow x = \dfrac{y}{1+y}$

Here, $y \in (-1, 0)$

So, x is defined for all values of y.

Also, $x \in (-\infty, 0)$ for all $y \in (-1, 0)$.

$\therefore$ f is onto.

27. We have, $f(0) = 1$ ...(1)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

We have, $-1 \le \sin\frac{1}{x} \le 1 \Rightarrow -x^2 \le x^2\sin\frac{1}{x} \le x^2$

$$\Rightarrow \lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \le 0 \Rightarrow \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0 \qquad \text{...(2)}$$

From (1) & (2), $\lim_{x\to 0} f(x) \ne f(0)$ $\therefore$ f is discontinuous at $x = 0$

OR

$$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$$

$$\Rightarrow y = \frac{ax^2 + bx(x-a) + c(x-a)(x-b) + (x-a)(x-b)(x-c)}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow y = \frac{x^3}{(x-a)(x-b)(x-c)} \Rightarrow \log y = \log\left\{\frac{x^3}{(x-a)(x-b)(x-c)}\right\}$$

$\Rightarrow \quad \log y = 3\log x - \{\log(x-a) + \log(x-b) + \log(x-c)\}$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \left\{\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\right\}$$

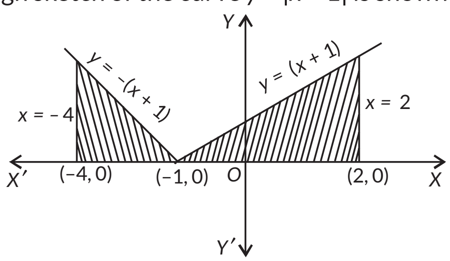
$$\Rightarrow \frac{dy}{dx} = y\left\{\left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right) + \left(\frac{1}{x} - \frac{1}{x-c}\right)\right\}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{-a}{x(x-a)} + \frac{(-b)}{x(x-b)} + \frac{(-c)}{x(x-c)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right\}$$

28. We have, $y = |x+1|$ $\therefore$ $y = \begin{cases} -(x+1) & x < -1 \\ (x+1) & x \ge -1 \end{cases}$

The rough sketch of the curve $y = |x+1|$ is shown in figure.



$$\therefore \int_{-4}^{2} |x+1|\,dx = \int_{-4}^{-1} -(x+1)\,dx + \int_{-1}^{2} (x+1)\,dx$$

$$= -\left[\frac{x^2}{2} + x\right]_{-4}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{2}$$

$$= -\left[\left(\frac{1}{2} - 1\right) - \left(\frac{16}{2} - 4\right)\right] + \left[\left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= -\left[-\frac{1}{2} - 4\right] + \left[4 + \frac{1}{2}\right] = \frac{9}{2} + \frac{9}{2} = 9$$

29. Let $A = \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix} = -2 + 12 = 10 \ne 0 \Rightarrow A^{-1}$ exists

$A_{11} = -2; A_{12} = -3; A_{21} = -(-4) = 4; A_{22} = 1$

$$\text{adj } A = \begin{bmatrix} -2 & -3 \\ 4 & 1 \end{bmatrix}' = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$$

The given matrix equation is

$AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow IX = A^{-1}B$

$$\Rightarrow X = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 32+28 & 12+8 \\ 48+7 & 18+2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 60 & 20 \\ 55 & 20 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ \frac{11}{2} & 2 \end{bmatrix}$$

OR

We have, $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$

$= (-1)(1-4) - (-2)(2+4) - 2(-4-2) = 3 + 12 + 12 = 27 \ne 0$

$\therefore$ A^{-1} exists

$A_{11} = -3, A_{12} = -6, A_{13} = -6, A_{21} = 6, A_{22} = 3, A_{23} = -6,$

$A_{31} = 6, A_{32} = -6, A_{33} = 3$

$$\therefore \text{adj } A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\therefore A(\text{adj } A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} = 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

30. (i) When A and B are mutually exclusive, then $A \cap B = \phi \Rightarrow P(A \cap B) = 0 \Rightarrow P(A \cup B) = P(A) + P(B)$

$\Rightarrow \quad \frac{3}{5}=\frac{1}{2}+p \Rightarrow p=\frac{3}{5}-\frac{1}{2}=\frac{6-5}{10}=\frac{1}{10}$

(ii) When A and B are independent, then
$P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

$\Rightarrow \quad \frac{3}{5}=\frac{1}{2}+p-\frac{1}{2}\cdot p \Rightarrow \frac{3}{5}-\frac{1}{2}=\frac{2p-p}{2}$

$\Rightarrow \quad \frac{p}{2}=\frac{6-5}{10} \Rightarrow p=\frac{2}{10}=\frac{1}{5}$

31. We have, $x=py+q \Rightarrow y=\frac{x-q}{p}$ ...(i)

and $z=ry+s \Rightarrow y=\frac{z-s}{r}$ ...(ii)

$\Rightarrow \quad \frac{x-q}{p}=\frac{y}{1}=\frac{z-s}{r}$ [From (i) and (ii)]

$\therefore \quad \langle a_1, b_1, c_1\rangle = \langle p, 1, r\rangle$ ...(iii)

Similarly, $\frac{x-q'}{p'}=\frac{y}{1}=\frac{z-s'}{r'}$ ...(iv)

$\therefore \quad \langle a_2, b_2, c_2\rangle = \langle p', 1, r'\rangle$

As we know, two lines are perpendicular to each other, if $a_1a_2+b_1b_2+c_1c_2=0 \Rightarrow pp'+rr'+1=0$,
which is the required condition.

OR

We have, $l+m+n=0$ ...(i)

or $\quad n=-(l+m)$ and $l^2+m^2-n^2=0$ ...(ii)

Substituting $n=-(l+m)$ in (ii), we get $l^2+m^2-(l+m)^2=0$

$\Rightarrow \quad l^2+m^2-l^2-m^2-2ml=0$

$\Rightarrow \quad 2lm=0$

$\Rightarrow \quad lm=0 \Rightarrow (-m-n)m=0$ $[\because l=-m-n]$

$\Rightarrow \quad (m+n)m=0 \Rightarrow m=-n$ or $m=0$

$\Rightarrow \quad l=0, l=-n$ (Using $l=-m-n$)

Hence, d.r.'s of two lines are proportional to 0, $-n$, n and $-n$, 0, n *i.e.*, 0, -1, 1 and -1, 0, 1.

Therefore, the vector parallel to these given lines are

$\vec{a}=-\hat{j}+\hat{k}$ and $\vec{b}=-\hat{i}+\hat{k}$

Now, $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} \Rightarrow \cos\theta=\frac{1}{2}=\cos\frac{\pi}{3}$

$\therefore \quad \theta=\frac{\pi}{3}$

32. Let $I=\int_0^{\pi}\frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$ ...(1)

$\Rightarrow \quad I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$

$\left[\text{Using } \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$\Rightarrow \quad I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2 x+b^2\sin^2 x}$ ...(2)

Adding (1) and (2), we get

$I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$

Let $f(x)=\frac{1}{a^2\cos^2 x+b^2\sin^2 x}$

$\Rightarrow \quad f(\pi-x)=\frac{1}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$

$\Rightarrow \quad f(\pi-x)=\frac{1}{a^2\cos^2 x+b^2\sin^2 x}=f(x)$

$\therefore \quad I=\frac{\pi}{2}\left(2\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}\right)$

$\left[\text{Using } \int_0^{2a} f(x)dx=2\int_0^a f(x)dx, \text{ if } f(2a-x)=f(x)\right]$

$\Rightarrow \quad I=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$.
Also when $x=0$, $t=\tan 0=0$;

And when $x=\frac{\pi}{2}$, $t=\tan\frac{\pi}{2}=\infty$

$\therefore I=\pi\int_0^{\infty}\frac{dt}{a^2+b^2t^2} \Rightarrow I=\frac{\pi}{b^2}\int_0^{\infty}\frac{dt}{\left(\frac{a}{b}\right)^2+t^2} \Rightarrow I=\frac{\pi}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$

$\Rightarrow \quad I=\frac{\pi}{ab}\left[\tan^{-1}\infty-\tan^{-1}0\right]=\frac{\pi^2}{2ab}$

33. (i) The given differential equation is

$(x+y)^2\frac{dy}{dx}=1$ ...(1)

Put $x+y=v \Rightarrow 1+\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$ ...(2)

From (1) and (2), we get

$v^2\left(\frac{dv}{dx}-1\right)=1 \Rightarrow \frac{dv}{dx}=1+\frac{1}{v^2}$

$\Rightarrow \quad \frac{v^2}{v^2+1}dv=dx \Rightarrow \int\frac{v^2+1-1}{v^2+1}dv=\int dx$

$\Rightarrow \quad \int\left(1-\frac{1}{v^2+1}\right)dv=\int dx \Rightarrow v-\tan^{-1}v=x+c$

$\Rightarrow \quad x+y-\tan^{-1}(x+y)=x+c$, which is the required solution.

(ii) The given differential equation can be written as

$\frac{dy}{dx}=\frac{1}{\cos(x+y)}$ ...(1)

Putting $x+y=v \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$

$\therefore \quad$ From (1), we have $\frac{dv}{dx}-1=\frac{1}{\cos v}$

$\Rightarrow \quad \frac{dv}{dx}=\frac{1+\cos v}{\cos v} \Rightarrow \frac{\cos v(1-\cos v)}{1-\cos^2 v}dv=dx$

$\Rightarrow \quad \left(\frac{\cos v}{\sin^2 v}-\frac{\cos^2 v}{\sin^2 v}\right)dv=dx$

$\Rightarrow \quad (\cot v\,\mathrm{cosec}\,v-\cot^2 v)dv=dx$

$\Rightarrow \int (\cot v \operatorname{cosec} v - \operatorname{cosec}^2 v + 1) dv = \int dx$

$\Rightarrow$ $-\operatorname{cosec} v + \cot v + v = x + C$

$\Rightarrow$ $-\operatorname{cosec}(x + y) + \cot(x + y) + (x + y) = x + C$

$$\Rightarrow -\frac{1}{\sin(x+y)} + \frac{\cos(x+y)}{\sin(x+y)} + y = C$$

$$\Rightarrow -\left(\frac{1-\cos(x+y)}{\sin(x+y)}\right) + y = C$$

$$\Rightarrow -\frac{2\sin^2\left(\frac{x+y}{2}\right)}{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)} + y = C$$

$\Rightarrow -\tan\left(\frac{x+y}{2}\right) + y = C$, which is required solution.

OR

(i) The given differential equation is $\frac{dy}{dx} = (x+y+1)^2$ ...(1)

Put $x + y + 1 = v$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$ ...(2)

From (1) and (2), we get

$$\frac{dv}{dx} - 1 = v^2 \Rightarrow \frac{dv}{dx} = v^2 + 1 \Rightarrow \int \frac{1}{v^2+1} dv = \int dx$$

$\Rightarrow$ $\tan^{-1} v = x + c$

$\Rightarrow$ $\tan^{-1}(x + y + 1) = x + c$, which is the required solution.

(ii) The given differential equation is

$$\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5} \quad ...(1)$$

Put $x - y = v \Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$ ...(2)

From (1) and (2), we get

$$1 - \frac{dv}{dx} = \frac{v+3}{2v+5} \Rightarrow \frac{dv}{dx} = 1 - \frac{v+3}{2v+5} = \frac{2v+5-v-3}{2v+5} = \frac{v+2}{2v+5}$$

$$\Rightarrow \int \frac{2v+5}{v+2} dv = \int dx \quad \Rightarrow \int \left(2 + \frac{1}{v+2}\right) dv = \int dx$$

$\Rightarrow$ $2v + \log|v + 2| = x + c$

$\Rightarrow$ $2(x - y) + \log|x - y + 2| = x + c$, which is the required solution.

34. Here we need to find, the equation of the line through the point (1, 1, –3) and perpendicular to the lines

$$\frac{x-2}{2} = \frac{y+3}{1} = \frac{z-0}{-3} \quad ...(i)$$

and $\frac{x-3}{1} = \frac{y+5}{1} = \frac{z-0}{1}$ ...(ii)

Let the direction ratios of required line are a, b, c. Then, equations of this line is given by

$$\frac{x-1}{a} = \frac{y-1}{b} = \frac{z+3}{c} \quad ...(iii)$$

Direction ratios of line (i) are 2, 1, –3 and line is perpendicular to line (iii) having direction ratios a, b, c

$\therefore$ $2a + b - 3c = 0$ ...(iv)

Similarly $a + b + c = 0$ ...(v)

On solving equation (iv) and (v), we get

$$\frac{a}{4} = \frac{-b}{5} = \frac{c}{1}$$

$\therefore$ From equation (iii), required equation of the line is $\frac{x-1}{4} = \frac{y-1}{-5} = \frac{z+3}{1}$.

Its vector equation is $\vec{r} = (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(4\hat{i} - 5\hat{j} + \hat{k})$

35. Maximize, $Z = 600x + 400y$

subject to the constraints :

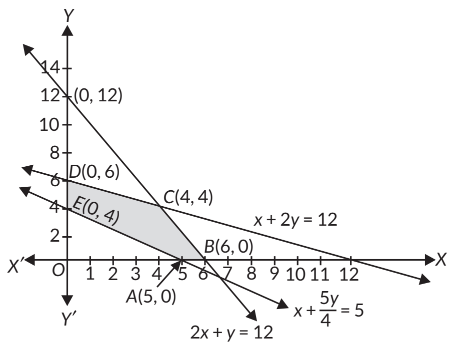
$x + 2y \le 12$ ...(i)

$2x + y \le 12$ ...(ii)

$x + \frac{5}{4}y \ge 5$ ...(iii)

$x, y \ge 0$ ...(iv)

Let us draw the graph of constraints (i) to (iv). $ABCDEA$ is the feasible region (shaded) as shown in the figure. Observe that the feasible region is bounded, and coordinates of the corner points A, B, C, D and E are (5, 0), (6, 0), (4, 4), (0, 6) and (0, 4) respectively.



Let us evaluate $Z = 600x + 400y$ at these corner points.

Corner Points	$Z = 600x + 400y$
$A(5, 0)$	3000
$B(6, 0)$	3600
$C(4, 4)$	4000 (Maximum)
$D(0, 6)$	2400
$E(0, 4)$	1600

We clearly see that the point (4, 4) is giving the maximum value of Z.

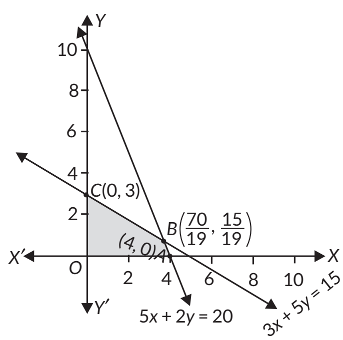
OR

Converting inequations into equations and drawing the corresponding lines.

$3x + 5y = 15,\ 5x + 2y = 20$ *i.e.* $\frac{x}{5} + \frac{y}{3} = 1,\ \frac{x}{4} + \frac{y}{10} = 1$

As $x \ge 0, y \ge 0$ solution lies in first quadrant.

Let us draw the graph of the above equations.



B is the point of intersection of the lines $3x + 5y = 15$ and $5x + 2y = 20$, *i.e.* $B = \left(\frac{70}{19}, \frac{15}{19}\right)$

We have points $O(0, 0)$, $A(4, 0)$, $B\left(\frac{70}{19}, \frac{15}{19}\right)$ and $C(0, 3)$.

Now $z = 3x + 2y$

$\therefore$ $z(O) = 3(0) + 2(0) = 0$

$z(A) = 3(4) + 2(0) = 12$

$$z(B) = 3\left(\frac{70}{19}\right) + 2\left(\frac{15}{19}\right) = 12.63$$

$z(C) = 3(0) + 2(3) = 6$

$\therefore$ z has maximum value 12.63 at only one point *i.e.* $B\left(\frac{70}{19}, \frac{15}{19}\right)$

36. (i) Let ₹ x, ₹ y and ₹ z be the cost incurred by the organisation for cities P, Q and R respectively, then the cost x, y and z will be given by the following matrix equation

$$\begin{bmatrix} 100 & 50 & 30 \\ 200 & 40 & 70 \\ 300 & 80 & 25 \end{bmatrix}\begin{bmatrix} 30 \\ 50 \\ 40 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3000+2500+1200 \\ 6000+2000+2800 \\ 9000+4000+1000 \end{bmatrix} = \begin{bmatrix} 6700 \\ 10800 \\ 14000 \end{bmatrix}$$

Cost incurred by organisation on city P = ₹ 6700

(ii) Cost incurred by organisation on city Q = ₹ 10800

Cost incurred by organisation on city R = ₹ 14000

$\therefore$ Cost incurred by organisation on city R than Q = ₹ 14000 – ₹ 10800 = ₹ 3200

(iii) Total number of tree plantation that can be expected in each city is given by the following matrix

$$\begin{matrix} P \\ Q \\ R \end{matrix}\begin{bmatrix} 100 & 50 & 30 \\ 200 & 40 & 70 \\ 300 & 80 & 25 \end{bmatrix}\begin{bmatrix} 3/100 \\ 10/100 \\ 20/100 \end{bmatrix} = \begin{bmatrix} 3+5+6 \\ 6+4+14 \\ 9+8+5 \end{bmatrix} = \begin{bmatrix} 14 \\ 24 \\ 22 \end{bmatrix}$$

$\therefore$ Total number of tree plantation in city P is 14.

OR

From above, we have total number of plantation in all the three cities = 14 + 24 + 22 = 60

37. (i) The given curve is $(x - 1)^2 + (y - 1)^2 = 1$ ...(i)

and line $y = 1$ ...(ii)

From (i) and (ii), we have

$(x - 1)^2 = 1$

$\Rightarrow$ $x^2 - 2x + 1 = 1$ $[\because y = 1]$

$\Rightarrow$ $x^2 - 2x = 0 \Rightarrow x(x - 2) = 0 \Rightarrow x = 0, 2$

$\therefore$ The points of intersection are (0, 1) and (2, 1).

(ii) Area of whole scenery = $\pi(1)^2 = \pi$ sq. units

(iii) Required area = $\int_0^2 \left(\sqrt{1-(x-1)^2} + 1\right) dx - \int_0^2 1 dx$

$$= \int_0^2 \sqrt{1-(x-1)^2} dx = \left[\frac{(x-1)}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^2$$

$$= \left[0 + \frac{1}{2}\sin^{-1}(1) - 0 - \frac{1}{2}\sin^{-1}(-1)\right]$$

$$= \left[\frac{1}{2} \times \frac{\pi}{2} + \frac{1}{2} \times \frac{\pi}{2}\right] = \frac{\pi}{2} \text{ sq. units}$$

OR

$$\int_0^4 \sqrt{4-(x-2)^2} = \int_0^4 \sqrt{(2)^2-(x-2)^2}$$

$$= \left[\frac{x-2}{2}\sqrt{(2)^2-(x-2)^2} + \frac{4}{2}\sin^{-1}\frac{x-2}{2}\right]_0^4$$

$= 0 + 2 \sin^{-1}(1) - 0 - 2 \sin^{-1}(-1)$

$$= 2 \times \frac{\pi}{2} - 2\left(\frac{-\pi}{2}\right) = \pi + \pi = 2\pi$$

38. (i) Direction ratios of the given lines are <1, 2, 4> and <3, 2, 1>

$\therefore$ Angle between them is given by

$$\cos\theta = \frac{(1)(3)+(2)(2)+(3)(1)}{\sqrt{(1)^2+(2)^2+(3)^2}\sqrt{(3)^2+(2)^2+(1)^2}}$$

$$= \frac{3+4+3}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7} \Rightarrow \theta = \cos^{-1}\frac{5}{7}.$$

(ii) Direction ratios of the line for helicopter H_1 are <3, 6, 9>.

Direction ratios of the line for helicopter H_2 are <9, 6, 3>.

$\therefore$ Direction ratios of the line joining H_1 and H_2 are <9 – 3, 6 – 6, 3 – 9> i.e., <6, 0, –6>

$\therefore$ Direction cosines are $< \frac{6}{\sqrt{(6)^2+(0)^2+(-6)^2}}, \frac{0}{\sqrt{(6)^2+(0)^2+(-6)^2}}, \frac{-6}{\sqrt{(6)^2+(0)^2+(-6)^2}} >$

i.e., $< \frac{6}{6\sqrt{2}}, \frac{0}{6\sqrt{2}}, \frac{-6}{6\sqrt{2}} >$ *i.e.*, $< \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} >$

PRACTICE PAPER 2

General Instructions :

1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub parts.

Time Allowed : 3 Hours **Maximum Marks :** 80

Section A

(Multiple Choice Questions)

Each question carries 1 mark

1. Find the value of k in the following probability distribution.

$X = x$	0.5	1	1.5	2
$P(X = x)$	k	k^2	$2k^2$	k

(a) -1 (b) 1/3 (c) 1/2 (d) 1/4

2. Find the values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

(a) $2\sqrt{2}$ (b) $\sqrt{2}$ (c) $\pm\sqrt{2}$ (d) $\pm 2\sqrt{2}$

3. Evaluate : $\int_{\pi/4}^{\pi/2} \cos 2x \, dx$

(a) 1/2 (b) $-1/2$ (c) 1/3 (d) $-1/3$

4. Determine the order and degree of

$$5\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}.$$

(a) 2, 2 (b) 2, 3 (c) 3, 2 (d) 3, 3

5. If $(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = 400$ and $|\vec{a}| = 4$, then find $|\vec{b}|$.

(a) 2 (b) 3 (c) 4 (d) 5

6. Find the principal value of $\cot^{-1}(-\sqrt{3})$.

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{4}$

7. If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ and $A^2 - kA - 5I = O$, then find the value of k.

(a) 3 (b) 4 (c) 2 (d) 5

8. If $x \sin(a + y) = \sin y$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\sin^2(a+y)}{\sin a}$ (b) $\frac{\sin a}{\sin^2(a+y)}$

(c) $\frac{\sin(a+y)}{\sin a}$ (d) $\frac{\sin a}{\sin(a+y)}$

9. The area bounded by the curve $2x^2 + y^2 = 2$ is

(a) π sq. units (b) $\sqrt{2}\pi$ sq. units

(c) $\frac{\pi}{2}$ sq. units (d) 2π sq. units

10. Which of the following statement is correct?

(a) Every LPP has at least one optimal solution.

(b) Every LPP has a unique optimal solution.

(c) If an LPP has two optimal solutions, then it has infinitely many solutions.

(d) None of these

11. Evaluate : $\int\left(5x^3 + 2x^{-5} - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x}\right)dx$

(a) $\frac{5x^4}{4} - \frac{1}{2x^4} - \frac{7x^2}{2} + 2\sqrt{x} + 5\log|x| + C$

(b) $\frac{5x^4}{4} + \frac{1}{2x^4} + \frac{7x^2}{2} + 2\sqrt{x} - 5\log|x| + C$

(c) $\frac{5x^4}{4} - \frac{1}{2x^4} + \frac{7x^2}{2} + 2\sqrt{x} - 5\log|x| + C$

(d) $\frac{5x^4}{4} - \frac{1}{2x^4} - \frac{7x^2}{2} + 2\sqrt{x} - 5\log|x| + C$

12. If $\begin{bmatrix} x+3y & y \\ 7-x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$, then find the values of x and y.

(a) $x = 7, y = 1$ (b) $x = -7, y = 1$

(c) $x = -7, y = -1$ (d) $x = 7, y = -1$

13. Find the integrating factor of the differential equation $\left\{\frac{e^{-2\sqrt{x}}}{\sqrt{x}}+\frac{y}{\sqrt{x}}\right\}\frac{dx}{dy}=1(x\neq 0)$.

(a) e^x (b) $e^{\sqrt{2x}}$ (c) $e^{-2\sqrt{x}}$ (d) $e^{-\sqrt{x}}$

14. If the function $f(x)=\begin{cases}\frac{\sin x^2}{x}; & x\neq 0\\ 0\ ; & x=0\end{cases}$, is differentiable at $x = 0$, then right hand derivative of $f(x)$ at $x = 0$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) -1

15. Evaluate : $\int_2^4 \frac{x}{x^2+1}dx$

(a) $\log\left(\frac{17}{5}\right)$ (b) $\frac{1}{2}\log\left(\frac{17}{5}\right)$

(c) $\frac{1}{2}\log\left(\frac{5}{17}\right)$ (d) $\log\left(\frac{5}{17}\right)$

16. If $f(x) = -\sqrt{25-x^2}$, then $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$ is equal to

(a) $\frac{1}{24}$ (b) $\frac{1}{5}$ (c) $-\sqrt{24}$ (d) $\frac{1}{\sqrt{24}}$

17. If the volume of a sphere is increasing at a constant rate, then the rate at which its radius is increasing, is

(a) a constant
(b) proportional to the radius
(c) inversely proportional to the radius
(d) inversely proportional to the surface area

18. The optimal value of the objective function is attained at the points

(a) on X-axis
(b) on Y-axis
(c) which are corner points of the feasible region
(d) none of these

ASSERTION-REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

19. **Assertion (A) :** The points $A(2, 9, 12)$, $B(1, 8, 8)$, $C(-2, 11, 8)$ and $D(-1, 12, 12)$ are the vertices of a square.

Reason (R) : In a quadrilateral if $AB = BC = CD = DA$ and $AC \neq BD$ then $ABCD$ is a square.

20. **Assertion (A) :** If the function $f(x)=\frac{ae^x+be^{-x}}{ce^x+de^{-x}}$ is increasing function of x, then $bc > ad$.

Reason (R) : A function $f(x)$ is increasing if $f'(x) > 0$ for all x.

Section B

This section comprises very short answer type questions (VSA) of 2 marks each

21. If there are two values of a which makes determinant, $\Delta=\begin{vmatrix}1 & -2 & 5\\ 2 & a & -1\\ 0 & 4 & 2a\end{vmatrix}=86$, then find the sum of these values.

22. Let k and K be the minimum and the maximum values of the function, $f(x)=\frac{(1+x)^{0.6}}{1+x^{0.6}}$ defined on $[0, 1]$, respectively. Find the ordered pair (k, K).

OR

Separate the interval $\left[0,\frac{\pi}{2}\right]$ into sub-intervals in which the function $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing.

23. Find $\int\frac{dx}{\sqrt{5-4x-2x^2}}$.

OR

Evaluate : $\int\frac{\sin 2x}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)}dx$

24. If the solution of the differential equation $\frac{dy}{dx}=\frac{ax+3}{2y+f}$ represents a circle, then find the value of 'a'.

25. If A and B are events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A\cap B) = \frac{1}{12}$, then find P(not A and not B).

Section C

This section comprises short answer type questions (SA) of 3 marks each

26. Let $f : A \to B$ be a function defined as $f(x)=\frac{2x+3}{x-3}$, where $A = R - \{3\}$ and $B = R - \{2\}$. Is the function f one-one and onto?

27. Find $\frac{dy}{dx}$, when $y=\sqrt{a+\sqrt{a+\sqrt{a+x^2}}}$, where a is a constant.

OR

Find the value of k, for which

$$f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if} -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if} 0\le x<1\end{cases}$$ is continuous

at $x = 0$.

28. Evaluate : $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

OR

Evaluate : $\int_1^3 |x^2-2x|\,dx$

29. Solve the differential equation

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1, x\ne 0.$$

30. The points $A(1, 2, 3)$, $B(-1, -2, -1)$ and $C(2, 3, 2)$ are three vertices of a parallelogram $ABCD$. Find the equation of CD.

OR

Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}$ $=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Also find their point of intersection.

31. Find graphically, the maximum value of $Z = 2x + 5y$, subject to constraints given below:
$2x + 4y \le 8, 3x + y \le 6, x + y \le 4, x \ge 0, y \ge 0$

Section D

This section comprises long answer type questions (LA) of 5 marks each

32. The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram $ABCD$. Find the vector and cartesian equations of the sides AB and BC and find coordinates of D.

OR

Find the coordinates of the points on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{6}$, which are at a distance of 2 units from the point $(-2, -1, 3)$.

33. If $A=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}, B=\begin{bmatrix}2 & 3\\ 3 & -4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$, then verify
(i) $(AB)C = A(BC)$ (ii) $A(B + C) = AB + AC$

34. If A and B are two independent events such that $P(\bar{A}\cap B)=\frac{2}{15}$ and $P(A\cap\bar{B})=\frac{1}{6}$, then find $P(A)$ and $P(B)$.

OR

A doctor claims that 60% of the patients he examines are corona negative. What is the probability that
(i) exactly 3 of his next 4 patients are corona negative?
(ii) none of his next 4 patients is corona negative?

35. For what value of a is the function f defined by

$$f(x)=\begin{cases}a\sin\frac{\pi}{2}(x+1), & x\le 0\\ \dfrac{\tan x-\sin x}{x^3}, & x>0\end{cases}$$ continuous at $x = 0$?

Section E

This section comprises of 3 case-study/passage-based questions of 4 marks each. First two case study questions have three sub-parts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two sub-parts of 2 marks each.

36. **Case-Study 1 :** Read the following passage and answer the questions given below.

A relation R on a set A is said to be an equivalence relation on A iff it is

- Reflexive *i.e.*, $(a, a) \in R\ \forall\ a \in A$.
- Symmetric *i.e.*, $(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A$.
- Transitive *i.e.*, $(a, b) \in R$ and $(b, c) \in R$ $\Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$.

(i) If the relation $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ defined on the set $A = \{1, 2, 3\}$, then find the relation R on set A.

(ii) If the relation $R = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$ defined on the set $A = \{1, 2, 3\}$, then find the relation R on set A.

(iii) If the relation R on the set N of all natural numbers defined as $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$, then find the relation R on set N.

OR

If the relation R on the set $A = \{1, 2, 3,......, 13, 14\}$ defined as $R = \{(x, y) : 3x - y = 0\}$, then find the relation R on set A.

37. **Case-Study 2:** Read the following passage and answer the questions given below.

Let $f(x)$ be a real valued function, then its Left Hand Derivative (L.H.D.) :

$$Lf'(a) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

Right Hand Derivative (R.H.D.) :

$$Rf'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$ exist and are equal.

For the function $f(x) = \begin{cases}|x-3|, & x\ge 1\\ \dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4}, & x<1\end{cases}$, answer the following questions.

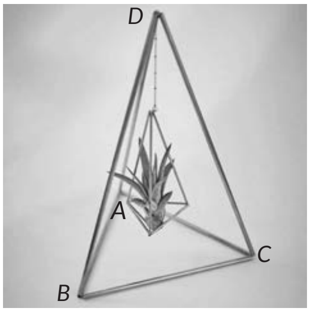
(i) Find the R.H.D. of $f(x)$ at $x = 1$.
(ii) Find the L.H.D. of $f(x)$ at $x = 1$.
(iii) Show that $f(x)$ is non-differentiable at $x = 3$.

OR

Find the value of $f''\left(\frac{1}{2}\right)$.

38. **Case-Study 3 :** Read the following passage and answer the questions given below.

Ginni purchased an air plant holder which is in the shape of a tetrahedron.
Let A, B, C and D are the coordinates of the air plant holder where $A \equiv (1, 1, 1)$, $B \equiv (2, 1, 3)$, $C \equiv (3, 2, 2)$ and $D \equiv (3, 3, 4)$.



(i) Find the magnitude of position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
(ii) Find the area of ΔBCD.

Detailed SOLUTIONS

1. **(b):** Since $P(X)$ is a probability distribution of X,

$\therefore \sum_{x_i=0.5}^{2} P(X=x)=1$

$\Rightarrow P(X = 0.5) + P(X = 1) + P(X = 1.5) + P(X = 2) = 1$

$\Rightarrow k + k^2 + 2k^2 + k = 1 \Rightarrow 3k^2 + 2k - 1 = 0$

$\Rightarrow (3k - 1)(k + 1) = 0 \Rightarrow k = \frac{1}{3}$ (k can't be negative)

2. **(d):** We have, $3 - x^2 = 3 - 8 \Rightarrow x^2 = 8$

$\Rightarrow x = \pm 2\sqrt{2}$

3. **(b):** We have, $I = \int_{\pi/4}^{\pi/2} \cos 2x\, dx = \left[\frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$

$= \left[\frac{\sin \pi}{2} - \frac{\sin\frac{\pi}{2}}{2}\right] = 0 - \frac{1}{2} = -\frac{1}{2}$

4. **(a):** The given differential equation can be written as

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = \left(5\frac{d^2y}{dx^2}\right)^2$$

Clearly, it can be observed that the order of differential equation is 2 and the degree is 2.

5. **(d):** We have, $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 400$ and $|\vec{a}| = 4$

We know that, $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

$\Rightarrow 400 = (4)^2|\vec{b}|^2 \Rightarrow 16|\vec{b}|^2 = 400 \Rightarrow |\vec{b}|^2 = 25 \Rightarrow |\vec{b}| = 5$

6. **(a):** Let $\cot^{-1}(-\sqrt{3}) = \theta$

$\Rightarrow \cot\theta = -\sqrt{3} = -\cot\frac{\pi}{6}$

$= \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6} \Rightarrow \theta = \frac{5\pi}{6} \in (0, \pi)$

$\therefore$ Principal value of $\cot^{-1}(-\sqrt{3})$ is $\frac{5\pi}{6}$.

7. **(d):** Given, $A^2 - kA - 5I = O$

$\Rightarrow kA = A^2 - 5I \Rightarrow kA = \begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 3\\3 & 4\end{bmatrix} - 5\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$= \begin{bmatrix}10 & 15\\15 & 25\end{bmatrix} - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix} = \begin{bmatrix}5 & 15\\15 & 20\end{bmatrix} = 5\begin{bmatrix}1 & 3\\3 & 4\end{bmatrix} = 5A$

$\Rightarrow kA = 5A \quad \therefore k = 5$

8. **(a):** Given, $x\sin(a + y) = \sin y \Rightarrow x = \frac{\sin y}{\sin(a+y)}$

Differentiating both sides w.r.t. x, we get

$$1 = \frac{\sin(a+y)\cos y\frac{dy}{dx} - \cos(a+y)\sin y\frac{dy}{dx}}{\sin^2(a+y)}$$

$\Rightarrow \sin^2(a + y) = \sin(a + y - y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

9. **(b):** We have, $2x^2 + y^2 = 2$

$\Rightarrow \frac{x^2}{1} + \frac{y^2}{2} = 1$, an ellipse

Here, $a = 1$ and $b = \sqrt{2}$

$\because$ Area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

$\therefore$ Required area $= \pi\sqrt{2}$ sq. units.

10. **(c) :** If optimal solution is obtained at two distinct points A and B (corners of the feasible region), then optimal solution is obtained at every point of segment AB.

11. **(a):** We have $\int\left(5x^3 + 2x^{-5} - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x}\right)dx$

$= 5\int x^3dx + 2\int x^{-5}dx - 7\int xdx + \int x^{-1/2}dx + 5\int\frac{1}{x}dx$

$= 5\cdot\frac{x^4}{4} + 2\cdot\frac{x^{-4}}{(-4)} - 7\cdot\frac{x^2}{2} + \frac{x^{1/2}}{(1/2)} + 5\log|x| + C$

$= \frac{5x^4}{4} - \frac{1}{2x^4} - \frac{7x^2}{2} + 2\sqrt{x} + 5\log|x| + C$

12. **(d):** Given $\begin{bmatrix}x+3y & y\\7-x & 4\end{bmatrix} = \begin{bmatrix}4 & -1\\0 & 4\end{bmatrix}$

$\Rightarrow y = -1$ and $7 - x = 0$

$\Rightarrow x = 7, y = -1$

13. **(c) :** We have, $\frac{dy}{dx} - \frac{1}{\sqrt{x}}\cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

The obtained equation is of the form $\frac{dy}{dx}+Py=Q,$ where $P=\frac{-1}{\sqrt{x}}$ and $Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$\text{I.F.}=e^{\int Pdx}=e^{\int\frac{-1}{\sqrt{x}}dx}=e^{-2\sqrt{x}}$.

14. (c) : At $x = 0$, right hand derivative

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{\frac{\sinh^2}{h}-0}{h}=\lim_{h\to 0}\frac{\sinh^2}{h^2}=1$$

15. (b): Let $I=\int_2^4\frac{x}{x^2+1}dx$

Put $x^2 + 1 = t \Rightarrow 2xdx = dt \Rightarrow xdx=\frac{1}{2}dt$

Also $x = 2 \Rightarrow t = 5$ and $x = 4 \Rightarrow t = 17$

$$\therefore\quad I=\frac{1}{2}\int_5^{17}\frac{dt}{t}=\frac{1}{2}[\log|t|]_5^{17}=\frac{1}{2}[\log 17-\log 5]=\frac{1}{2}\log\left(\frac{17}{5}\right)$$

16. (d): $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=f'(1)$

Now, $f'(x)=-\frac{1}{2}\cdot\frac{-2x}{\sqrt{25-x^2}}$

$$\therefore\quad f'(1)=-\frac{1}{2}\cdot\frac{-2\times 1}{\sqrt{25-1^2}}=\frac{1}{\sqrt{24}}$$

17. (d): Given that, $\frac{dV}{dt}=k$ (say)

$$\because\quad V=\frac{4}{3}\pi R^3\Rightarrow\frac{dV}{dt}=4\pi R^2\frac{dR}{dt}\Rightarrow\frac{dR}{dt}=\frac{k}{4\pi R^2}$$

$\Rightarrow$ Rate of increase of radius is inversely proportional to its surface area.

18. (c) : When we solve an L.P.P. graphically, the optimal (or optimum) value of the objective function is attained at corner points of the feasible region.

19. (c) : We have

$$AB=\sqrt{(2-1)^2+(9-8)^2+(12-8)^2}=\sqrt{18}=3\sqrt{2}$$
$$BC=\sqrt{(1+2)^2+(8-11)^2+(8-8)^2}=\sqrt{18}=3\sqrt{2}$$
$$CD=\sqrt{(-2+1)^2+(11-12)^2+(8-12)^2}=\sqrt{18}=3\sqrt{2}$$
$$DA=\sqrt{(-1-2)^2+(12-9)^2+(12-12)^2}=\sqrt{18}=3\sqrt{2}$$
$$AC=\sqrt{(2+2)^2+(9-11)^2+(12-8)^2}=\sqrt{36}=6$$

and $BD=\sqrt{(1+1)^2+(8-12)^2+(8-12)^2}=\sqrt{36}=6$

So, $AB = BC = CD = DA$ and $AC = BD$

Thus, $ABCD$ is a square.

Hence, Statement-I is true but Statement-II is false.

20. (d): $f'(x)=\frac{2(ad-bc)}{(ce^x+de^{-x})^2}$

and $f(x)$ is an increasing function

$$\therefore\quad f'(x)>0\Rightarrow\frac{2(ad-bc)}{(ce^x+de^{-x})^2}>0$$

$\Rightarrow 2(ad - bc) > 0 \Rightarrow ad > bc \Rightarrow bc < ad$

21. Given, $\Delta=\begin{vmatrix}1 & -2 & 5\\ 2 & a & -1\\ 0 & 4 & 2a\end{vmatrix}=86$

Expanding along C_1, we get

$1(2a^2 + 4) - 2(-4a - 20) = 86$

$\Rightarrow 2a^2 + 4 + 8a + 40 = 86$

$\Rightarrow 2a^2 + 8a - 42 = 0$

$\Rightarrow a^2 + 4a - 21 = 0$

$\Rightarrow (a + 7)(a - 3) = 0 \Rightarrow a = 3, -7$

Now, sum of values of a is $3 + (-7) = 3 - 7 = -4$

22. We have, $f(x)=\frac{(1+x)^{3/5}}{1+x^{3/5}}$

$$f'(x)=\frac{\left(1+x^{3/5}\right)\frac{3}{5}(1+x)^{-2/5}-(1+x)^{3/5}\cdot\frac{3}{5}x^{-2/5}}{(1+x^{3/5})^2}$$

Clearly $f'(x) = 0 \Rightarrow x = 1$

Also, $f(0)=1$, and $f(1)=\frac{2^{0.6}}{2}=2^{-0.4}\quad\therefore\ f(x)\in[2^{-0.4},1]$

Thus, $(k, K) = (2^{-0.4}, 1)$

OR

Here $f(x) = \sin^4x + \cos^4x$ ...(i)

On differentiating (i) w.r.t. x, we get

$f'(x) = 4\sin^3x\cos x - 4\cos^3x\sin x = 4\sin x\cos x(\sin^2x - \cos^2x)$

$= -2\sin 2x\cos 2x = -\sin 4x$

Put $f'(x) = 0 \Rightarrow x=\frac{\pi}{4}$

Intervals	Sign of $f'(x)$	Conclusion
$\left(0,\frac{\pi}{4}\right)$	$-$ve as $0 < 4x < \pi$	f is decreasing in $\left(0,\frac{\pi}{4}\right)$
$\left(\frac{\pi}{4},\frac{\pi}{2}\right)$	+ve as $\pi < 4x < 2\pi$	f is increasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

23. Let $I=\int\frac{dx}{\sqrt{5-4x-2x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{7}{2}-1-2x-x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}}$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right)+C=\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right]+C$$

OR

Let $I=\int\frac{\sin 2x}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)}dx$

$$=\int\frac{\sin\left\{\left(x-\frac{\pi}{3}\right)+\left(x+\frac{\pi}{3}\right)\right\}}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)}dx$$

$$=\int \frac{\left\{\sin\left(x-\frac{\pi}{3}\right)\cos\left(x+\frac{\pi}{3}\right)+\cos\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)\right\}}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)}dx$$

$$=\int\left\{\cot\left(x+\frac{\pi}{3}\right)+\cot\left(x-\frac{\pi}{3}\right)\right\}dx$$

$$=\log\left|\sin\left(x+\frac{\pi}{3}\right)\right|+\log\left|\sin\left(x-\frac{\pi}{3}\right)\right|+C$$

24. We have, $\frac{dy}{dx}=\frac{ax+3}{2y+f}$

$\Rightarrow$ $(ax+3)dx=(2y+f)dy$

$\Rightarrow$ $a\frac{x^2}{2}+3x=y^2+fy+C$ (Integrating both sides)

$\Rightarrow$ $-\frac{a}{2}x^2+y^2-3x+fy+C=0$

This will represent a circle, if $\frac{-a}{2}=1\Rightarrow a=-2$

[$\because$ Coefficient of x^2 should be equal to coefficient of y^2]

25. Here, $P(A)\cdot P(B)=\frac{1}{3}\times\frac{1}{4}=\frac{1}{12}=P(A\cap B)$

$\Rightarrow$ Events A and B are independent.

$\Rightarrow$ Events $\bar{A}$ and $\bar{B}$ are also independent.

Now, $P(\bar{A}\cap\bar{B})=P(\bar{A})P(\bar{B})$

($\because$ $\bar{A}$ and $\bar{B}$ are independent events)

$=(1-P(A))(1-P(B))$

$$=\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=\frac{2}{3}\times\frac{3}{4}=\frac{1}{2}$$

26. Let $y=f(x)=\frac{2x+3}{x-3}$ ...(i)

Let $x_1, x_2\in A=R-\{3\}$ such that $f(x_1)=f(x_2)$

$\Rightarrow$ $\frac{2x_1+3}{x_1-3}=\frac{2x_2+3}{x_2-3}$

$\Rightarrow$ $(2x_1+3)(x_2-3)=(2x_2+3)(x_1-3)$

$\Rightarrow$ $2x_1x_2-6x_1+3x_2-9=2x_1x_2-6x_2+3x_1-9$

$\Rightarrow$ $-6x_1+3x_2=-6x_2+3x_1\Rightarrow 9x_1=9x_2\Rightarrow x_1=x_2$

Now, $f(x_1)=f(x_2)\Rightarrow x_1=x_2$

So $f(x)$ is one-one.

For onto, let $y=\frac{2x+3}{x-3}$

$\Rightarrow$ $xy-3y=2x+3$ $\Rightarrow$ $xy-2x=3y+3$

$\Rightarrow$ $x(y-2)=3(y+1)$

$\Rightarrow$ $x=\frac{3(y+1)}{(y-2)}$ ...(ii)

Equation (ii) is defined for all real values of y except 2 which is same as given set $B=R-\{2\}$.

Thus, for every $y\in B$, there exists $x=\frac{3(y+1)}{y-2}\in A$ such that $f(x)=y$

Hence, function f is onto.

27. We have, $y=\sqrt{a+\sqrt{a+\sqrt{a+x^2}}}$, where a is a constant.

$$\Rightarrow y=\left[a+\sqrt{a+\sqrt{a+x^2}}\right]^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[a+\sqrt{a+\sqrt{a+x^2}}\right]^{-\frac{1}{2}}\cdot\frac{d}{dx}\left[a+\sqrt{a+\sqrt{a+x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[a+\sqrt{a+\sqrt{a+x^2}}\right]^{-\frac{1}{2}}\left[\frac{1}{2}\left(a+\sqrt{a+x^2}\right)^{-\frac{1}{2}}\times\frac{d}{dx}\left(a+\sqrt{a+x^2}\right)\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[a+\sqrt{a+\sqrt{a+x^2}}\right]^{-\frac{1}{2}}\left[\frac{1}{2}\left(a+\sqrt{a+x^2}\right)^{-\frac{1}{2}}\right]\times\left[\frac{1}{2}(a+x^2)^{-\frac{1}{2}}\cdot 2x\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{4}x\left[\left(a+\sqrt{a+\sqrt{a+x^2}}\right)\cdot\left(a+\sqrt{a+x^2}\right)\cdot(a+x^2)\right]^{-\frac{1}{2}}$$

OR

Since, $f(x)$ is continuous at $x=0$

$\therefore$ $\lim_{x\to 0^+}f(x)=f(0)=\lim_{x\to 0^-}f(x)$ ...(i)

Here, $f(0)=\frac{2\times 0+1}{0-1}=-1$ ...(ii)

$\lim_{x\to 0^+}f(x)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}\frac{2h+1}{h-1}=-1$ ...(iii)

and $\lim_{x\to 0^-}f(x)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}$

$$=\lim_{h\to 0}\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}\times\frac{\sqrt{1-kh}+\sqrt{1+kh}}{\sqrt{1-kh}+\sqrt{1+kh}}$$

$$=\lim_{h\to 0}\frac{(1-kh)-(1+kh)}{-h[\sqrt{1-kh}+\sqrt{1+kh}]}$$

$=\lim_{h\to 0}\frac{2k}{\sqrt{1-kh}+\sqrt{1+kh}}=\frac{2k}{2}=k$ ...(iv)

Now, from (i), (ii), (iii) and (iv), we get $k=-1$

28. Let $I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ ...(i)

Then, $I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$

$\Rightarrow$ $I=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{(x+\pi-x)\sin x}{1+\cos^2 x}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

$\Rightarrow$ $I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$ ...(iii)

Put $z=\cos x\Rightarrow dz=-\sin x\,dx$

Also, when $x=0$, $z=1$ and when $x=\pi$, $z=-1$

$\therefore$ From (iii),

$$I=\frac{\pi}{2}\int_1^{-1}\frac{-dz}{1+z^2}=-\frac{\pi}{2}\int_1^{-1}\frac{1}{1+z^2}dz=-\frac{\pi}{2}[\tan^{-1}z]_1^{-1}$$

$$=-\frac{\pi}{2}[\tan^{-1}(-1)-\tan^{-1}(1)]=-\frac{\pi}{2}\left[-\frac{\pi}{4}-\frac{\pi}{4}\right]=\frac{\pi^2}{4}$$

OR

Let $I=\int_1^3 |x^2-2x|dx$

$\because \quad |x^2-2x|=\begin{cases}-(x^2-2x), & \text{when } 1\le x<2\\ (x^2-2x), & \text{when } 2\le x\le 3\end{cases}$

$\Rightarrow \quad I=\int_1^2 |x^2-2x|dx+\int_2^3 |x^2-2x|dx$

$=\int_1^2 -(x^2-2x)dx+\int_2^3 (x^2-2x)dx$

$=-\left[\frac{x^3}{3}-x^2\right]_1^2+\left[\frac{x^3}{3}-x^2\right]_2^3=-\left(-\frac{4}{3}+\frac{2}{3}\right)+\left(\frac{4}{3}\right)=\frac{6}{3}=2$

29. We have, $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

$\Rightarrow \quad \frac{dy}{dx}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}} \Rightarrow \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$, with $P=\frac{1}{\sqrt{x}}$ and $Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$\text{I.F.}=e^{\int P\,dx}=e^{\int\frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$

$\therefore$ The solution is given by $ye^{2\sqrt{x}}=\int e^{2\sqrt{x}}\cdot\frac{e^{-2\sqrt{x}}}{\sqrt{x}}dx+C$

$\Rightarrow \quad ye^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+C \quad \Rightarrow \quad ye^{2\sqrt{x}}=2\sqrt{x}+C$

$\Rightarrow \quad y=(2\sqrt{x}+C)e^{-2\sqrt{x}}$, which is the required solution.

30. Given, $A(1, 2, 3)$, $B(-1, -2, -1)$ and $C(2, 3, 2)$. Let coordinates of D be (α, β, γ). Since $ABCD$ is a parallelogram, diagonals AC and BD bisect each other *i.e.*, mid-point of segment AC is same as mid-point of segment BD.

$\Rightarrow \quad \left(\frac{1+2}{2},\frac{2+3}{2},\frac{3+2}{2}\right)=\left(\frac{\alpha-1}{2},\frac{\beta-2}{2},\frac{\gamma-1}{2}\right)$

$\Rightarrow \quad \alpha-1=3, \beta-2=5, \gamma-1=5 \Rightarrow \alpha=4, \beta=7, \gamma=6.$

Thus, the point D is $(4, 7, 6)$.

Now, we have $C(2, 3, 2)$ and $D(4, 7, 6)$.

$\therefore$ Equation of line CD is

$\frac{x-2}{4-2}=\frac{y-3}{7-3}=\frac{z-2}{6-2} \Rightarrow \frac{x-2}{2}=\frac{y-3}{4}=\frac{z-2}{4}$

i.e. $\frac{x-2}{1}=\frac{y-3}{2}=\frac{z-2}{2}$ is the required equation of line.

OR

Any point on the line $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=r$ (say) ...(i)

is $(3r-1, 5r-3, 7r-5)$.

Any point on the line $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=k$ (say) ...(ii)

is $(k+2, 3k+4, 5k+6)$

For lines (i) and (ii) to intersect, we must have

$3r-1=k+2,\ 5r-3=3k+4,\ 7r-5=5k+6$

On solving these, we get $r=\frac{1}{2}, k=-\frac{3}{2}$

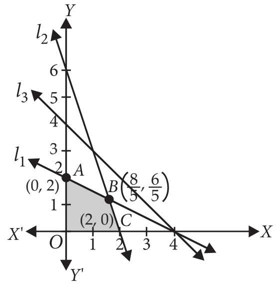
$\therefore$ Lines (i) and (ii) intersect and their point of intersection is $\left(\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\right)$

31. Convert the inequation into equation, we get

$l_1: 2x+4y=8,\ l_2: 3x+y=6,\ l_3: x+y=4;\ x=0,\ y=0$

Solving l_1 and l_2 we get $B\left(\frac{8}{5},\frac{6}{5}\right)$

Let us draw the graph of these equation.



The shaded portion $OABC$ is the feasible region, where coordinates of the corner points are $O(0, 0)$, $A(0, 2)$, $B\left(\frac{8}{5},\frac{6}{5}\right)$, $C(2, 0)$.

The value of objective function at these points are :

Corner Points	Value of $Z = 2x + 5y$
$O(0, 0)$	$2\times0+5\times0=0$
$A(0, 2)$	$2\times0+5\times2=10$ (Maximum)
$B\left(\frac{8}{5},\frac{6}{5}\right)$	$2\times\frac{8}{5}+5\times\frac{6}{5}=9.2$
$C(2, 0)$	$2\times2+5\times0=4$

$\therefore$ The maximum value of Z is 10, which is attained at point $A(0, 2)$.

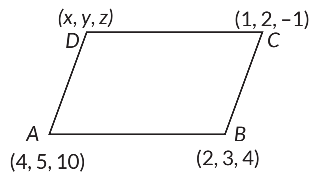
32. In a parallelogram, diagonals bisect each other.

$\therefore$ Mid point of BD = Mid point of AC

$\Rightarrow \quad \left(\frac{x+2}{2},\frac{y+3}{2},\frac{z+4}{2}\right)=\left(\frac{4+1}{2},\frac{5+2}{2},\frac{10-1}{2}\right)$

$\Rightarrow \quad x+2=5,\ y+3=7,\ z+4=9$

$\Rightarrow \quad x=3,\ y=4,\ z=5$



So coordinates of D are $(3, 4, 5)$.

Cartesian equation of side AB is $\frac{x-4}{2-4}=\frac{y-5}{3-5}=\frac{z-10}{4-10}$

$\Rightarrow \quad \frac{x-4}{-2}=\frac{y-5}{-2}=\frac{z-10}{-6} \Rightarrow \frac{x-4}{1}=\frac{y-5}{1}=\frac{z-10}{3}$

and vector equation of side AB is

$\vec{r}=4\hat{i}+5\hat{j}+10\hat{k}+\lambda(\hat{i}+\hat{j}+3\hat{k})$

Again, cartesian equation of side BC is

$\frac{x-2}{1-2}=\frac{y-3}{2-3}=\frac{z-4}{-1-4} \Rightarrow \frac{x-2}{-1}=\frac{y-3}{-1}=\frac{z-4}{-5}$

$\Rightarrow \quad \frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{5}$

and vector equation of side BC is

$\vec{r}=2\hat{i}+3\hat{j}+4\hat{k}+\lambda(\hat{i}+\hat{j}+5\hat{k})$

OR

$\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{6}$ is the given line ...(i)

Let $P(-2, -1, 3)$ lies on the line.

The direction ratios of line (i) are 3, 2, 6

$\therefore$ The direction cosines of line are $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

Equation (i) may be written as

$$\frac{x+2}{\frac{3}{7}} = \frac{y+1}{\frac{2}{7}} = \frac{z-3}{\frac{6}{7}} \quad ...(ii)$$

Coordinates of any point on the line (ii) may be taken as

$$\left(\frac{3}{7}r-2, \frac{2}{7}r-1, \frac{6}{7}r+3\right)$$

Let $Q = \left(\frac{3}{7}r-2, \frac{2}{7}r-1, \frac{6}{7}r+3\right)$

Given $|r| = 2$, $\therefore$ $r = \pm 2$

Putting the value r, we have

$$Q = \left(\frac{-8}{7}, \frac{-3}{7}, \frac{33}{7}\right) \text{ or } Q = \left(\frac{-20}{7}, \frac{-11}{7}, \frac{9}{7}\right)$$

33. We have, $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

(i) $AB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 2+6 & 3-8 \\ -4+3 & -6-4 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix}$

and $(AB)C = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 8+5 & 0 \\ -1+10 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 9 & 0 \end{bmatrix}$ ... (i)

Now, $BC = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2-3 & 0 \\ 3+4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 7 & 0 \end{bmatrix}$

and $A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} -1+14 & 0 \\ 2+7 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 9 & 0 \end{bmatrix}$ ... (ii)

Hence, $(AB)C = A(BC)$ [From (i) and (ii)]

(ii) $B + C = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & -4 \end{bmatrix}$

Now, $A \cdot (B+C) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} 3+4 & 3-8 \\ -6+2 & -6-4 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -4 & -10 \end{bmatrix}$ ... (i)

Also, $AB = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix}$

and $AC = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -3 & 0 \end{bmatrix}$

Now, $AB + AC = \begin{bmatrix} 8 & -5 \\ -1 & -10 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -4 & -10 \end{bmatrix}$ ... (ii)

From (i) and (ii), we get

$A(B + C) = AB + AC$.

34. It is given that A and B are independent events and $P(\bar{A} \cap B) = \frac{2}{15}$

$\Rightarrow$ $P(\bar{A})P(B) = \frac{2}{15}$ ...(i)

Also, $P(A \cap \bar{B}) = \frac{1}{6} \Rightarrow P(A)P(\bar{B}) = \frac{1}{6}$ ...(ii)

Let $p = P(A) \Rightarrow P(\bar{A}) = 1 - P(A) = 1 - p$

and $q = P(B) \Rightarrow P(\bar{B}) = 1 - P(B) = 1 - q$

Now, from (i) and (ii), we get

$(1 - p)q = \frac{2}{15}$ ...(iii)

and $p(1 - q) = \frac{1}{6}$ ...(iv)

Subtracting (iii) from (iv), we get

$$p - q = \frac{1}{30} \Rightarrow p = q + \frac{1}{30}$$

Putting this value of p in (iii), we get

$$\left(1 - q - \frac{1}{30}\right)q = \frac{2}{15} \Rightarrow \frac{29}{30}q - q^2 = \frac{2}{15}$$

$\Rightarrow$ $30q^2 - 29q + 4 = 0 \Rightarrow 30q^2 - 24q - 5q + 4 = 0$

$\Rightarrow$ $6q(5q - 4) - 1(5q - 4) = 0 \Rightarrow (5q - 4)(6q - 1) = 0$

$\Rightarrow$ $q = \frac{4}{5}$ or $\frac{1}{6}$

For $q = \frac{4}{5}$, using (iv), we have

$$p\left(1 - \frac{4}{5}\right) = \frac{1}{6} \Rightarrow p\left(\frac{1}{5}\right) = \frac{1}{6} \Rightarrow p = \frac{5}{6}$$

For $q = \frac{1}{6}$, using (iv), we have

$$p\left(1 - \frac{1}{6}\right) = \frac{1}{6} \Rightarrow p\left(\frac{5}{6}\right) = \frac{1}{6} \Rightarrow p = \frac{1}{5}$$

$\therefore$ $P(A) = \frac{5}{6}, P(B) = \frac{4}{5}$ or $P(A) = \frac{1}{5}, P(B) = \frac{1}{6}$

OR

Let us consider,

A = First patient is corona negative,

B = Second patient is corona negative,

C = Third patient is corona negative,

D = Fourth patient is corona negative,

Clearly, A, B, C, D are independent events, such that

$P(A) = P(B) = P(C) = P(D) = \frac{60}{100} = \frac{3}{5}$

Also, $P(\bar{A}) = P(\bar{B}) = P(\bar{C}) = P(\bar{D}) = 1 - \frac{3}{5} = \frac{2}{5}$

(i) Required Probability $= P[(\bar{A} \cap B \cap C \cap D) \cup (A \cap \bar{B} \cap C \cap D) \cup (A \cap B \cap \bar{C} \cap D) \cup (A \cap B \cap C \cap \bar{D})]$

$= P(\bar{A} \cap B \cap C \cap D) + P(A \cap \bar{B} \cap C \cap D) + P(A \cap B \cap \bar{C} \cap D) + P(A \cap B \cap C \cap \bar{D})]$

$= P(\bar{A})\, P(B)\, P(C)\, P(D) + P(A)\, P(\bar{B})\, P(C)\, P(D) + P(A)\, P(B)\, P(\bar{C})\, P(D) + P(A)\, P(B)\, P(C)\, P(\bar{D})$

$= \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}$

$= \frac{4 \times 2 \times 3 \times 3 \times 3}{5 \times 5 \times 5 \times 5} = \frac{216}{625}$.

(ii) Required Probability $= P(\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D})$

$= P(\bar{A})\, P(\bar{B})\, P(\bar{C})\, P(\bar{D}) = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{16}{625}$

35. $\because$ $f(x)$ is continuous at $x = 0$,

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ ...(i)

Here, $f(0) = a\sin\frac{\pi}{2} = a$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{h\to 0} a\sin\left(\frac{\pi}{2}(-h+1)\right)$

$= \lim_{h\to 0} a\sin\left(\frac{\pi}{2} - h\frac{\pi}{2}\right) = a \quad \lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{\tan h - \sin h}{h^3}$

$= \lim_{h\to 0}\frac{\frac{\sin h}{\cos h} - \sin h}{h^3} = \lim_{h\to 0}\frac{\sin h\left(\frac{1}{\cos h} - 1\right)}{h^3}$

$= \lim_{h\to 0}\frac{\sin h}{h} \times \lim_{h\to 0}\frac{1-\cos h}{\cos h \cdot h^2}$

$= 1 \times \lim_{h\to 0}\frac{1}{\cos h} \times \lim_{h\to 0}\frac{2\sin^2\left(\frac{h}{2}\right)}{4\times\left(\frac{h}{2}\right)^2} = 1\times\frac{1}{2} = \frac{1}{2}$

From (i), $a = \frac{1}{2}$

Hence, $f(x)$ is continuous at $x = 0$, if $a = \frac{1}{2}$.

36. **(i)** Clearly, (1, 1), (2, 2), (3, 3), $\in R$. So, R is reflexive on A.
Since, $(1, 2) \in R$ but $(2, 1) \notin R$. So, R is not symmetric on A.
Since, $(2, 3), \in R$ and $(3, 1) \in R$ but $(2, 1) \notin R$. So, R is not transitive on A.

(ii) Since, (1, 1), (2, 2) and (3, 3) are not in R.
So, R is not reflexive on A.
Now, $(1, 2) \in R \Rightarrow (2, 1) \in R$
and $(1, 3) \in R \Rightarrow (3, 1) \in R$.
So, R is symmetric
Clearly, $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$.
So, R is not transitive on A.

(iii) We have, $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$, where $x, y \in N$.
$\therefore$ $R = \{(1, 6), (2, 7), (3, 8)\}$
Clearly, (1, 1), (2, 2) etc. are not in R. So, R is not reflexive.
Since, $(1, 6) \in R$ but $(6, 1) \notin R$. So, R is not symmetric.
Since, $(1, 6) \in R$ and there is no order pair in R which has 6 as the first element. Same is the case for (2, 7) and (3, 8). So, R is transitive.

OR

We have, $R = \{(x, y) : 3x - y = 0\}$,
where $x, y \in A = \{1, 2, ..., 14\}$
$\therefore$ $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$
Clearly, $(1, 1) \notin R$. So, R is not reflexive on A.
Since, $(1, 3) \in R$ but $(3, 1) \notin R$. So, R is not symmetric on A.
Since, $(1, 3) \in R$ and $(3, 9) \in R$ but $(1, 9) \notin R$. So, R is not transitive on A.

37. We have, $f(x) = \begin{cases} x-3 & , x \ge 3 \\ 3-x & , 1 \le x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & , x < 1 \end{cases}$

(i) $Rf'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$

$= \lim_{h\to 0}\frac{3-(1+h)-2}{h} = \lim_{h\to 0} -\frac{h}{h} = -1$

(ii) $Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0}\frac{-1}{h}\left[\frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} - 2\right]$

$= \lim_{h\to 0}\left(\frac{1+h^2-2h-6+6h+13-8}{-4h}\right)$

$= \lim_{h\to 0}\left(\frac{h^2+4h}{-4h}\right) = -1$

(iii) $Rf'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$

$= \lim_{h\to 0}\frac{3+h-3-0}{h} = \lim_{h\to 0}\frac{h}{h} = 1$

$Lf'(3) = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}$

$= \lim_{h\to 0}\frac{3-(3-h)-0}{-h} = \lim_{h\to 0} -\frac{h}{h} = -1$

$\because$ $Lf'(3) \ne Rf'(3)$

Hence, $f(x)$ is not differentiable at $x = 3$.

OR

Given, $f(x) = \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, x < 1$

$\Rightarrow f'(x) = \frac{2x}{4} - \frac{3}{2} + 0 \Rightarrow f''(x) = \frac{1}{2}$

$\Rightarrow f''\left(\frac{1}{2}\right) = \frac{1}{2}$

38. **(i)** $\overrightarrow{AB} = (2-1)\hat{i} + (1-1)\hat{j} + (3-1)\hat{k} = \hat{i} + 2\hat{k}$

$\therefore |\overrightarrow{AB}| = \sqrt{1^2+2^2} = \sqrt{5}$ units

$\overrightarrow{AC} = (3-1)\hat{i} + (2-1)\hat{j} + (2-1)\hat{k} = 2\hat{i} + \hat{j} + \hat{k}$

$\therefore |\overrightarrow{AC}| = \sqrt{2^2+1^2+1^2} = \sqrt{4+1+1} = \sqrt{6}$ units

(ii) $\overrightarrow{BC} = (3-2)\hat{i} + (2-1)\hat{j} + (2-3)\hat{k} = \hat{i} + \hat{j} - \hat{k}$

and $\overrightarrow{CD} = (3-3)\hat{i} + (3-2)\hat{j} + (4-2)\hat{k} = \hat{j} + 2\hat{k}$

$\therefore \overrightarrow{BC} \times \overrightarrow{CD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\therefore$ Area of $\Delta BCD = \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{CD}| = \frac{1}{2}\sqrt{3^2+2^2+1^2}$

$= \frac{1}{2}\times\sqrt{14} = \frac{\sqrt{14}}{2}$ sq. units

PRACTICE PAPER 3

General Instructions :

1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub parts.

Time Allowed : 3 Hours | **Maximum Marks :** 80

Section A

(Multiple Choice Questions)

Each question carries 1 mark

1. Let R be the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2 \text{ divides } (a - b)\}$. Write the equivalence class [0].
   (a) $\{0, 2\}$ (b) $\{0, 2, 4\}$
   (c) $\{0, 4\}$ (d) $\{0, 1, 4\}$

2. The order and degree of $y = px + \sqrt{a^2p^2 + b^2}$, where $p = \frac{dy}{dx}$, are respectively
   (a) 1, 2 (b) 2, 1
   (c) 3, 1 (d) 1, 3

3. Find the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$.
   (a) 1 (b) 2 (c) 3 (d) 4

4. The value of $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$, is
   (a) $\tan x + \cot x + C$ (b) $\tan x - \cot x + C$
   (c) $\sec x + \tan x + C$ (d) $\sec x - \tan x + C$

5. The direction cosines of the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$, are
   (a) $\frac{2}{7}, \frac{6}{7}, \frac{3}{7}$ (b) $\frac{-2}{7}, \frac{6}{7}, \frac{3}{7}$
   (c) $\frac{2}{7}, \frac{-6}{7}, \frac{3}{7}$ (d) $\frac{2}{7}, \frac{6}{7}, \frac{-3}{2}$

6. The value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$ is
   (a) $\pi/3$ (b) $\pi/2$
   (c) $\pi/6$ (d) $\pi/4$

7. If the vectors $3\hat{i} + 2\hat{j} - \hat{k}$ and $6\hat{i} - 4x\hat{j} + y\hat{k}$ are parallel, then the values of x and y, are respectively
   (a) $-1, -2$ (b) $1, 2$
   (c) $1, -2$ (d) $-1, 2$

8. Let $f : [2, \infty) \to R$ be the function defined by $f(x) = x^2 - 4x + 5$, then find the range of f.
   (a) $(1, \infty)$ (b) $[1, \infty)$
   (c) $(-\infty, 1)$ (d) $(-\infty, 1]$

9. The value of $\int_{\pi/6}^{\pi/4} (\sec^2 x + \text{cosec}^2 x)\, dx$, is
   (a) 2/3 (b) $2/\sqrt{3}$
   (c) 3/2 (d) $3/\sqrt{3}$

10. The random variable X has a probability distribution $P(X)$ of the following form, where 'k' is some number,
$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$
   Determine the value of 'k'.
   (a) 1/3 (b) 1/2 (c) 1/5 (d) 1/6

11. The area of the region bounded between the line $x = 2$ and the parabola $y^2 = 8x$, is
   (a) $\frac{32}{3}$ sq. units (b) $\frac{31}{3}$ sq. units
   (c) $\frac{31}{2}$ sq. units (d) $\frac{33}{2}$ sq. units

12. The integrating factor of the differential equation $\frac{dy}{dx} + (\sec x)y = \tan x$, is
   (a) $|\sec x - \tan x|$ (b) $|\sec x + \cot x|$
   (c) $|\sec x + \tan x|$ (d) $|\sec x|$

13. $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)+4\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{3\pi}{4}$

14. If $y=\tan^{-1}(\sqrt{x})-\tan^{-1}(x)$, then $y'(1)$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) -1 (d) $-\frac{1}{4}$

15. The value of $\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$, is

(a) $\tan x + C$ (b) $\sec x + C$
(c) $\cos x + C$ (d) $\operatorname{cosec} x + C$

16. If $\sqrt{x+y}+\sqrt{y-x}=a$, then $\frac{dy}{dx}=$

(a) $\frac{\sqrt{x+y}-\sqrt{y-x}}{\sqrt{y-x}+\sqrt{x+y}}$

(b) $\frac{2\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$

(c) $\frac{x+y+\sqrt{xy}}{\sqrt{x+y}}$

(d) $\frac{x^2+y^2+2xy}{x^2+y^2}$

17. The function $f(x) = x + \cos x$ is
(a) always increasing
(b) always decreasing
(c) increasing for certain range of x
(d) None of these

18. If $y = ax^2 + b$, then $\frac{dy}{dx}$ at $x = 2$ is equal to

(a) $4a$ (b) $3a$
(c) $2a$ (d) none of these

ASSERTION–REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

19. **Assertion (A) :** The pair of lines given by $\vec{r}=\hat{i}-\hat{j}+\lambda(2\hat{i}+\hat{k})$ and $\vec{r}=2\hat{i}-\hat{k}+\mu(\hat{i}+\hat{j}-\hat{k})$ intersect.

Reason (R) : Two lines intersect each other, if they are not parallel and shortest distance = 0.

20. Let $H_1, H_2, \ldots, H_n$ be mutually exclusive and exhaustive events with $P(H_i) > 0$, $i = 1, 2, \ldots, n$.

Let E be any other event with $0 < P(E) < 1$

Assertion (A) : $P(H_i / E) > P(E / H_i) \times P(H_i)$ for $i = 1, 2, \ldots, n$

Reason (R) : $\sum_{i=1}^{n} P(H_i)=1.$

Section B

This section comprises of very short answer type questions (VSA) of 2 marks each.

21. Find cofactors of a_{21} and a_{31} of the matrix

$$A=[a_{ij}]=\begin{bmatrix}1 & 3 & -2\\ 4 & -5 & 6\\ 3 & 5 & 2\end{bmatrix}.$$

OR

If $A=\begin{bmatrix}1 & 3\\ 3 & 10\end{bmatrix}$, then find A^{-1}.

22. Find the order and degree of the differential equation given by

$$\begin{vmatrix} x^3 & y^2 & 3\\ 2x^2 & 3y\frac{dy}{dx} & 0\\ 5x & 2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right) & 0\end{vmatrix}=0.$$

23. Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b}=-\hat{i}+3\hat{k}$.

OR

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ having the same length $\sqrt{2}$ and their scalar product is -1.

24. Find the equation of a line passing through the point $(-3, 2, -4)$ and equally inclined to the axes.

25. A coin is tossed and then a die is thrown. Find the probability of obtaining a '6' given that head came up.

Section C

This section comprises of short answer type questions (SA) of 3 marks each.

26. Evaluate : $\int\frac{x^2+9}{x^4+81}dx$

27. If $\begin{bmatrix}x+3 & z+4 & 2y-7\\ 4x+6 & a-1 & 0\\ b-3 & 3b & z+2c\end{bmatrix}=\begin{bmatrix}0 & 6 & 3y-2\\ 2x & -3 & 2c+2\\ 2b+4 & -21 & 0\end{bmatrix}$,

then find the values of a, b, c, x, y and z.

OR

Find x, y, a and b if $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$.

28. Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY and $|\overrightarrow{OA}| = 10$ units.

OR

Using direction numbers, show that the points $A(-2, 4, 7)$, $B(3, -6, -8)$ and $C(1, -2, -2)$ are collinear.

29. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$, respectively of a ΔABC. Find the length of the median through A.

30. If $(ax + b)\, e^{y/x} = x$, then show that $x^3 \dfrac{d^2y}{dx^2} = \left(x\dfrac{dy}{dx} - y\right)^2$.

OR

Find $\dfrac{dy}{dx}$, when $x = a\left\{\cos t + \dfrac{1}{2}\log \tan^2 \dfrac{t}{2}\right\}$ and $y = a \sin t$.

31. Solve the differential equation :

$(1+x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$.

Section D

This section comprises of long answer type questions (LA) of 5 marks each.

32. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, then find AB. Hence, solve the system of equations :

$x - y = 6,\ 2x + 3y + 4z = 34,\ y + 2z = 14$

33. Solve the following LPP graphically.

Maximize $Z = 50x + 40y$

Subject to constraints :

$1000x + 1200y \le 7600$

$12x + 8y \le 72$

$x, y \ge 0$

OR

Solve the following LPP graphically.

Minimize $Z = 5x + 7y$

Subject to constraints :

$2x + y \le 8$

$x + 2y \ge 10$ and $x, y \ge 0$

34. Evaluate : $\displaystyle\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\,dx$

OR

Evaluate : $\displaystyle\int_1^2 \frac{dx}{x(1+x^2)}$

35. An open box with a square base is to be made out of a given quantity of cardboard of area c^2 square units. Show that the maximum volume of the box is $\dfrac{c^3}{6\sqrt{3}}$ cubic units.

Section E

This section comprises of 3 case-study/passage-based questions of 4 marks each. First two case study questions have three sub-parts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two sub-parts of 2 marks each.

36. **Case-Study 1 :** Read the following passage and answer the questions given below.

Neelam and Ved appeared for first round of an interview for two vacancies. The probability of Neelam's selection is 1/6 and that of Ved's selection is 1/4.

(i) Find the probability that both of them are selected.

(ii) Find the probability that none of them is selected.

(iii) Find the probability that only one of them is selected.

OR

Find the probability that atleast one of them is selected.

37. **Case-Study 2 :** Read the following passage and answer the questions given below.

Sonam wants to prepare a sweet box for Diwali at home. For making lower part of box, she takes a square piece of cardboard of side 18 cm.

(i) If x cm be the length of each side of the square cardboard which is to be cut off from corner of the square piece of side 18 cm, then find the value of x.

(ii) Find the expression for the volume of the open box formed by folding up the cutting corners.

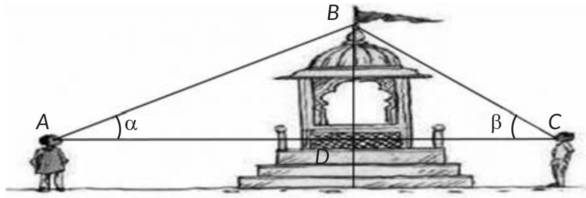
(iii) Find the value(s) of x for which $\frac{dV}{dx} = 0$.

OR

Sonam is interested in maximising the volume of the box. So, what should be the side of the square to be cut off so that the volume of the box is maximum?

38. **Case-Study 3 :** Read the following passage and answer the questions given below.

Two men on either side of a temple, which is 30 metres high above the stairs, observe its top at the angles of elevation α and β respectively (as shown in the figure). It is known that $\alpha = \sin^{-1}\left(\frac{1}{2}\right)$ and $\beta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.



(i) Find the angle B.

(ii) Suppose C moves by certain distance then angle β changes to $\tan^{-1}x$. Find domain and range of $\tan^{-1}x$.

Detailed SOLUTIONS

1. (b) : Here, $R = \{(a, b) \in A \times A : 2 \text{ divides } (a - b)\}$, which is an equivalence relation, where $A = \{0, 1, 2, 3, 4, 5\}$.

Clearly, $[0] = \{a \in A : aR0\}$

$= \{a \in A : 2 \text{ divides } (a - 0)\} = \{a \in A : 2 \text{ divides } a\} = \{0, 2, 4\}$

$\therefore$ Equivalence class [0] is {0, 2, 4}.

2. (a) : Given, $y - px = \sqrt{a^2p^2 + b^2}$

$\Rightarrow (y - px)^2 = a^2p^2 + b^2 \Rightarrow (x^2 - a^2)p^2 - 2xyp + (y^2 - b^2) = 0$

$\Rightarrow (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + (y^2 - b^2) = 0$

Hence, order is 1 and degree is 2.

3. (b) : Given, $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$

Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are $\pm\left(\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}\right)$.

So, there are two unit vectors perpendicular to the given vectors.

4. (a) : We have,

$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int (\sec^2 x - \operatorname{cosec}^2 x) dx = \tan x + \cot x + C$

5. (c) : The given equation of the line is

$\frac{4 - x}{2} = \frac{y}{6} = \frac{1 - z}{3}$ ...(i)

$\Rightarrow \frac{x - 4}{-2} = \frac{y}{6} = \frac{z - 1}{-3} \Rightarrow \frac{x - 4}{2} = \frac{y}{-6} = \frac{z - 1}{3}$

Now, as $\sqrt{2^2 + (-6)^2 + 3^2} = 7$

$\therefore$ D.c's. of (i) are $\frac{2}{7}, \frac{-6}{7}, \frac{3}{7}$.

6. (d) : We have, $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\left(\frac{1}{2}\right)\right)\right\}$

$= \tan^{-1}\left\{2\cos\left(2 \times \frac{\pi}{6}\right)\right\}$ $\left[\because \sin^{-1}\frac{1}{2} = \frac{\pi}{6}\right]$

$= \tan^{-1}\left\{2\cos\frac{\pi}{3}\right\} = \tan^{-1}\left[2 \times \frac{1}{2}\right] = \tan^{-1} 1 = \frac{\pi}{4}$

7. (a) : Let $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 6\hat{i} - 4x\hat{j} + y\hat{k}$

Since, $\vec{a}$ and $\vec{b}$ are parallel $\therefore$ $\vec{a} = m\vec{b}$, for some $m \in R$

$\Rightarrow 3\hat{i} + 2\hat{j} - \hat{k} = m(6\hat{i} - 4x\hat{j} + y\hat{k}) \Rightarrow 3 = 6m \Rightarrow m = \frac{1}{2}$

Also, $-4xm = 2 \Rightarrow \frac{-4x}{2} = 2 \Rightarrow x = -1$

and $ym = -1 \Rightarrow \frac{y}{2} = -1 \Rightarrow y = -2$

8. (b) : Given, $f(x) = x^2 - 4x + 5$

Let $y = x^2 - 4x + 5 \Rightarrow y = (x - 2)^2 + 1$

$\Rightarrow (x - 2)^2 = y - 1 \Rightarrow x - 2 = \sqrt{y - 1}$ $[\because x \in [2, \infty)]$

$\Rightarrow x = \sqrt{y - 1} + 2$

For range $y - 1 \geq 0 \Rightarrow y \geq 1$ $\therefore$ Range is $[1, \infty)$.

9. (b) : $\int_{\pi/6}^{\pi/4} (\sec^2 x + \operatorname{cosec}^2 x) dx = [\tan x - \cot x]_{\pi/6}^{\pi/4}$

$= (1 - 1) - \left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = \frac{2}{\sqrt{3}}$

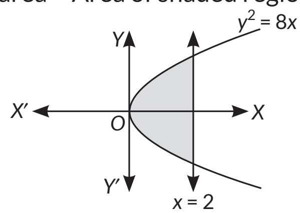
10. (d) : We have, $P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$

Since, $\Sigma P(x_i) = 1$, therefore $k + 2k + 3k = 1$

$\Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$

11. (a) : We have, $y^2 = 8x$ and $x = 2$

$\therefore$ Required area = Area of shaded region



$= 2 \cdot \int_0^2 \sqrt{8x}\, dx = 4\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2 = \frac{32}{3}$ sq. units.

12. (c) : Given, $\frac{dy}{dx}+(\sec x)y=\tan x$, which is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$

Here, $P = \sec x$ and $Q = \tan x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \sec x\, dx} = e^{\log|\sec x+\tan x|} = |\sec x + \tan x|$

13. (c) : $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)+4\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$=\frac{\pi}{3}+2\cdot\frac{\pi}{6}+4\cdot\frac{\pi}{6}=\frac{\pi}{3}+\frac{\pi}{3}+\frac{2\pi}{3}=\frac{4\pi}{3}$

14. (d) : Given, $y=\tan^{-1}(\sqrt{x})-\tan^{-1}(x)$

Differentiating w.r.t. x, we get

$y'=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}-\frac{1}{1+x^2} \Rightarrow y'(1)=\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}=-\frac{1}{4}$

15. (a) : Let $I=\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$

$=\int\frac{\cos^2 x-\sin^2 x+2\sin^2 x}{\cos^2 x}dx$

$=\int\frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx = \int\frac{1}{\cos^2 x}dx=\int\sec^2 x dx = \tan x + C$

16. (a) : We have, $\sqrt{x+y}+\sqrt{y-x}=a$

On differentiating w.r.t. x, we get

$\frac{1}{2\sqrt{x+y}}\left(1+\frac{dy}{dx}\right)+\frac{1}{2\sqrt{y-x}}\left(\frac{dy}{dx}-1\right)=0$

$\Rightarrow \frac{1}{\sqrt{x+y}}-\frac{1}{\sqrt{y-x}}=-\left(\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y-x}}\right)\frac{dy}{dx}$

$\Rightarrow \frac{-(\sqrt{y-x}-\sqrt{x+y})}{(\sqrt{y-x}+\sqrt{x+y})}=\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{\sqrt{x+y}-\sqrt{y-x}}{\sqrt{y-x}+\sqrt{x+y}}$

17. (a) : Given, $f(x) = x + \cos x$

On differentiating, we get $f'(x) = 1 - \sin x$

$f'(x) > 0$ for all values of x ($\because$ $\sin x$ is lying between -1 to 1)

$\therefore$ $f(x)$ is always increasing.

18. (a) : We have, $y = ax^2 + b \Rightarrow \frac{dy}{dx} = 2ax$

$\left[\frac{dy}{dx}\right]_{x=2} = 2a \times 2 = 4a$

19. (a) : Here, $\vec{a}_1=\hat{i}-\hat{j}, \vec{b}_1=2\hat{i}+\hat{k}$

$\vec{a}_2=2\hat{i}-\hat{k}$ and $\vec{b}_2=\hat{i}+\hat{j}-\hat{k}$

Now, $\vec{a}_2-\vec{a}_1=(2\hat{i}-\hat{k})-(\hat{i}-\hat{j})=\hat{i}+\hat{j}-\hat{k}$

and $\vec{b}_1\times\vec{b}_2=-\hat{i}+3\hat{j}+2\hat{k}$

$\Rightarrow |\vec{b}_1\times\vec{b}_2|=\sqrt{(-1)^2+(3)^2+(2)^2}=\sqrt{1+9+4}=\sqrt{14}$

$\therefore$ S.D. $=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right| = \left|\frac{(\hat{i}+\hat{j}-\hat{k})\cdot(-\hat{i}+3\hat{j}+2\hat{k})}{\sqrt{14}}\right| = 0$

Hence, two lines intersect each other.

Two lines intersect each other, if they are not parallel and shortest distance = 0.

20. (d) : $P(H_i / E) > P(E / H_i) \times P(H_i)$

$\Rightarrow \frac{P(H_i\cap E)}{P(E)} > P(E\cap H_i) \Rightarrow P(H_i\cap E)(1 - P(E)) > 0$

$\Rightarrow P(H_i \cap E) > 0$

This leads to a contradiction $0 > 0$ if $H_i \cap E = \phi$ for any i.

21. Let M_{ij} and C_{ij} respectively denote the minor and cofactor of element a_{ij} in A. Then,

$M_{21}=\begin{vmatrix}3 & -2\\5 & 2\end{vmatrix}=6+10=16 \Rightarrow C_{21}=-M_{21}=-16$

$M_{31}=\begin{vmatrix}3 & -2\\-5 & 6\end{vmatrix}=18-10=8 \Rightarrow C_{31}=M_{31}=8$

OR

Let $A=\begin{bmatrix}1 & 3\\3 & 10\end{bmatrix}$ $\therefore$ $|A| = 10 - 9 = 1 \neq 0$

So, A^{-1} exists.

Now, adj $A=\begin{bmatrix}10 & -3\\-3 & 1\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{|A|}(\text{adj } A)=\begin{bmatrix}10 & -3\\-3 & 1\end{bmatrix}$.

22. We have,

$$\begin{vmatrix}x^3 & y^2 & 3\\ 2x^2 & 3y\frac{dy}{dx} & 0\\ 5x & 2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right) & 0\end{vmatrix}=0$$

Expanding along C_3, we get

$3\left[4x^2\left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}-15xy\frac{dy}{dx}\right]=0$

$\Rightarrow 4x^2y\frac{d^2y}{dx^2}+4x^2\left(\frac{dy}{dx}\right)^2-15xy\frac{dy}{dx}=0$

Now, highest order derivative is $\frac{d^2y}{dx^2}$. So, its order is 2 and degree is 1.

23. We are given, $\vec{a}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b}=-\hat{i}+3\hat{k}$

$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\1 & 3 & -2\\-1 & 0 & 3\end{vmatrix}$

$=(9-0)\hat{i}-(3-2)\hat{j}+(0+3)\hat{k}=9\hat{i}-\hat{j}+3\hat{k}$

$\therefore |\vec{a}\times\vec{b}|=\sqrt{9^2+(-1)^2+3^2}=\sqrt{81+1+9}=\sqrt{91}$

OR

Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$.

We have, $|\vec{a}|=|\vec{b}|=\sqrt{2}$ and $\vec{a}\cdot\vec{b}=-1$

$\therefore \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \cos\theta=\frac{-1}{\sqrt{2}\times\sqrt{2}}=-\frac{1}{2}$

$\Rightarrow \cos\theta=\cos\frac{2\pi}{3} \Rightarrow \theta=\frac{2\pi}{3}$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

24. Since, the line is equally inclined to the axes.

$\therefore \quad l = m = n$ ...(i)

The required equation of line is

$$\frac{x+3}{l} = \frac{y-2}{l} = \frac{z+4}{l} \quad \text{[using (i)]}$$

$$\Rightarrow \frac{x+3}{1} = \frac{y-2}{1} = \frac{z+4}{1} \Rightarrow x+3 = y-2 = z+4$$

25. The sample space S associated to the given random experiment is given by

$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$ and

Let the event $B = \{(H, 6), (T, 6)\}$ and

$A = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)\}$

$$\therefore \quad \text{Required probability} = P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{12}}{\frac{6}{12}} = \frac{1}{6}$$

26. Let $I = \int \frac{x^2+9}{x^4+81}dx \Rightarrow I = \int \frac{1+9/x^2}{x^2+\frac{81}{x^2}}dx$

$$\Rightarrow I = \int \frac{1+9/x^2}{x^2+\left(\frac{9}{x}\right)^2 - 18 + 18}dx = \int \frac{1+9/x^2}{\left(x-\frac{9}{x}\right)^2+18}dx$$

Put $x - \frac{9}{x} = t \Rightarrow \left(1+\frac{9}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2+18} \Rightarrow I = \int \frac{dt}{t^2+(3\sqrt{2})^2}$$

$$\Rightarrow I = \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{t}{3\sqrt{2}}\right)+c$$

$$\Rightarrow I = \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x^2-9}{3\sqrt{2}x}\right)+c$$

27. We have,

$$\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$$

Equating the corresponding elements of two matrices, we get

$x + 3 = 0 \Rightarrow x = -3$

$z + 4 = 6 \Rightarrow z = 2$

$2y - 7 = 3y - 2 \Rightarrow y = -5$

$a - 1 = -3 \Rightarrow a = -2$

$b - 3 = 2b + 4 \Rightarrow b = -7$

$2c + 2 = 0 \Rightarrow c = -1$

Thus, $a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$.

OR

We have, $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$

Equating the corresponding elements of two matrices, we get

$3x + 4y = 2$ ... (i)

$x - 2y = 4$ ... (ii)

$a + b = 5$ ... (iii)

$2a - b = -5$ ... (iv)

Multiplying (ii) by 2 and then adding to (i), we get

$3x + 4y + 2x - 4y = 2 + 8$

$\Rightarrow \quad 5x = 10 \Rightarrow x = 2$

From (i), $3(2) + 4y = 2 \Rightarrow 4y = -4 \Rightarrow y = -1$

Solving (iii) and (iv), we get

$3a = 0 \Rightarrow a = 0$

From (iii), $b = 5$

Hence, $x = 2, y = -1, a = 0, b = 5$

28. We have, $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY. If $\overrightarrow{OA}$ makes angle γ with OZ.

Then, $\cos^2 60° + \cos^2 45° + \cos^2\gamma = 1 \quad [\because l^2+m^2+n^2=1]$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$$

$$\Rightarrow \frac{1}{4}+\frac{1}{2}+\cos^2\gamma = 1 \Rightarrow \cos^2\gamma = 1-\left(\frac{1}{2}+\frac{1}{4}\right)$$

$$\Rightarrow \cos^2\gamma = \frac{1}{4} \Rightarrow \cos\gamma = \frac{1}{2} = \cos 60° \quad \therefore \gamma = 60°$$

Now, $\overrightarrow{OA} = |\overrightarrow{OA}|\left(\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{2}\hat{k}\right)$

$$= 10\left(\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{2}\hat{k}\right) = 5\hat{i}+5\sqrt{2}\hat{j}+5\hat{k} \quad [\because |\overrightarrow{OA}| = 10]$$

OR

Direction numbers of the line AB are

$< 3 + 2, -6 - 4, -8 - 7 >$ *i.e.*, $< 5, -10, -15 >$ *i.e.*, $< 1, -2, -3 >$

and direction numbers of the line AC are

$< 1 + 2, -2 - 4, -2 - 7 >$ *i.e.*, $< 3, -6, -9 >$ *i.e.*, $< 1, -2, -3 >$.

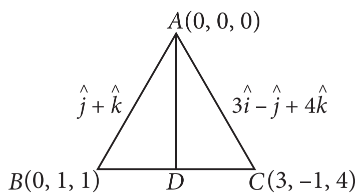
Clearly, direction numbers of the two lines AB and AC are proportional, therefore these lines are parallel.

But the lines AB and AC have a common point A.

Hence, the points A, B and C are collinear.

29. Take A to be as origin (0, 0, 0).

$\therefore$ Coordinates of B are (0, 1, 1) and coordinates of C are (3, −1, 4).



Let D be the mid point of BC and AD is a median of ΔABC.

$\therefore$ Coordinates of D are $\left(\frac{3}{2}, 0, \frac{5}{2}\right)$

$$\text{So, length of } AD = \sqrt{\left(\frac{3}{2}-0\right)^2+(0)^2+\left(\frac{5}{2}-0\right)^2}$$

$$= \sqrt{\frac{9}{4}+\frac{25}{4}} = \frac{\sqrt{34}}{2} \text{ units.}$$

30. Given, $(ax + b)e^{y/x} = x \Rightarrow e^{y/x} = \frac{x}{ax+b}$

Taking log on both sides, we get

$$\frac{y}{x}\cdot\log e = \log\frac{x}{ax+b}$$

$\Rightarrow \quad \frac{y}{x} = \log x - \log(ax+b)$ $\qquad (\because \log e = 1)$

Differentiating w.r.t. x, we get

$$\frac{x\cdot\frac{dy}{dx} - y\cdot 1}{x^2} = \frac{1}{x} - \frac{1}{ax+b}\cdot a \Rightarrow x\frac{dy}{dx} - y = x^2\cdot\frac{ax+b-ax}{x(ax+b)}$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = \frac{bx}{ax+b} \qquad ...(i)$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 - \frac{dy}{dx} = \frac{(ax+b)\cdot b - bx\cdot a}{(ax+b)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{b^2}{(ax+b)^2} \Rightarrow x^3\frac{d^2y}{dx^2} = \left(\frac{bx}{ax+b}\right)^2$$

$$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2 \qquad \text{(Using (i))}$$

OR

We have, $x = a\left\{\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right\}$ and $y = a\sin t$

$$\Rightarrow \quad x = a\left\{\cos t + \frac{1}{2}\cdot 2\log\tan\frac{t}{2}\right\} \Rightarrow x = a\left\{\cos t + \log\tan\frac{t}{2}\right\}$$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\tan t/2}\sec^2\frac{t}{2}\cdot\frac{1}{2}\right\} \text{ and } \frac{dy}{dt} = a\cos t$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{2\sin(t/2)\cos(t/2)}\right\}$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\sin t}\right\} \Rightarrow \frac{dx}{dt} = a\left\{\frac{-\sin^2 t + 1}{\sin t}\right\}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \tan t$$

31. We have, $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$ where $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$$

So, the required solution is given by

$$y(1+x^2) = \int\frac{4x^2}{1+x^2}(1+x^2)\,dx + C$$

$$\Rightarrow \quad y(1+x^2) = 4\int x^2 dx + C \Rightarrow y(1+x^2) = \frac{4x^3}{3} + C$$

Given that $y(0) = 0$

$\therefore \quad 0(1+0) = 0 + C \Rightarrow C = 0$

Thus, $y = \frac{4x^3}{3(1+x^2)}$ is the required solution.

32. $AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I_3$$

$$\Rightarrow \quad A\left(\frac{1}{6}B\right) = I_3 \Rightarrow A^{-1} = \frac{1}{6}B \qquad \text{(By definition of inverse)}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

The given system of equations is

$x - y + 0z = 6$

$2x + 3y + 4z = 34$

$0x + y + 2z = 14$

This system of equations can be written as

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 34 \\ 14 \end{bmatrix} \text{ or } AX = C$$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 6 \\ 34 \\ 14 \end{bmatrix}$

As A^{-1} exists, therefore $X = A^{-1}C$

$$\Rightarrow \quad X = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 34 \\ 14 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12+68-56 \\ -24+68-56 \\ 12-34+70 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 24 \\ -12 \\ 48 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 8 \end{bmatrix}$$

$\Rightarrow \quad x = 4, y = -2, z = 8$

Hence, the solution of the given system of equations is $x = 4, y = -2, z = 8$.

33. The given problem can be written as

Maximize $Z = 50x + 40y$

subject to constraints :

$5x + 6y \le 38$

$3x + 2y \le 18$

$x \ge 0, y \ge 0$

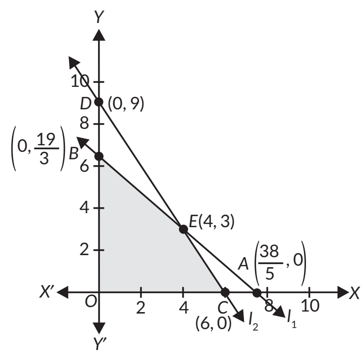
Now, let us draw the lines

$l_1 : 5x + 6y = 38$

or $\frac{x}{\left(\frac{38}{5}\right)} + \frac{y}{\left(\frac{19}{3}\right)} = 1$

$l_2 : 3x + 2y = 18$

or $\frac{x}{6} + \frac{y}{9} = 1$



Lines l_1 and l_2 intersect at $E(4, 3)$.
The shaded region $OCEB$ is the feasible region which is bounded.

Corner points of the feasible region are $O(0, 0)$, $C(6, 0)$, $E(4, 3)$ and $B\left(0, \frac{19}{3}\right)$.

The value of the objective function $Z = 50x + 40y$ at corner points are given below:
At $O(0, 0)$, $Z = 50 \times 0 + 40 \times 0 = 0$
At $C(6, 0)$, $Z = 50 \times 6 + 40 \times 0 = 300$
At $E(4, 3)$, $Z = 50 \times 4 + 40 \times 3 = 320$ (Maximum)
At $B\left(0, \frac{19}{3}\right)$, $Z = 50 \times 0 + 40 \times \frac{19}{3} = 253.33$
Clearly, the maximum value is 320 at point $E(4, 3)$.

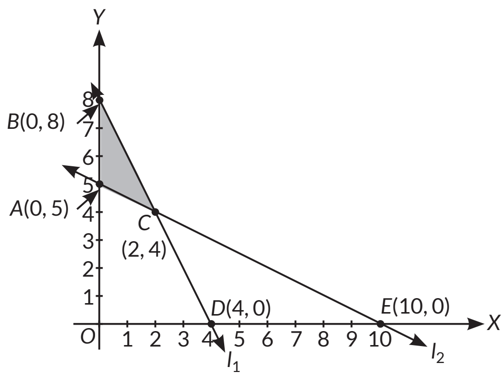
OR

The given problem is minimize $Z = 5x + 7y$
subject to constraints : $2x + y \le 8$
$x + 2y \ge 10$ and $x, y \ge 0$
To solve this LPP graphically, we first convert the inequations into equations to obtain the following line

$l_1 : 2x + y = 8$, or $\frac{x}{4} + \frac{y}{8} = 1$

$l_2 : x + 2y = 10$, or $\frac{x}{10} + \frac{y}{5} = 1$

Lines l_1 and l_2 intersect at $C(2, 4)$.



The coordinates of the corner points of the feasible region ABC are $A(0, 5)$, $B(0, 8)$ and $C(2, 4)$.
The values of the objective function $Z = 5x + 7y$ at the corner points of the feasible region are given in the following table.

Corner Points	Value of $Z = 5x + 7y$
$A(0, 5)$	$5 \times 0 + 7 \times 5 = 35$ (Minimum)
$B(0, 8)$	$5 \times 0 + 7 \times 8 = 56$
$C(2, 4)$	$5 \times 2 + 7 \times 4 = 38$

Thus, Z is minimum at point A (0, 5).

34. Let $I = \int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x) dx$

$= \int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x) dx$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

Also, when $x = 0$, then $\Rightarrow t = 0$ and when $x = \frac{\pi}{2}$, then $t = 1$

$\therefore\ I = \int_0^1 2t \cdot \tan^{-1} t\, dt = 2\int_0^1 t \cdot \tan^{-1} t\, dt$

Integrating by parts, we have

$$I = 2\left[\frac{t^2}{2}\tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{t^2}{2(1+t^2)} dt$$

$$= 2\left[\frac{1}{2}\tan^{-1} 1 - 0\right] - \int_0^1 \frac{t^2}{(1+t^2)} dt$$

$$= 2\left[\frac{1}{2} \times \frac{\pi}{4}\right] - \int_0^1 \frac{t^2+1-1}{(1+t^2)} dt = \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{(1+t^2)}\right) dt$$

$$= \frac{\pi}{4} - [t]_0^1 + [\tan^{-1} t]_0^1 = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

OR

Let $I = \int_1^2 \frac{dx}{x(1+x^2)}$

Consider, $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx + C}{1+x^2}$

$\Rightarrow\ 1 = A(1+x^2) + (Bx + C) \cdot x \Rightarrow 1 = x^2(A + B) + Cx + A$

On equating the coefficients of x^2, x and the constant term from both sides, we get $A = 1$, $B = -1$ and $C = 0$

$\therefore\ I = \int_1^2 \frac{1}{x} dx + \int_1^2 \frac{-x}{1+x^2} dx$

$$= [\log x]_1^2 - \int_1^2 \frac{x}{1+x^2} dx = \log 2 - \int_1^2 \frac{x\, dx}{1+x^2}$$

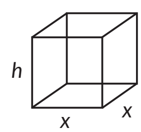
Put $1 + x^2 = t \Rightarrow 2x\, dx = dt$
When $x = 1$, $t = 2$ and when $x = 2$, $t = 5$

$\therefore\ I = \log 2 - \frac{1}{2}\int_2^5 \frac{1}{t} dt = \log 2 - \frac{1}{2}[\log t]_2^5$

$$= \log 2 - \frac{1}{2}[\log 5 - \log 2] = \log 2 - \frac{1}{2}\log\left(\frac{5}{2}\right) = \frac{1}{2}\log\left(\frac{8}{5}\right)$$

35. Let h be height and x be the side of the square base of the open box.

Then, its area $= x^2 + 4hx = c^2$ (given)



$\Rightarrow\ h = \frac{c^2 - x^2}{4x}$

Now, V = volume of the box

$$= x^2 h = x^2 \cdot \frac{c^2 - x^2}{4x} = \frac{1}{4}(c^2 x - x^3)$$

$\Rightarrow\ \frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2)$ and $\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2}x$

For maxima or minima, put $\frac{dV}{dx} = 0 \Rightarrow x^2 = \frac{c^2}{3}$

$\Rightarrow\ x = \frac{c}{\sqrt{3}}$ ($\because x$ can't be negative)

For this value of x, $\frac{d^2V}{dx^2} < 0$

$\Rightarrow$ V is maximum at $x = \frac{c}{\sqrt{3}}$ and its maximum volume is,

$$V = \frac{1}{4}x(c^2 - x^2) = \frac{1}{4}\cdot\frac{c}{\sqrt{3}}\left(c^2 - \frac{c^2}{3}\right) = \frac{c^3}{6\sqrt{3}} \text{ cubic units.}$$

36. Let A be the event that Neelam is selected and B be the event that Ved is selected. Then, we have, $P(A) = \frac{1}{6}$

$\Rightarrow$ $P(\bar{A}) = 1 - \frac{1}{6} = \frac{5}{6} = P$ (Neelam is not selected)

$P(B) = \frac{1}{4}$

$\Rightarrow$ $P(\bar{B}) = 1 - \frac{1}{4} = \frac{3}{4} = P$ (Ved is not selected)

(i) P (both are selected) $= P(A \cap B) = P(A) \cdot P(B)$

$$= \frac{1}{6} \times \frac{1}{4} = \frac{1}{24}$$

(ii) P (both are rejected) $= P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

$$= \frac{5}{6} \times \frac{3}{4} = \frac{5}{8}$$

(iii) P (only one of them is selected)

$= P(A \cap \bar{B}) + P(\bar{A} \cap B) = P(A) \cdot P(\bar{B}) + P(\bar{A}) \cdot P(B)$

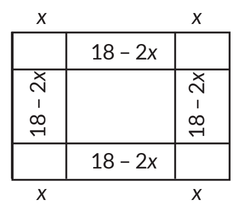
$$= \frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4} = \frac{3}{24} + \frac{5}{24} = \frac{8}{24} = \frac{1}{3}$$

OR

P (at least one of them is selected)

$= 1 - P$ (Both are rejected) $= 1 - \frac{5}{8} = \frac{3}{8}$

37. (i) Since, side of square is of length 18 cm, therefore $x \in (0, 9)$.



(ii) Clearly, height of open box = x cm

Length of open box = $18 - 2x$

and width of open box = $18 - 2x$

$\therefore$ Volume (V) of the open box = $x \times (18 - 2x) \times (18 - 2x)$

(iii) We have, $V = x(18 - 2x)^2$

$$\therefore \quad \frac{dV}{dx} = x \cdot 2(18 - 2x)(-2) + (18 - 2x)^2$$

$$= (18 - 2x)(-4x + 18 - 2x) = (18 - 2x)(18 - 6x)$$

Now, $\frac{dV}{dx} = 0$

$\Rightarrow$ $18 - 2x = 0$ or $18 - 6x = 0 \Rightarrow x = 9$ or 3

Here $x = 9$ is not possible so, $x = 3$.

OR

We have, $V = x(18 - 2x)^2$

and $\frac{dV}{dx} = (18 - 2x)(18 - 6x)$

$$\Rightarrow \frac{d^2V}{dx^2} = (18 - 2x)(-6) + (18 - 6x)(-2)$$

$$= (-2)[54 - 6x + 18 - 6x]$$

$$= (-2)[72 - 12x] = 24x - 144$$

For $x = 3$, $\frac{d^2V}{dx^2} < 0$

So, volume will be maximum when $x = 3$.

38. (i) Here, $\alpha = \sin^{-1}\frac{1}{2} \Rightarrow \sin\alpha = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow \alpha = 30°$

and $\beta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \tan\beta = \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) \Rightarrow \beta = 30°$

$\therefore \angle ABC = 180° - (30° + 30°) = 120°$

(ii) Domain and range of $\tan^{-1}x$ are $(-\infty, \infty)$ and $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ respectively.

PRACTICE PAPER 4

General Instructions :

1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub parts.

Time Allowed : 3 hours **Maximum Marks :** 80

Section A

(Multiple Choice Questions)

Each question carries 1 mark

1. If A and B are two independent events such that $P(A \cup B) = 0.6$ and $P(A) = 0.2$, then find $P(B)$.
(a) 0.2 (b) 0.3 (c) 0.5 (d) 0.7

2. Find the solution of the differential equation $\frac{dy}{dx} = x^2 e^{-y}$
(a) $3e^y = x^3 + C$ (b) $3e^y = x^{-3} + C$
(c) $2e^{2y} = x^2 + C$ (d) $2e^{2y} = x^5 + C$

3. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, then find the value of 'a' and 'b'.
(a) $a = 2$ and $b = 2$ (b) $a = -2$ and $b = 3$
(c) $a = 2$ and $b = -3$ (d) $a = 2$ and $b = -2$

4. Differentiate the function $\frac{2\tan x}{\tan x + \cos x}$ w.r.t. x.
(a) $\frac{2\tan x(\sec x + \tan x)}{(\tan x + \cos x)^2}$
(b) $\frac{2(\sec x + \tan x \cdot \sin x)}{(\tan x + \cos x)^2}$
(c) $\frac{8\tan x(\sec x + \tan x)}{(\tan x + \cos x)^2}$
(d) $\frac{8\tan x(\sec x + \tan x \sin x)}{\tan x + \cos x}$

5. Evaluate : $\int \frac{\sqrt{x}}{\sqrt{x^2 - x}} dx$
(a) $2\sqrt{x} + C$ (b) $2\sqrt{x-1} + C$
(c) $2\sqrt{x+1} + C$ (d) $2\sqrt{x-2} + C$

6. If A and B are two events such that $P(A) = 0.2$, $P(B) = 0.4$ and $P(A \cup B) = 0.5$, then find the value of $P(A/B)$.
(a) 0.15 (b) 0.25 (c) 0.35 (d) 0.45

7. The function $f(x) = -\tan^{-1} x + x$ is increasing on
(a) $(-1, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty)$ (d) None of these

8. The area between the curve $y = 4 + 3x - x^2$ and x-axis is
(a) $\frac{125}{6}$ sq. units (b) $\frac{125}{3}$ sq. units
(c) $\frac{125}{2}$ sq. units (d) none of these

9. Find the solution of $y' = y \cot 2x$.
(a) $y = c\sqrt{\sin x}$ (b) $y = c\sqrt{\sin 3x}$
(c) $y = c\sqrt{\cos x}$ (d) $y = c\sqrt{\sin 2x}$

10. If $f(x) = x^2 - 4x + 1$, find $f(A)$, where $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.
(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

11. $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx$ is equal to
(a) $10^x - x^{10} + C$ (b) $10^x + x^{10} + C$
(c) $(10^x - x^{10})^{-1} + C$ (d) $\log_e(10^x + x^{10}) + C$

12. The function $f(x) = |x - 4|$ is
(a) continuous at $x = 4$
(b) differentiable at $x = 4$
(c) not differentiable at $x = 4$
(d) Both (a) and (c)

13. Area bounded by the curves $y = \sin x$, the line $x = 0$ and the line $x = \frac{\pi}{2}$, is equal to

(a) π sq. units (b) 1 sq. unit

(c) $\frac{\pi}{2}$ sq. units (d) 2 sq. units

14. Solve the differential equation $\frac{dy}{dx} = 1 - x + y - xy$.

(a) $\log|1+y| = \frac{x^2}{2} + C$ (b) $\log|1+y| = x + \frac{x^2}{2} + C$

(c) $\log|1+y| = -x + \frac{x^2}{2} + C$ (d) $\log|1+y| = x - \frac{x^2}{2} + C$

15. A bag contains 3 white and 6 black balls while another bag contains 6 white and 3 black balls. A bag is selected at random and a ball is drawn. Find the probability that the ball drawn is of white colour.

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

16. $2x^3 - 6x + 5$ is an increasing function, if

(a) $0 < x < 1$ (b) $-1 < x < 1$

(c) $x < -1$ or $x > 1$ (d) $-1 < x < -\frac{1}{2}$

17. Let $y = t^{10} + 1$ and $x = t^8 + 1$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{5}{2}t$ (b) $20t^8$

(c) $\frac{5}{16t^6}$ (d) None of these

18. For what value of a, $f(x) = -x^3 + 4ax^2 + 2x - 5$ is decreasing $\forall\, x$?

(a) ± 5 (b) 3

(c) 0 (d) Cannot say

ASSERTION-REASON BASED QUESTIONS

In the following questions, a statement of assertion (A) is followed by a statement of reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true.

19. **Assertion (A) :** If the relation R defined in $A = \{1, 2, 3\}$ by aRb, if $|a^2 - b^2| \le 5$, then $R^{-1} = R$.
Reason (R) : For above relation, domain of R^{-1} = Range of R.

20. For any square matrix A with real number entries, consider the following statements.
Assertion (A) : $A + A'$ is a symmetric matrix.
Reason (R) : $A - A'$ is a skew-symmetric matrix.

Section B

This section comprises of very short answer type questions (VSA) of 2 marks each

21. Find the number of triplets (x, y, z) satisfying the equation $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$.

OR

Let A be any non-empty set, then prove that identify function on set A is a bijection.

22. Find the value of $\tan(\cos^{-1} x)$ and hence evaluate $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$.

23. Find : $\int \frac{x+1}{(x+2)(x+3)}dx$

24. Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 6\hat{j} + 2\hat{k}$.

OR

Find co-ordinates of the points on the line $x - 2 = \frac{y+3}{-2} = \frac{z+5}{2}$, which are on either side of the point $A(2, -3, -5)$ at a distance of 3 units from it.

25. A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.

Section C

This section comprises of short answer type questions (SA) of 3 marks each

26. Write minor and cofactor of the element a_{11}, a_{13} and a_{31} of the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

27. Evaluate : $\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

28. If $x, y, z \in [-1, 1]$ such that $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then find the value of $xy + yz + zx$.

OR

If $\operatorname{cosec}^{-1} x + \operatorname{cosec}^{-1} y + \operatorname{cosec}^{-1} z = -\frac{3\pi}{2}$, find the value of $\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

29. Determine the values of a, b and c for which the function $f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x}, & x < 0 \\ c\ , & x = 0 \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{b\sqrt{x^3}}, & x > 0 \end{cases}$ may be continuous at $x = 0$.

OR

If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

30. If $P(B) = \frac{3}{5}$, $P(A|B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then find $P(A \cup B)' + P(A' \cup B)$.

31. Find the coordinates of a point on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ at a distance of $\frac{6}{\sqrt{2}}$ units from the point (1, 2, 3).

OR

Find the vector and cartesian equations of the line through the point (1, 2, –4) and perpendicular to the two lines

$\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$ and

$\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$.

Section D

This section comprises of long answer type questions (LA) of 5 marks each

32. Solve the following problem graphically.

Minimize $Z=\frac{4x}{1000}+\frac{6y}{1000}$

subject to constraints :

$0.1x + 0.05y \le 50$; $0.25x + 0.5y \ge 200$; $x, y \ge 0$

OR

Solve the following problem graphically.

Minimize $Z = 150x + 200y$

subject to constraints :

$6x + 10y \ge 60$; $4x + 4y \le 32$; $x, y \ge 0$

33. Find the general solution of differential equation $\frac{dy}{dx}=\frac{y(x+2y)}{x(2x+y)}$.

OR

Find the particular solution of the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$ for $x = 1, y = 1$.

34. If the shortest distance between the lines $L_1:\frac{x-1}{1}=\frac{y}{-1}=\frac{z}{2}$ and $L_2:\frac{x+1}{2}=\frac{y}{2}=\frac{z-3}{\lambda}$ is unity, then find the value of λ.

35. For $x > 0$, let $f(x) = \int_1^x \frac{\log_e t}{1+t}dt$. Find the function $f(x)+f\left(\frac{1}{x}\right)$ and show that $f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}$.

Section E

This section comprises of 3 case-study/passage-based questions of 4 marks each. First two case study questions have three sub-parts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two sub-parts of 2 marks each.

36. **Case-Study 1 :** Read the following passage and answer the questions given below.

Three schools *A*, *B* and *C* organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold by school *A*, *B*, *C* are given below.

Article \ School	A	B	C
Fans	40	25	35
Mats	50	40	50
Plates	20	30	40

(i) If *P* be a 3 × 3 matrix represents the sale of handmade fans, mats and plates by three schools *A*, *B* and *C*, then find the matrix *P*.

(ii) If *Q* be a 3 × 1 matrix represents the sale prices (in ₹) of given products per unit, then find the matrix *Q*.

(iii) If school A sold all the three articles, then what amount is collected by school for funds?

OR

If all the three articles sold at ₹ 40 each, then find the amount collected by school *C* ?

37. **Case-Study 2:** Read the following passage and answer the questions given below.

A mirror in the shape of an ellipse represented by $\frac{x^2}{9}+\frac{y^2}{4}=1$ was hanging on the wall. Arun and his sister were playing with ball inside the house, even their mother refused to do so. All of sudden, ball hit the mirror and got a scratch in the shape of line represented by $\frac{x}{3}+\frac{y}{2}=1$.

(i) Find the point(s) of intersection of mirror (ellipse) and scratch (straight line) on it.

(ii) Represent graphically the given equation of mirror and scratch on it.

(iii) Find the area of smaller region bounded by the ellipse and line.

OR

If the equation of the mirror is of the form $\frac{x^2}{25}+\frac{y^2}{9}=1$, then find its area.

38. **Case-Study 3 :** Read the following passage and answer the questions given below.

Two motorcycles A and B are running at the speed more than allowed speed on the road along the lines $\vec{r}=\lambda(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{r}=3\hat{i}+3\hat{j}+\mu(2\hat{i}+\hat{j}+\hat{k})$, respectively.

(i) If both the motorcycles unfortunately met with an accident. Then, find the point of intersection using given equation of lines.

(ii) Find the D.R.'s and D.C.'s of line along which motorcycle A is running.

Detailed SOLUTIONS

1. **(c) :** If A and B are two independent events, then $P(A\cap B)=P(A)\times P(B)$

It is given that $P(A\cup B)=0.6$, $P(A)=0.2$

$\therefore\quad P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow\quad P(A\cup B)=P(A)+P(B)-P(A)\times P(B)$

$\Rightarrow\quad 0.6=0.2+P(B)(1-0.2)$

$\Rightarrow\quad 0.4=P(B)(0.8)$

$\Rightarrow\quad P(B)=\dfrac{0.4}{0.8}\Rightarrow P(B)=\dfrac{1}{2}=0.5$

2. **(a):** We have, $\dfrac{dy}{dx}=x^2e^{-y}$

$\Rightarrow\quad e^y\,dy=x^2dx$

On integrating, we get $e^y=\dfrac{x^3}{3}+C'$

$\Rightarrow\quad 3e^y=x^3+C$, where $C=3C'$

3. **(b) :** Since, matrix A is skew symmetric matrix.

$\therefore\quad A'=-A$ ...(i)

As $A=\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$ $\quad\therefore\quad A'=\begin{bmatrix}0 & 2 & b\\ a & 0 & 1\\ -3 & -1 & 0\end{bmatrix}$

From (i), $A+A'=O$

$$\Rightarrow\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}+\begin{bmatrix}0 & 2 & b\\ a & 0 & 1\\ -3 & -1 & 0\end{bmatrix}=O$$

$$\Rightarrow\begin{bmatrix}0 & 2+a & b-3\\ a+2 & 0 & 0\\ b-3 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore\quad a+2=0$ and $b-3=0\Rightarrow a=-2$ and $b=3$

4. **(b):** Let $y=\dfrac{2\tan x}{\tan x+\cos x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=2\left(\frac{\sec^2x(\tan x+\cos x)-\tan x(\sec^2x-\sin x)}{(\tan x+\cos x)^2}\right)$$

$$=2\left(\frac{\sec x+\tan x\cdot\sin x)}{(\tan x+\cos x)^2}\right)$$

5. **(b):** Let $I=\displaystyle\int\frac{\sqrt{x}}{\sqrt{x^2-x}}dx$

$$=\int\frac{\sqrt{x}}{\sqrt{x(x-1)}}dx=\int\frac{dx}{\sqrt{x-1}}=2\sqrt{x-1}+C$$

6. **(b):** We have, $P(A)=0.2$, $P(B)=0.4$ and $P(A\cup B)=0.5$

$\because\quad P(A\cap B)=P(A)+P(B)-P(A\cup B)=0.2+0.4-0.5=0.1$

$\therefore\quad P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{0.1}{0.4}=\dfrac{1}{4}=0.25$

7. **(c) :** Given, $f(x)=-\tan^{-1}x+x$

$\Rightarrow\quad f'(x)=\dfrac{-1}{1+x^2}+1\Rightarrow f'(x)=\dfrac{x^2}{1+x^2}\geq 0,\forall x\in R$

$\therefore\quad f(x)$ is increasing in $(-\infty,\infty)$.

8. **(a) :** We have, $y=4+3x-x^2$, a parabola with vertex at $\left(\dfrac{3}{2},\dfrac{25}{4}\right)$.

Putting $y=0$, we get $x^2-3x-4=0$

$\Rightarrow\quad (x-4)(x+1)=0\Rightarrow x=-1$ or $x=4$

$\therefore\quad$ Required area $=\displaystyle\int_{-1}^{4}(4+3x-x^2)dx$

$$=\left[4x+\frac{3x^2}{2}-\frac{x^3}{3}\right]_{-1}^{4}=\frac{125}{6}\text{ sq. units}$$

9. **(d):** We have, $y'=y\cot 2x$

$\Rightarrow\quad \dfrac{dy}{dx}=y\cot 2x$

$\Rightarrow\quad \dfrac{dy}{y}=\cot 2x\,dx$

Integrating both sides, we get

$$\int \frac{dy}{y} = \int \cot 2x\, dx$$

$\Rightarrow \quad \log|y| = \frac{1}{2}\log|\sin 2x| + \log c$

$\Rightarrow \quad \log|y| = \log\left|c\sqrt{\sin 2x}\right|$

$\Rightarrow \quad y = c\sqrt{\sin 2x}$

10. (c) : We have, $f(x) = x^2 - 4x + 1$

$\Rightarrow f(A) = A^2 - 4A + I$

$\because \quad A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad \therefore \quad A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$\therefore \quad f(A) = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

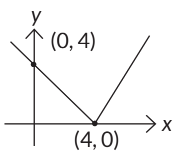
11. (d) : Putting $10^x + x^{10} = t$

$\Rightarrow \quad (10^x \log_e 10 + 10x^9)dx = dt$

$\therefore \quad \int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}}dx = \int \frac{dt}{t}$

$= \log_e t + C = \log_e(10^x + x^{10}) + C$

12. (d) : $|x - 4|$ is continuous but not differentiable at $x = 4$.

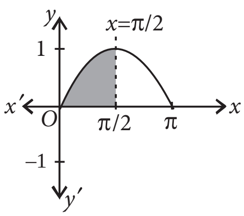


13. (b) : Required area

$= \int_0^{\pi/2} (\sin x)dx = [-\cos x]_0^{\pi/2}$

$= -\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= -[0 - 1] = 1$ sq. unit



14. (d): We have, $\frac{dy}{dx} = (1-x)(1+y) \Rightarrow \frac{dy}{1+y} = (1-x)dx$

$\Rightarrow \quad \int \frac{dy}{1+y} = \int (1-x)dx \Rightarrow \log|1+y| = x - \frac{x^2}{2} + C$

15. (a): Let E_1 be the event that bag I is selected, E_2 be the event that bag II is selected and E be the event that the ball drawn is of white colour.

By rule of total probability,

$P(E) = P(E_1)\cdot P(E|E_1) + P(E_2)\cdot P(E|E_2) = \frac{1}{2}\cdot\frac{3}{9} + \frac{1}{2}\cdot\frac{6}{9} = \frac{9}{18} = \frac{1}{2}$

16. (c) : Let $f(x) = 2x^3 - 6x + 5$

Differentiating w.r.t. x, we get $f'(x) = 6x^2 - 6$

Since, it is increasing function.

$\Rightarrow \quad 6x^2 - 6 > 0 \Rightarrow (x-1)(x+1) > 0 \Rightarrow x > 1$ or $x < -1$

17. (c) : We have, $y = t^{10} + 1$, $x = t^8 + 1$

$\Rightarrow \quad \frac{dy}{dt} = 10t^9, \ \frac{dx}{dt} = 8t^7$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{10t^9}{8t^7} = \frac{5}{4}t^2$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{5}{4}(2t)\frac{dt}{dx} = \frac{5}{4}\times 2t \times \frac{1}{8t^7} = \frac{5}{16t^6}$

18. (d) : $\because f(x) = -x^3 + 4ax^2 + 2x - 5$

$\therefore \quad f'(x) = -3x^2 + 8ax + 2$

Since, $f(x)$ is decreasing $\forall\, x$, therefore

$f'(x) < 0$

$\Rightarrow \quad -3x^2 + 8ax + 2 < 0$

From above, it is clear that decreasingness of $f(x)$ will be depend on the value of a and x.

19. (b) : Assertion (A) :

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

$R^{-1} = \{(y, x) : (x, y) \in R\}$

$= \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\} = R$

Reason (R) : Domain of $R^{-1} = \{1, 2, 3\}$

Range of $R = \{1, 2, 3\}$

Hence, both A and R are true but R is not the correct explanation of A.

20. (b) : Let $B = A + A'$, then

$B' = (A + A')' = A' + (A')' = A' + A = A + A' = B$

Therefore, $B = A + A'$ is a symmetric matrix.

Now, let $C = A - A'$

$C' = (A - A')' = A' - (A')' = A' - A = -(A - A') = -C$

Therefore, $C = A - A'$ is a skew-symmetric matrix.

Hence, both A and R are true but R is not the correct explanation of A.

21. We have, $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

$\because \quad -\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}, \frac{-\pi}{2} \le \sin^{-1}y \le \frac{\pi}{2}$ and $\frac{-\pi}{2} \le \sin^{-1}z \le \frac{\pi}{2}$

$\therefore$ The above condition will true if

$\sin^{-1}x = \sin^{-1}y = \sin^{-1}z = \frac{\pi}{2} \Rightarrow x = y = z = 1$

Thus, there is only one triplet.

OR

The identify function $I_A : A \to A$ is defined as $I_A(x) = x \ \forall\, x \in A$

Let $x_1, x_2 \in A$, then $I_A(x_1) = I_A(x_2) \Rightarrow x_1 = x_2$

$\therefore I_A$ is one-one or injective.

Let $y \in A$ be any arbitrary element, then there exists $x = y \in A$ such that, $I_A(x) = x = y \ \therefore \ I_A$ is onto or surjective.

22. Let $\cos^{-1}x = \theta$, then $\cos\theta = x$, where $\theta \in [0, \pi]$

$\therefore \quad \tan(\cos^{-1}x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}$

Hence, $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right) = \frac{\sqrt{1-(8/17)^2}}{8/17} = \frac{15}{8}$

23. Let $I = \int \frac{x+1}{(x+2)(x+3)}dx$

Also let, $\frac{x+1}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$

$\Rightarrow \quad x + 1 = A(x + 3) + B(x + 2)$ ...(i)

Putting $x = -3$ in (i), we get

$-B = -3 + 1 = -2 \Rightarrow B = 2$

Putting $x = -2$ in (i), we get

$A = -2 + 1 = -1$

$\therefore I = \int \frac{-1}{(x+2)}dx + 2\int \frac{1}{(x+3)}dx$

$= -\log|x + 2| + 2\log|x + 3| + C$

24. Given, $\vec{a} = 5\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 6\hat{j} + 2\hat{k}$

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} = 24\hat{i} - 24\hat{j} - 12\hat{k}$

$\therefore$ Unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ are given by

$\hat{n} = \pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \pm\frac{24\hat{i} - 24\hat{j} - 12\hat{k}}{\sqrt{576+576+144}}$

$= \pm\frac{12(2\hat{i} - 2\hat{j} - \hat{k})}{36} = \pm\frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$

OR

The given line is $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+5}{2}$ ...(i)

Let $A(2, -3, -5)$ lies on the line.

Direction ratios of line (i) are 1, −2, 2

$\therefore$ Direction cosines of line are $\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$

$\therefore$ (i) may be written as $\frac{x-2}{\frac{1}{3}} = \frac{y+3}{-\frac{2}{3}} = \frac{z+5}{\frac{2}{3}}$ ...(ii)

Coordinates of any point on the line (ii), may be taken as

$\left(\frac{1}{3}r+2, \frac{-2}{3}r-3, \frac{2}{3}r-5\right)$

Let $Q = \left(\frac{1}{3}r+2, \frac{-2}{3}r-3, \frac{2}{3}r-5\right)$

Given $|r| = 3$, $\therefore \quad r = \pm 3$

Putting the values of r, we have

$Q \equiv (3, -5, -3)$ or $Q \equiv (1, -1, -7)$

25. Let p be the probability that B gets selected.

P(Exactly one of A, B is selected) = 0.6 (given)

P(A is selected, B is not selected; B is selected, A is not selected) = 0.6

$\Rightarrow \quad P(A \cap B') + P(A' \cap B) = 0.6$

$\Rightarrow \quad P(A)\,P(B') + P(A')\,P(B) = 0.6$

$\Rightarrow \quad (0.7)(1 - p) + (0.3)p = 0.6 \Rightarrow p = 0.25$

Thus, the probability that B gets selected is 0.25.

26. We have, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

Minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$

Minor of $a_{31} = M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}a_{23} - a_{13}a_{22}$

Also, cofactor of $a_{11} = A_{11} = (-1)^{1+1}(a_{22}a_{33} - a_{23}a_{32})$

$= a_{22}a_{33} - a_{23}a_{32}$

Cofactor of $a_{13} = A_{13} = (-1)^{1+3}(a_{21}a_{32} - a_{22}a_{31})$

$= a_{21}a_{32} - a_{22}a_{31}$

Cofactor of $a_{31} = A_{31} = (-1)^{3+1}(a_{12}a_{23} - a_{13}a_{22})$

$= a_{12}a_{23} - a_{13}a_{22}$

27. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$= \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\cdot 2\sin x\cos x}} = \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^{\frac{7}{2}} x\cdot\sin^{\frac{1}{2}} x}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^{\frac{7}{2}} x\cdot\tan^{\frac{1}{2}} x\cdot\cos^{\frac{1}{2}} x}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^4 x\sqrt{\tan x}} = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x}}dx$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

Also $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{4} \Rightarrow t = 1$

$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{(1+t^2)dt}{\sqrt{t}} = \frac{1}{2}\int_0^1 (t^{-\frac{1}{2}} + t^{\frac{3}{2}})\,dt = \frac{1}{2}\left[\frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2}\right]_0^1 = \frac{6}{5}$

28. We have, $x, y, z \in [-1, 1]$

$\Rightarrow \quad -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1$

$\Rightarrow \quad 0 \le \cos^{-1} x \le \pi, 0 \le \cos^{-1} y \le \pi, 0 \le \cos^{-1} z \le \pi$

Given, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi + \pi + \pi$

$\Rightarrow \quad \cos^{-1}x = \pi, \cos^{-1}y = \pi, \cos^{-1}z = \pi \Rightarrow x = -1, y = -1, z = -1$

$\therefore \quad xy + yz + zx = (-1)\times(-1) + (-1)\times(-1) + (-1)\times(-1)$

$= 1 + 1 + 1 = 3$

OR

We know that the minimum value of $\text{cosec}^{-1}x$ is $-\frac{\pi}{2}$ which is attained at $x = -1$.

$\therefore \quad \text{cosec}^{-1}x + \text{cosec}^{-1}y + \text{cosec}^{-1}z = -\frac{3\pi}{2}$

$\Rightarrow \quad \text{cosec}^{-1}x + \text{cosec}^{-1}y + \text{cosec}^{-1}z = \left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right)$

$\Rightarrow \quad \text{cosec}^{-1}x = -\frac{\pi}{2}, \text{cosec}^{-1}y = -\frac{\pi}{2}, \text{cosec}^{-1}z = -\frac{\pi}{2}$

$\Rightarrow \quad x = -1, y = -1, z = -1$

$\therefore \quad \frac{x}{y}+\frac{y}{z}+\frac{z}{x} = \frac{(-1)}{(-1)}+\frac{(-1)}{(-1)}+\frac{(-1)}{(-1)} = 3$

29. Here, $f(0) = c$

L.H.L. at $x = 0$

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\frac{\sin(a+1)x+\sin x}{x}$$

$$= \lim_{x\to0^-}\left(\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right)$$

$$= \lim_{x\to0^-}\left\{\frac{\sin(a+1)x}{(a+1)x}\right\}\cdot(a+1)+\lim_{x\to0^-}\frac{\sin x}{x} = (a+1)+1 = a+2$$

R.H.L. at $x = 0$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{\sqrt{x+bx^2}-\sqrt{x}}{b\sqrt{x^3}} = \lim_{x\to0^+}\frac{\sqrt{x}\{\sqrt{1+bx}-1\}}{bx\sqrt{x}}$$

$$= \lim_{x\to0^+}\left(\frac{\sqrt{1+bx}-1}{bx}\right)\times\left(\frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}\right)$$

$$= \lim_{x\to0^+}\frac{1+bx-1}{bx(\sqrt{1+bx}+1)} = \lim_{x\to0^+}\frac{1}{\sqrt{1+bx}+1} = \frac{1}{2}$$

Now, f is continuous at $x = 0$ if

$$\lim_{x\to0^-} f(x) = f(0) = \lim_{x\to0^+} f(x)$$

i.e., if $a + 2 = c = \frac{1}{2} \Rightarrow a = -\frac{3}{2}$ and $c = \frac{1}{2}$

Hence, for $f(x)$ to be continuous at $x = 0$, we must have $a = -\frac{3}{2}, c = \frac{1}{2}$; b may have any real value.

OR

We have, $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$

Let $x = \sin A$, $y = \sin B$

$\therefore \quad \sqrt{1-\sin^2 A}+\sqrt{1-\sin^2 B} = a(\sin A-\sin B)$

$\Rightarrow \quad \cos A + \cos B = a(\sin A - \sin B)$

$\Rightarrow \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2a\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{A-B}{2}\right) = a\sin\left(\frac{A-B}{2}\right)$

$\Rightarrow \quad \cot\left(\frac{A-B}{2}\right) = a \Rightarrow \frac{A-B}{2} = \cot^{-1}a$

$\Rightarrow \quad A - B = 2\cot^{-1}a$

$\Rightarrow \quad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

Differentiating w.r.t. x, we get

$\Rightarrow \quad \frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

30. We have, $P(B) = \frac{3}{5}, P(A|B) = \frac{1}{2}$ and $P(A\cup B) = \frac{4}{5}$

$\because \quad P(A|B) = \frac{P(A\cap B)}{P(B)}$

$\Rightarrow \quad P(A\cap B) = P(A|B)\cdot P(B) = \frac{1}{2}\times\frac{3}{5} = \frac{3}{10}$

Since, $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow \quad P(A) = \frac{4}{5}-\frac{3}{5}+\frac{3}{10} = \frac{1}{2}$

$\therefore \quad P(A\cup B)' = 1 - P(A\cup B) = 1-\frac{4}{5} = \frac{1}{5}$

We know, $P(A\cap B) + P(A'\cap B) = P(B)$

[As $A\cap B$ and $A'\cap B$ are mutually exclusive events]

$\Rightarrow \quad P(A'\cap B) = \frac{3}{5}-\frac{3}{10} = \frac{3}{10}$

Now, $P(A'\cup B) = P(A') + P(B) - P(A'\cap B)$

$= \frac{1}{2}+\frac{3}{5}-\frac{3}{10} = \frac{5+6-3}{10} = \frac{4}{5}$

$\Rightarrow \quad P(A\cup B)'+P(A'\cup B) = \frac{1}{5}+\frac{4}{5} = \frac{5}{5} = 1$

31. Given equation of line is $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda$ (say)

$\Rightarrow \quad x = 3\lambda - 2, y = 2\lambda - 1, z = 2\lambda + 3$

$\therefore$ Coordinates of any point on the line are

$(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$ ...(i)

Let the distance between this point and (1, 2, 3) be $\frac{6}{\sqrt{2}}$ units

$\therefore \quad \sqrt{(3\lambda-2-1)^2+(2\lambda-1-2)^2+(2\lambda+3-3)^2} = \frac{6}{\sqrt{2}}$

$\Rightarrow \quad (3\lambda-3)^2+(2\lambda-3)^2+(2\lambda)^2 = \frac{36}{2}$

$\Rightarrow 9\lambda^2+9-18\lambda+4\lambda^2+9-12\lambda+4\lambda^2 = 18 \Rightarrow 17\lambda^2-30\lambda = 0$

$\Rightarrow \quad \lambda(17\lambda-30) = 0 \Rightarrow \lambda = 0, \frac{30}{17}$

Substituting the values of λ in (i), we get the required point as $(-2, -1, 3)$ or $\left(\frac{56}{17},\frac{43}{17},\frac{111}{17}\right)$.

OR

The given lines are

$\vec{r} = (8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$

and $\vec{r} = (15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$

Equation of any line through $(1, 2, -4)$ with d.r's l, m, n is

$\vec{r} = (\hat{i}+2\hat{j}-4\hat{k})+t(l\hat{i}+m\hat{j}+n\hat{k})$ ...(i)

Since, the required line is perpendicular to both the given lines. $\therefore$ $3l - 16m + 7n = 0$ and $3l + 8m - 5n = 0$

$\Rightarrow \quad \frac{l}{80-56} = \frac{m}{21+15} = \frac{n}{24+48} \Rightarrow \frac{l}{2} = \frac{m}{3} = \frac{n}{6}$

$\therefore$ From (i), the required line is

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + t(2\hat{i} + 3\hat{j} + 6\hat{k})$

Here, the position vector of passing point is

$\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$ and parallel vector is $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\therefore$ Cartesian equation is given by

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

32. The given problem is

Minimize $Z = \frac{4x}{1000} + \frac{6y}{1000}$

Subject to constraints:

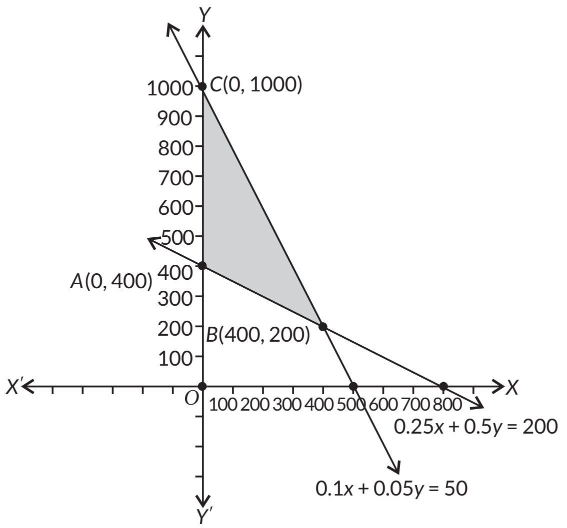
$0.1\,x + 0.05\,y \le 50$

$0.25\,x + 0.5\,y \ge 200$

$x, y \ge 0$

Convert the inequations into equations and draw the graph of lines :

$0.1x + 0.05y = 50$; $0.25x + 0.5y = 200$



As $x \ge 0, y \ge 0$ $\therefore$ Solution lies in first quadrant

Here, the shaded region is the feasible region. Now, we find the value of Z at each corner point.

Corner Points	Value of Z
A (0, 400)	2.4 ← Minimum
B (400, 200)	2.8
C (0, 1000)	6

Thus, Z has minimum value 2.4, when $x = 0$ and $y = 400$

OR

The given problem is

Minimize $Z = 150x + 200y$

subject to constraints

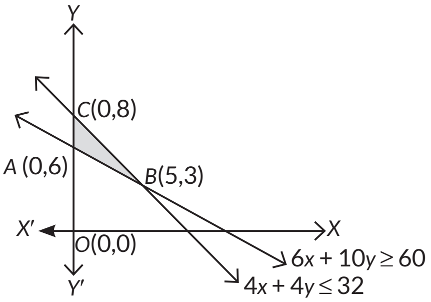
$6x + 10y \geq 60$;

$4x + 4y \le 32$;

$x \ge 0, y \ge 0$

As $x \ge 0, y \ge 0$

$\therefore$ Solution lies in first quadrant



Convert the inequations into equations and draw the graph of lines : $6x + 10y = 60$; $4x + 4y = 32$

Here, shaded region is the feasible region.

Corner points of feasible region are $A(0, 6)$, $B(5, 3)$ and $C(0, 8)$.

Value of Z at these corner points are:

Corner Points	Value of Z
$A(9, 6)$	1200 (minimum)
$B(5, 3)$	1350
$C(0, 8)$	1600

Thus, Z has minimum value 1200 when $x = 0$ and $y = 6$.

33. We have, $\frac{dy}{dx} = \frac{y(x+2y)}{x(2x+y)}$ ...(i),

which is a homogeneous differential equation.

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ Equation (i) becomes

$$v + x\frac{dv}{dx} = \frac{vx(x+2vx)}{x(2x+vx)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(1+2v)}{(2+v)} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v+2v^2-2v-v^2}{2+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2-v}{2+v} \Rightarrow \int\frac{2+v}{v^2-v}dv = \int\frac{dx}{x} \quad \text{...(ii)}$$

Now, $\frac{2+v}{v(v-1)} = \frac{A}{v} + \frac{B}{v-1}$

$$\Rightarrow \frac{2+v}{v(v-1)} = \frac{A(v-1)+Bv}{v(v-1)}$$

$\Rightarrow$ $2 + v = (A + B)v - A$

Comparing the coefficients of like powers of v, we get

$A = -2$; $A + B = 1$

$\Rightarrow$ $-2 + B = 1 \Rightarrow B = 3$

$\therefore$ Equation (ii) becomes

$$\int\frac{-2}{v}dv + 3\int\frac{1}{v-1}dv = \int\frac{dx}{x}$$

$\Rightarrow$ $-2\log|v| + 3\log|v - 1| = \log|x| + \log|c|$

$$\Rightarrow \frac{(v-1)^3}{v^2} = cx \Rightarrow \frac{(y-x)^3}{x^3} = \frac{y^2}{x^2}\cdot cx$$

$\Rightarrow$ $(y - x)^3 = cx^2y^2$, which is required solution.

OR

We have, $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{3xy+y^2}{x^2+xy} = -\frac{3y/x+(y/x)^2}{1+y/x} \quad ...(i)$$

This is a homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v\cdot 1 + x\cdot\frac{dv}{dx}$

$\therefore$ Equation (i) becomes,

$$v + x\frac{dv}{dx} = -\frac{3x\cdot vx + v^2x^2}{x^2 + x\cdot vx} = -\frac{3v+v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = -v - \frac{3v+v^2}{1+v} = \frac{-2v^2-4v}{v+1}$$

$$\Rightarrow \frac{v+1}{2v^2+4v}dv + \frac{dx}{x} = 0$$

Integrating both sides, we get

$$\frac{1}{4}\log|2v^2+4v| + \log|x| = \log C'$$

$$\Rightarrow (2v^2+4v)^{\frac{1}{4}}x = C'$$

$$\Rightarrow \left(\frac{2y^2}{x^2}+\frac{4y}{x}\right)^{\frac{1}{4}}x = C' \Rightarrow (2x^2y^2+4x^3y)^{\frac{1}{4}} = C'$$

$\Rightarrow$ $2x^2y^2 + 4x^3y = C$ [where $C = (C')^4$] ...(ii)

Put $x = 1, y = 1$ in (ii), we get $C = 6$

Hence, $2x^2y^2 + 4x^3y = 6$

$\Rightarrow$ $x^2y^2 + 2x^3y = 3$ is the required particular solution.

34. The lines are $\frac{x-1}{1} = \frac{y}{-1} = \frac{z}{2}$ ...(i)

and $\frac{x+1}{2} = \frac{y}{2} = \frac{z-3}{\lambda}$ ...(ii)

Here, $x_1 = 1, y_1 = 0, z_1 = 0$

$a_1 = 1, b_1 = -1, c_1 = 2$

$x_2 = -1, y_2 = 0, z_2 = 3$

$a_2 = 2, b_2 = 2, c_2 = \lambda$

Now, $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -2 & 0 & 3 \\ 1 & -1 & 2 \\ 2 & 2 & \lambda \end{vmatrix}$

$= -2(-\lambda - 4) + 0 + 3(2 + 2) = 2\lambda + 8 + 12 = 2\lambda + 20$

and $(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2$

$= (-\lambda - 4)^2 + (4 - \lambda)^2 + (2 + 2)^2$

$= \lambda^2 + 16 + 8\lambda + 16 + \lambda^2 - 8\lambda + 16 = 2\lambda^2 + 48$

Shortest distance between lines $= \frac{|2\lambda+20|}{\sqrt{2\lambda^2+48}} = 1$ [Given]

$\Rightarrow$ $2\lambda + 20 = \sqrt{2\lambda^2+48}$

$\Rightarrow$ $4\lambda^2 + 400 + 80\lambda = 2\lambda^2 + 48$

$\Rightarrow$ $2\lambda^2 + 80\lambda + 352 = 0 \Rightarrow \lambda^2 + 40\lambda + 176 = 0$

$$\Rightarrow \lambda = \frac{-40\pm\sqrt{1600-4(1)(176)}}{2} = \frac{-40\pm\sqrt{1600-704}}{2}$$

$$= \frac{-40\pm\sqrt{896}}{2} = \frac{-40\pm 2\sqrt{224}}{2} = -20\pm\sqrt{224} \text{ units}$$

35. We have, $f(x) = \int_1^x \frac{\log_e t}{1+t}dt$ ...(i)

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^{1/x}\frac{\log_e t}{1+t}dt$$

Let $t = \frac{1}{u}$. Then, $dt = -\frac{1}{u^2}du$.

When $t = 1 \Rightarrow u = 1$; when $t = \frac{1}{x} \Rightarrow \frac{1}{u} = \frac{1}{x} \Rightarrow u = x$

$$\therefore f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e(1/u)}{1+\frac{1}{u}}\times\frac{-1}{u^2}du$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e u}{(1+u)u}du$$

$$\Rightarrow f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e t}{(1+t)t}dt \quad ...(ii)$$

Adding (i) and (ii), we get

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left\{\frac{\log_e t}{1+t} + \frac{\log_e t}{(1+t)t}\right\}dt$$

$$\Rightarrow f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e t}{1+t}\left(\frac{1+t}{t}\right)dt \Rightarrow f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e t}{t}dt$$

$$\Rightarrow f(x) + f\left(\frac{1}{x}\right) = \int_1^{\log_e x} v\,dv, \text{where } v = \log_e t \text{ and } \frac{1}{t}dt = dv$$

$$\Rightarrow f(x) + f\left(\frac{1}{x}\right) = \left[\frac{v^2}{2}\right]_0^{\log_e x} = \frac{1}{2}(\log_e x)^2 - 0 = \frac{1}{2}(\log_e x)^2$$

Putting $x = e$, we get

$$f(e) + f\left(\frac{1}{e}\right) = \frac{(\log_e e)^2}{2} = \frac{1}{2}$$

36. (i) Clearly, $P = \begin{matrix} & \text{Fans} & \text{Mats} & \text{Plates} \\ A & 40 & 50 & 20 \\ B & 25 & 40 & 30 \\ C & 35 & 50 & 40 \end{matrix}$

(ii) Since, Q is a 3×1 matrix. Therefore, $Q = \begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix} \begin{matrix} \text{Fans} \\ \text{Mats} \\ \text{Plates} \end{matrix}$

(iii) Clearly, total amount collected by each school for funds is given by the matrix

$$PQ = \begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 1000+5000+1000 \\ 625+4000+1500 \\ 875+5000+2000 \end{bmatrix} = \begin{bmatrix} 7000 \\ 6125 \\ 7875 \end{bmatrix}$$

$\therefore$ Thus, amount collected by school A is ₹ 7000.

OR

Now, $\begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix}\begin{bmatrix} 40 \\ 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 1600 & 2000 & 800 \\ 100 & 1600 & 1200 \\ 1400 & 2000 & 1600 \end{bmatrix} = \begin{bmatrix} 4400 \\ 3800 \\ 5000 \end{bmatrix}$

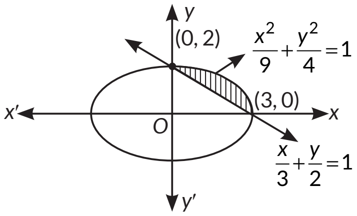
Thus, amount collected by school C is ₹ 5000.

37. (i) Given, $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ ...(i)

and $\dfrac{x}{3}+\dfrac{y}{2}=1$ ...(ii)

On solving (i) and (ii), we have the point of intersection as (3, 0) and (2, 0).

(ii)



(iii) Area of smaller region bounded by the mirror and scratch

$$=\frac{2}{3}\cdot\int_0^3\sqrt{9-x^2}\,dx-2\int_0^3\left(1-\frac{x}{3}\right)dx = I_1 - I_2$$

Now, $I_1 = \dfrac{2}{3}\displaystyle\int_0^3\sqrt{9-x^2}\,dx=\dfrac{2}{3}\int_0^3\sqrt{(3)^2-x^2}\,dx$

$$=\frac{2}{3}\left[\frac{1}{2}x\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$=\frac{2}{3}\left[\frac{3}{2}\sqrt{0}+\frac{9}{2}\sin^{-1}(1)-\frac{1}{2}(0)-\frac{9}{2}\sin^{-1}(0)\right]$$

$$=\frac{2}{3}\left[\frac{9}{2}\cdot\frac{\pi}{2}\right]=\frac{3\pi}{2} \quad \therefore \quad I_1=\frac{3\pi}{2} \quad ...(1)$$

Let $I_2 = 2\displaystyle\int_0^3\left(1-\frac{x}{3}\right)dx=2\left[x-\frac{x^2}{6}\right]_0^3$

$$=2\left(3-\frac{9}{6}-0-0\right)=2\times\frac{3}{2}=3 \quad \because I_2 = 3 \quad ...(2)$$

From (1) and (2) we have

$$=\frac{3\pi}{2}-3=3\left(\frac{\pi}{2}-1\right)\text{sq. units.}$$

OR

We have, $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1 \Rightarrow \dfrac{y^2}{9}=1-\dfrac{x^2}{25} \Rightarrow y=\pm\dfrac{3}{5}\sqrt{25-x^2}$

Required area = $\left|4\displaystyle\int_0^5 y\,dx\right| = \left|4\int_0^5\frac{3}{5}\sqrt{25-x^2}\,dx\right|$

$$=\frac{12}{5}\left[\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^5$$

$$=\frac{12}{5}\left[0+\frac{25}{2}\times\sin^{-1}1-0\right]=\frac{12}{5}\left[\frac{25}{2}\times\frac{\pi}{2}\right]=\frac{12}{5}\times\frac{25}{2}\times\frac{\pi}{2}$$

= 15 π sq. units

38. (i) Given equation of lines can be written in cartesion form as

$\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{-1}$ (λ say) ...(1)

$\dfrac{x-3}{2}=\dfrac{y-3}{1}=\dfrac{z}{1}$ (μ say) ...(2)

Any point on (1) is $(\lambda, 2\lambda, -\lambda)$ and point on (2) is $(2\mu + 3, \mu + 3, \mu)$

Both the lines will intersect iff these two points coincide.

$\therefore$ $\lambda = 2\mu + 3$ ...(i)

$2\lambda = \mu + 3$ ...(ii)

$-\lambda = \mu$

On solving above equations, we get $\lambda = 1, \mu = -1$

$\therefore$ Point of intersection is (1, 2, –1).

(ii) Clearly, D.R.'s of the required line are < 1, 2, –1 >

$\therefore$ D.C.'s are

$$<\frac{1}{\sqrt{1^2+2^2+(-1)^2}},\frac{2}{\sqrt{1^2+2^2+(-1)^2}},\frac{-1}{\sqrt{1^2+2^2+(-1)^2}}>$$

i.e., $<\dfrac{1}{\sqrt{6}},\dfrac{2}{\sqrt{6}},\dfrac{-1}{\sqrt{6}}>$

PRACTICE PAPER 5

General Instructions :

1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub parts.

Time Allowed : 3 Hours **Maximum Marks :** 80

Section A

(Multiple Choice Questions)
Each question carries 1 mark

1. The sum of the vectors $\vec{a}=\hat{i}-2\hat{j}+\hat{k},\ \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$ is

(a) $4\hat{j}-\hat{k}$ (b) $-4\hat{j}-\hat{k}$

(c) $4\hat{j}+\hat{k}$ (d) $-4\hat{j}+\hat{k}$

2. Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

(a) $\frac{1}{21}$ (b) $\frac{1}{11}$

(c) $\frac{1}{10}$ (d) $\frac{1}{23}$

3. If lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$ are mutually perpendicular, then find the value of k.

(a) $\frac{-10}{7}$ (b) $\frac{-11}{7}$

(c) $\frac{-13}{7}$ (d) $\frac{10}{7}$

4. Let $f: R \to R$ be defined by $f(x) = \begin{cases} 2x : x>3 \\ x^2 : 1<x\le 3 \\ 3x : x\le 1 \end{cases}$.
Find $f(-1)+f(2)+f(4)$.

(a) 7 (b) 8

(c) 9 (d) 10

5. Evaluate : $\int \frac{dx}{\sqrt{1-2x-x^2}}$

(a) $\sin^{-1}\frac{x}{\sqrt{2}}+c$ (b) $\sin^{-1}\left(\frac{1+x}{\sqrt{2}}\right)+c$

(c) $\cos^{-1}\frac{x}{\sqrt{2}}+c$ (d) $\cos^{-1}\left(\frac{1+x}{\sqrt{2}}\right)+c$

6. If $y = ae^x + be^{-x} + c$, where a, b, c are parameters, then y' is equal to

(a) $ae^x - be^{-x}$ (b) $ae^x + be^{-x}$

(c) $-(ae^x + be^{-x})$ (d) $ae^x - be^x$

7. $\sin^{-1}\left(\frac{1}{2}\right)+2\cos^{-1}\left(\frac{1}{2}\right)+4\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is equal to

(a) $\frac{13\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{3\pi}{4}$

8. Evaluate : $\int(2\tan x-3\cot x)^2 dx$

(a) $-4\tan x - 9\cot x - 25x + C$

(b) $4\tan x - 9\cot x - 25x + C$

(c) $-4\tan x + 9\cot x + 25x + C$

(d) $4\tan x + 9\cot x + 25x + C$

9. The function $f(x)=\log(1+x)-\frac{2x}{2+x}$ is increasing on

(a) $(-1, \infty)$ (b) $(-\infty, 0)$

(c) $(-\infty, \infty)$ (d) None of these

10. The area of the region bounded by the lines $y = x + 1$ and $x = 2$, $x = 3$ is

(a) $\frac{9}{2}$ sq. units (b) $\frac{7}{2}$ sq. units

(c) $\frac{11}{2}$ sq. units (d) $\frac{13}{2}$ sq. units

11. The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^{2/3} + 4 - 3\frac{dy}{dx} = 0 \text{ is}$$

(a) 2 (b) 1
(c) 3 (d) None of these

12. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of points *A*, *B*, *C* respectively such that $5\vec{a} - 3\vec{b} - 2\vec{c} = \vec{0}$, then find the ratio in which *C* divides *AB* externally.

(a) 1 : 3 (b) 2 : 5 (c) 3 : 5 (d) 5 : 2

13. Domain of $\cos^{-1}[x]$ (where [ ] denotes G.I.F.) is

(a) $[-1, 2]$ (b) $[-1, 2)$
(c) $(-1, 2]$ (d) None of these

14. If $y = \log_{10}x + \log_e y$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{y-1}$ (b) $\frac{y}{x}$
(c) $\frac{\log_{10}e}{x}\left(\frac{y}{y-1}\right)$ (d) None of these

15. Let *R* be a relation on the set *N* of natural numbers denoted by $nRm \Leftrightarrow n$ is a factor of *m* (*i.e.*, $n \mid m$). Then, *R* is

(a) Reflexive and symmetric
(b) Transitive and symmetric
(c) Equivalence relation
(d) Reflexive, transitive but not symmetric

16. Evaluate : $\int (e^{x\log a} + e^{a\log x} + e^{a\log a})dx$

(a) $\frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a x + C$

(b) $\frac{a^x}{\log a} + \frac{x^{a+1}}{a-1} + ax^a + C$

(c) $\frac{a^x}{\log a} + \frac{x^a}{a+1} + ax^a + C$

(d) $\frac{a^x}{\log x} + \frac{x^{a+1}}{a+1} + a^a x + C$

17. If *f* is a real-valued differentiable function satisfying $|f(x) - f(y)| \le (x-y)^2, x, y \in R$ and $f(0) = 0$, then $f(1)$ equals

(a) 1 (b) 2
(c) 0 (d) −1

18. The order of the differential equation whose general solution is given by

$$y = (C_1 + C_2)\cos(x + C_3) - C_4 e^{x + C_5}$$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants, is

(a) 5 (b) 4
(c) 3 (d) 2

ASSERTION-REASON BASED QUESTIONS

In the following questions, a statement of assertion (A) is followed by a statement of reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true but R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

19. **Assertion (A) :** The unit vector in the direction of sum of the vectors $\hat{i} + \hat{j} + \hat{k}, 2\hat{i} - \hat{j} - \hat{k}$ and $2\hat{j} + 6\hat{k}$ is $-\frac{1}{7}(3\hat{i} + 2\hat{j} + 6\hat{k})$.

Reason (R) : Let $\vec{a}$ be a non-zero vector then $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector parallel to $\vec{a}$.

20. Let E_1 and E_2 be any two events associated with an experiment, then

Assertion (A) : $P(E_1) + P(E_2) \le 1$.

Reason (R) : $P(E_1) + P(E_2) = P(E_1 \cup E_2) + P(E_1 \cap E_2)$.

Section B

This section comprises of very short answer type questions (VSA) of 2 marks each.

21. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and *A* adj *A* = AA^T, then find the value of $5a + b$.

OR

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then find $(AB)^{-1}$.

22. Prove that the area enclosed between the *x*-axis and the curve $y = x^2 - 1$ is $\frac{4}{3}$ sq. units.

23. If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$, then find $\vec{a} \times \vec{b}$ and $|\vec{a} \times \vec{b}|$.

OR

Prove by vector method that the area of ΔABC is $\frac{a^2 \sin B \sin C}{2\sin A}$.

24. Find the vector equation of the line passing through the point *A*(1, 2, −1) and parallel to the line $5x - 25 = 14 - 7y = 35z$.

25. A random variable X has the following distribution.

X	1	2	3	4	5	6	7	8
$P(X)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the event $E = \{X \text{ is prime number}\}$ and $F = \{X < 4\}$, find $P(E \cup F)$.

Section C

This section comprises of short answer type questions (SA) of 3 marks each.

26. Find the inverse of matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

OR

Given $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}$

Is $(AB)' = B'A'$?

27. If $y = \sin^{-1}x$, then show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$.

OR

If $f(x)$ is continuous at $x = 0$, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & \text{for } x > 0 \\ \dfrac{4(1-\sqrt{1-x})}{x}, & \text{for } x < 0 \end{cases}$$, then find $f(0)$.

28. Evaluate : $\displaystyle\int_0^1 \tan^{-1} x\, dx$

29. Find the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y(1+x)}{x(y-1)}$.

OR

If $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$, $y(0) = 0$, then find $\tan\left(\dfrac{x+y}{2}\right)$.

30. A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3, – 6. find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with X-axis.

31. Find the foot of perpendicular from the point (2, 3, –8) to the line $\dfrac{4-x}{2} = \dfrac{y}{6} = \dfrac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.

Section D

This section comprises of long answer type questions (LA) of 5 marks each.

32. Solve the following LPP graphically.
Maximize, $Z = 150x + 250y$
Subject to the constraints, $x + y \le 35$
$1000x + 2000y \le 50000$; $x, y \ge 0$

OR

Find the number of point(s) at which the objective function $Z = 4x + 3y$ can be minimum subjected to the constraints $3x + 4y \le 24$, $8x + 6y \le 48$, $x \le 5$, $y \le 6$; $x, y \ge 0$.

33. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

34. Find : $\displaystyle\int \sin^3 x \cos\frac{x}{2}\, dx$

OR

Evaluate: $\displaystyle\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}\, dx$

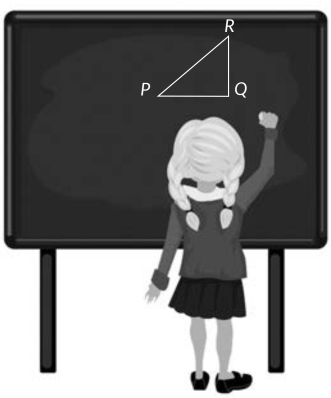
35. Show that $y = \log(x + \sqrt{x^2 + a^2})^2$ satisfies the differential equation $(x^2 + a^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$.

Section E

This section comprises of 3 case-study/passage-based questions of 4 marks each. First two case study questions have three sub-parts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two sub-parts of 2 marks each.

36. **Case-Study 1 :** Read the following passage and answer the questions given below.

In the math class, the teacher asked a student to construct a triangle on a black board and name it as PQR. Two angles P and Q were given to be equal to $\tan^{-1}\left(\dfrac{1}{3}\right)$ and $\tan^{-1}\left(\dfrac{1}{2}\right)$ respectively.



(i) Find the value of cos $(P + Q + R)$.

(ii) Find the value of $\cos P + \sin P$.

(iii) Find the value of $\sin^2 P + \sin^2 Q$.

OR

If $P = \cos^{-1} x$, then find the value of $10x^2$.

37. **Case-Study 2 :** Read the following passage and answer the questions given below.

A night before sleep, grandfather gave a puzzle to Rohan and Payal. The probability of solving this specific puzzle independently by Rohan and Payal are $\frac{1}{2}$ and $\frac{1}{3}$ respectively.

(i) Find the probability that both solved the puzzle.

(ii) Find the probability that puzzle is solved by Rohan but not by Payal.

(iii) Find the probability that puzzle is solved.

OR

Find the probability that exactly one of them solved the puzzle.

38. **Case-Study 3 :** Read the following passage and answer the questions given below.

A concert is organised every year in the stadium that can hold 42000 spectators. With ticket price of ₹ 10, the average attendance has been 27000. Some financial expert estimated that price of a ticket should be determined by the function $p(x) = 19 - \frac{x}{3000}$, where x is the number of tickets sold.

(i) Find the range of x and revenue R as a function of x.

(ii) Find the value of x for which revenue is maximum.

Detailed SOLUTIONS

1. **(b) :** The given vectors are

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k},\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k},\ \vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$

$\therefore$ Required sum $= \vec{a} + \vec{b} + \vec{c}$

$= (\hat{i} - 2\hat{j} + \hat{k}) + (-2\hat{i} + 4\hat{j} + 5\hat{k}) + (\hat{i} - 6\hat{j} - 7\hat{k}) = -4\hat{j} - \hat{k}$.

2. **(a) :** Let D_1, D_2 be the events that we find a defective fuse in the first and second test respectively.

$\therefore$ Required probability $= P(D_1 \cap D_2)$

$= P(D_1) \cdot P(D_2 | D_1) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21}$

3. **(a) :** Lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$

will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow\ -3(3k) + 2k + 2(-5) = 0$

$\Rightarrow\ -9k + 2k - 10 = 0$

$\Rightarrow\ k = -\frac{10}{7}$

4. **(c) :** Clearly $f(-1) = 3(-1) = -3$;

$f(2) = (2)^2 = 4$ and $f(4) = 2(4) = 8$

$\therefore\ f(-1) + f(2) + f(4) = -3 + 4 + 8 = 9$

5. **(b) :** We have, $I = \int \frac{dx}{\sqrt{1-(x^2+2x)}} = \int \frac{dx}{\sqrt{2-(x^2+2x+1)}}$

$= \int \frac{dx}{\sqrt{2-(1+x)^2}} = \int \frac{dx}{\sqrt{(\sqrt{2})^2 - (1+x)^2}}$

Put $1 + x = z \Rightarrow dx = dz$

$\therefore I = \int \frac{dz}{\sqrt{(\sqrt{2})^2 - z^2}} = \sin^{-1}\frac{z}{\sqrt{2}} + c = \sin^{-1}\left(\frac{1+x}{\sqrt{2}}\right) + c$

6. **(a) :** Given, $y = ae^x + be^{-x} + c$

Differentiating w.r.t. x, we get

$y' = ae^x - be^{-x}$

7. **(a) :** $\sin^{-1}\left(\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{1}{2}\right) + 4\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6} + 2 \cdot \frac{\pi}{3} + 4 \cdot \frac{\pi}{3} = \frac{\pi}{6} + \frac{2\pi}{3} + \frac{4\pi}{3} = \frac{13\pi}{6}$

8. **(b) :** We have, $\int (2\tan x - 3\cot x)^2 dx$

$= \int (4\tan^2 x + 9\cot^2 x - 12\tan x \cot x)dx$

$= \int \{4(\sec^2 x - 1) + 9(\operatorname{cosec}^2 x - 1) - 12\} dx$

$= \int (4\sec^2 x + 9\operatorname{cosec}^2 x - 25) dx$

$= 4 \tan x - 9 \cot x - 25x + C$

9. (a) : Given, $f(x) = \log(1+x) - \frac{2x}{2+x}$

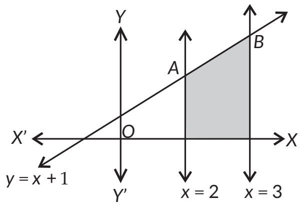
$\Rightarrow f'(x) = \frac{1}{1+x} - \frac{(2+x)2 - 2x}{(2+x)^2} \Rightarrow f'(x) = \frac{x^2}{(x+1)(x+2)^2}$

Clearly, $f'(x) > 0$ for all $x > -1$, hence $f(x)$ is increasing on $(-1, \infty)$.

10. (b) : We have, $y = x + 1$ and lines $x = 2$, $x = 3$.

$\therefore$ Points of intersection are $A(2, 3)$ and $B(3, 4)$.

$\therefore$ Required area of the shaded region $= \int_2^3 (x+1)dx$



$= \left[\frac{x^2}{2} + x\right]_2^3$

$= \left[\frac{9}{2} + 3 - \frac{4}{2} - 2\right] = \frac{7}{2}$ sq. units.

11. (a) : $\left(\frac{d^2y}{dx^2}\right)^{2/3} = 3\frac{dy}{dx} - 4 \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(3\frac{dy}{dx} - 4\right)^3$

$\therefore$ Degree is 2.

12. (c) : Given, $5\vec{a} - 3\vec{b} - 2\vec{c} = \vec{0} \Rightarrow 2\vec{c} = 5\vec{a} - 3\vec{b}$

$\Rightarrow \vec{c} = \frac{5\vec{a} - 3\vec{b}}{2} = \frac{5\vec{a} - 3\vec{b}}{5-3} = \frac{3\vec{b} - 5\vec{a}}{3-5}$

So, C divides AB externally in the ratio 3 : 5.

13. (b) : Clearly, $-1 \le [x] \le 1$

$\Rightarrow -1 \le x < 2 \Rightarrow x \in [-1, 2)$

14. (c) : We have, $y = \log_{10} x + \log_e y$

$\therefore \frac{dy}{dx} = \frac{1}{x}\log_{10} e + \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\log_{10} e}{x}\left(\frac{y}{y-1}\right)$

15. (d) : Reflexive : $n \mid n$ for all $n \in N$ $\therefore$ R is reflexive.

Symmetric : 2 | 6 but 6 does not divides 2

$\therefore$ R is not symmetric.

Transitive : Let nRm and mRp

$\Rightarrow n \mid m$ and $m \mid p$

$\Rightarrow n \mid p \Rightarrow nRp$. So, R is transitive.

16. (a) : We have, $\int (e^{x\log a} + e^{a\log x} + e^{a\log a}) dx$

$= \int (e^{\log a^x} + e^{\log x^a} + e^{\log a^a}) dx$

$= \int (a^x + x^a + a^a) dx$ $[\because e^{\log \lambda} = \lambda]$

$= \int a^x dx + \int x^a dx + \int a^a dx = \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a x + C$

17. (c) : Since, $\lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \le \lim_{x \to y} |x - y|$

$\Rightarrow |f'(y)| \le 0 \Rightarrow f'(y) = 0 \Rightarrow f(y) = \text{constant}$

As $f(0) = 0 \Rightarrow f(y) = 0 \Rightarrow f(1) = 0$

18. (c) : The given equation can be written as

$y = A\cos(x + C_3) - Be^x$

where $A = C_1 + C_2$ and $B = C_4 e^{C_5}$

So, there are three independent variables, (A, B, C_3).

Hence, the differential equation will be of order 3.

19. (d) : Sum of the given vectors

$= (\hat{i} + \hat{j} + \hat{k}) + (2\hat{i} - \hat{j} - \hat{k}) + (2\hat{j} + 6\hat{k}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$\therefore$ The unit vector in the direction of the sum of the given vectors

$= \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{9 + 4 + 36}} = \frac{1}{7}(3\hat{i} + 2\hat{j} + 6\hat{k})$

Hence, Assertion is false.

Also, $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector parallel to $\vec{a}$.

Reason is true.

20. (d) : Reason is a standard result. It is addition theorem of probability. However, the first result is untrue as we can have

$P(E_1) + P(E_2) > 1$

For example, when a dice is rolled once and

E_1 : 'a number < 5' shows up,

E_2 : 'a number > 1' shows up

then $P(E_1) = \frac{4}{6} = \frac{2}{3}$ and $P(E_2) = \frac{5}{6}$.

Here, $P(E_1) + P(E_2) > 1$.

Hence, A is false but R is true.

21. We have, $AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$= \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$

and $A \cdot (\text{adj}A) = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix} = \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix}$

$\because A \cdot (\text{adj}\,A) = AA^T$ is known, so equating the two expressions, we get

$\begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix} = \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix}$

We have, $10a + 3b = 13$ and $15a - 2b = 0$

On solving, we get $a = 2/5$ and $b = 3$

Thus, $5a + b = 2 + 3 = 5$.

OR

Given, $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$\therefore |AB| = \begin{vmatrix} -1 & 5 \\ 5 & -14 \end{vmatrix} = 14 - 25 = -11 \neq 0$

$\therefore \quad (AB)^{-1}$ exists. $\therefore \quad (AB)^{-1} = \dfrac{1}{|AB|}[\text{adj}(AB)]$

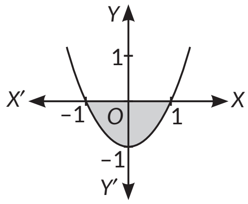
$$= \frac{-1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

22. The equation $y = x^2 - 1$ represents a upward parabola with vertex at (0, –1).

It cuts x-axis where $y = 0$

i.e., $x^2 - 1 = 0 \Rightarrow x = \pm 1$

$\therefore$ Required area $= \left| \int_{-1}^{1} (x^2 - 1)\,dx \right|$



$$= \left| \left[\frac{x^3}{3}\right]_{-1}^{1} - [x]_{-1}^{1} \right|$$

$$= \left| \frac{1}{3}(1+1) - (1+1) \right| = \left| \frac{2}{3} - 2 \right| = \frac{4}{3} \text{ sq. units}$$

23. Given, $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{vmatrix}$

$= (9+2)\hat{i} - (6+1)\hat{j} + (4-3)\hat{k} = 11\hat{i} - 7\hat{j} + \hat{k}.$

$\therefore \ |\vec{a} \times \vec{b}| = \sqrt{(11)^2 + (-7)^2 + (1)^2} = \sqrt{171}$

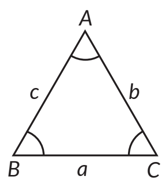
OR

Area of the triangle ABC

$= \dfrac{1}{2}|\overline{BC} \times \overline{BA}| = \dfrac{1}{2}|ac\sin B\,\hat{n}|$

$= \dfrac{1}{2}ac\sin B = \dfrac{1}{2}a\dfrac{c}{\sin C}\sin B \sin C$

$= \dfrac{1}{2}a\dfrac{a}{\sin A}\sin B \sin C$ [By sine rule]



$= \dfrac{a^2 \sin B \sin C}{2\sin A}$

24. Vector equation of the line passing through (1, 2, –1) and parallel to the line

$5x - 25 = 14 - 7y = 35z$

i.e., $\dfrac{x-5}{1/5} = \dfrac{y-2}{-1/7} = \dfrac{z}{1/35}$ or $\dfrac{x-5}{7} = \dfrac{y-2}{-5} = \dfrac{z}{1}$

is $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$.

25. Clearly, $P(E) = P(X = 2) + P(X = 3) + P(X = 5) + P(X = 7)$

$= 0.23 + 0.12 + 0.20 + 0.07 = 0.62$

$P(F) = P(X = 1) + P(X = 2) + P(X = 3)$

$= 0.15 + 0.23 + 0.12 = 0.50$

$P(E \cap F) = P(X = 2) + P(X = 3) = 0.23 + 0.12 = 0.35$

$\therefore \quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$= 0.62 + 0.50 - 0.35 = 0.77.$

26. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1, \forall\, \theta$

Since, $|A| \neq 0 \quad \therefore \quad A^{-1}$ exists

$\text{adj}\,A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^{-1} = \dfrac{1}{|A|}\text{adj}\,A = \dfrac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

OR

We have, $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 9 & 6 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 1 & 3 \end{bmatrix}_{3\times2}$

Now, $AB = \begin{bmatrix} 2+8+0 & 8+32+0 \\ 3+18+6 & 12+72+18 \end{bmatrix} = \begin{bmatrix} 10 & 40 \\ 27 & 102 \end{bmatrix}$

$\Rightarrow \quad (AB)' = \begin{bmatrix} 10 & 27 \\ 40 & 102 \end{bmatrix}$ ... (i)

Also, $B' = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 3 \end{bmatrix}_{2\times3}$ and $A' = \begin{bmatrix} 2 & 3 \\ 4 & 9 \\ 0 & 6 \end{bmatrix}_{3\times2}$

$\therefore \quad B'A' = \begin{bmatrix} 2+8+0 & 3+18+6 \\ 8+32+0 & 12+72+18 \end{bmatrix} = \begin{bmatrix} 10 & 27 \\ 40 & 102 \end{bmatrix}$ ... (ii)

From (i) and (ii), we get, $(AB)' = B'A'$.

27. We have, $y = \sin^{-1} x$.

On differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = 1$$

Again, differentiating w.r.t. x, we have

$$\frac{d}{dx}\left(\sqrt{1-x^2} \cdot \frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right) = 0$$

$$\Rightarrow \quad \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} \cdot \frac{2x}{2\sqrt{1-x^2}} = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

OR

Since, $f(x)$ is continuous at $x = 0$, therefore

$f(0) = \lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^+} f(x)$ ...(i)

$$\lim_{x\to0^-} f(x) = \lim_{x\to0}\left(\frac{4(1-\sqrt{1-x})}{x}\right)$$

$$= 4\lim_{x\to0}\left(\frac{1-(1-x)}{x(1+\sqrt{1-x})}\right) = 4\lim_{x\to0}\left(\frac{x}{x(1+\sqrt{1-x})}\right)$$

$$= 4\left(\frac{1}{(1+1)}\right) = \frac{4}{2} = 2$$

From (i), we get $f(0) = 2$.

28. Let $I=\int_0^1 \tan^{-1}x\,dx = \int_0^1 \tan^{-1}x\cdot 1\,dx$

$= [\tan^{-1}x\cdot x]_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot x\,dx$

$= \left(\frac{\pi}{4}-0\right)-\frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx = \frac{\pi}{4}-\frac{1}{2}I_1$ ...(i)

Consider $I_1 = \int_0^1 \frac{2x}{1+x^2}dx$

Put $1 + x^2 = t$

$\Rightarrow$ $2x\,dx = dt$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$\therefore$ $I_1 = \int_1^2 \frac{1}{t}dt = [\log t]_1^2 = \log 2 - \log 1 = \log 2$ ...(ii)

$\Rightarrow$ $I = \frac{\pi}{4} - \log\sqrt{2}$ [From (i) and (ii)]

29. We have, $\frac{dy}{dx} = \frac{y(1+x)}{x(y-1)} \Rightarrow \left(\frac{y-1}{y}\right)dy = \left(\frac{1+x}{x}\right)dx$

$\Rightarrow$ $\int\left(1-\frac{1}{y}\right)dy = \int\left(\frac{1}{x}+1\right)dx + C_1$

$\Rightarrow$ $y - \log|y| = \log|x| + x + C_1 \Rightarrow x - y + \log|xy| = C$ (where $C = -C_1$)

OR

Substitute $x + y = z$ $\Rightarrow$ $\frac{dy}{dx} = \frac{dz}{dx} - 1$

So, given equation becomes

$\frac{dz}{dx} = 1 + \sin z + \cos z$

$\Rightarrow$ $\int dx = \int \frac{dz}{1+\sin z+\cos z} = I_1$ (say) ...(i)

Now, $I_1 = \int \frac{dz}{1+\frac{2\tan(z/2)}{1+\tan^2(z/2)}+\frac{1-\tan^2(z/2)}{1+\tan^2 z/2}}$

$\Rightarrow$ $I_1 = \int \frac{(1+\tan^2(z/2))dz}{2(1+\tan(z/2))}$ $\Rightarrow$ $I_1 = \frac{1}{2}\int \frac{\sec^2(z/2)dz}{1+\tan(z/2)}$

Substitute $\tan\frac{z}{2} = t$, we get

$I_1 = \int \frac{dt}{t+1} = \ln(t+1) + c$

Now, from (i), we have

$x = \ln\left(1+\tan\frac{x+y}{2}\right)+c$

$\because$ $x = 0, y = 0 \Rightarrow c = 0 \Rightarrow e^x - 1 = \tan\left(\frac{x+y}{2}\right)$.

30. Let $\vec{r} = a\hat{i}+b\hat{j}+c\hat{k}$

Also, $\frac{a}{2} = \frac{b}{3} = \frac{c}{-6} = k$(say)

$\Rightarrow$ $a = 2k, b = 3k$ and $c = -6k$ $\therefore$ $\vec{r} = k(2\hat{i}+3\hat{j}-6\hat{k})$

Now, $|\vec{r}| = 14$

$\Rightarrow$ $4k^2 + 9k^2 + 36k^2 = 196$

$\Rightarrow$ $49k^2 = 196 \Rightarrow k^2 = 4$

$\Rightarrow$ $k = 2$ [$\because \vec{r}$ makes an acute angle with X-axis]

$\therefore$ $\vec{r} = 4\hat{i}+6\hat{j}-12\hat{k}$

Now, $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{4\hat{i}+6\hat{j}-12\hat{k}}{14} = \frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}$

$\therefore$ Direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$

and components of $\vec{r} = 4\hat{i}, 6\hat{j}, -12\hat{k}$.

31. We have, equation of line as $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$

$\Rightarrow$ $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = \lambda$ and dr's are $-2, 6, -3$.

$\Rightarrow$ $x = -2\lambda + 4, y = 6\lambda$ and $z = -3\lambda + 1$

If the coordinates of L be $(4 - 2\lambda, 6\lambda, 1 - 3\lambda)$ and P be $(2, 3, -8)$ then direction ratios of PL are proportional to $(4 - 2\lambda - 2, 6\lambda - 3, 1 - 3\lambda + 8)$ *i.e.*, $(2 - 2\lambda, 6\lambda - 3, 9 - 3\lambda)$.

Since, PL is perpendicular to given line

$\therefore$ $-2(2 - 2\lambda) + 6(6\lambda - 3) - 3(9 - 3\lambda) = 0$

$\Rightarrow$ $-4 + 4\lambda + 36\lambda - 18 - 27 + 9\lambda = 0$

$\Rightarrow$ $49\lambda = 49 \Rightarrow \lambda = 1$

Hence, the coordinates of L are $(4 - 2, 6, 1 - 3)$ *i.e.*, $(2, 6, -2)$

Now, $PL = \sqrt{(2-2)^2+(6-3)^2+(-2+8)^2}$

$= \sqrt{0+9+36} = 3\sqrt{5}$ units

32. The given problem is

Maximize, $Z = 150x + 250y$

Subject to the constraints

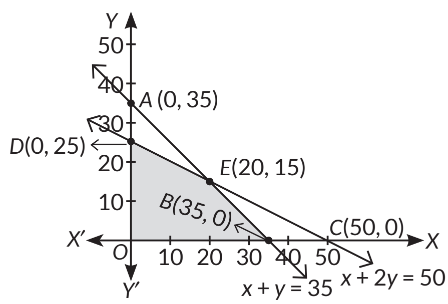
$x + y \le 35, x + 2y \le 50$ and $x, y \ge 0$

To solve graphically, we convert the inequations into equations to obtain the following lines :

$x + y = 35, x + 2y = 50, x = 0, y = 0$

The point of intersection of these lines is (20, 15).

Let us draw the graph of these lines as shown below.



The feasible region is the shaded region. We observe that the region is bounded.

The corner points of the feasible region $OBED$ are $O(0, 0)$, $B(35, 0)$, $E(20, 15)$ and $D(0, 25)$.

The value of the objective function at corner points of the feasible region are :

Corner points	Value of $Z = 150x + 250y$
$O(0, 0)$	0
$B(35, 0)$	5250
$E(20, 15)$	6750 (Maximum)
$D(0, 25)$	6250

Clearly, Z is maximum at $x = 20$, $y = 15$.

OR

The given problem is minimize $z = 4x + 3y$

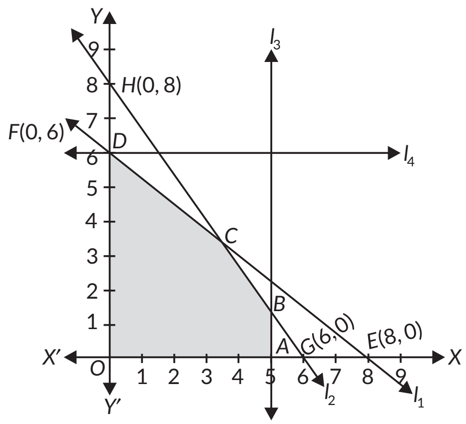
Subject to the constraints $3x + 4y \leq 24$, $8x + 6y \leq 48$, $x \leq 5$, $y \leq 6$; $x, y \geq 0$

To solve graphically, we convert the inequations into equations to obtain the following lines :

Let $l_1 : 3x + 4y = 24$, $l_2 : 8x + 6y = 48$, $l_3 : x = 5$, $l_4 : y = 6$,

For B : On solving l_2 and l_3, we get $B(5, 4/3)$

For C : On solving l_1 and l_2, we get $C\left(\frac{24}{7}, \frac{24}{7}\right)$



From the graph it is clear that Shaded portion $OABCD$ is the feasible region, where $O(0, 0)$, $A(5, 0)$, $B(5, 4/3)$, $C\left(\frac{24}{7}, \frac{24}{7}\right)$ and $D(0, 6)$ are

Corner points	Value of $Z = 4x + 3y$
$O(0, 0)$	0 (Minimum)
$A(5, 0)$	20
$B\left(5, \frac{4}{3}\right)$	24
$C\left(\frac{24}{7}, \frac{24}{7}\right)$	24
$D(0, 6)$	18

33. Here, $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\Rightarrow$ $|A| = -11$ and $|B| = 1$

$\therefore$ $A^{-1} = \frac{1}{|A|}\text{adj } A = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow$ R.H.S. $= B^{-1}A^{-1}$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \cdot \left(-\frac{1}{11}\right)\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = \left(-\frac{1}{11}\right)\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} \quad \text{...(i)}$$

Now, $A \cdot B = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

$\Rightarrow$ $|AB| = 14 - 25 = -11$

$\therefore$ L.H.S. $= (AB)^{-1} = \left(-\frac{1}{11}\right)\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$ ...(ii)

From (i) and (ii), $(AB)^{-1} = B^{-1}A^{-1}$.

34. $\because$ We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

Now, $\int \sin^3 x \cos\frac{x}{2}dx = \int \frac{3\sin x - \sin 3x}{4} \cdot \cos\frac{x}{2}dx$

$$= \frac{3}{4}\int \sin x \cos\frac{x}{2}dx - \frac{1}{4}\int \sin 3x \cos\frac{x}{2}dx$$

$$= \frac{3}{8}\int 2\sin x \cos\frac{x}{2}dx - \frac{1}{8}\int 2\sin 3x \cos\frac{x}{2}dx$$

$$= \frac{3}{8}\int\left(\sin\frac{3x}{2} + \sin\frac{x}{2}\right)dx - \frac{1}{8}\int\left(\sin\frac{7x}{2} + \sin\frac{5x}{2}\right)dx$$

[$\because$ $2\sin A\cos B = \sin(A + B) + \sin(A - B)$]

$$= \frac{3}{8}\int \sin\frac{3x}{2}dx + \frac{3}{8}\int \sin\frac{x}{2}dx - \frac{1}{8}\int \sin\frac{7x}{2}dx - \frac{1}{8}\int \sin\frac{5x}{2}dx.$$

$$= \frac{3}{8}\left(-\frac{\cos 3x/2}{\frac{3}{2}}\right) + \frac{3}{8}\cdot\left(-\frac{\cos x/2}{\frac{1}{2}}\right) - \frac{1}{8}\left(-\frac{\cos 7x/2}{\frac{7}{2}}\right) - \frac{1}{8}\left(-\frac{\cos 5x/2}{\frac{5}{2}}\right) + C$$

$$= -\frac{1}{4}\cos\frac{3x}{2} - \frac{3}{4}\cos\frac{x}{2} + \frac{1}{28}\cos\frac{7x}{2} + \frac{1}{20}\cos\frac{5x}{2} + C$$

OR

Let $I = \int \frac{1}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}dx$

$$= \int \frac{\sec^4 x\,dx}{\tan^4 x + \tan^2 x + 1} = \int \frac{(1 + \tan^2 x)\sec^2 x\,dx}{\tan^4 x + \tan^2 x + 1}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore \quad I = \int \frac{1 + t^2}{t^4 + t^2 + 1}dt = \int \frac{1 + \frac{1}{t^2}}{t^2 + 1 + \frac{1}{t^2}}dt$$

$$\Rightarrow \quad I = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 3}dt$$

Put $t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right)dt = dy$

$\therefore \quad I=\int \frac{dy}{y^2+(\sqrt{3})^2}=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right)+C$

$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{3}}+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{3}}\right)+C$

35. We have, $y=\log\left(x+\sqrt{x^2+a^2}\right)^2$ ...(i)

On differentiating (i), w.r.t. 'x', we get

$$\frac{dy}{dx}=\frac{1}{\left(x+\sqrt{x^2+a^2}\right)^2}2\left(x+\sqrt{x^2+a^2}\right)\left(1+\frac{1}{2\sqrt{x^2+a^2}}(2x)\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{(x+\sqrt{x^2+a^2})}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \sqrt{x^2+a^2}\frac{dy}{dx}=2 \quad \text{...(ii)}$$

On differentiating (ii) w.r.t. 'x', we get

$$\sqrt{x^2+a^2}\frac{d^2y}{dx^2}+\frac{1}{2\sqrt{x^2+a^2}}2x\frac{dy}{dx}=0$$

$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$, which is given differential equation.

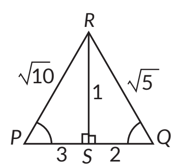
$\therefore \quad y=\log(x+\sqrt{x^2+a^2})^2$ is the solution of given differential equation.

36. **(i)** Since, PQR is a triangle

$\therefore \quad \angle P+\angle Q+\angle R=180°$

$\Rightarrow \quad \cos(P+Q+R)=\cos 180°=-1$

(ii) Since, $\angle P=\tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan P=\frac{1}{3}$



$\Rightarrow \quad \sin P=\frac{1}{\sqrt{10}}$ and $\cos P=\frac{3}{\sqrt{10}}$

$\therefore \quad$ Value of $\cos P+\sin P=\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{10}}=\frac{4}{\sqrt{10}}$

(iii) Given, $\angle P=\tan^{-1}\left(\frac{1}{3}\right)$ and $\angle Q=\tan^{-1}\left(\frac{1}{2}\right)$

$\tan P=\frac{1}{3}$ and $\tan Q=\frac{1}{2} \quad \therefore \quad \sin P=\frac{1}{\sqrt{10}}$ and $\sin Q=\frac{1}{\sqrt{5}}$

$\Rightarrow \quad \sin^2 P+\sin^2 Q=\frac{1}{10}+\frac{1}{5}=\frac{1+2}{10}=\frac{3}{10}$

OR

Given, $P=\cos^{-1}x$ ...(1)

Also, $\angle P=\tan^{-1}\left(\frac{1}{3}\right)$

$\Rightarrow \quad \tan P=\frac{1}{3} \quad \therefore \quad \cos P=\frac{3}{\sqrt{10}}$

$\Rightarrow \quad P=\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$ ...(2)

From (1) and (2), we get

$x=\frac{3}{\sqrt{10}} \Rightarrow x^2=\frac{9}{10} \Rightarrow 10x^2=9$

37. Let E_1 be the event that Rohan solved the puzzle and E_2 be the event that Payal solved the puzzle.

Then, $P(E_1)=1/2$ and $P(E_2)=1/3$

(i) Since, E_1 and E_2 are independent events

$\therefore \quad P$(both solved the puzzle) = $P(E_1 \cap E_2)$

$=P(E_1)\cdot P(E_2)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

(ii) P(puzzle is solved by Rohan but not by Payal)

$=P(\bar{E}_2)P(E_1)=\left(1-\frac{1}{3}\right)\cdot\frac{1}{2}=\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3}$

(iii) P(puzzle is solved) = $P(E_1$ or $E_2)$

$=P(E_1\cup E_2)=P(E_1)+P(E_2)-P(E_1\cap E_2)$

$=\frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$

OR

P(Exactly one of them solved the puzzle)

$=P((E_1$ and $\bar{E}_2)$ or $(E_2$ and $\bar{E}_1))$

$=P(E_1\cap\bar{E}_2)+P(E_2\cap\bar{E}_1)$

$=P(E_1)\times P(\bar{E}_2)+P(E_2)\times P(\bar{E}_1)$

$=\frac{1}{2}\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{2}$ $\quad [\because P(\bar{E}_1)=1-P(E_1)]$

$=\frac{1}{3}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$

38. **(i)** Since, more than 42000 tickets cannot be sold.

So, range of x is [0, 42000].

Revenue function, $R(x)=xP(x)$

$=x\left(19-\frac{x}{3000}\right)=19x-\frac{x^2}{3000}$

(ii) We have, $R(x)=19x-\frac{x^2}{3000}$

$\Rightarrow \quad R'(x)=19-\frac{x}{1500}$

For maxima/minima, put $R'(x)=0$

$\Rightarrow \quad 19-\frac{x}{1500}=0 \Rightarrow x=28500$

Also, $R''(x)=-\frac{1}{1500}<0.$

$\therefore \quad$ Maximum revenue will be at $x=28500$

CBSE SAMPLE PAPER 2023-24

(Based on CBSE Circular released on 31st March 2023)

General Instructions :

1. This question paper contains - five sections A, B, C, D and E. Each section is compulsory. However, there are internal choices in some questions.
2. Section A has 18 MCQs and 2 Assertion-Reason based questions of 1 mark each.
3. Section B has 5 Very Short Answer (VSA)-type questions of 2 marks each.
4. Section C has 6 Short Answer (SA)-type questions of 3 marks each.
5. Section D has 4 Long Answer (LA)-type questions of 5 marks each.
6. Section E has 3 source based/case based/passage based/integrated units of assessment (4 marks each) with sub-parts.

Time Allowed : 3 Hours **Maximum Marks :** 80

Section A

(Multiple Choice Questions)
Each question carries 1 mark

1. If $A=[a_{ij}]$ is a square matrix of order 2 such that $a_{ij}=\begin{cases}1, \text{when } i\neq j\\0, \text{when } i=j\end{cases}$, then A^2 is

(a) $\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$

(c) $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

2. If A and B are invertible square matrices of the same order, then which of the following is not correct?

(a) adj $A = |A|\cdot A^{-1}$ (b) $\det(A)^{-1} = [\det(A)]^{-1}$

(c) $(AB)^{-1} = B^{-1}A^{-1}$ (d) $(A+B)^{-1} = B^{-1} + A^{-1}$

3. If the area of the triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq units, then the value/s of k is will be

(a) 9 (b) ±3 (c) –9 (d) 6

4. If $f(x)=\begin{cases}\frac{kx}{|x|}, \text{if } x<0\\3, \text{if } x\geq 0\end{cases}$ is continuous at x = 0, then the value of k is

(a) –3 (b) 0

(c) 3 (d) any real number

5. The lines $\vec{r}=\hat{i}+\hat{j}-\hat{k}+\lambda(2\hat{i}+3\hat{j}-6\hat{k})$ and $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(6\hat{i}+9\hat{j}-18\hat{k})$; (where λ an μ are scalars) are

(a) coincident (b) skew

(c) intersecting (d) parallel

6. The degree of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}=\frac{d^2y}{dx^2} \text{ is}$$

(a) 4 (b) $\frac{3}{2}$ (c) 2 (d) Not defined

7. The corner points of the bounded feasible region determined by a system of linear constraints are (0, 3), (1, 1) and (3, 0). Let $Z = px + qy$, where $p, q > 0$,. The condition on p and q so that the minimum of Z occurs at (3, 0) and (1, 1) is

(a) $p = 2q$ (b) $p=\frac{q}{2}$

(c) $p = 3q$ (d) $p = q$

8. $ABCD$ is a rhombus whose diagonals intersects at E. Then $\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$ equals to

(a) $\vec{0}$ (b) $\overrightarrow{AD}$ (c) $2\overrightarrow{BD}$ (d) $2\overrightarrow{AD}$

9. For any integer n, the value of $\int_0^{\pi} e^{\sin^2 x}\cos^3(2n+1)x\,dx$ is

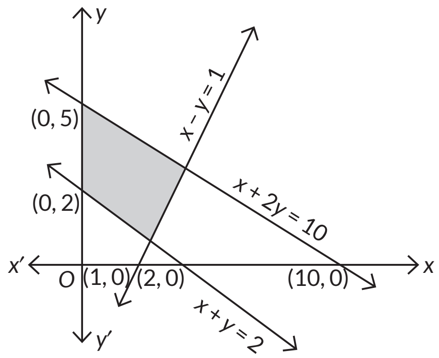
(a) –1 (b) 0 (c) 1 (d) 2

10. The value of $|A|$, if $A=\begin{bmatrix}0 & 2x-1 & \sqrt{x}\\1-2x & 0 & 2\sqrt{x}\\-\sqrt{x} & -2\sqrt{x} & 0\end{bmatrix}$, where $x\in\mathbb{R}^+$, is

(a) $(2x + 1)^2$ (b) 0

(c) $(2x + 1)^3$ (d) None of these

11. The feasible region corresponding to the linear constraints of a Linear Programming Problem is given below



Which of the following is not a constraint to the given Linear Programming Problem ?

(a) $x + y \geq 2$ (b) $x + 2y \leq 10$

(c) $x - y \geq 1$ (d) $x - y \leq 1$

12. If $\vec{a} = 4\hat{i} + 6\hat{j}$ and $\vec{b} = 3\hat{j} + 4\hat{k}$, then the vector form of the component of $\vec{a}$ along $\vec{b}$ is

(a) $\frac{18}{5}(3\hat{i} + 4\hat{k})$ (b) $\frac{18}{25}(3\hat{j} + 4\hat{k})$

(c) $\frac{18}{5}(3\hat{i} + 4\hat{k})$ (d) $\frac{18}{25}(2\hat{i} + 4\hat{j})$

13. Given that A is a square matrix of order 3 and $|A| = -2$, then $|\text{adj}\,(2A)|$ is equal to

(a) -2^6 (b) 4

(c) -2^8 (d) 2^8

14. A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. If the events of their solving the problem are independent then the probability that the problem will be solved, is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

15. The general solution of the differential equation $ydx - xdy = 0$; (Given $x, y > 0$), is of the form

(a) $xy = c$ (b) $x = cy^2$

(c) $y = cx$ (d) $y = cx^2$

(Where 'c' is an arbitrary positive constant of integration)

16. The value of λ for which two vectors $2\hat{i} - \hat{j} + 2\hat{k}$ and $3\hat{i} + \lambda\hat{j} + \hat{k}$ are perpendicular is

(a) 2 (b) 4

(c) 6 (d) 8

17. The set of all points where the function $f(x) = x + |x|$ is differentiable, is

(a) $(0, \infty)$ (b) $(-\infty, 0)$

(c) $(-\infty, 0) \cup (0, \infty)$ (d) $(-\infty, \infty)$

18. If the direction cosines of a line are $< \frac{1}{c}, \frac{1}{c}, \frac{1}{c} >$, then

(a) $0 < c < 1$ (b) $c > 2$

(c) $c = \pm\sqrt{2}$ (d) $c = \pm\sqrt{3}$

Assertion -Reason Based Questions

In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R).

Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

19. Let $f(x)$ be a polynomial function of degree 6 such that $\frac{d}{dx}(f(x)) = (x-1)^3(x-3)^2$, then

Assertion (A) : $f(x)$ has minimum at $x = 1$.

Reason (R) : When $\frac{d}{dx}(f(x)) < 0, \forall x \in (a-h, a)$ and $\frac{d}{dx}(f(x)) > 0, \forall x \in (a, a+h)$; where '$h$' is an infinitesimally small positive quantity, then $f(x)$ has a minimum at $x = a$, provided $f(x)$ is continuous at $x = a$.

20. **Assertion (A) :** The relation $f : \{1, 2, 3, 4\} \to \{x, y, z, p\}$ defined by $f = \{(1, x), (2, y), (3, z)\}$ is a bijective function.

Reason (R) : The function $f : \{1, 2, 3\} \to \{x, y, z, p\}$ such that $f = \{(1, x), (2, y), (3, z)\}$ is one-one.

Section B

This section comprises of very short answer type questions (VSA) of 2 marks each.

21. Find the value of $\sin^{-1}\left(\cos\left(\frac{33\pi}{5}\right)\right)$.

OR

Find the domain of $\sin^{-1}(x^2 - 4)$.

22. Find the interval/s in which the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = xe^x$, is increasing.

23. If $f(x) = \frac{1}{4x^2 + 2x + 1}; x \in \mathbb{R}$, then find the maximum value of $f(x)$.

OR

Find the maximum profit that a company can make, if the profit function is given by $P(x) = 72 + 42x - x^2$, where x is the number of units and P is the profit in rupees.

24. Evaluate : $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right)dx$.

25. Check whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + x$, has any critical point/s or not ? If yes, then find the point/s.

Section C

This section comprises of short answer type questions (SA) of 3 marks each.

26. Evaluate : $\int \frac{2x^2 + 3}{x^2(x^2 + 9)}dx; x \neq 0$.

27. The random variable X has a probability distribution $P(X)$ of the following form, where 'k' is some real number.

$$P(X)=\begin{cases}k, & \text{if} \quad x=0\\ 2k, & \text{if} \quad x=1\\ 3k, & \text{if} \quad x=2\\ 0, & \text{otherwise}\end{cases}$$

(i) Determine the value of k.
(ii) Find $P(X<2)$.
(iii) Find $P(X>2)$.

28. Evaluate : $\int\sqrt{\frac{x}{1-x^3}}\,dx;\ x\in(0,1)$.

OR

Evaluate : $\int_0^{\frac{\pi}{4}}\log_e(1+\tan x)\,dx$.

29. Solve the differential equation :

$ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy,\ (y\neq 0)$.

OR

Solve the differential equation :

$(\cos^2 x)\frac{dy}{dx}+y=\tan x;\left(0\leq x<\frac{\pi}{2}\right)$.

30. Solve the following Linear Programming Problem graphically :
Minimize : $z = x + 2y$,
subject to the constraints : $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x, y \geq 0$.

OR

Solve the following Linear Programming Problem graphically :
Maximize : $z = -x + 2y$,
subject to the constraints : $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$

31. If $(a+bx)e^{\frac{y}{x}}=x$ then prove that $x\frac{d^2y}{dx^2}=\left(\frac{a}{a+bx}\right)^2$.

Section D

This section comprises of long answer type questions (LA) of 5 marks each.

32. Make a rough sketch of the region $\{(x, y) : 0 \leq y \leq x^2 + 1, 0 \leq y \leq x + 1, 0 \leq x \leq 2\}$ and find the area of the region, using the method of integration.

33. Let $\mathbb{N}$ be the set of all natural numbers and R be a relation on $\mathbb{N}\times\mathbb{N}$ defined by
$(a, b)\ R(c, d) \Leftrightarrow ad = bc$ for all $(a, b), (c, d) \in \mathbb{N}\times\mathbb{N}$. Show that R is an equivalence relation on $\mathbb{N}\times\mathbb{N}$. Also, find the equivalence class of (2, 6), *i.e.*, [(2, 6)].

OR

Show that the function $f:\mathbb{R}\to\{x\in\mathbb{R}:-1<x<1\}$ defined by $f(x)=\frac{x}{1+|x|}, x\in\mathbb{R}$ is one-one and onto function.

34. Using the matrix method, solve the following system of linear equations :

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,\quad \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,\quad \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2.$$

35. Find the coordinates of the image of the point (1, 6, 3) with respect to the line $\vec{r}=(\hat{j}+2\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$; where '$\lambda$' is a scalar. Also, find the distance of the image from the y-axis.

OR

An aeroplane is flying along the line $\vec{r}=\lambda(\hat{i}-\hat{j}+\hat{k})$; where '$\lambda$' is a scalar and another aeroplane is flying along the line $\vec{r}=\hat{i}-\hat{j}+\mu(-2\hat{j}+\hat{k})$; '$\mu$' is a scalar. At what points on the lines should they reach, so that the distance between them is the shortest? Find the shortest possible distance between them.

Section E

This section comprises of 3 case-study/passage-based questions of 4 marks each with sub parts. The first two case study questions have three sub-parts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two sub parts of 2 marks each.

36. Read the following passage and answer the questions given below.

In an office three employees James, Sophia and Oliver process incoming copies of a certain form. James processes 50% of the forms, Sophia processes 20% and Oliver the remaining 30% of the forms. James has an error rate of 0.06, Sophia has an error rate of 0.04 and Oliver has an error has an error rate of 0.03. Based on the above information, answer the following questions.

(i) Find the probability that Sophia processed the form and committed an error.
(ii) Find the total probability of committing an error in processing the form.
(iii) The manager of the company wants to do a quality check. During inspection, he selects a form at random from the days output of processed form. If the form selected at random has an error, find the probability that the form is not processed by James.

OR

Let E be the event of committing an error in processing the form and let E_1, E_2 and E_3 be the events that James, Sophia and Oliver processed the form. Find the value of $\sum_{i=1}^{3} P(E_i/E)$.

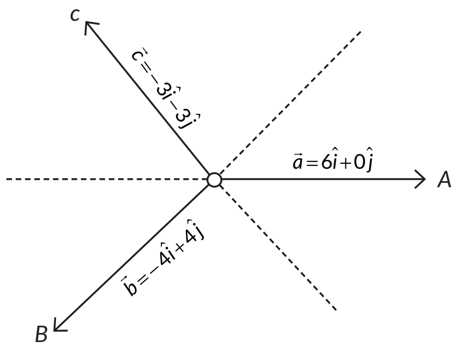
37. Read the following passage and answer the questions given below :

Teams A, B, C went for playing a tug of war game. Teams A, B, C have attached a rope to a metal ring and is trying to pull the ring into their own area.

Team A pulls with force $F_1 = 6\hat{i} + 0\hat{j}\, kN$,

Team B pulls with force $F_2 = -4\hat{i} + 4\hat{j}\, kN$,

Team C pulls with force $F_3 = -3\hat{i} - 3\hat{j}\, kN$,



(i) What is the magnitude of the force of Team A?

(ii) Which team will win the game?

(iii) Find the magnitude of the resultant force exerted by the teams.

OR

In which direction is the ring getting pulled?

38. Read the following passage and answer the questions given below :

The relation between the height of the plant ('y' in cm) with respect to its exposure to the sunlight is governed by the following equation $y = 4x - \frac{1}{2}x^2$, where 'x' is the number of days exposed to the sunlight, for $x \le 3$.

(i) Find the rate of growth of the plant with respect to the number of days exposed to the sunlight.

(ii) Does the rate of growth of the plant increase or decrease in the first three days? What will be the height of the plant after 2 days?

Detailed SOLUTIONS

1. **(d) :** Given, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. **(d) :** $(A+B)^{-1} \ne B^{-1} + A^{-1}$

3. **(b) :** Area $= \left| \frac{1}{2} \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} \right|$, given that the area

$= 9$ sq. unit.

$\Rightarrow \quad \pm 9 = \frac{1}{2} \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$; expanding along C_2, we get

$\Rightarrow \quad \pm 18 = -k(-3-3) \Rightarrow \pm 18 = 6k \Rightarrow k = \pm 3$

4. **(a) :** $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases}$

Since, f is continuous at $x = 0$,

$\Rightarrow$ L.H.L = R.H.L. $= f(0) \Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\Rightarrow \quad \lim_{x\to 0^-} \frac{-kx}{x} = \lim_{x\to 0^+} 3 = 3 \Rightarrow k = -3.$

5. **(d) :** Vectors $2\hat{i} + 3\hat{j} - 6\hat{k}$ and $6\hat{i} + 9\hat{j} - 18\hat{k}$ are parallel and the fixed point $\hat{i} + \hat{j} - \hat{k}$ on the line $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(2\hat{i} + 3\hat{j} - 6\hat{k})$ does not satisfy the other line $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(6\hat{i} + 9\hat{j} - 18\hat{k})$; where λ and μ are scalars.

6. **(c) :** Squaring the given differential equation, we get

$$\left[\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}\right]^2 = \left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2 \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

$\Rightarrow$ degree = 2

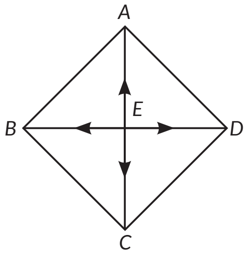
7. **(b) :**

Corner point	Value of $Z = px + qy$; $p, q > 0$
(0, 3)	$p \times 0 + q \times 3 = 3q$
(1, 1)	$p \times 1 + q \times 1 = p + q$
(3, 0)	$p \times 3 + q \times 0 = 3p$

The minimum of Z occurs at (3, 0) and (1, 1)

$\therefore \quad p+q=3p \Rightarrow p=\frac{q}{2}$

8. (a) : Given, $ABCD$ is a rhombus whose diagonals bisect each other. $|\vec{EA}|=|\vec{EC}|$ and $|\vec{EB}|=|\vec{ED}|$ but since they are opposite to each other so they are of opposite signs



$\Rightarrow \quad \vec{EA}=-\vec{EC}$ and $\vec{EB}=-\vec{ED}$.

$\Rightarrow \quad \vec{EA}+\vec{EC}=\vec{O}$ ...(i) and

$\vec{EB}+\vec{ED}=\vec{O}$ ...(ii)

Adding (i) and (ii), we get $\vec{EA}+\vec{EB}+\vec{EC}+\vec{ED}=\vec{O}$.

9. (b) : Let $f(x)=e^{\sin^2 x}\cos^3(2n+1)x$

$f(\pi-x)=e^{\sin^2(\pi-x)}\cos^3(2n+1)(\pi-x)$

$=-e^{\sin^2 x}\cos^3(2n+1)x=-f(x)$

$\therefore \quad \int_0^{\pi} e^{\sin^2 x}\cos^3(2n+1)x\,dx=0,$ as if f is integrable in $[0, 2a]$ and $f(2a-x)=-f(x)$ then $\int_0^{2a} f(x)\,dx=0$.

10. (b) : Given,

$$A=\begin{bmatrix}0 & 2x-1 & \sqrt{x}\\ 1-2x & 0 & 2\sqrt{x}\\ -\sqrt{x} & -2\sqrt{x} & 0\end{bmatrix}, A'=\begin{bmatrix}0 & 1-2x & -\sqrt{x}\\ 2x-1 & 0 & -2\sqrt{x}\\ \sqrt{x} & 2\sqrt{x} & 0\end{bmatrix}$$

Since, $A'=-A$,
Matrix A is a skew symmetric matrix of odd order.
$\therefore \quad |A|=0$

11. (c) : We observe, (0, 0) does not satisfy the inequality $x-y\geq 1$
So, the half plane represented by the above inequality will not contain origin therefore, it will not contain the shaded feasible region.

12. (b) : Given, $\vec{a}=4\hat{i}+6\hat{j}$ and $\vec{b}=3\hat{j}+4\hat{k}$

$\vec{a}\cdot\vec{b}=(4\hat{i}+6\hat{j})\cdot(3\hat{j}+4\hat{k})=4\times 0+6\times 3+0\times 4=18$

$|\vec{b}|=\sqrt{(3)^2+(4)^2}=\sqrt{9+16}=5$

$\therefore \quad |\vec{b}|^2=5^2=25$

Vector component of $\vec{a}$ along $\vec{b}=\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}=\frac{18}{25}(3\hat{j}+4\hat{k})$.

13. (d) : $|adj(2A)|=|(2A)|^2=(2^3|A|)^2=2^6|A|^2=2^6\times(-2)^2=2^8$.

14. (d) : Let A, B, C be the respective events of solving the problem. Then, $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$ and $P(C)=\frac{1}{4}$. Here, A,B,C are independent events.
Problem is solved if at least one of them solves the problem.
Required probability is $=P(A\cup B\cup C)=1-P(\bar{A})P(\bar{B})P(\bar{C})$

$=1-\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=1-\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}=1-\frac{1}{4}=\frac{3}{4}$.

15. (c) : Given, $ydx-xdy=0\Rightarrow ydx=xdy\Rightarrow \frac{dy}{y}=\frac{dx}{x}$;

Integrating both sides, we get $\int\frac{dy}{y}=\int\frac{dx}{x}$

$\log_e|y|=\log_e|x|+\log_e|c|$
Since $x, y, c>0$, we write $\log_e y=\log_e x+\log_e c\Rightarrow y=cx$.

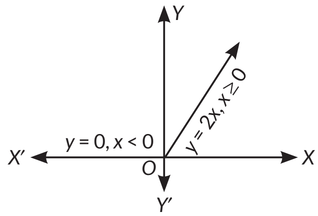
16. (d) : Dot product of two mutually perpendicular vectors is zero.

$(2\hat{i}-\hat{j}+2\hat{k})\cdot(3\hat{i}+\lambda\hat{j}+\hat{k})=0$

$\Rightarrow \quad 2\times 3+(-1)\lambda+2\times 1=0\Rightarrow 6-\lambda+2=0\Rightarrow\lambda=8$

17. (c) : Given,

$f(x)=x+|x|=\begin{cases}2x, x\geq 0\\ 0, x<0\end{cases}$



There is a sharp corner at $x=0$,
$\Rightarrow \quad f(x)$ is not differentiable at $x=0$.
Hence, $f(x)$ is differentiable when $x\in(-\infty, 0)\cup(0,\infty)$.

18. (d) : We know, $l^2+m^2+n^2=1$

$\Rightarrow \quad \left(\frac{1}{c}\right)^2+\left(\frac{1}{c}\right)^2+\left(\frac{1}{c}\right)^2=1\Rightarrow\frac{1}{c^2}+\frac{1}{c^2}+\frac{1}{c^2}=1$

$\Rightarrow \quad \frac{3}{c^2}=1\Rightarrow c=\pm\sqrt{3}$.

19. (a) : Given, $\frac{d}{dx}(f(x))=(x-1)^3(x-3)^2$

Assertion : $f(x)$ has minimum at $x=1$ is true as

$\frac{d}{dx}(f(x))<0, \forall\, x\in(1-h,1)$ and $\frac{d}{dx}(f(x))>0, \forall\, x\in(1,1+h)$; where '$h$' is an infinitesimally small positive quantity, which is in accordance with the Reason statement.

$\therefore$ Both (A) and (R) are true and (R) is the correct explanation of (A).

20. (d) : The element 4 has no image under $f\Rightarrow$ relation f is not a function. So, Assertion is false.
The given function $f:\{1, 2, 3\}\to\{x, y, z, p\}$ is one – one as for each element of {1, 2, 3}, there is different image in $\{x, y, z, p\}$ under f.
$\therefore$ Reason is true.

21. Given,

$\sin^{-1}\left(\cos\left(\frac{33\pi}{5}\right)\right)=\sin^{-1}\left[\cos\left(6\pi+\frac{3\pi}{5}\right)\right]=\sin^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right]$

($\because \cos(2\pi+x)=\cos x$)

$=\frac{\pi}{2}-\cos^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right]$

$=\frac{\pi}{2}-\frac{3\pi}{5}$ ($\because \cos^{-1}(\cos x)=x$)

$=-\frac{\pi}{10}$.

OR

Given, $\sin^{-1}(x^2-4)$
Since the domain of $\sin^{-1}x$ is $[-1, 1]$
$-1 \le (x^2-4) \le 1$
$\Rightarrow\quad 3 \le x^2 \le 5 \Rightarrow \sqrt{3} \le |x| \le \sqrt{5}$
$\Rightarrow\quad x \in \left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, \sqrt{5}\right].$
Hence, the required domain is $\left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, \sqrt{5}\right]$.

22. We have, $f(x) = xe^x \Rightarrow f'(x) = xe^x + e^x = e^x(x+1)$
For $f(x)$ to be increasing, we have $f'(x) = e^x(x+1) \ge 0$
$\Rightarrow\quad x \ge -1$ as $e^x > 0, \forall x \in \mathbb{R}$
Hence, the required interval where $f(x)$ increases is $[-1, \infty)$.

23. Given, $f(x) = \dfrac{1}{4x^2+2x+1}$
On differentiating w.r.t. x, we get
$$f'(x) = \frac{(4x^2+2x+1)\cdot 0 - 1\cdot(8x+2)}{(4x^2+2x+1)^2} = \frac{-(8x+2)}{(4x^2+2x+1)^2} \quad \text{...(i)}$$
For maxima or minima, we put $f'(x) = 0$,
$$\Rightarrow \frac{-(8x+2)}{(4x^2+2x+1)^2} = 0 \Rightarrow 8x+2=0 \Rightarrow x = -\frac{1}{4}.$$
Again differentiating equation (i) w.r.t x, we get
$$f''(x) = -\left\{\frac{(4x^2+2x+1)^2(8) - (8x+2)2\times(4x^2+2x+1)(8x+2)}{(4x^2+2x+1)^4}\right\}$$
At $x = -\frac{1}{4}, f''\left(-\frac{1}{4}\right) < 0$

$f(x)$ is maximum at $x = -\frac{1}{4}$.
$\therefore$ maximum value of $f(x)$ is
$$f\left(-\frac{1}{4}\right) = \frac{1}{4\left(-\frac{1}{4}\right)^2 + 2\left(-\frac{1}{4}\right) + 1} = \frac{4}{3}.$$

OR

For maxima and minima, $P'(x) = 0 = \dfrac{d}{dx}(72+42x-x^2) = 0$
$\Rightarrow\quad 42 - 2x = 0 \Rightarrow x = 21$ and $P''(x) = -2 < 0$
So, $P(x)$ is maximum at $x = 21$.
The maximum value of $P(x) = 72 + 42 \times 21 - (21)^2 = 513$ i.e., the maximum profit is ₹ 513.

24. Let $f(x) = \log_e\left(\dfrac{2-x}{2+x}\right)$

We have, $f(-x) = \log_e\left(\dfrac{2+x}{2-x}\right) = \log_e\left(\dfrac{2-x}{2+x}\right) = -f(x)$

$\Rightarrow\quad f(x)$ is an odd function.
By the property
$$\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(-x) = -f(x) \\ 2\int_0^a f(x),dx & \text{if } (-x) = f(x) \end{cases}$$
$$\therefore \int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right)dx = 0.$$

25. $f(x) = x^3 + x$, for all $x \in \mathbb{R}$.
On differentiating w.r.t. 'x', we get
$f'(x) = 3x^2 + 1$; for all $x \in \mathbb{R}, x^2 \ge 0$
$\Rightarrow\quad f'(x) > 0$
Hence, there is no point where the function is undefined and its derivative is zero or undefined.
Hence, no critical point exists.

26. Given, $\dfrac{2x^2+3}{x^2(x^2+9)}$. Now, let $x^2 = t$.
The form of the partial fraction decomposition
$$\Rightarrow \frac{2t+3}{t(t+9)} = \frac{A}{t} + \frac{B}{t+9} \Rightarrow \frac{2t+3}{t(t+9)} = \frac{A(t+9)+Bt}{t(t+9)}$$
$\Rightarrow\quad 2t + 3 = t(A+B) + 9A \therefore A + B = 2, 9A = 3$
On solving, we get $A = \dfrac{1}{3}$ and $B = \dfrac{5}{3}$
$$\int \frac{2x^2+3}{x^2(x^2+9)}dx = \frac{1}{3}\int\frac{dx}{x^2} + \frac{5}{3}\int\frac{dx}{x^2+9} = -\frac{1}{3x} + \frac{5}{9}\tan^{-1}\left(\frac{x}{3}\right) + c,$$
Where 'c' is an arbitrary constant of integration.

27. We have, (i) $\sum P(X_i) = 1 \Rightarrow k + 2k + 3k = 1 \Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$.

(ii) $P(X<2) = P(X=0) + P(X=1) = k + 2k = 3k = 3\times\frac{1}{6} = \frac{1}{2}$.

(iii) $P(X > 2) = 0$.

28. Let $x^{\frac{3}{2}} = t \Rightarrow dt = \frac{3}{2}x^{\frac{1}{2}}dx$
The integration becomes
$$\int\sqrt{\frac{x}{1-x^3}}\,dx = \int\frac{\sqrt{x}}{\sqrt{1-t^2}} \times \frac{2}{3\sqrt{x}}\,dt$$
$$\Rightarrow \int\sqrt{\frac{x}{1-x^3}}\,dx = \frac{2}{3}\int\frac{dt}{\sqrt{1-t^2}} = \frac{2}{3}\sin^{-1}(t) + c$$
$= \frac{2}{3}\sin^{-1}\left(x^{\frac{3}{2}}\right) + c$, where, '$c$' is an arbitrary constant of integration.

OR

Let $I = \int_0^{\frac{\pi}{4}} \log_e(1+\tan x)\,dx$ ...(i)
$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log_e\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx,$$
$$\left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right)$$
$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log_e\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$
$$= \int_0^{\frac{\pi}{4}} \log_e\left(\frac{2}{1+\tan x}\right)dx = \int_0^{\frac{\pi}{4}} \log_e 2\,dx - I \quad \text{[Using ...(i)]}$$
$$\Rightarrow 2I = \frac{\pi}{4}\log_e 2 \Rightarrow I = \frac{\pi}{8}\log_e 2.$$

29. We have, $ye^{x/y}dx = (xe^{x/y} + y^2)dy \Rightarrow \dfrac{dx}{dy} = \dfrac{xe^{\frac{x}{y}} + y^2}{y\cdot e^{\frac{x}{y}}}$
$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + \frac{y}{e^{\frac{x}{y}}} \quad \text{...(i)}$$

Let $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$;

So, equation (i) becomes $v + y\frac{dv}{dy} = v + \frac{y}{e^v}$

$\Rightarrow \quad y\frac{dv}{dy} = \frac{y}{e^v} \Rightarrow e^v\, dv = dy$

On integrating we get, $\int e^v dv = \int dy \Rightarrow e^v = y + c \Rightarrow e^{x/y} = y + c$

Where 'c' is an arbitrary constant of integration.

OR

The given differential equation is

$(\cos^2 x)\frac{dy}{dx} + y = \tan x$

Dividing both the sides by $\cos^2 x$, we get

$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$

$\frac{dy}{dx} + y(\sec^2 x) = \tan x(\sec^2 x)$ ...(i)

Clearly, it is a linear differential equation.

On comparing with $\frac{dy}{dx} + Py = Q$, we get

$P = \sec^2 x$, $Q = \tan x. \sec^2 x$

The integrating factor is, $I.F. = e^{\int P dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

On multiplying the equation (i) by $e^{\tan x}$, we get

$\frac{d}{dx}(y.e^{\tan x}) = e^{\tan x} \tan x \sec^2 x \Rightarrow d(y.e^{\tan x}) = e^{\tan x} \tan x \sec^2 x\, dx$

On integrating we get, $y.e^{\tan x} = \int t.e^t\, dt + c_1$; where,

$t = \tan x$; $dt = \sec^2 x dx$

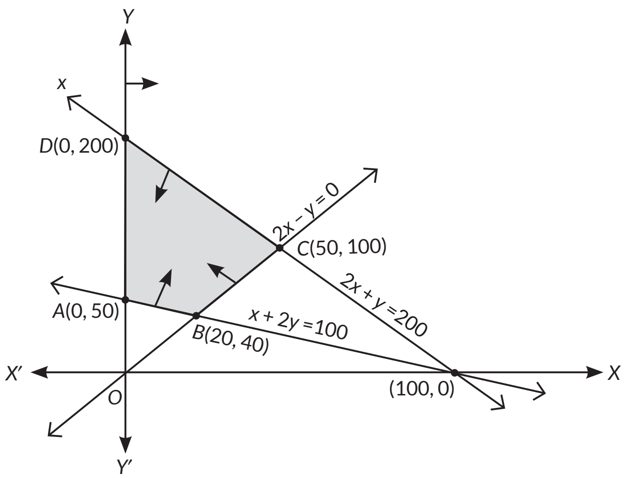
$= te^t - e^t + c = (\tan x)\, e^{\tan x} - e^{\tan x} + c$

$\therefore \quad y = \tan x - 1 + c.\ (e^{-\tan x})$, where '$c_1$' & '$c$' are arbitrary constants of integration.

30. Given, First, convert the inequation into equation

$x + 2y = 100$, $2x - y = 0$, $2x + y = 200$, $x, y = 0$

Now, draw the graph.



$A(0, 50)$, $B(20, 40)$, $C(50, 100)$ and $D(0, 200)$ are the corner points of the feasible region.

The values of Z at these corner points are given below :

Corner point	Corresponding value of $Z = x + 2y$	
A (0, 50)	100	Minimum
B (20, 40)	100	Minimum
C(50, 100)	250	
D(0, 200)	400	

Since, the minimum value is at two points *i.e.*, $A(0, 50)$ and $B(20, 40)$

$\Rightarrow$ The minimum value of Z is 100 at all the points on the line segment joining the points $A(0, 50)$ and $B(20, 40)$.

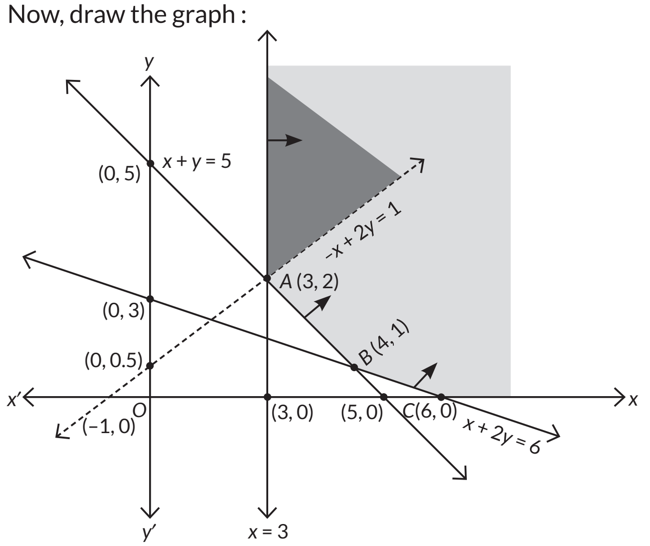
OR

Given, $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$.

First, covert the inequations into equation.

$x = 3$, $x + y = 5$, $x + 2y = 6$, $y = 0$,

Now, draw the graph :



Here, it can be seen that the feasible region is unbounded. The values of Z at corner points $A(3, 2)$, $B(4, 1)$ and $C(6, 0)$ are given below.

Corner point	Corresponding value of $Z = -X + 2y$
A (3, 2)	1 (may or may not be the maximum value)
B (4, 1)	–2
C (6, 0)	–6

Since the feasible region is unbounded, $Z = 1$ may or may not be the maximum value.

Now, we draw the graph of the inequality, $-x + 2y > 1$, and we check whether the resulting open half-plane has any point/s, in common with the feasible region or not.

Here, the resulting open half plane has points in common with the feasible region.

Hence, $Z = 1$ is not the maximum value. We conclude, Z has no maximum value.

31. Given, $(a+bx)e^{y/x} = x \Rightarrow e^{y/x} = \dfrac{x}{(a+bx)}$

Taking log on both sides, we get

$\dfrac{y}{x} = \log_e\left(\dfrac{x}{a+bx}\right) = \log_e x - \log_e(a+bx)$

On differentiating w.r.t. 'x', we get

$$\Rightarrow \quad \frac{x\frac{dy}{dx}-y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}\frac{d}{dx}(a+bx) = \frac{1}{x} - \frac{b}{a+bx}$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = x^2\left(\frac{1}{x} - \frac{b}{a+bx}\right) \Rightarrow x\frac{dy}{dx} - y = x^2\left(\frac{a+bx-bx}{x(a+bx)}\right)$$

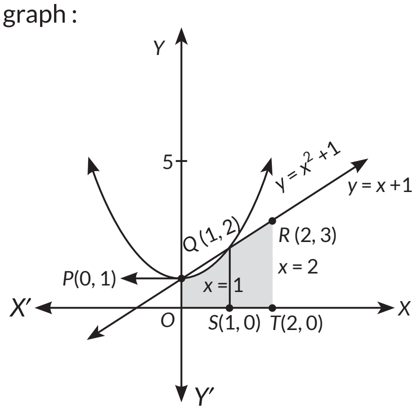
On differentiating again w.r.t. 'x', we get

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax(b)}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \left(\frac{a}{a+bx}\right)^2$$

32. Given, $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

Draw the graph :



$y = x^2 + 1$ ...(i) ; $\quad y = x + 1$ ..(ii)

By (i) and (ii), we get

$x^2 + 1 = x + 1 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0, 1.$

So, the point of intersections $P(0, 1)$ and $Q(1, 2)$.

Area of the shaded region $OPQRTSO$

= Area of the region $OSQPO$ + Area of the region $STRQS$

$$= \int_0^1 (x^2+1)\,dx + \int_1^2 (x+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^1 + \left[\frac{x^2}{2}+x\right]_1^2$$

$$= \left[\frac{1}{3}+1-0\right] + \left[(2+2) - \left(\frac{1}{2}+1\right)\right] = \frac{23}{6}$$

Hence, the required area is $\dfrac{23}{6}$ sq. unit.

33. Let (a, b) be an arbitrary element of $\mathbb{N} \times \mathbb{N}$. Then $(a, b) \in \mathbb{N} \times \mathbb{N}$ and $a, b \in \mathbb{N}$

We have, $ab = ba$; (As $a, b \in \mathbb{N}$ and multiplication is commutative on $\mathbb{N}$)

$\Rightarrow$ $(a, b)\ R\ (a, b)$, according to the definition of the relation R on $\mathbb{N} \times \mathbb{N}$

Thus $(a, b)\ R\ (a, b)$, $\forall\ (a, b) \in \mathbb{N} \times \mathbb{N}$

So, R is reflexive relation of $\mathbb{N} \times \mathbb{N}$

Let (a, b), (c, d) be arbitrary elements of $\mathbb{N} \times \mathbb{N}$ such that $(a, b)\ R(c, d)$.

Then, $(a, b)\ R(c, d) \Rightarrow ad = bc \Rightarrow bc = ad$; (changing LHS and RHS)

$\Rightarrow cb = da$; (As $a, b, c, d \in \mathbb{N}$ and multiplication is commutative on $\mathbb{N}$)

$\Rightarrow$ $(c, d)\ R(a, b)$; according to the definition of the relation R on $\mathbb{N} \times \mathbb{N}$

Thus $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$

So, R is symmetric relation on $\mathbb{N} \times \mathbb{N}$

Let (a, b), (c, d), (e, f) be arbitrary elements of $\mathbb{N} \times \mathbb{N}$ such that

$(a, b)\ R(c, d)$ and $(c, d)\ R(e, f)$.

Then $\left.\begin{matrix}(a,b)R(c,d) \Rightarrow ad = bc \\ (c,d)R(e,f) \Rightarrow cf = de\end{matrix}\right\} \Rightarrow (ad)(cf) = (bc)(de) \Rightarrow af = be$

$\Rightarrow$ $(a, b)R(e, f)$; (according to the definition of the relation R on $\mathbb{N} \times \mathbb{N}$)

Thus $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$

So, R is transitive relation of $\mathbb{N} \times \mathbb{N}$.

As the relation R is reflexive, symmetric and transitive so, it is equivalence relation on $\mathbb{N} \times \mathbb{N}$.

$[(2, 6)] = \{(x, y) \in \mathbb{N} \times \mathbb{N} : (x, y)\ R\ (2, 6)\}$

$= \{(x, y) \in \mathbb{N} \times \mathbb{N} : 3x = y\} = \{(x, 3x) : x \in \mathbb{N}\} = \{(1, 3), (2, 6), (3, 9), ...\}$

OR

We have, $f(x) = \begin{cases} \dfrac{x}{1+x}, \text{if } x \ge 0 \\ \dfrac{x}{1-x}, \text{if } x < 0 \end{cases}$

Now, we consider the following cases

Case 1 : when $x \ge 0$, we have $f(x) = \dfrac{x}{1+x}$

Injectivity : let $x, y \in \mathbb{R}^+ \cup \{0\}$ such that $f(x) = f(y)$, then

$\Rightarrow \dfrac{x}{1+x} = \dfrac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$

So, f is injective function.

Surjectivity : when $x \ge 0$, we have $f(x) = \dfrac{x}{1+x} \ge 0$ and $f(x) = 1 - \dfrac{1}{1+x} < 1$, as $x \ge 0$

Let $y \in [0, 1)$, thus for each $y \in [0, 1)$, there exists $x = \dfrac{y}{1-y} \ge 0$

such that $f(x) = \dfrac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = y.$

So, f is onto function on $[0, \infty)$ to $[0, 1)$.

Case 2 : when $x < 0$, we have $f(x) = \dfrac{x}{1-x}$

Injectivity : Let $x, y \in \mathbb{R}^-$ i.e., $x, y < 0$, such that $f(x) = f(y)$, then

$\Rightarrow \dfrac{x}{1-x} = \dfrac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$

So, f is injective function.

Surjectivity : $x < 0$, we have $f(x) = \dfrac{x}{1-x} < 0$ also, $f(x) = \dfrac{x}{1-x} = -1 + \dfrac{1}{1-x} > -1$; $-1 < f(x) < 0$

Let $y \in (-1, 0)$, be an arbitrary real number and there exists

$x = \dfrac{y}{1+y} < 0$ such that, $f(x) = f\left(\dfrac{y}{1+y}\right) = \dfrac{\frac{y}{1+y}}{1-\frac{y}{1+y}} = y.$

So, for $y \in (-1, 0)$, there exists $x = \frac{y}{1+y} < 0$ such that $f(x) = y$.

Hence, f is onto function on $(-\infty, 0)$ to $(-1, 0)$

Case 3 : Injectivity : Let $x > 0$ & $y < 0$ such that $f(x) = f(y)$

$\Rightarrow \frac{x}{1+x} = \frac{y}{1-y}$

$\Rightarrow$ $x - xy = y + xy \Rightarrow x - y = 2xy$, here LHS > 0 but RHS < 0, which is inadmissible.

Hence, $f(x) \neq f(y)$ when $x \neq y$.

Hence f is one-one and onto function.

34. The given system of equations can be written in the form $AX = B$,

Where, $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Now,

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120-45) - 3(-80-30) + 10(36+36)$$

$= 2(75) - 3(-110) + 10(72) = 150 + 330 + 720 = 1200 \neq 0$

$\therefore A^{-1}$ exists.

$$\therefore adj\ A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

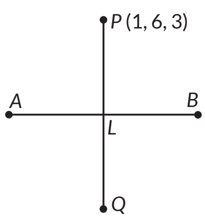
$$\text{Hence, } A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\text{As, } AX = B \Rightarrow X = A^{-1}B \Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300 & 150 & 150 \\ 440 & -100 & 60 \\ 288 & 0 & -48 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

Thus, $\frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{5}$ Hence, $x = 2, y = 3, z = 5$.

35. Let $P(1, 6, 3)$ be the given point, and let 'L' be the foot of the perpendicular from 'P' to the given line AB (as shown in the figure below).



The coordinates of a general point on the given line are given by

$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda; \lambda$ is a scalar,

i.e., $x = \lambda$, $y = 2\lambda + 1$ and $z = 3\lambda + 2$

Let the coordinates of L be $(\lambda, 2\lambda + 1, 3\lambda + 2)$.

So, direction ratios of PL are $\lambda - 1$, $2\lambda + 1 - 6$ and $3\lambda + 2 - 3$, i.e. $\lambda - 1$, $2\lambda - 5$ and $3\lambda - 1$.

Direction ratios of the given line are 1, 2 and 3, which is perpendicular to PL.

Therefore, $(\lambda - 1)\,1 + (2\lambda - 5)\,2 + (3\lambda - 1)3 = 0$

$\Rightarrow \lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0 \Rightarrow 14\lambda - 14 = 0 \Rightarrow \lambda = 1$

So, coordinates of L are $(1, 3, 5)$.

Let $Q(x_1, y_1, z_1)$ be the image of $P(1, 6, 3)$ in the given line.

Then, L is the mid-point of PQ.

Therefore,

$$\frac{(x_1+1)}{2} = 1, \frac{(y_1+6)}{2} = 3 \text{ and } \frac{(z_1+3)}{2} = 5 \Rightarrow x_1 = 1, y_1 = 0 \text{ and } z_1 = 7$$

Hence, the image of $P(1, 6, 3)$ in the given line is $(1, 0, 7)$.

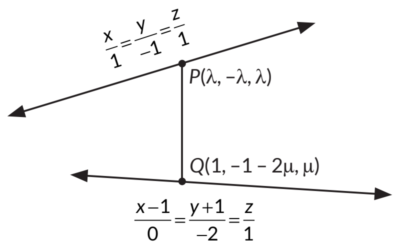
Now, the distance of the point $(1, 0, 7)$ from the y-axix is $\sqrt{1^2 + 7^2} = \sqrt{50}$ units.

OR

Given, an aeroplane is flying along the line $\vec{r} = \lambda(\hat{i} - \hat{j} + \hat{k})$.

Another aeroplane is flying along the line

$\vec{r} = \hat{i} - \hat{j} + \mu(-2\hat{j} + \hat{k})$



The equation of two given straight lines in the Cartesian form are $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$ ...(i) and $\frac{x-1}{0} = \frac{y+1}{-2} = \frac{z}{1}$ ... (ii)

The lines are not parallel as direction ratios are not proportional. Let P be a point on straight line (i) and Q be a point on straight line (ii) such that line PQ is perpendicular to both of the lines.

Let the coordinates of P be $(\lambda, -\lambda, \lambda)$ and that of Q be $(1, -2\mu - 1, \mu)$; where 'λ' and 'μ' are scalars.

Then the direction ratios of the line PQ are $(\lambda - 1, -\lambda + 2\mu + 1, \lambda - \mu)$

Since PQ is perpendicular to straight line (i), we have,

$(\lambda - 1).1 + (-\lambda + 2\mu + 1).(-1) + (\lambda - \mu).1 = 0$

$\Rightarrow \quad 3\lambda - 3\mu = 2$ ...(iii)

Since PQ is perpendicular to straight line (ii), we have,

$0.(\lambda - 1) + (-\lambda + 2\mu + 1).(-2) + (\lambda - \mu).1 = 0 \Rightarrow 3\lambda - 5\mu = 2$ ...(iv)

Solving (iii) and (iv), we get $\mu = 0$, $\lambda = \frac{2}{3}$

Therefore, the Coordinates of P are $\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)$ and that of Q are $(1, -1, 0)$

So, the required shortest distance

$$= \sqrt{\left(1 - \frac{2}{3}\right)^2 + \left(-1 + \frac{2}{3}\right)^2 + \left(0 - \frac{2}{3}\right)^2} = \sqrt{\frac{2}{3}} \text{ units}$$

36. Let E_1, E_2, E_3 be the events that James, Sophia and Oliver processed the form, which are clearly pairwise mutually exclusive and exhausive set of events.

Then $P(E_1)=\frac{50}{100}=\frac{5}{10}, P(E_2)=\frac{20}{100}=\frac{1}{5}$ and $P(E_3)=\frac{30}{100}=\frac{3}{10}$.

Also, let E be the event of committing an error.
We have, $P(E|E_1) = 0.06$, $P(E|E_2) = 0.04$, $P(E|E_3) = 0.03$.
(i) The probability that Sophia processed the form and committed an error is given by

$$P(E\cap E_2)=P(E_2)\cdot P(E|E_2)=\frac{1}{5}\times 0.04=0.008.$$

(ii) The total probability of committing an error in processing the form is given by
$P(E) = P(E_1).P(E|E_1) + P(E_2).P(E|E_2) + P(E_3).P(E|E_3)$

$$P(E)=\frac{50}{100}\times 0.06+\frac{20}{100}\times 0.04+\frac{30}{100}\times 0.03=0.047.$$

(iii) The probability that the form is processed by James given that form has an error is given by

$$P(E_1|E)=\frac{P(E|E_1)\times P(E_1)}{P(E|E_1)\cdot P(E_1)+P(E|E_2)\cdot P(E_2)+P(E|E_3)\cdot P(E_3)}$$

$$=\frac{0.06\times\frac{50}{100}}{0.06\times\frac{50}{100}+0.04\times\frac{20}{100}+0.03\times\frac{30}{100}}=\frac{30}{47}$$

Therefore, the required probability that the form is not processed by James given that form has an error

$$= P(\bar{E}_1|E)=1-P(E_1|E)=1-\frac{30}{47}=\frac{17}{47}.$$

OR

$$\sum_{i=1}^{3}P(E_i|E)=P(E_1|E)+P(E_2|E)+P(E_3|E)=1$$

Since, sum of the posterior probability is 1.

(We have, $\sum_{i=1}^{3}P(E_i|E)=P(E_1|E)+P(E_2|E)+P(E_3|E)$

$$=\frac{P(E\cap E_1)+P(E\cap E_2)+P(E\cap E_3)}{P(E)}$$

$$=\frac{P(E\cap E_1)\cup P(E\cap E_2)\cup P(E\cap E_3)}{P(E)}$$ as E_i & $E_j; i\neq j$, are mutually exclusive events

$$=\frac{P(E\cap(E_1\cup E_2\cup E_3))}{P(E)}=\frac{P(E\cap S)}{P(E)}=\frac{P(E)}{P(E)}=1;$$

'S' being the sample space)

37. We have,

$$|\vec{F}_1|=\sqrt{6^2+0}=6\,kN, |\vec{F}_2|=\sqrt{(-4)^2+4^2}=\sqrt{32}=4\sqrt{2}kN,$$

$$|\vec{F}_3|=\sqrt{(-3)^2+(-3)^2}=\sqrt{18}=3\sqrt{2}kN$$

(i) Magnitude of force of Team A = 6kN.
(ii) Since, 6 kN is largest so, team A will win the game.
(iii) $\vec{F}=\vec{F}_1+\vec{F}_2+\vec{F}_3=6\hat{i}+0\hat{j}-4\hat{i}+4\hat{j}-3\hat{i}-3\hat{j}=-\hat{i}+\hat{j}$

$$\therefore |\vec{F}|=\sqrt{(-1)^2+(1)^2}=\sqrt{2}\,kN.$$

OR

$$\vec{F}=-\hat{i}+\hat{j}$$

$\therefore\ \theta=\pi-\tan^{-1}\left(\frac{1}{1}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$; where '$\theta$' is the angle made by the resultant force with the +ve direction of the x-axis.

38. (i) $y=4x-\frac{1}{2}x^2$

To find the growth rate, differentiate the given equation w.r.t. 'x', we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(4x-\frac{1}{2}x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=4-\frac{2}{2}x \Rightarrow \frac{dy}{dx}=4-x$$

$\therefore$ The rate of growth of the plant with respect to the number of days exposed to sunlight is given by $\frac{dy}{dx}=4-x$.

(ii) Let rate of growth be represented by the function $g(x)=\frac{dy}{dx}$.

Now, $g'(x)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-1<0$

$\Rightarrow g(x)$ decreases
So, the rate of growth of the plant decreases for the first three days.

Height of the plant after 2 days is $y=4\times 2-\frac{1}{2}(2)^2=6$ cm.