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2023-24 • Fully Solved
Released on 31st March, 2023

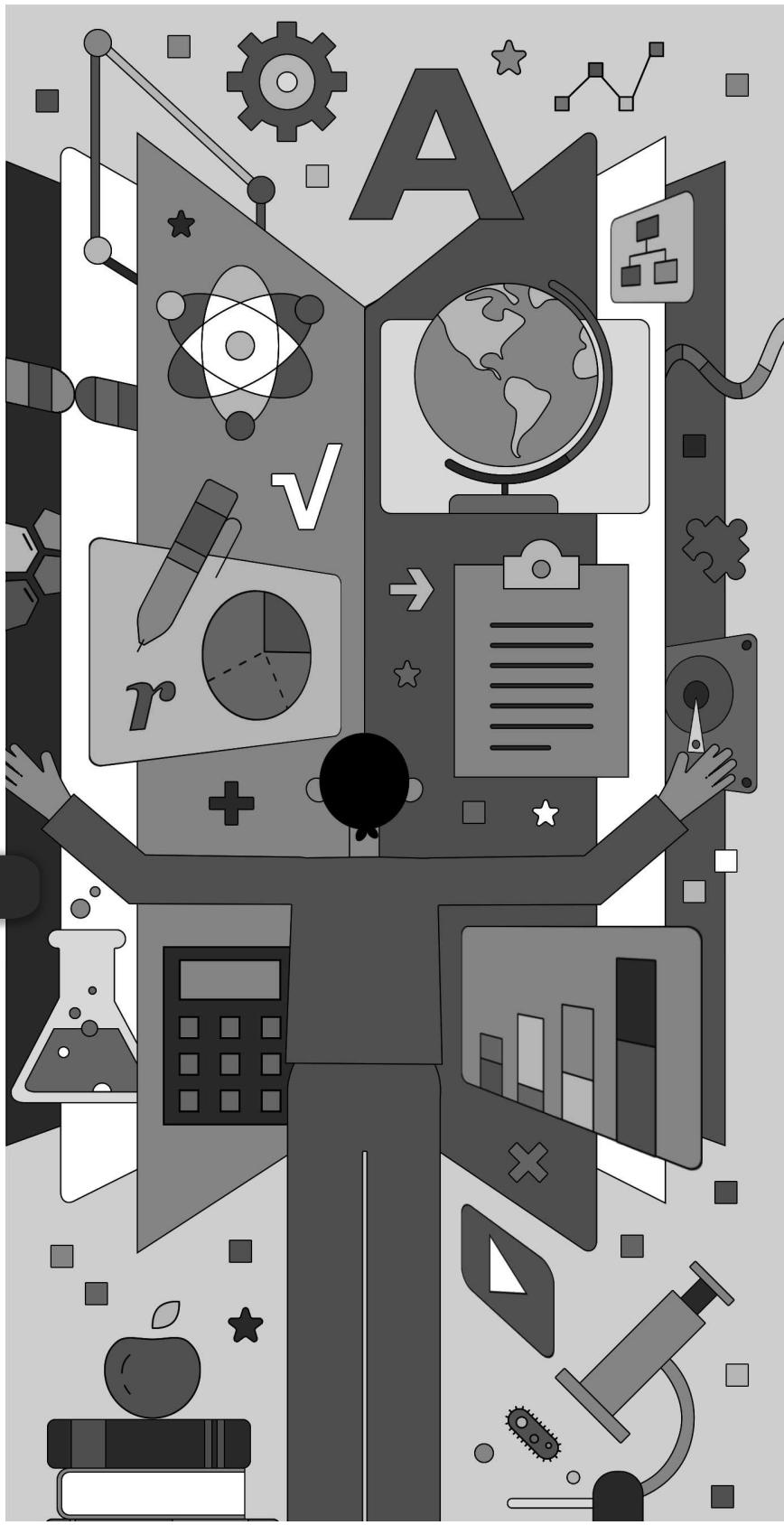
CBSE

Chapterwise & Topicwise

SOLVED PAPERS CLASS 12 QUESTION BANK

APPLIED MATHEMATICS

Strictly as per the latest CBSE Syllabus released on 31st March, 2023 (CBSE Cir No. Acad-39/2023)



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3rd EDITION

YEAR 2023-24

ISBN "9789356349506"

SYLLABUS COVERED

CENTRAL BOARD OF SECONDARY EDUCATION DELHI

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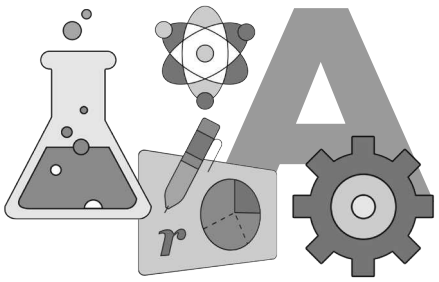

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In each chapter, for better understanding, questions have been classified the typology issued by CBSE as :

R - Remembering, U - Understanding, OEB Oswaal Editorial Board

A - Applying,

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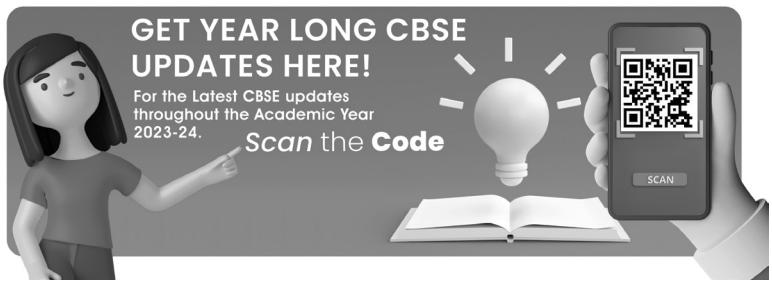
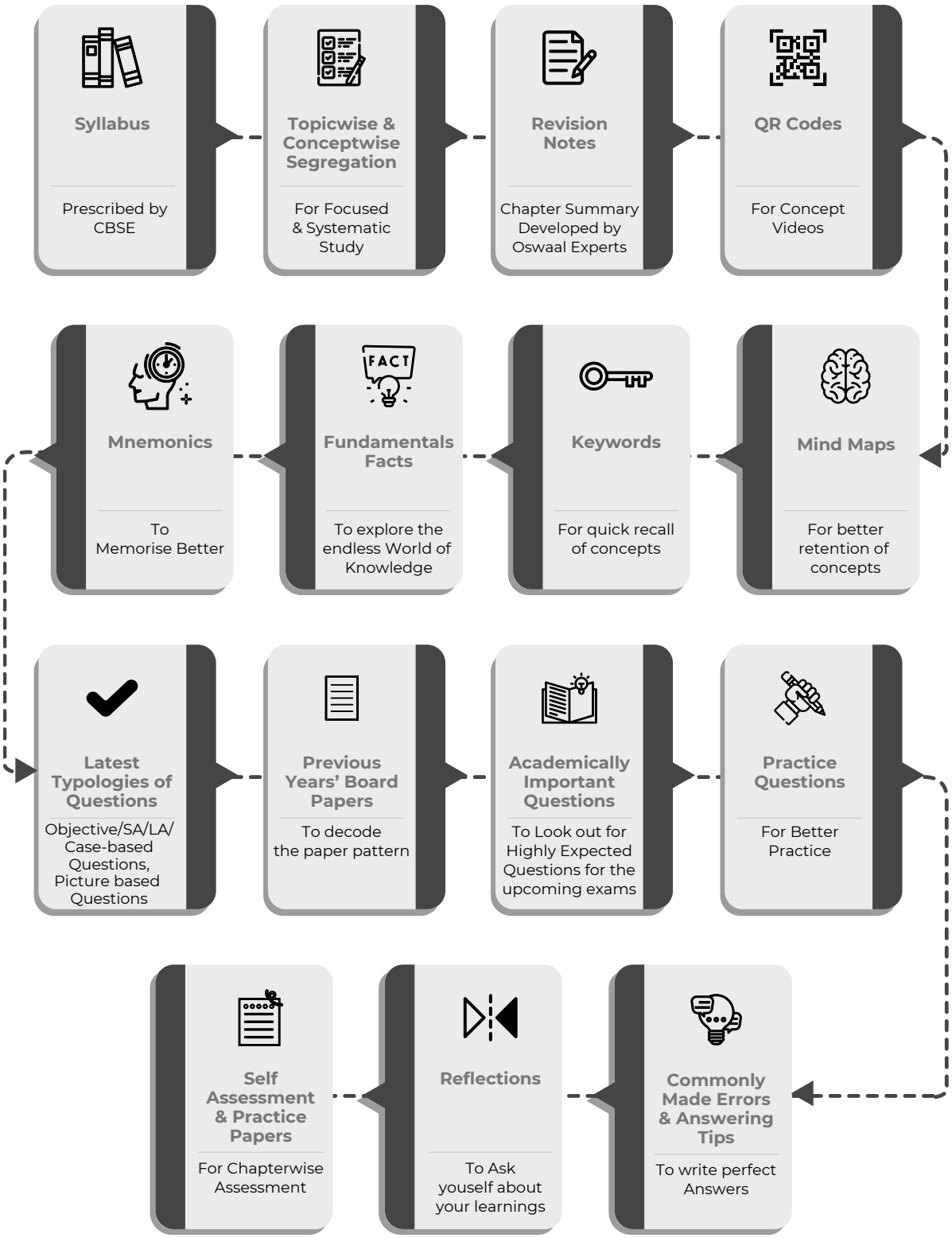


How to use this Book

Chapter Navigation Tools



What is on your wishlist for this Academic Year?

- Do better than the previous year.
- Perfect every concept, every topic, and every question from the very beginning.

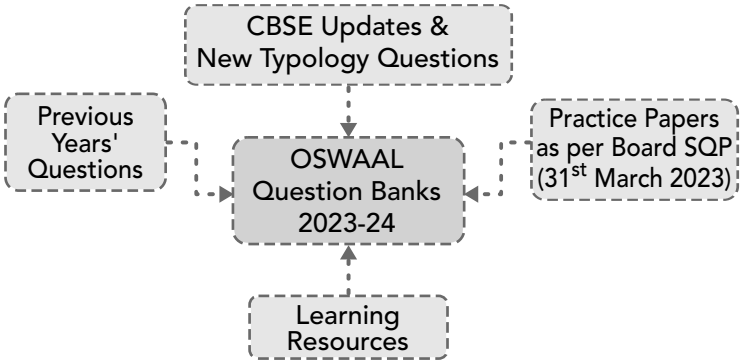
You said it, we heard it!

Practice means to perform, repeatedly in the face of all obstacles, some act of vision, of faith, of desire. Practice is a means of inviting the perfection desired.

-Martha Graham

As we usher into a brand-new Academic Year 2023-24, Oswaal Books, with its all-new Question Banks, empowers you to perfect your learning, consistently!

These Question Banks have been updated for 2023-24 with utmost care. They are a unique blend of all the **CBSE Board Updates, Previous Years' Questions,** and specially curated Questions as per the **Latest Typologies** along with best-in-class **Learning Resources**.



All these together will charge you with the much-needed confidence to face the boards and emerge champions. But what makes it so Unique?

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For those who are looking to ramp up their preparation and to 'PERFECT' every nuance of concepts studied, these Question Banks are a must in your Boards arsenal. This is the perfect time to start your exciting journey with these Question Banks and fill in learning gaps, throughout the year with utmost ease & confidence.

This Question Bank would not have been made possible without the valuable contributions of the esteemed members of the Oswaal Editorial Board-Authors, Editors, Subject matter experts, Proofreaders & DTP operators who worked day and night to bring this incredible book to you. We are also highly grateful to our dear students for all their valuable and impeccable inputs in the making of this one-of-a-kind exam preparation tool.

All the best Students!! Be the perfectionist that you are!

Team Oswaal Books

CBSE CIRCULAR 2023-24

केन्द्रीय माध्यमिक शिक्षा बोर्ड
(शिक्षा मंत्रालय भारत सरकार के अधीन एक स्वायत्त संगठन)
CENTRAL BOARD OF SECONDARY EDUCATION
(An Autonomous Organisation Under the Ministry of Education, Govt. of India)

F.1001/CBSE-Acad/Curriculum/2023 **March 31, 2023**

Cir No: Acad-39/2023

All Heads of Institutions affiliated to CBSE

Subject : Secondary and Senior School Curriculum and Sample Question Papers for the session 2023-24

1. CBSE annually provides the curriculum for classes IX to XII containing academic content, syllabus for examinations with learning outcomes, pedagogical practices, and assessment guidelines.
2. It is important that schools ensure curriculum transactions as per the directions given in the initial pages of the Curriculum document. The subjects should be taught as per the curriculum given by suitably incorporating strategies such as Art-Integrated Education, Experiential Learning, Pedagogical Plans, etc. wherever possible.
3. As CBSE has adopted National Curriculum Framework for Foundational Stage – 2022, schools offering foundational or preparatory education are advised to adhere to the recommendations regarding curriculum, pedagogy, assessment, and other areas described in detail in the NCFFS-2022 and guidelines of the Board issued from time to time.
4. Sample Question Papers with detailed designs of the Question Paper are also available on CBSE's website to reflect the impact of changes made in the curriculum. SQPs also provide students with an idea of the exam pattern and the type of questions that may be asked in the actual examination in order to ensure transparency and reduce stress. Students will also get a clear understanding of the weightage of different topics and the marking scheme to be followed by going through these Sample Question Papers.

   Schools are requested to share the Curriculum and Sample Question Papers available on **www.cbseacademic nic.in** at the given links, with all students and teachers –

   - Secondary & Senior Secondary School Curriculum –
     https://cbseacademic.nic.in/curriculum_2024.html
   - Sample Question Papers – Secondary –
     https://cbseacademic.nic.in/SQP_CLASSX_2023-24.html
   - Sample Question Papers – Senior Secondary –
     https://cbseacademic.nic.in/SQP_CLASSXII_2023-24.html

(Dr. Joseph Emmanuel)
Director (Academics)

Syllabus

Latest Syllabus-2023-24

(Issued by CBSE on 31st March 2023)

APPLIED MATHEMATICS (Code No.- 241)
CLASS-XII

Number of Paper : **1**
Total number of Periods : **240 (35 Minutes Each)**
Time : **3 Hours**
Max Marks : **80**

No.	Units	No. of Periods	Marks
I	Numbers, Quantification and Numerical Applications	30	11
II	Algebra	20	10
III	Calculus	50	15
IV	Probability Distributions	35	10
V	Inferential Statistics	10	05
VI	Index Numbers and Time-based data	30	06
VII	Financial Mathematics	50	15
VIII	Linear Programming	15	08
	Total	**240**	**80**
	Internal Assessment		**20**

SI. No.	Contents	Learning Outcomes : Students will be able to	Notes / Explanation
UNIT - 1 NUMBERS, QUANTIFICATION AND NUMERICAL APPLICATIONS			
1.1	Modulo Arithmetic	• Define modulus of an integer • Apply arithmetic operations using modular arithmetic rules	• Definition and meaning • Introduction to modulo operator • Modular addition and subtraction
1.2	Congruence Modulo	• Define congruence modulo • Apply the definition in various problems	• Definition and meaning • Solution using congruence modulo • Equivalence class
1.4	Alligation and Mixture	• Understand the rule of alligation to produce a mixture at a given price • Determine the mean price of a mixture • Apply rule of alligation	• Meaning and Application of rule of alligation • Mean price of a mixture
1.5	Numerical Problems	Solve real life problems mathematically	
	Boats and Streams (upstream and downstream)	• Distinguish between upstream and downstream • Express the problem in the form of an equation	• Problems based on speed of stream and the speed of boat in still water
	Pipes and Cisterns	• Determine the time taken by two or more pipes to fill or empty the tank	• Calculation of the portion of the tank filled or drained by the pipe(s) in unit time
	Races and Games	• Compare the performance of two players w.r.t. time, distance	• Calculation of the time taken/ distance covered / speed of each player

Syllabus

1.6	Numerical Inequalities	• Describe the basic concepts of numerical inequalities • Understand and write numerical inequalities	• Comparison between two statements/situations which can be compared numerically • Application of the techniques of numerical solution of algebraic inequations
UNIT - 2 ALGEBRA			
2.1	Matrices and types of matrices	• Define matrix • Identify different kinds of matrices • Find the size / order of matrices	• The entries, rows and columns of matrices • Present a set of data in a matrix form
2.2	Equality of matrices, Transpose of a matrix, Symmetric and Skew symmetric matrix	• Determine equality of two matrices • Write transpose of given matrix • Define symmetric and skew symmetric matrix	• Examples of transpose of matrix • A square matrix as a sum of symmetric and skew symmetric matrix • Observe that diagonal elements of skew symmetric matrices are always zero
2.3	Algebra of Matrices	• Perform operations like addition & subtraction on matrices of same order • Perform multiplication of two matrices of appropriate order • Perform multiplication of a scalar with matrix	• Addition and Subtraction of matrices • Multiplication of matrices (It can be shown to the students that Matrix multiplication is similar to multiplication of two polynomials) • Multiplication of a matrix with a real number
2.4	Determinants	• Find determinant of a square matrix • Use elementary properties of determinants	• Singular matrix, Non singular matrix • $\|AB\| = \|A\|\|B\|$ • Simple problems to find determinant value
2.5	Inverse of a matrix	• Define the inverse of a square matrix • Explain elementary row operations and use to it find the inverse of a matrix • Apply properties of inverse of matrices	• Inverse of a matrix using: (a) cofactors If A and B are invertible square matrices of same size, (i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(A^{-1})^{-1} = A$ (iii) $(AT)^{-1} = (A^{-1})^{T}$
2.6	Solving system of simultaneous equations using matrix method, Cramer's rule and row reduction method	• Solve the system of simultaneous equations using (i) Cramer's Rule (ii) Inverse of coefficient matrix • Formulate real life problems into a system of simultaneous linear equations and solve it using these methods	• Solution of system of simultaneous equations upto three variables only (non- homogeneous equations)

Syllabus

UNIT - 3 CALCULUS			
Differentiation and its Applications			
3.1	Higher Order Derivatives	• Determine second and higher order derivatives • Understand differentiation of parametric functions and implicit functions	• Simple problems based on higher order derivatives • Differentiation of parametric functions and implicit functions (upto 2nd order)
3.2	Application of Derivatives	• Determine the rate of change of various quantities • Understand the gradient of tangent and normal to a curve at a given point • Write the equation of tangents and normal to a curve at a given point	• To find the rate of change of quantities such as area and volume with respect to time or its dimension • Gradient / Slope of tangent and normal to the curve • The equation of the tangent and normal to the curve (simple problems only)
3.3	Marginal Cost and Marginal Revenue using derivatives	• Define marginal cost and marginal revenue • Find marginal cost and marginal revenue	• Examples related to marginal cost, marginal revenue, etc.
3.4	Increasing / Decreasing Functions	• Determine whether a function is increasing or decreasing • Determine the conditions for a function to be increasing or decreasing	• Simple problems related to increasing and decreasing behaviour of a function in the given interval
3.5	Maxima and Minima	• Determine critical points of the function • Find the point(s) of local maxima and local minima and corresponding local maximum and local minimum values • Find the absolute maximum and absolute minimum value of a function • Solve applied problems	• A point x= c is called the critical point of f if f is defined at c and f '(c) = 0 or f is not differentiable at c • To find local maxima and local minima by: (i) First Derivative Test (ii) Second Derivative Test • Contextualized real life problems
Integration and its Applications			
3.6	Integration	• Understand and determine indefinite integrals of simple functions as anti-derivative	• Integration as a reverse process of differentiation • Vocabulary and Notations related to Integration
3.7	Indefinite Integrals as family of curves	• Evaluate indefinite integrals of simple algebraic functions by method of: (i) substitution (ii) partial fraction (iii) by parts	• Simple integrals based on each method (non-trigonometric function)
3.8	Definite Integrals as area under the curve	• Define definite integral as area under the curve • Understand fundamental theorem of Integral calculus and apply it to evaluate the definite integral • Apply properties of definite integrals to solve the problems	• Evaluation of definite integrals using properties

Syllabus

3.9	Application of Integration	• Identify the region representing C.S. and P.S. graphically • Apply the definite integral to find consumer surplus-producer surplus	Problems based on finding • Total cost when Marginal Cost is given • Total Revenue when Marginal Revenue is given • Equilibrium price and equilibrium quantity and hence consumer and producer surplus
Differential Equations and Modeling			
3.10	Differential Equations	• Recognize a differential equation • Find the order and degree of a differential equation	• Definition, order, degree and examples
3.11	Formulating and Solving Differential Equations	• Formulate differential equation • Verify the solution of differential equation • Solve simple differential equation	• Formation of differential equation by eliminating arbitrary constants • Solution of simple differential equations (direct integration only)
3.12	Application of Differential Equations	• Define Growth and Decay Model • Apply the differential equations to solve Growth and Decay Models	• Growth and Decay Model in Biological sciences, Economics and business, etc.
UNIT - 4 PROBABILITY DISTRIBUTIONS			
4.1	Probability Distribution	• Understand the concept of Random Variables and its Probability Distributions • Find probability distribution of discrete random variable	• Definition and example of discrete and continuous random variable and their distribution
4.2	Mathematical Expectation	• Apply arithmetic mean of frequency distribution to find the expected value of a random variable	• The expected value of discrete random variable as summation of product of discrete random variable by the probability of its occurrence.
4.3	Variance	• Calculate the Variance and S.D. of a random variable	• Questions based on variance and standard deviation
4.4	Binomial Distribution	• Identify the Bernoulli Trials and apply Binomial Distribution • Evaluate Mean, Variance and S.D of a binomial distribution	• Characteristics of the binomial distribution • Binomial formula: $P(r) = {}^{n}C_{r}\, p^{r}\, q^{n-r}$ Where n = number of trials p = probability of success q = probability of failure Mean = np Variance = npq Standard Deviation = $\sqrt{npq}$
4.5	Poison Distribution	• Understand the Conditions of Poisson Distribution • Evaluate the Mean and Variance of Poisson distribution	• Characteristics of Poisson Probability distribution Poisson formula: $P(x) = \frac{\lambda^{x}.e^{-\lambda}}{x!}$ • Mean = Variance = λ

4.6	Normal Distribution	• Understand normal distribution is a Continuous distribution • Evaluate value of Standard normal variate • Area relationship between Mean and Standard Deviation	• Characteristics of a normal probability distribution • Total area under the curve = total probability = 1 • Standard Normal Variate: $Z = \frac{x-\mu}{\sigma}$ where x = value of the random variable μ = mean σ = S.D.
UNIT - 5 INFERENTIAL STATISTICS			
5.1	Population and Sample	• Define Population and Sample • Differentiate between population and sample • Define a representative sample from a population • Differentiate between a representative and non-representative sample • Draw a representative sample using simple random sampling • Draw a representative sample using and systematic random sampling	• Population data from census, economic surveys and other contexts from practical life • Examples of drawing more than one sample set from the same population • Examples of representative and non-representative sample • Unbiased and biased sampling • Problems based on random sampling using simple random sampling and systematic random sampling (sample size less than 100)
5.2	Parameter and Statistics and Statistical Interferences	• Define Parameter with reference to Population • Define Statistics with reference to Sample • Explain the relation between Parameter and Statistic • Explain the limitation of Statistic to generalize the estimation for population • Interpret the concept of Statistical Significance and Statistical Inferences • State Central Limit Theorem • Explain the relation between Population-Sampling Distribution-Sample	• Conceptual understanding of Parameter and Statistics • Examples of Parameter and Statistic limited to Mean and Standard deviation only • Examples to highlight limitations of generalizing results from sample to population • Only conceptual understanding of Statistical Significance/Statistical Inferences • Only conceptual understanding of Sampling Distribution through simulation and graphs
5.3	t-Test (one sample t-test and two independent groups t-test)	• Define a hypothesis • Differentiate between Null and Alternate hypothesis • Define and calculate degree of freedom • Test Null hypothesis and make inferences using t-test statistic for one group / two independent groups	• Examples and non-examples of Null and Alternate hypothesis (only non-directional alternate hypothesis) • Framing of Null and Alternate hypothesis • Testing a Null Hypothesis to make Statistical Inferences for small sample size • (for small sample size: t- test for one group and two independent groups • Use of t-table

Syllabus

UNIT - 6 INDEX NUMBERS AND TIME BASED DATA			
6.4	Time Series	• Identify time series as chronological data	• Meaning and Definition
6.5	Components of Time Series	• Distinguish between different components of time series	• Secular trend • Seasonal variation • Cyclical variation • Irregular variation
6.6	Time Series analysis for univariate data	• Solve practical problems based on statistical data and Interpret the result	• Fitting a straight line trend and estimating the value
6.7	Secular Trend	• Understand the long term tendency	• The tendency of the variable to increase or decrease over a long period of time
6.8	Methods of Measuring trend	• Demonstrate the techniques of finding trend by different methods	• Moving Average method • Method of Least Squares
UNIT - 7 FINANCIAL MATHEMATICS			
7.1	Perpetuity, Sinking Funds	• Explain the concept of perpetuity and sinking fund • Calculate perpetuity • Differentiate between sinking fund and saving account	• Meaning of Perpetuity and Sinking Fund • Real life examples of sinking fund • Advantages of Sinking Fund • Sinking Fund vs. Savings account
7.3	Calculation of EMI	• Explain the concept of EMI • Calculate EMI using various methods	• Methods to calculate EMI: (i) Flat-Rate Method (ii) Reducing-Balance Method • Real life examples to calculate EMI of various types of loans, purchase of assets, etc.
7.4	Calculation of Returns, Nominal Rate of Return	• Explain the concept of rate of return and nominal rate of return • Calculate rate of return and nominal rate of return	• Formula for calculation of Rate of Return, Nominal Rate of Return
7.5	Compound Annual Growth Rate	• Understand the concept of Compound Annual Growth Rate • Differentiate between Compound Annual Growth Rate and Annual Growth Rate • Calculate Compound Annual Growth Rate	• Meaning and use of Compound Annual Growth Rate • Formula for Compound Annual Growth Rate
7.7	Linear method of Depreciation	• Define the concept of linear method of Depreciation • Interpret cost, residual value and useful life of an asset from the given information • Calculate depreciation	• Meaning and formula for Linear Method of Depreciation • Advantages and disadvantages of Linear Method

Syllabus

UNIT - 8 LINEAR PROGRAMMING			
8.1	Introduction and related terminology	• Familiarize with terms related to Linear Programming Problem	• Need for framing linear programming problem • Definition of Decision Variable, Constraints, Objective function, Optimization and Non Negative conditions
8.2	Mathematical formulation of Linear Programming Problem	• Formulate Linear Programming Problem	• Set the problem in terms of decision variables, identify the objective function, identify the set of problem constraints, express the problem in terms of inequations
8.3	Different types of Linear Programming Problems	• Identify and formulate different types of LPP	• Formulate various types of LPP's like Manufacturing Problem, Diet Problem, Transportation Problem, etc.
8.4	Graphical method of solution for problems in two variables	• Draw the Graph for a system of linear inequalities involving two variables and to find its solution graphically	• Corner Point Method for the Optimal solution of LPP • Iso-cost/ Iso-profit Method
8.5	Feasible and Infeasible Regions	• Identify feasible, infeasible, bounded and unbounded regions	• Definition and Examples to explain the terms
8.6	Feasible and infeasible solutions, optimal feasible solution	• Understand feasible and infeasible solutions • Find optimal feasible solution	• Problems based on optimization • Examples of finding the solutions by graphical method

Practical: Use of spreadsheet

Graphs of an exponential function, demand and supply functions on Excel and study the nature of function at various points, maxima/minima

Matrix operations using Excel

Suggested practical using the spreadsheet

(i) Plot the graphs of functions on excel and study the graph to find out the point of maxima/minima
(ii) Probability and dice roll simulation
(iii) Matrix multiplication and the inverse of a matrix
(iv) Stock Market data sheet on excel
(v) Collect the data on weather, price, inflation, and pollution analyze the data and make meaningful inferences
(vi) Collect data from newspapers on traffic, sports activities and market trends and use excel to study future trends

List of Suggested projects (Class XI /XII)

(i) Use of prime numbers in coding and decoding of messages
(ii) Prime numbers and divisibility rules
(iii) Logarithms for financial calculations such as interest, present value, future value, profit/loss etc. with large values)
(iv) The cardinality of a set and orders of infinity
(v) Comparing sets of Natural numbers, rational numbers, real numbers and others
(vi) Use of Venn diagram in solving practical problems
(vii) Fibonacci sequence: Its' history and presence in nature
(viii) Testing the validity of mathematical statements and framing truth tables
(ix) Investigating Graphs of functions for their properties

(x) Visit the census site of India http://www.censusindia.gov.in/Census_Data_2001/Census_Data_Online/Language/State ment3.htm Depict the information given there in a pictorial form

(xi) Prepare a questionnaire to collect information about money spent by your friends in a month on activities like travelling, movies, recharging of the mobiles, etc. and draw interesting conclusions

(xii) Check out the local newspaper and cut out examples of information depicted by graphs. Draw your own conclusions from the graph and compare it with the analysis given in the report

(xiii) Analysis of population migration data – positive and negative influence on urbanization

(xiv) Each day newspaper tells us about the maximum temperature, minimum temperature, and humidity. Collect the data for a period of 30 days and represent it graphically. Compare it with the data available for the same time period for the previous year

(xv) Analysis of career graph of a cricketer (batting average for a batsman and bowling average for a bowler). Conclude the best year of his career. It may be extended for other players also – tennis, badminton, athlete

(xvi) Vehicle registration data – correlating with pollution and the number of accidents

(xvii) Visit a village near Delhi and collect data of various crops over the past few years from the farmers. Also, collect data about temperature variation and rain over the period for a particular crop. Try to find the effect of temperature and rain variations on various crops

(xviii) Choose any week of your ongoing semester. Collect data for the past 10 – 15 years for the amount of rainfall received in Delhi during that week. Predict the amount of rainfall for the current year

(xix) Weather prediction (prediction of monsoon from past data)

(xx) Visit Kirana shops near your home and collect the data regarding the sales of certain commodities over a month. Try to figure out the stock of a particular commodity which should be in the store in order to maximize the profit

(xxi) Stock price movement

(xxii) Risk assessments by insurance firms from data

(xxiii) Predicting stock market crash

(xxiv) Predicting the outcome of an election – exit polls

(xxv) Predicting mortality of infants

Assessment Plan

1. Overall Assessment of the course is out of 100 marks.
2. The assessment plan consists of an External Exam and Internal Assessment.
3. External Exam will be of 03 hours duration Pen/ Paper Test consisting of 80 marks.
4. The weightage of the Internal Assessment is 20 marks. Internal Assessment can be a combination of activities spread throughout the semester/ academic year. Internal Assessment activities include projects and excel based practical. Teachers can choose activities from the suggested list of practical or they can plan activities of a similar nature. For data-based practical, teachers are encouraged to use data from local sources to make it more relevant for students.
5. Weightage for each area of internal assessment may be as under:

Sl. No.	Area and Weightage	Assessment Area	Marks allocated
1	Project work (10 marks)	Project work and record	5
		Year-end Presentation/ Viva of the Project	5
2	Practical work (10 marks)	Performance of practical and record	5
		Year-end test of any one practical	5
		Total	**20**

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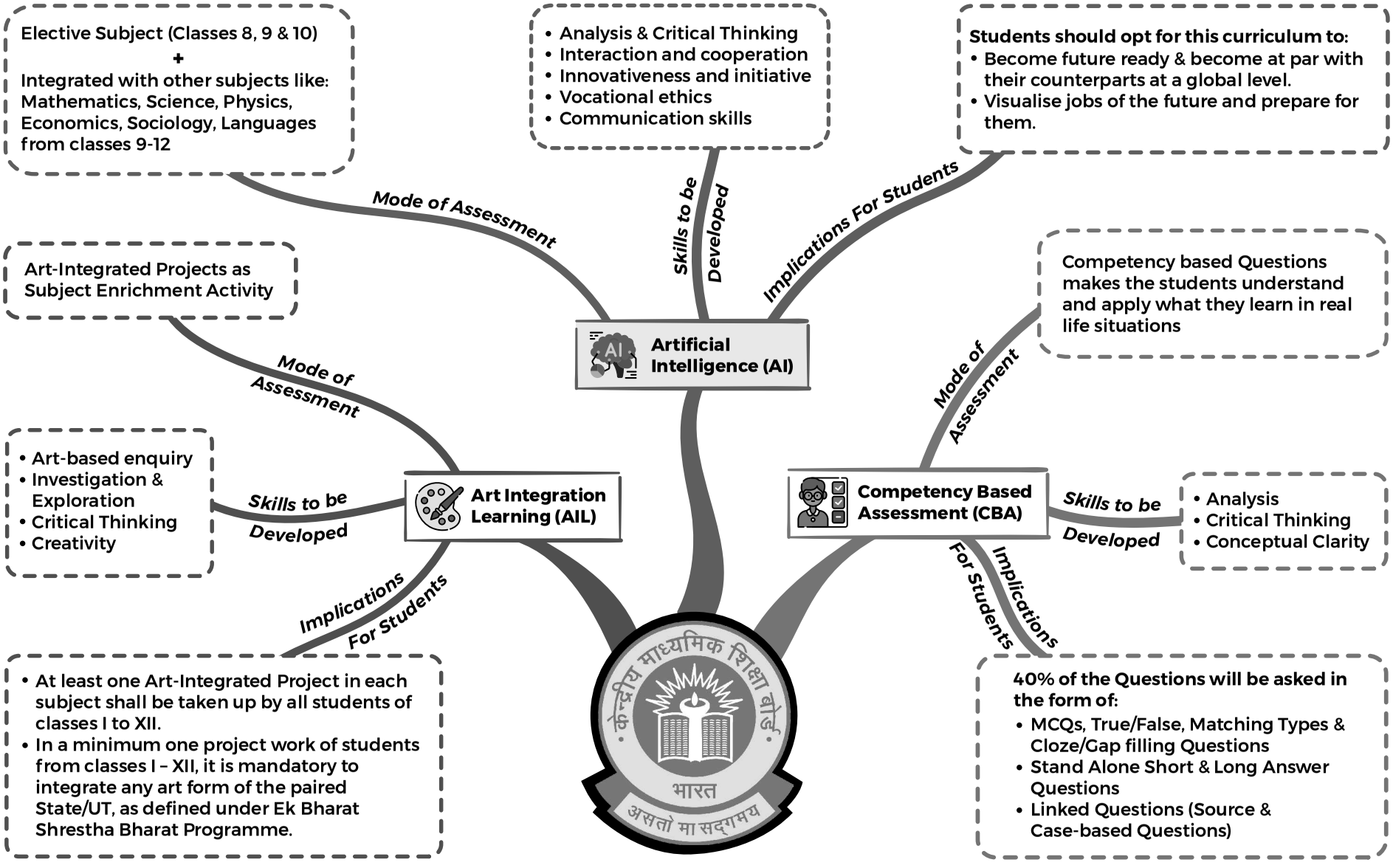
NEP Derived Learning Resources Prescribed by CBSE for Year 2023-24



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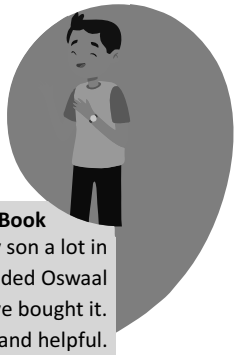

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Sample Question Paper

(Issued by Board on 31st March, 2023)

APPLIED MATHEMATICS

Class- XII
Session – 2023-24

SOLVED

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions:

(i) *This Question paper contains - **five sections** A,B,C,D and E. Each section is compulsory. However, there is some internal choice in some questions.*

(ii) ***Section A** has **18 MCQ's and 02** Assertion Reason based questions of 1 mark each.*

(iii) ***Section B** has 5 **Very Short Answer(VSA)** questions of 2 marks each.*

(iv) ***Section C** has 6 **Short Answer(SA)** questions of 3 marks each.*

(v) ***Section D** has 4 **Long Answer(LA)** questions of 5 marks each.*

(vi) ***Section E** has 3 **source based/case based/passage based/integrated units of assessment** (04 marks each) with sub parts.*

(vii) *Internal Choice is provided **in 2 questions in Section-B, 2 questions in Section-C, 2 Questions in Section-D**. You have to attempt only one alternatives in all such questions.*

Section - A

(All Questions are compulsory. No internal choice is provided in this section)

1. The value of –70 mod 13 is

(a) 5 **(b)** -5

(c) 8 **(d)** -8

2. If $\frac{x+1}{x+2} \geq 1$, then

(a) $x \in [-\infty, 2]$ **(b)** $x \in (-\infty, -2)$

(c) $x \in (-\infty, 2]$ **(d)** $x \in (-\infty, 2)$

3. Which of the following is a statistic ?

(a) μ **(b)** $\bar{x}$

(c) σ^2 **(d)** None

4. In one sample test, the estimation for population mean is

(a) $\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$ **(b)** $\frac{\bar{x}-\mu}{\frac{s}{n}}$

(c) $\frac{\bar{x}-\mu}{\frac{s^2}{n}}$ **(d)** None

5. A man can row 6 km/h in still water. It takes him twice as long to row up as to row down the river. Then the rate of the stream is

(a) 2 km/h **(b)** 4 km/h

(c) 6 km/h **(d)** 8 km/h

6. If random variable X represents the number of heads when a coin is tossed twice then mathematical expectation of X is

(a) 0 **(b)** $\frac{1}{4}$

(c) $\frac{1}{2}$ **(d)** 1

7. The least non-negative remainder when 3^{50} is divided by 7 is

(a) 4 **(b)** 3

(c) 30 **(d)** 1

8. If the cash equivalent of a perpetuity of ₹ 300 payable at the end of each quarter is ₹ 24000 then rate of interest converted quarterly is

(a) 5% **(b)** 4%

(c) 3% **(d)** 2%

9. The value of $\int \frac{\log x}{x} dx$ is

(a) $\frac{\log x}{2} + C$ **(b)** $\frac{(\log x)^2}{2} + C$

(c) $\log x + C$ **(d)** None

10. The supply of finished good was delayed for a month due to landslide in hilly terrain. Under which trend oscillation does this situation fall

(a) Seasonal **(b)** Cyclical

(c) Secular **(d)** Irregular

11. A machine costing ₹ 30,000 is expected to have a useful life of 4 years and a final scrap value of ₹ 4000. The annual depreciation is

(a) ₹ 5500 **(b)** ₹ 6500

(c) ₹ 7500 (d) ₹ 8500

12. The effective rate of interest equivalent to the nominal rate 6% compounded semi-annually is

(a) 6.05% **(b)** 6.07%

(c) 6.09% **(d)** None

13. If the investment of ₹ 20000 in the mutual fund in 2015 increased to ₹ 32000 in year 2020, then CAGR (Compound Annual Growth rate is) is [Given$(1.6)^{1/5} = 1.098$]

(a) 9.08% **(b)** 9.8%

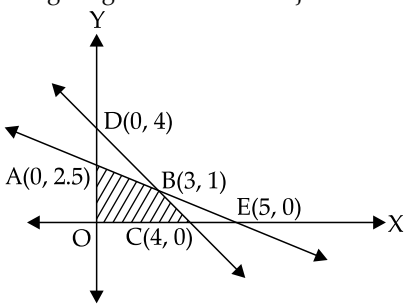
(c) 0.098% **(d)** 0.09%

14. The integrating factor of the differential equation $x\frac{dy}{dx} + 2y = x^3 (x \neq 0)$ is

(a) x **(b)** $\log x$

(c) x^2 **(d)** $\frac{1}{x^2}$

15. Besides non negativity constraint the figure given below is subject to which of the following constraints



(a) $x + 2y \leq 5$; $x + y \leq 4$ **(b)** $x + 2y \geq 5$; $x + y \leq 4$

(c) $x + 2y \geq 5$; $x + y \geq 4$ **(d)** $x + 2y \leq 5$; $x + y \geq 4$

16. If X is a Poisson variate such that $3P(X = 2) = 2P(X = 1)$ then the mean of the distribution is equal to

(a) $\frac{4}{3}$ **(b)** $\frac{3}{4}$

(c) $-\frac{4}{3}$ **(d)** $-\frac{3}{4}$

17. For the given five values 35, 70, 36, 59, 64, the three years moving averages are given by

(a) 47, 53, 55 **(b)** 53, 47, 45

(c) 47, 55, 53 **(d)** 45, 55, 57

18. The data point of a normal variate with mean 12, standard deviation 4 and Z – score 5 is

(a) 28 **(b)** 304

(c) 34 **(d)** 32

ASSERTION REASON BASED QUESTIONS

In the following questions, a statement of Assertion(A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true and R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false and R is true.

19. **Assertion (A):** The maximum profit that a company makes if profit function is given by $P(x) = 41 + 24x - 8x^2$; where 'x' is the number of units and P is the profit in rupees is 59

Reason (R): The profit is maximum at $x = a$ if $P'(a) = 0$ and $P''(a) > 0$

20. **Assertion (A):** The probability of getting 6 heads when a unbiased coin is tossed 10 times is $C(10, 6)\left(\frac{1}{2}\right)^{10}$.

Reason (R): In a Binomial distribution the probability is given by
$P(X = r) = C(n, r)(p)^r(q)^{n-r}$

Section - B

(All Questions are compulsory. In case of internal Choice, attempt any one question only)

21. At what rate of interest will the present value of perpetuity of ₹ 1500 payable at the end of every 6 months be ₹ 20,000 ?

22. If A is a square matrix $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ such that $A^2 = pA$, then find the value of p.

OR

If $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is skew -symmetric, matrix, then find value of $a + b + c$

23. A Cooperative Society of farmers has 10 hectares of land to grow two crops A and B. To control weeds, pesticide has to be used for crops A and B at the rate of 30 grams and 15 grams per hectare. Further, not more than 750 grams of pesticide should be used. The profit from crops A and B per hectare are estimated as ₹ 8000 and ₹ 9500. Formulate the above problem as LPP, in order to allocate land to each crop for maximum use.

24. A boatman takes twice as long as to go upstream to a point as to return downstream to the starting point. If the speed of a boat in still water is 15 km/h, what is the speed of the stream.

OR

'A' can run 40 meters while 'B' runs 50 meters in the same time. In a 1000 m race, find by how much distance 'B' beats 'A'.

25. A machine produces washers of thickness 0.50 mm. To determine whether the machine is in proper working order, a sample of 10 washers is chosen for which the mean thickness is 0.53 mm and the standard deviation is 0.03 mm. Test the hypothesis at 5% level of significance that the machine is working in proper order. [Given $t_{0.025} = 2.262$ at 9 degree of freedom]

Section - C

(All Questions are compulsory. In case of internal Choice, attempt any one question only)

26. Evaluate: $\int \frac{x^3}{(x+2)} dx$

OR

Evaluate: $\int (x^2+1)\log_e x \, dx$

27. Cost of two toys A and B are ₹ 50 and ₹ 75. On a particular Sunday shopkeeper P sells 7 toys of type A and 10 toys of type B whereas shopkeeper Q sells 8 toys of type A and 6 toys of type B. Find income of both shopkeepers using matrix Algebra.

28. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x - 5$ is increasing or decreasing.

29. The demand and supply functions under the pure market competition are $p_d = 16 - x^2$ and $p_s = 2x^2 + 4$ respectively, where p is the price and x is the quantity of the commodity. Using integrals find Consumer's surplus.

OR

The demand and supply functions under the pure market competition are $p_d = 56 - x^2$ and $p_s = 8 + \frac{x^2}{3}$ respectively, where p is the price and x is the quantity of the commodity. Using integrals find Producer's surplus.

30. Mr Surya borrows a sum of ₹ 5,00,000 with total interest paid ₹ 2,00,000 (flat) and he is paying an EMI of ₹ 12,500. Calculate loan tenure.

31. Mr Sharma wants to send his daughter abroad for higher studies after 10 years. He sets up a sinking fund in order to have ₹ 5,00,000 after 10 years. How much should he set aside bi-annually into an account paying 5% per annum compounded annually.
[Use $(1.025)^{20} = 1.6386$]

Section - D

(This section comprises of long answer type questions (LA) of 5 marks each)

32. On doing the proof reading of a book on an average 4 errors in 10 pages were detected. Using Poisson's distribution find the probability of (i) No error and (ii) one error in 1000 pages of first printed edition of the book. (Given $e^{-0.4} = 0.6703$)

OR

How many time must Sunil toss a fair coin so that the probability of getting at least one head is more than 90 %?

33. A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for at least 5 hours a day. He produces only two items M and N, each requiring the use of all the three machines. The number of hours required for producing 1 unit of M and N on three machines are given in the following table:

Items	Number of hours required on machines		
	I	II	III
M	1	2	1
N	2	1	1.25

He makes a profit of ₹ 600 and ₹ 400 on one unit of items M and N respectively. How many units of each item be produced so as to maximize the profit . What is the maximum Profit?

34. A company produces a certain commodity with ₹ 2400 fixed cost. The variable cost is estimated to be 25% of the total revenue received on selling the product at a rate of ₹ 8 per unit. Find the following
(i) Cost Function. **(ii)** Revenue Function
(iii) Breakeven Point **(iv)** Profit Function

OR

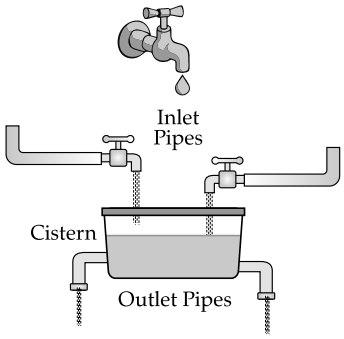
The production manager of a company plans to include 180 sq. cms. of actual printed matter in each page of a book under production. Each page should have a 2.5 cm wide margin along the top and bottom and 2 cm wide margin along the sides. What are the most economical dimensions of each printed page?

35. The management committee of a Welfare Club decided to award some of its members (say x) for sincerity, some (say y) for helping others selflessly and some others (say z) for effective management. The sum of all the awardees is 12. Three times the sum of all awardees for helping others selflessly and effective management added to two times the number of awardees for sincerity is 33. If the sum of the number of awardees for sincerity and effective management is twice the number of awardees for helping others, use matrix method to find the number of awardees of each category.

Section - E

(This section comprises of 3 source based questions (Case Studies) of 4 marks each)

36. **Case Study 1:** Pipes and Cisterns (Mark 2 + 1 + 1) (Internal choice is in the iii part) A, B and C are three pipes connected to a tank. A and B together fill the tank in 6 hours B and C together fill the tank in 10 hours. A and C together fill the tank in $7\frac{1}{2}$ hours. Based on above information answer the following questions.



(i) In how much time will A, B and C fill the tank?
(ii) In how much time will A separately fill the tank?
(iii) In how much time will B separately fill the tank?

OR

In how much time will C separately fill the tank?

37. **Case Study 2:** Read the following passage and answer the questions below (Internal Choice is in option iii.) (Mark 1 + 1 + 2)
Let X denote the number of hours a person watches television during a randomly selected day. The probability that X can take the values x_i, has the following form, where 'k' is some unknown constant.

$$P(X = x_i) = \begin{cases} 0.2, & \text{if } x_i = 0 \\ kx_i, & \text{if } x_i = 1 \text{ or } 2 \\ k(5 - x_i), & \text{if } x_i = 3 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the value of k.
(ii) What is the probability that a person watches two hours of television on a selected day?
(iii) What is the probability that the person watches at least two hours of television on a selected day ?

OR

(iv) What is the probability that the person watches at most two hours of television on a selected day ?

38. **Case Study 3:** When observed over a long period of time, a time series data can predict trend that can forecast increase or decrease or stagnation of a variable under consideration. Such analytical studies can benefit a business for forecasting or prediction of future estimated sales or production.
The table below shows the welfare expenses(in lakh ₹) of Steel Industry during 2001-2005. Fit a straight line trend by the method of least squares and estimate the trend for the year 2008.

Year	2001	2002	20003	2004	2005
Welfare expenses	160	185	220	300	510

OR

The annual rainfall (in mm) was recorded in Cherrapunji, Meghalaya.

Year	2001	2002	20003	2004	2005	2006	2007	2008	2009
Rainfall (in mm)	1.2	1.9	2	1.4	2.1	1.3	1.8	1.1	1.3

Determine the trend of rainfall by three years moving average and draw the moving averages graph.

❑❑

SOLUTIONS

All solutions are as per the CBSE Board Marking Scheme 2023-24

Section - A

1. **Option (c) is correct.** **1**

Explanation: Here, $X < 0$ and $Y > 0$, hence $-70 \bmod 13$ is $8 \because 8 > 0$.

2. **Option (b) is correct.** **1**

Explanation: $x \in (-\infty, -2)$

$$\frac{x+1}{x+2} \geq 1$$

$$\Rightarrow \quad \frac{x+1}{x+2} - 1 \geq 0$$

$$\Rightarrow \quad \frac{x+1-x-2}{x+2} \geq 0$$

$$\Rightarrow \quad \frac{-1}{x+2} \geq 0$$

$$\Rightarrow \quad x + 2 < 0$$

$$\left[\because \frac{a}{b} > 0 \text{ and } a < 0 \Rightarrow b < 0\right]$$

$$\Rightarrow \quad x < -2$$

3. **Option (b) is correct.** **1**

Explanation: $\bar{x}$ is a statistic.

4. **Option (a) is correct.** **1**

5. **Option (a) is correct.** **1**

Explanation: Let man's rate upstream = x km/h

Let man's rate downstream = $2x$ km/h

Hence, man's rate in still water $= \frac{1}{2}(x + 2x)$

$$= \frac{3x}{2} \text{km / h}$$

Therefore, $\frac{3x}{2} = 6 \Rightarrow x = 4$ km/h

Man's rate downstream = 8 km/h

Hence, rate of stream $= \frac{1}{2}(8 - 4) = 2$ km/h

6. **Option (d) is correct.** **1**

Explanation:

x_i	Sample Event	$P(x_i) = p_i$	$x_i p_i$
0	TT	$\frac{1}{4}$	0
1	HT, TH	$\frac{1}{2}$	$\frac{1}{2}$
2	HH	$\frac{1}{4}$	$\frac{1}{2}$

Mathematical Expectation $E(X) = \Sigma p_i x_i = 1$.

7. **Option (c) is correct.** **1**

Explanation: $3^1 = 3 \pmod 7$

$$\Rightarrow \quad 3^2 = 3 \times 3 = 2 \pmod 7$$

$$\Rightarrow \quad 3^3 = 3 \times 2 = 6 = -1 \pmod 7$$

$$\Rightarrow \quad (3^3)^{16} = (-1)^{16} \pmod 7$$

$$\Rightarrow \quad (3^3)^{16} = 1 \pmod 7$$

$$\Rightarrow \quad (3^3)^{16} \times 3^2 = 1 \times 3^2 \pmod 7$$

$$\Rightarrow \quad 3^{50} = 2 \pmod 7$$

8. **Option (a) is correct.** **1**

Explanation: $i = \frac{r}{400}$

$$P = \frac{R}{i}$$

$$\Rightarrow \quad 24000 = \frac{300 \times 400}{r}$$

$$\Rightarrow \quad r = \frac{120}{24} = 5\%$$

9. **Option (b) is correct.** **1**

Explanation: $\int \frac{\log x}{x} dx$

Put $\log x = t$

Differentiating $\frac{1}{x} dx = dt$

Hence, $\int \frac{\log x}{x} dx = \int t dt$

$$= \frac{t^2}{2} + C = \frac{(\log x)^2}{2} + C$$

10. **Option (d) is correct.** **1**

11. **Option (b) is correct.** **1**

Explanation: $D = \frac{c - s}{n}$

$$\Rightarrow \quad D = \frac{30,000 - 4,000}{4}$$

$$= \frac{26,000}{4} = 6,500$$

Hence, the depreciation is ₹ 6,500.

12. **Option (c) is correct.** **1**

Explanation: $r_{eff} = \left[\left(1+\frac{r}{m}\right)^m - 1\right]\times 100$

$r_{eff} = [(1.03)^2 - 1] \times 100$
$= (1.0609 - 1) \times 100$
$= 6.09\%.$

13. **Option (b) is correct.** **1**

Explanation: $CAGR = \left[\left(\frac{EV}{SV}\right)^{\frac{1}{5}} - 1\right]\times 100$

$= \left[\left(\frac{32000}{20000}\right)^{\frac{1}{5}} - 1\right]\times 100$

$= \left[(1.6)^{\frac{1}{5}} - 1\right]\times 100$

$= [1.098 - 1] \times 100$
$= 0.098 \times 100 = 9.8\%$

14. **Option (c) is correct.** **1**

Explanation: $\frac{xdy}{dx} + 2y = x^3$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{x^3}{x}$

$\frac{dy}{dx} + \frac{2y}{x} = x^2$

$I.F. = e^{\int \frac{2}{x} dx}$
$= e^{2lnx} = e^{lnx^2} = x^2$

15. **Option (a) is correct.** **1**

Explanation:

Line 1: $\frac{x}{4} + \frac{y}{4} \le 1$ or $x + y \le 4$

Line 2: $\frac{x}{5} + \frac{y}{2.5} \le 1$ or $x + 2y \le 5$

16. **Option (a) is correct.** **1**

Explanation: $3P(X = 2) = 2P(X = 1)$

$\Rightarrow \quad 3\frac{m^2 e^{-m}}{2!} = 2\frac{me^{-m}}{1!}$

$\Rightarrow \quad m = \frac{4}{3}$

17. **Option (c) is correct.** **1**

Explanation:

Values	3-years moving total	3-years moving average
35	—	
70	35 + 70 + 36 = 141	$\frac{141}{3} = 47$
36	70 + 36 + 59 = 165	$\frac{165}{3} = 55$
59	36 + 59 + 64 = 159	$\frac{159}{3} = 53$
64	—	

18. **Option (d) is correct.** **1**

Explanation: $Z = \frac{x - \mu}{\sigma}$

$\Rightarrow \quad 5 = \frac{x - 12}{4}$

$\Rightarrow \quad x = 32$

19. **Option (c) is correct.** **1**

Explanation: $P(x) = 41 + 24x - 8x^2$

$P'(x) = 24 - 16x$

$P'(x) = 0$

$\Rightarrow \quad 24 - 16x = 0$

$\Rightarrow \quad x = \frac{24}{16} = \frac{3}{2}$

$P''(x) = -16 < 0$

$\Rightarrow \quad x = \frac{3}{2}$ is a point of maxima

Max Profit = P

$= 41 + 24 \times \frac{3}{2} - 8 \times \frac{9}{4}$

$= 41 + 36 - 18 = 59$

Assertion is true but Reason is false, for Maximum $P'(x) = 0$ and $P''(x) < 0$.

20. **Option (a) is correct.** **1**

Explanation: Given, $n = 10$

Probability of getting head = $p = 1/2$

Probability of getting tail = $q = 1 - \frac{1}{2} = \frac{1}{2}$

The probability of getting 6 heads when a unbiased coin is tossed 10 times is

$P(X = 6) = C(10, 6)\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{10-6}$

$= C(10, 6)\left(\frac{1}{2}\right)^{10}$

[Using $P(X = r) = C(n, r)(p)^r(q)^{n-r}$]

Section - B

21. Let rate of interest be $r\%$ per annum, then $i = \frac{r}{200}$

Given $R = ₹\ 1500$ and $P = ₹\ 20{,}000$

$P = \frac{R}{i}$

$\Rightarrow \quad i = \frac{R}{P}$

$= \frac{1500}{20000}$ **1**

$\Rightarrow \quad \frac{r}{200} = \frac{1500}{20000}$

$\Rightarrow \quad r = 15\%$ **1**

22. $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$A^2 = A.A$

$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$ **1**

$A^2 = pA$

$\Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 2p & -2p \\ -2p & 2p \end{bmatrix}$

$\Rightarrow p = 4$ **1**

OR

$\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix}$ **1**

Comparing $a = -2$; $b = -b \Rightarrow 2b = 0 \Rightarrow b = 0$ and $c = -3$ **1½**

Hence, $a + b + c = -2 + 0 - 3 = -5$. ½

23. Let 'x' hectares and 'y' hectares of land be allocated to crop A and Crop B.

Max $Z = 8000x + 9500y$ ½

Subject to $x + y \le 10$; $2x + y \le 50$; $x \ge 0$ and $y \ge 0$. ½

[CBSE Marking Scheme, 2023]

Detailed Solution:

Let 'x' hectares and 'y' hectares of land be allocated to crop A and Crop B.

Crop A	Crop B	
₹ 8000	₹ 9500	Profit
x	y	10 hectares
30 grams/hectares	15 grams/hectares	750 grams atmost

1

Max $Z = 8000\,x + 9500\,y$

Subject to $x + y \le 10$

$2x + y \le 50$

$x \ge 0$ and $y \ge 0$ **1**

24. $\frac{\text{Time taken upstream}}{\text{Time taken downstream}} = \frac{2}{1}$ **1**

Let speed of boat = 15 km/h and speed of stream = y km/h

Hence, $\frac{15+y}{15-y} = \frac{2}{1}$ ½

$\Rightarrow 15 + y = 30 - 2y$

$\Rightarrow 3y = 15$

$\Rightarrow y = 5$ km/h ½

OR

When B runs 50 m, A runs 40 m ½

When B runs 1 m, A runs $= \frac{40}{50} = \frac{4}{5}$ ½

When B runs 1000 m, A runs $= \frac{4}{5} \times 1000 = 800$ m ½

Hence, B beats A by 200 m. ½

25. Define Null hypothesis H_0, Alternate hypothesis H_1 as follows :

$H_0 : \mu = 0.50$ mm

$H_1 : \mu \ne 0.50$ mm

Thus, a two-tailed test is applied under hypothesis H_0, we have

$t = \frac{\bar{x} - \mu}{s}\sqrt{n-1} = \frac{0.53 - 0.50}{0.03} \times 3 = 3$ **1**

Since, the calculated value of t i.e., $t_{cal}(= 3) > t_{tab}$ $(= 2.262)$, the null hypothesis H_0 can be rejected. Hence, we conclude that machine is not working properly.

Section - C

26. $\int \frac{x^3}{(x+2)}dx = \left(\int x^2 - 2x + 4 - \frac{8}{x+2}\right)dx$ **2**

[By Performing Division]

$= \frac{x^3}{3} - x^2 + 4x - 8\,ln(x+2) + C$ **1**

(where C is an arbitrary constant of integration)

OR

$\int (x^2 + 1) ln\, x dx$

Integrating by parts

$ln\,x\left(\frac{x^3}{3} + x\right) - \int \frac{1}{x}\left(\frac{x^3}{3} + x\right)dx$ **2**

$ln\,x\left(\frac{x^3}{3} + x\right) - \int \left(\frac{x^2}{3} + 1\right)dx$

$ln\,x\left(\frac{x^3}{3} + x\right) - \left(\frac{x^3}{9} + x\right) + C$ **1**

27. Toy A Toy B

$\begin{bmatrix} 7 & 10 \\ 8 & 6 \end{bmatrix}$ Here Row 1 and Row 2 indicate Shopkeeper 1 and Shopkeeper 2

Cost Matrix $= \begin{bmatrix} 50 \\ 75 \end{bmatrix}$ **1**

Amount $= \begin{bmatrix} 7 & 10 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 50 \\ 75 \end{bmatrix}$

$= \begin{bmatrix} 350 + 750 \\ 400 + 450 \end{bmatrix} = \begin{bmatrix} 1100 \\ 850 \end{bmatrix}$

Income of shopkeeper P is ₹ 1100 and shopkeeper Q is ₹ 850. **2**

28. $f(x) = 2x^3 - 9x^2 + 12x - 5$

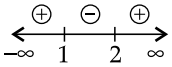
$f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$f'(x) = 6(x-1)(x-2)$ **1**

$f'(x) = 0$

$\Rightarrow \quad x = 1$



and $x = 2$ are the critical points.

The intervals are $(-\infty, 1)$; $(1, 2)$; $(2, \infty)$ ½

Increasing in $(-\infty, 1) \cup (2, \infty)$ Decreasing in $(1, 2)$ **1**

29. Under pure competition

$p_d = p_s$

$\Rightarrow \quad 16 - x^2 = 2x^2 + 4$

$\Rightarrow \quad 3x^2 = 12$

$\Rightarrow \quad x = 2, -2;$

since x can't be –ve, so $x = 2$ **1**

When $x_0 = 2$; $p_0 = 12$

Hence, Consumer's surplus $= \int_0^2 p_d dx - p_0 x_0$ ½

$= \int_0^2 (16 - x^2) dx - 12 \times 2$

$= 16/3$ units **1**

OR

$p_d = p_s$

$\Rightarrow \quad 56 - x^2 = 8 + \frac{x^2}{3}$

$\Rightarrow \quad \frac{4}{3}x^2 = 48$

$\Rightarrow \quad x^2 = 36$

$\Rightarrow \quad x = 6, -6;$

since, x can't *be – ve*, so $x = 6$ **1**

When $x_0 = 6$; $p_0 = 20$

Hence, Producer's surplus $= p_0 x_0 - \int_0^6 p_s dx$

$= 6 \times 20 - \int_0^6 \left(8 + \frac{x^2}{3}\right) dx$

$= 120 - [48 + 24]$

$= 48$ units.

[CBSE Marking Scheme, 2023]

30. Here $P = 5{,}00{,}000;$

$I = 2{,}00{,}000;$

$EMI = 12{,}500$

$EMI = \frac{P + I}{n}$ ½

$12{,}500 = \frac{5{,}00{,}000 + 2{,}00{,}000}{n}$

$\Rightarrow \quad n = \frac{7{,}00{,}000}{12{,}500}$

$= 56$ months. **1½**

31. Let ₹ R be set aside biannually for 10 years in order to have ₹ 5,00,000 after 10 years.

Here, S = 5,00,000; $n = 10 \times 2 = 20$

$i = \frac{5}{2 \times 100} = 0.025$ ½

$R = \frac{iS}{(1+i)^n - 1}$

$= \frac{0.025 \times 5{,}00{,}000}{(1.025)^{20} - 1}$

$= \frac{12.500}{1.6386 - 1}$

$= 19{,}574.07$ **2½**

Section - D

32. Here $m = 0.4$

$P(X = r) = \frac{e^{-m}.m^r}{r!}$

$= \frac{e^{-0.4} \times (0.4)^r}{r!}$

$= \frac{0.6703 \times (0.4)^r}{r!}$

In 1000 pages error $= 1000 \times \frac{0.6703 \times (0.4)^r}{r!}$ ½

For zero error $P(X = 0) = 1000 \times \frac{e^{-m}.m^0}{0!}$

$= \frac{e^{-0.4} \times (0.4)^0}{0!}$

$= 1000 \times 0.6703$

$= 670.3$ **1½**

For one error $P(X = 1) = 1000 \times \frac{e^{-m}.m^1}{1!}$

$= \frac{e^{-0.4} \times (0.4)^1}{1!}$

$= 670.3 \times 0.4$

$= 268.12$ **2**

OR

Here, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$P(X = r) = C(n, r) p^r q^{n-r}$

$1 - P(r = 0) > \frac{90}{100}$ **1**

$1 - C(n,0)\left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^n > \frac{9}{10}$

$\Rightarrow \quad \frac{n!}{0!(n-0)!}\left(\frac{1}{2}\right)^n < \frac{1}{10}$ **2**

$\Rightarrow \quad \left(\frac{1}{2}\right)^n < \frac{1}{10}$ **2**

$\Rightarrow 2^n > 10$

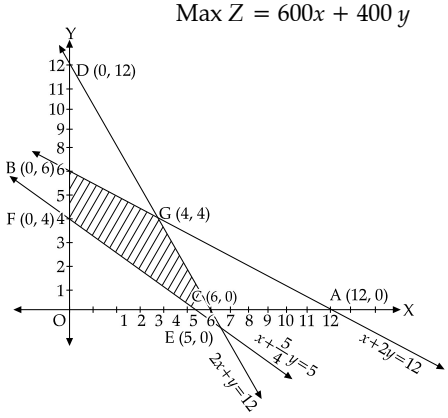
$\Rightarrow n$ is 4 or more times. **2**

33. Let 'x' and 'y' be the number of units of items M and N respectively. We have : $x \geq 0, y \geq 0$

$$x + 2y \leq 12;$$

$$2x + y \leq 12;\ x + \frac{5}{4}y \geq 5.$$

$$\text{Max } Z = 600x + 400y$$



1½

Corner Point	$Z = 600x + 400y$
E : (5, 0)	3000
C : (6, 0)	3600
G : (4, 4)	4000 (Maximum)
B : (0, 6)	2400
F : (0, 4)	1600

Hence, maximum profit is ₹ 4000 when 4 units of each of the items M and N are produced.

34. Let 'x' units of product be produced and sold. As selling price of one unit is ₹ 8 total revenue on 'x' units = ₹ $8x$.

(i) Cost Function $C(x)$ = Fixed Cost + 25% of $8x$

$$= 24000 + \frac{25}{100} \times 8x$$

$$= 24000 + 2x \qquad \textbf{1½}$$

(ii) Revenue Function = $8x$ **1**

(iii) Breakeven Point $8x = 24000 + 2x$

$$x = 4000 \qquad \textbf{1½}$$

(iv) Profit function = $R(x) - C(x) = 6x - 24000$ **1**

OR

Let x and y be the dimension of the printed pages, then $x.y = 180$

A = Area of the page $= (x + 4)(y + 5)$

$$= xy + 5x + 4y + 20$$

$$= 180 + 5x + 4 \times \left(\frac{180}{x}\right) + 20$$

$$= 200 + 5x + \frac{720}{x} \qquad \textbf{2½}$$

For most economical dimension $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 5 - \frac{720}{x^2} = 0$$

$$\Rightarrow \quad x^2 = 144$$

$$\Rightarrow \quad x = 12$$

Now $\quad \frac{d^2A}{dx^2} = \frac{1440}{x^3}$

$$\left(\frac{d^2A}{dx^2}\right)_{x=12} = \frac{1440}{12^3} > 0$$

$\therefore$ A is minimum

Hence, the most economical dimensions are 16 cm and 20 cm. **2½**

[CBSE Marking Scheme, 2023]

Detailed Solution:

(iii) At Breakeven Point, $R(x) = C(x)$

$$8x = 24000 + 2x$$

$$6x = 24000$$

$$x = 4000$$

OR

(iv) Let x and y be the dimension of the printed pages.

ATQ, $\quad x.y = 180 \qquad \ldots(i)$

Dimension of length of printed matter = $x + 4$

Dimension of width of printed matter = $y + 5$

A = Area of the page

$= (x + 4)(y + 5)$

$$A = xy + 5x + 4y + 20$$

$$A = 180 + 5x + 4 \times \frac{180}{x} + 20$$

[From eq. (i)]

$$A = 200 + 5x + \frac{720}{x}$$

Now, $\quad \frac{dA}{dx} = 5 - \frac{720}{x^2}$

For most economical dimension, put $\frac{dA}{dx} = 0$

i.e., $\quad 5 - \frac{720}{x^2} = 0$

$$\Rightarrow \quad x^2 = 144$$

$$\Rightarrow \quad x = 12$$

Now, $\quad \frac{d^2A}{dx^2} = \frac{2 \times 720}{x^3} = \frac{1440}{x^3}$

$$\frac{d^2A}{dx^2}_{\text{at } x=12} = \frac{1440}{12^3} > 0$$

Thus, A is minimum at $x = 12$.

Substituting value of x in equation (i), we get

$$y = \frac{180}{12} = 15$$

Hence, $x = 12$ cm and $y = 15$ cm.

Therefore,

Dimension of length of printed matter

$= 12 + 4 = 16$ cm

Dimension of width of printed matter

$= 15 + 5 = 20$ cm

35.

$$x + y + z = 12$$
$$2x + 3y + 3z = 33$$
$$x - 2y + z = 0$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$
$$|A| = 3 \neq 0$$
$$\text{adj } A = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$
$$X = A^{-1}B$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$
$$X = A^{-1}B$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 108 - 99 + 0 \\ 12 + 0 - 0 \\ -84 + 99 + 0 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Hence $x = 3, y = 4, z = 5$.

Section - E

36. Case Study-I

(i) $A + B$ fill the tank in 6 hours

$B + C$ fill the tank in 10 hours

$A + C$ fill the tank in $\frac{15}{2}$ hours

$$2(A + B + C) = \frac{6 \times 10 \times \frac{15}{2}}{6 \times 10 + 6 \times \frac{15}{2} + 10 \times \frac{15}{2}}$$
$$= \frac{450}{60 + 45 + 75}$$
$$= \frac{450}{180}$$
$$= \frac{5}{2} \text{ hours}$$

Hence A, B and C together will fill the tank in 5 hours.

(ii) A will fill in $[(A + B + C) - (B + C)]$

$$= \frac{10 \times 5}{10 - 5} = 10 \text{ hours}$$

(iii) B will fill in $\frac{\frac{15}{2} \times 5}{\frac{15}{2} - 5} = 15$ hours

OR

C will fill in $\frac{5 \times 6}{6 - 5} = 30$ hours

37. Case Study-II

x_i	0	1	2	3	4	5
$P(X = x_i)$	0.2	k	$2k$	$2k$	0	0

(i) Since, $\Sigma P = 1 \Rightarrow 0.2 + k + 2k + 2k = 1 \Rightarrow 0.2 + 5k = 1 \Rightarrow 5k = 1 - 0.2$

$\Rightarrow k = \frac{4}{25}$ **1½**

(ii) $P(X = 2) = 2k = \frac{8}{25}$ **1**

(iii) $P(X \geq 2) = 4k = \frac{16}{25}$ **1½**

OR

$P(X \leq 2) = 0.2 + 3k = \frac{17}{25}$

38. Case Study-III

Year	Y	X = Year – 2003	X^2	XY
2001	160	– 2	4	– 320
2002	185	– 1	1	– 185
2003	220	0	0	0
2004	300	1	1	300
2005	510	2	4	1020
	1375		10	815

2

$$a = \frac{\Sigma Y}{n} = \frac{1375}{5} = 275$$ ½

$$b = \frac{\Sigma XY}{\Sigma x^2} = \frac{815}{10} = 81.5$$ ½

$Y_c = a + bX$

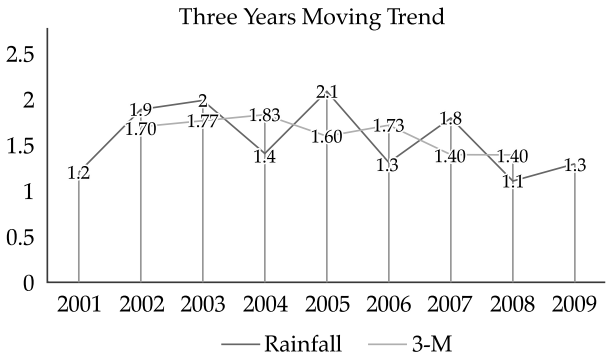
$Y_c = 275 + 81.5X$

The estimated value for 2008 will be $275 + 151.5 \times 5 = 275 + 757.5 = 1032.5$ **1**

OR

Year	Rainfall (in cm)	3 years moving total	3 years moving average
2001	1.2		—
2002	1.9	5.1	1.70
2003	2	5.3	1.77

2004	1.4	5.5	1.83
2005	2.1	4.8	1.60
2006	1.3	5.2	1.73
2007	1.8	4.2	1.40
2008	1.1	4.2	1.40
2009	1.3		—



2½

❑❑

CBSE
Solved Paper 2023
Applied Mathematics
Class-XII
(Delhi/Outside Delhi Sets)

Time : 3 Hours **Max. Marks : 80**

General Instructions:

Read the following instructions very carefully and strictly follow them:

*(i) This question paper contains **38** questions. **All** questions are compulsory.*

*(ii) This question paper is divided into **five** Sections - **A, B, C, D** and **E**.*

*(iii) In **Section A,** Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.*

*(iv) In **Section B,** Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.*

*(v) In **Section C,** Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.*

*(vi) In **Section D,** Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.*

*(vii) In **Section E,** Questions no. **36** to **38** are case study based questions carrying **4** marks each.*

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 2 questions in Section C, 2 questions in Section D and 3 questions in Section E.

*(ix) Use of calculators is **not** allowed.*

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SECTION — A

This section comprises multiple choice questions (MCQs) of 1 mark each

1. The last (unit) digit of $(22)^{12}$ is :

(a) 2 **(b)** 4

(c) 6 **(d)** 8

2. The least non-negative remainder, when 3^{15} is divided by 7 is:

(a) 1 **(b)** 5

(c) 6 **(d)** 7

3. If $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 10 \\ -10 & -5 \end{bmatrix}$, then AB is:

(a) $\begin{bmatrix} -5 & 10 \\ 0 & -5 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & -5 \\ 25 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 10 & -25 \\ -5 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} -5 & 10 \\ 0 & -25 \end{bmatrix}$

4. If $\begin{bmatrix} x+y & x+2 \\ 2x-y & 16 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 1 & 3y+1 \end{bmatrix}$, then the values of x and y are:

(a) $x = 3, y = 5$ **(b)** $x = 5, y = 3$

(c) $x = 2, y = 7$ **(d)** $x = 7, y = 2$

5. The ratio in which a grocer mixes two varieties of pulses costing ₹ 85 per kg and ₹ 100 per kg respectively so as to get a mixture worth ₹ 92 per kg, is:

(a) 7 : 8 **(b)** 8 : 7

(c) 5 : 7 **(d)** 7 : 5

6. If $\frac{|x+1|}{x+1} > 0,\ x \in R$, then:

(a) $x \in [-1, \infty)$ **(b)** $x \in (-1, \infty)$

(c) $x \in (-\infty, -1)$ **(d)** $x \in (-\infty, -1]$

7. A and B are square matrices each of order 3 such that $|A| = -1$ and $|B| = 3$. What is the value of $|3AB|$?
(a) -9 **(b)** -18
(c) -27 **(d)** -81

8. If $\begin{vmatrix} 2 & 3 & 2 \\ x & x & x \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0$, then the value of x is:
(a) -1 **(b)** 0
(c) 1 **(d)** 3

9. The relation between 'Marginal cost' and 'Average cost' of producing 'x' units of a product is:
(a) $\frac{d(AC)}{dx} = x(MC - AC)$ **(b)** $\frac{d(AC)}{dx} = x(AC - MC)$
(c) $\frac{d(AC)}{dx} = \frac{1}{x}(AC - MC)$ **(d)** $\frac{d(AC)}{dx} = \frac{1}{x}(MC - AC)$

10. $\int (x-1)e^{-x}\,dx$ is equal to:
(a) $(x-2)e^{-x} + C$ **(b)** $xe^{-x} + C$
(c) $-xe^{-x} + C$ **(d)** $(x+1)e^{-x} + C$

11. The solution of the differential equation $\frac{dx}{x} + \frac{dy}{y} = 0$ is:
(a) $\frac{1}{x} + \frac{1}{y} = C$ **(b)** $xy = C$
(c) $\log x \log y = C$ **(d)** $x + y = C$

12. If X is a Poisson variable such that $P(X = 1) = 2P(X = 2)$, then $P(X = 0)$ is:
(a) e **(b)** $\frac{1}{e}$
(c) 1 **(d)** e^2

13. If the calculated value of $|t| < t_v(\alpha)$, then the null hypothesis is:
(a) rejected **(b)** accepted
(c) cannot be determined **(d)** neither accepted nor rejected

14. For testing the significance of difference between the means of two independent samples, the degree of freedom (v) is taken as:
(a) $n_1 - n_2 + 2$ **(b)** $n_1 - n_2 - 2$
(c) $n_1 + n_2 - 2$ **(d)** $n_1 + n_2 - 1$

15. The straight line trend is represented by the equation:
(a) $y_c = a + bx$ **(b)** $y_c = a - bx$
(c) $y_c = na + b\Sigma x$ **(d)** $y_c = na - b\Sigma x$

16. The present value of a perpetuity of ₹ R payable at the end of each payment period, when the money is worth i per period, is given by:
(a) Ri **(b)** $R + \frac{R}{i}$
(c) $\frac{R}{i}$ **(b)** $R - Ri$

17. The effective rate which is equivalent to nominal rate of 10% p.a. compounded quarterly is:
(a) 10.25% **(b)** 10.38%
(c) 10.47% **(d)** 10. 53%

18. Region represented by $x \geq 0, y \geq 0$ lies in:
(a) I quadrant **(b)** II quadrant
(c) III quadrant **(d)** IV quadrant

Questions number **19** and **20** are Assertion and Reason based questions carrying 1 mark each. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (a), (b), (c) and (d) as given below.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(b) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(c) Assertion (A) is true and Reason (R) is false.
(d) Assertion (A) is false and Reason (R) is true.

19. Assertion (A): The function $f(x) = (x + 2)\,e^{-x}$ is increasing in the interval $(-1, \infty)$.
Reason (R): A function $f(x)$ is increasing, if $f'(x) > 0$.

20. **Assertion (A):** The differential equation representing the family of parabolas $y^2 = 4ax$, where 'a' is a parameter, is $x\frac{dy}{dx} - 2y = 0$.

Reason (R): If the given family of curves has n parameters, then it is to be differentiated n times to eliminate the parameter and obtain the n^{th} order differential equation.

SECTION — B

This section comprises very short answer (VSA) type questions of 2 marks each.

21. (a) Two pipes A and B can fill a tank in 24 minutes and 32 minutes respectively. If both the pipes are opened simultaneously, after how much time should B be closed so that the tank is full in 18 minutes?

OR

(b) In a one-kilometre race, A beats B by 30 seconds and B beats C by 15 seconds. If A beats C by 180 metres, then find the time taken by A to run 1 kilometre.

22. Solve for x: $\frac{x+3}{x-2} \leq 2$.

23. (a) Solve the following system of equations by Cramer's rule:

$$2x - y = 17,\ 3x + 5y = 6$$

OR

(b) Determine the integral value(s) of x for which the matrix A is singular:

$$A = \begin{bmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{bmatrix}$$

24. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the ordinate is changing 8 times as fast as abscissa.

25. Suppose 2% of the items made by a factory are defective. Find the probability that there are 3 defective items in a sample of 100 items selected at random. (Given $e^{-2} = 0.135$)

SECTION — C

This section comprises short answer (SA) type questions of 3 marks each.

26. (a) A bottle is full of dettol. One-third of its dettol is taken away and an equal amount of water is poured into the bottle to fill it again. This operation is repeated three times. Find the final ratio of dettol to water in the bottle.

OR

(b) A pipe A can fill a tank in 3 hours. There are two outlet pipes B and C from the tank which can empty it in 7 and 10 hours respectively. If all the three pipes are opened simultaneously, how long will it take to fill the tank?

27. Find all the points of local maxima and local minima for the function $f(x) = x^3 - 6x^2 + 9x - 8$.

28. An unbiased die is thrown again and again until three sixes are obtained. Find the probability of obtaining a third six in the sixth throw of the die.

29. The mean weekly sales of a four-wheeler were 50 units per agency in 20 agencies. After an advertising campaign, the mean weekly sales increased to 55 units per agency with standard deviation of 10 units. Test whether the advertising campaign was successful.
(Use $t_{0.005} = 1.729$ for 19 *d.f.*)

30. (a) An asset costs ₹ 4,50,000 with an estimated useful life of 5 years and a scrap value of ₹ 1,00,000. Using linear depreciation method, find the annual depreciation of the asset and construct a yearly depreciation schedule.

OR

(b) Amrita bought a car worth ₹ 12,50,000 and makes a down payment of ₹ 3,00,000. The balance amount is to be paid in 4 years by equal monthly instalments at an interest rate of 15% p.a. Find the EMI that Amrita has to pay for the car.
{Given $(1.0125)^{-48} = 0.5508565$}

31. Maximise $z = 300x + 190y$
subject to constraints:

$x + y \leq 24$,
$2x + y \leq 32$,
$x \geq 0, y \geq 0$.

SECTION — D

This section comprises long answer (LA) type questions of 5 marks each.

32. (a) Find the inverse of the matrix:

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

and hence show that $AA^{-1} = I$.

OR

(b) Using matrix method, solve the following system of equations for x, y and z:
$x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

33. (a) Divide a number 15 into two parts such that the square of one part multiplied with the cube of the other part is maximum.

OR

(b) Find a point on the curve $y^2 = 2x$ which is nearest to the point (1, 4).

34. Fit a straight line trend by method of least squares to the following data and find the trend values:

Year:	2010	2012	2013	2014	2015	2016	2019
Sales (in lakh ₹):	65	68	70	72	75	67	73

35. Define Compound Annual Growth Rate (CAGR) and give the formula for calculating CAGR. Using the formula, calculate CAGR of Vikas's investment given below:
Vikas invested ₹ 10,000 in a stock of a company for 6 years. The value of his investment at the end of each year is given below:

Year 1	Year 2	Year 3	Year 4	Year 5	Year 6
₹ 11,000	₹ 11,500	₹ 11,650	₹ 11,800	₹ 12,200	₹14,000

[Use $(1.4)^{1/6} = 1.058$]

SECTION — E

This section comprises 3 case study based questions of 4 marks each.

Case Study-1

36. A factory produces bulbs, of which 6% are defective bulbs in a large bulk of bulbs.
Based on the above information, answer the following questions:

(i) Find the probability that in a sample of 100 bulbs selected at random, none of the bulbs is defective. (Use: $e^{-6} = 0.0024$) **1**

(ii) Find the probability that the sample of 100 bulbs has exactly two defective bulbs. **1**

(iii) (a) Find the probability that the sample of 100 bulbs will include not more than one defective bulb. **2**

OR

(iii) (b) Find the mean and the variance of the distribution of number of defective bulbs in a sample of 100 bulbs. **2**

Case Study-2

37. A factory manufactures tennis rackets and cricket bats. A tennis racket takes $1\frac{1}{2}$ hours of machine time and 3 hours of craftsmanship in its making; while a cricket bat takes 3 hours of machine time and 1 hour of craftsmanship. In a day, the factory has availability of not more than 42 hours of machine time and 24 hours of craftsmanship. Profit on a racket and on a bat are ₹ 20 and ₹ 10 respectively.

Based on the above information, answer the following questions:

(i) If x and y are the numbers of bats and rackets manufactured by the factory, then write the expression of total profit. **1**

(ii) Write the constraint that relates the number of craftsmanship hours. **1**

(iii) (a) Determine the maximum profit (in ₹) earned by the factory. **2**

OR

(iii) (b) How many bats and rackets respectively, are manufactured to earn maximum profit ? **2**

Case Study-3

38. In the year 2010, Mr. Aggarwal took a home loan of ₹ 30,00,000 from State Bank of India at 7.5% p.a. compounded monthly for 20 years.
Based on the above information, answer the following questions:

(i) Determine the EMI. **1**

(ii) Find the principal paid by Mr. Aggarwal in the 150^{th} instalment. **1**

(iii) (a) Find the total interest paid by Mr. Aggarwal. **2**

OR

(iii) (b) How much was paid by Mr. Aggarwal to repay the entire amount of home loan ? **2**
[Use $(1.00625)^{240} = 4.4608$; $(1.00625)^{91} = 1.7629$]

❑❑

SOLUTIONS

SECTION — A

1. Option (c) is correct.

***Explanation*:** We have $(22)^{12}$,

Here, base number is 22 and its units digit is 2.

Also, exponent is 12, which is divisible by 2 (units digit).

i.e., $12 \div 2 = 6$

So, last digit of $(22)^{12}$ is 6.

2. Option (c) is correct.

***Explanation*:** Since, $3^3 = 27 \pmod 7$

$$3^3 = -1 \pmod 7$$
$$(3^3)^5 = (-1)^5 \pmod 7$$
$$3^{15} = 6 \pmod 7$$

3. Option (d) is correct.

***Explanation*:** $AB = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -5 & 10 \\ -10 & -5 \end{bmatrix}$

$$= \begin{bmatrix} -5-0 & 10-0 \\ 10-10 & -20-5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 10 \\ 0 & -25 \end{bmatrix}$$

4. Option (a) is correct.

***Explanation*:** $\begin{bmatrix} x+y & x+2 \\ 2x-y & 16 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 1 & 3y+1 \end{bmatrix}$

Equating corresponding elements of equal matrices, we get

$x + y = 8$ and $x + 2 = 5$

$\because$ $x + 2 = 5 \Rightarrow x = 3$

Also, $x + y = 8 \Rightarrow 3 + y = 8 \Rightarrow y = 5$

Thus, $x = 3$ and $y = 5$

5. Option (b) is correct.

***Explanation*:** Let the amount of pulse of cost ₹ 85 = x

Let the amount of pulse of cost ₹ 100 = y

Then the total amount of mixture = $85x + 100y$

But the cost of per kg of mixture =₹ 92

ATQ,

$$92(x + y) = 85x + 100y$$
$$(92 - 85)x = (100 - 92)y$$
$$7x = 8y$$
$$\frac{x}{y} = \frac{8}{7}$$
$$x : y = 8 : 7$$

6. Option (b) is correct.

***Explanation*:** If $\frac{|x+1|}{x+1} > 0$, then to make the given fraction, greater than zero denominator should always be greater than zero.

So, $x + 1 > 0 \Rightarrow x > -1$

or, $x \in (-1, \infty)$

7. Option (d) is correct.

***Explanation*:** $|3AB| = (3)^3 |A||B|$

$= 27(-1)(3)$

$= -81$

[$\because$ $|mAB| = m^n |A||B|$ where n is the order of square matrices A and B]

8. Option (a) is correct.

***Explanation*:** $\begin{vmatrix} 2 & 3 & 2 \\ x & x & x \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0$

$$x\begin{vmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0 \qquad \text{[Taking } x \text{ common]}$$

$$x\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0 \qquad [R_1 \to R_1 - 2R_2]$$

$x[0(1-9) - 1(1-4) + 0(9-4)] + 3 = 0$ [Expanding along R_1]

$$3x + 3 = 0$$
$$\Rightarrow \quad x = -1$$

9. Option (d) is correct.

10. Option (c) is correct.

***Explanation*:** Let $I = \int (x-1)e^{-x}\, dx$

Let $-x = t \Rightarrow -dx = dt$

$\therefore$ $I = -\int e^t(-t-1)\, dt$

$= \int e^t(t+1)\, dt$

[$\because$ $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$]

$= te^t + c$

$I = -xe^{-x} + c$

11. Option (b) is correct.

***Explanation*:** Given differential equation is

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating both sides, we get

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$
$$\log x = -\log y + \log c$$
$$\log x + \log y = \log c$$
$$\log(xy) = \log c$$
$$xy = c$$

12. Option (a) is correct.

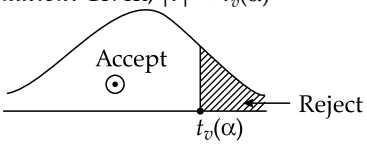
***Explanation*:** Given,

$$P(x = 1) = 2P(x = 2)$$
$$\Rightarrow \quad \frac{e^{-m}m^1}{1!} = 2\frac{e^{-m}m^2}{2!}$$
$$\Rightarrow \quad m = 1$$

Now, $P(x = 0) = \frac{e^{-m}m^0}{0!} = \frac{e^1 1}{1} = e$

13. Option (b) is correct.

***Explanation*:** Given, $|t| < t_v(\alpha)$



14. Option (c) is correct.

15. Option (a) is correct.

16. Option (c) is correct.

***Explanation*:** Present value at the end of each payment period is $\frac{R}{i}$.

17. Option (b) is correct.

***Explanation*:** Amount of ₹ 100 for 1 year when compounded quarterly

$$= ₹\left[100\times\left(1+\frac{2.5}{100}\right)^4\right]$$
$$= ₹\,[100 \times (1 + 0.025)^4]$$
$$= ₹\,(100 \times 1.1038)$$
$$= ₹\,110.38$$

$\therefore$ Effective rate $= (110.38 - 100)\% = 10.38\%$

[Note : Effective rate which is equivalent to nominal rate of 10%. p. a.

When compounded semi-annually = 10.250 %

When compounded Quarterly = 10.381 %

When compounded monthly = 10.471%

When compounded daily = 10.516%

18. Option (a) is correct.

***Explanation*:** Since x and y both take positive values, so region lies in I quadrant.

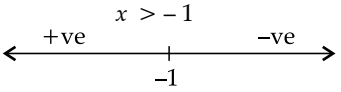
19. Option (d) is correct.

***Explanation*:** We have assertion,

$$f(x) = (x + 2)e^{-x}$$
$\therefore$ $$f'(x) = e^{-x} - (x + 2)\,e^{-x}$$
$$= e^{-x}\,(1 - x - 2)$$
$$= -e^{-x}\,(x + 1)$$

Here, e^{-x} will be always positive, so the function will increasing when $x + 1 > 0$

or $$x > -1$$



So, $f'(x) > 0$ when $x \in (-\infty, -1)$.

Thus, Assertion is incorrect.

Here, Reason is the base correct condition for a function be an increasing function.

20. Option (d) is correct.

***Explanation*:** For assertion,

$$y^2 = 4ax \quad ...(i)$$
$\therefore$ $$2yy' = 4a$$
or $$yy' = 2a$$
or $$a = \frac{yy'}{2}$$

Substituting value of a in eq (i), we get

$$y^2 = 4\left(\frac{yy'}{2}\right)x$$

or $$y^2 = 2yy'x$$
or $$y = 2y'x$$
or $$2x\frac{dy}{dx} - y = 0$$ is the differential equation

representing the family of parabolas $y^2 = 4ax$ where a is the parameter.

Thus, assertion is incorrect.

Reason is correct.

SECTION — B

21. (a) Here capacity of the tank be 96 units (LCM of 24 & 32)

Pipe A fills $= \frac{96}{24} = 4$ units/minute

Pipe B fills $= \frac{96}{32} = 3$ units/minute **1**

Let the time for which B is opened is x minutes.

ATQ,

$$4 \times 18 + 3 \times x = 96$$
$\Rightarrow$ $$3x = 96 - 72$$
$\Rightarrow$ $$x = 8 \text{ minutes}$$ **1**

The pipe B should be closed after 8 minutes

OR

(b) Given,

A beats B by 30 seconds

B beats C by 15 seconds

So, A beats C by (30 + 15) seconds = 45 seconds

Now, time taken by C to travel 180 m = 45 seconds

$\therefore$ Time taken by C to cover the distance of 1 km (1000 m) $= \frac{45}{180}\times 1000 = 250$ seconds **1**

Thus, time taken by A to cover the distance of 1 km $= (250 - 45) = 205$ seconds. **1**

22. We have,

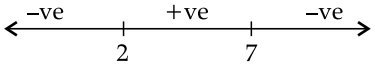
$$\frac{x+3}{x-2} \le 2$$ **1**

$$\frac{x+3}{x-2} - 2 \le 0$$

$$\frac{x+3-2x+4}{x-2} \le 0$$

$$\frac{-x+7}{x-2} \le 0$$

$\therefore$ $$-x + 7 = 0 \text{ and } x - 2 = 0$$
$$x = 7 \text{ and } x = 2$$



Thus, solution set is $(-\infty, 2) \cup [7, \infty)$. **1**

23. (a) Given,

$$2x - y = 17$$
$$3x + 5y = 6$$

Here, $$\Delta = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$$
$$= 10 + 3 = 13$$

$$\Delta_x = \begin{vmatrix} 17 & -1 \\ 6 & 5 \end{vmatrix} = 85 + 6 = 91$$

$$\Delta_y = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix} = 12 - 51 = -39 \quad \mathbf{1}$$

Using Cramer's rule,

$$x = \frac{\Delta x}{\Delta} = \frac{91}{13} = 7$$

and $$y = \frac{\Delta y}{\Delta} = \frac{-39}{13} = -3 \quad \mathbf{1}$$

OR

(b) We have,

$$A = \begin{bmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{bmatrix} \text{ is singular}$$

$$\therefore \quad |A| = 0$$

$$\Rightarrow \begin{vmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0$$

So, $[(x+1)\{(x+2)(x-6)-2\} + 3\{-5(x-6)-8\} + 4\{-5-4(x+2)\}] = 0$ **1**

$[(x+1)\{(x+2)(x-6)-2\} + 3\{-5(x-6)-8\} + 4\{-5-4(x+2)\}] = 0$

$[(x+1)(x^2-4x-14) + 3(-5x+22) + 4(-13-4x)] = 0$

$x^3 - 4x^2 - 14x + x^2 - 4x - 14 - 15x + 66 - 52 - 16x = 0$

$x^3 - 3x^2 - 49x = 0$

$x(x^2 - 3x - 49) = 0$

$\therefore \quad x = 0$ or $x^2 - 3x - 49 = 0$

$$x = 0 \text{ or } x = \frac{3 \pm \sqrt{9 + 4(49)}}{2}$$

$$x = 0 \text{ or } x = \frac{1}{2}(3 \pm \sqrt{205}) \quad \mathbf{1}$$

24. Let (x_1, y_1) be the point on the curve $6y = x^3 + 2$ whose y-coordinate (ordinate) is changing 8 times as fast as the x-coordinate (abscissa).

Then,

$$\left(\frac{dy}{dt}\right)_{\text{at}(x_1, y_1)} = 8\left(\frac{dx}{dt}\right)_{\text{at}(x_1, y_1)} \quad \text{...(i) } ½$$

Differentiating $6y = x^3 + 2$ w.r.t. 't', we get

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$

$$2\frac{dy}{dt} = x^2\frac{dx}{dt} \quad ½$$

$$2 \times 8\left(\frac{dx}{dt}\right)_{\text{at}(x_1, y_1)} = x_1^2\left(\frac{dx}{dt}\right)_{\text{at}(x_1, y_1)} \quad \text{[Using eq (i)]}$$

$$\therefore \quad x_1^2 = 16 \Rightarrow x_1 = \pm 4 \quad ½$$

Now, (x_1, y_1) lies on the curve $6y = x^3 + 2$

$$\therefore \quad 6y_1 = x_1^3 + 2$$

$$6y_1 = (-4)^3 + 2, \text{ when } x_1 = -4$$

$$y_1 = \frac{-31}{3}$$

and $$6y_1 = (4)^3 + 2, \text{ when } x_1 = 4$$

$$y_1 = 11 \quad ½$$

Hence, the required points on the curve are $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$.

25. Given, $n = 100$

Mean $(\lambda) = 2\%$ of $100 = 2$

$$\because \quad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \mathbf{1}$$

$$P(X = 3) = \frac{e^{-2}2^3}{3!}$$

$$= \frac{0.135 \times 8}{6}$$

$$= \frac{1.08}{6} = 0.18 \quad \mathbf{1}$$

SECTION — C

26. **(a)** Let original quantity of dettol in the bottle be x litres.

Then initially quantity of water in bottle = 0 litres

After the 1st operation,

Quantity of dettol in bottle

$$= \left(x - \frac{x}{3}\right) \text{ litres} = \frac{2x}{3}\text{litres} \quad ½$$

After the 2nd operation,

Quantity of dettol in bottle $= \left(\frac{2x}{3} - \frac{1}{3} \times \frac{2x}{3}\right)$litres

$$= \frac{4x}{9}\text{litres} \quad ½$$

After the 3rd operation,

Quantity of dettol in bottle $= \left(\frac{4x}{9} - \frac{1}{3} \times \frac{4x}{9}\right)$litres

$$= \frac{8x}{27}\text{litres} \quad ½$$

Now, Quantity of water in the bottle after 1st operation $= \frac{2x}{3}$litres

Quantity of water in the bottle after 2nd operation

$$= \left(\frac{x}{3} + \frac{2x}{9}\right)\text{litres}$$

$$= \frac{5x}{9}\text{litres}$$

Quantity of water in the bottle offer 3rd

$$= \left(\frac{x}{3} + \frac{2x}{9} + \frac{4x}{27}\right)\text{litres}$$

$$= \frac{19x}{27}\text{litres} \quad \mathbf{1}$$

$\therefore$ Required ratio $= \frac{8x}{27} : \frac{19x}{27}$

$= 8 : 19 \quad ½$

OR

(b) Pipe A can fill the part of tank in 1 hour = $\frac{1}{3}$part ½

Pipe B can empty the part of tank in 1 hour = $\frac{1}{7}$part ½

Pipe C can empty the part of tank in 1 hour = $\frac{1}{10}$part ½

Net part filled in 1 hour = $\frac{1}{3}-\left(\frac{1}{7}+\frac{1}{10}\right)$

$= \frac{1}{3}-\frac{17}{10}$

$= \frac{19}{210}$ **1**

Thus, tank to will be filled in $\frac{210}{19}$ hour *i.e.* $11\frac{1}{9}$ hours or 11 hours. ½

27. Given, $y = f(x) = x^3 - 6x^2 + 9x - 8$

$\therefore \quad y' = 3x^2 - 12x + 9$

Now, put $y' = 0$

$\therefore \quad 3x^2 - 12x + 9 = 0$

$\Rightarrow \quad x = \frac{12 \pm \sqrt{144 - 4\times 3\times 9}}{6}$

$\Rightarrow \quad x = \frac{12 \pm \sqrt{36}}{6}$

$\Rightarrow \quad x = \frac{12 \pm 6}{6}$

$\Rightarrow \quad x = \frac{12+6}{6}$ and $x = \frac{12-6}{6}$

$\Rightarrow \quad x = 3$ and $x = 1$ **1**

Now, $y'' = 6x - 12$

At $x = 3$, $y'' = 6.3 - 12 = 6 > 0$

At $x = 1$, $y'' = 6.1 - 12 = -6 < 0$ **1**

Thus, at $x = 3$, the function is at its local minimum and at $x = 1$, the function is at its local maximum. **1**

28. Let p be the probability of obtaining a 'six' in a single throw of the die.

Then, $p = \frac{1}{6}$ and $q = 1 - \frac{1}{6} = \frac{5}{6}$

Obtaining 3rd six in the 6th throw of the die means that in first five throws there are 2 sixes and the 3rd six is obtained in 6th throw.

Therefore, required probability $= {}^5c_2\, p^2 q^{5-2} \times p$ **1**

$= {}^5c_2\left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^3 \times \frac{1}{6}$

$= \frac{5!}{3!2!} \times \frac{1}{36} \times \frac{125}{216} \times \frac{1}{6}$

$= \frac{5\times 4\times 3!\times 125}{3!\times 2\times 216\times 216}$

$= \frac{20\times 125}{2\times 216\times 216}$

$= \frac{10\times 125}{216\times 216}$

$= \frac{5\times 125}{108\times 216}$

$= \frac{625}{23328}$ **2**

29. Sample size, $n = 20$ agencies

Sample mean $\bar{x} = 55$ units

Sample SD, $\sigma = 10$ units

Population mean, $\mu = 50$ units

Null Hypothesis : The advertising campaign is not successful

i.e., $H_o : \mu = 50$ (There is no significant difference between the mean weekly Sales of units in agencies before and after advertising campaign)

Alternate Hypothesis, $H_1 : \mu > 50$. The advertising campaign was successful. **1**

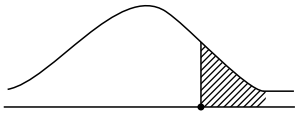
Test statistic, $z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n-1}}}$

$z = \frac{55-50}{\frac{10}{\sqrt{20-1}}} = \frac{5\times\sqrt{20-1}}{10}$

$= \frac{5\times\sqrt{19}}{10}$

$= \frac{5\times 4.358}{10}$

$= \frac{21.794}{10} = 2.179$ **1**



$z > t_{0.05} = 1.729.$

Thus, the advertising campaign was successful. **1**

30. (a) We know that,

Amount of annual depreciation

$= \frac{\text{Cast of asset-scrap value of asset}}{\text{Useful life in years}}$ ½

$= \frac{4,50,000 - 1,00,000}{5}$

$= \frac{3,50,000}{5} = ₹\,70,000$ **1**

Depreciation Schedule

Year	Book Value (Beginning of the Year)	Depreciation	Book Value (End of the of Year)
1.	₹ 4,50,000	₹ 70,000	₹ 3,80,000
2.	₹ 3,80,000	₹ 70,000	₹ 3,10,000
3.	₹ 3,10,000	₹ 70,000	₹ 2,40,000
4.	₹ 2,40,000	₹ 70,000	₹ 1,70,000
5.	₹ 1,70,000	₹ 70,000	₹ 1,00,000

1½

OR

(b) We have,

Cost of car purchased by Amrita = ₹ 12,50,000

Amount of down payment made by Amrita = ₹ 3,00,000

So, balanced amount = 12,50,000 – 3,00,000 = ₹ 9,50,000

Here, P = ₹ 9,50,000

$$i = \frac{15}{1200} = 0.0125$$

and $n = 4$ years $= 4 \times 12$ months $= 48$ months **1**

Let E be the monthly installment Amrita has to pay.

Then $$E = \frac{Pi}{1-(1+i)^{-n}}$$ **1**

$$= \frac{9,50,000 \times 0.0125}{1-(1+0.0125)^{-48}}$$

$$= \frac{11,875}{1-(1.0125)^{-48}}$$

$$= \frac{11,875}{1-0.5508565}$$

$[\because$ Given $(1.0125)^{-48} = 0.5508565]$

$$= \frac{11,875}{0.4491435}$$

= ₹ 26,439.2115 ~ ₹ 26,439. **1**

31. Given, LPP is

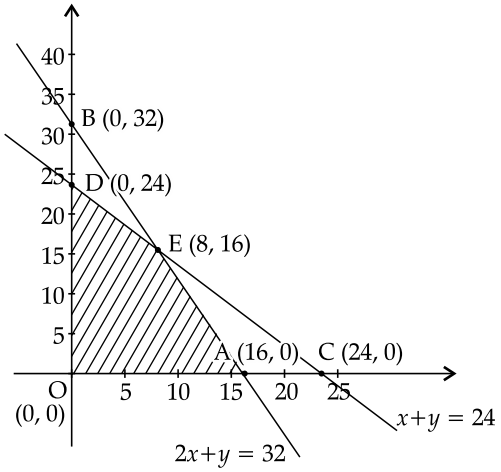
Max Z $= 300x + 190y$

Subject to constraints : $x + y \le 24$

$2x + y \le 32$

$x \ge 0, y \ge 0$

The feasible region of the system of inequation given in constraints is shown in the figure given below.



2

Corner points of feasible region are O(0, 0), A(16, 0), D(0, 24) and E(8, 16).

The values of the objective function z at the corner points are given in the following table:

Comer point (x, y)	Value of $z = 300x + 190y$
A(16, 0)	Z = 300 × 16 + 190 × 0 = 4800
D(0, 24)	Z = 300 × 0 + 190 × 24 = 4560
E(8, 16)	Z = 300 × 8 +190 × 16 = 2400 + 3040 = 5440 (Maximum)

Thus, maximum value of z is 5440 at E (8, 16). **1**

SECTION — D

32. Given metrix is

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

A^{-1} exist if $|A| \neq 0$

Now, $$|A| = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{vmatrix}$$

$= -1(-4-3) - 1(12+1) + 2(9-1)$

$= -1(-7) - 1(13) + 2(8)$

$= 7 - 13 + 16$

$= 10 \neq 0$ **1**

Now, $$A^{-1} = \frac{\text{adj}A}{|A|}$$

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}'$$

where, cofactor $c_{ij} = (-1)^{i+j} M_{ij}$

$$c_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} = -13$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2$$

$$\text{adj } A = \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}' \quad \textbf{2}$$

$$= \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{10}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix} \quad \textbf{1}$$

$$\text{Now, } AA^{-1} = \frac{1}{10}\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 7-13+16 & -2-2+4 & -3+7-4 \\ -21+13+8 & 6+2+2 & 9-7-2 \\ +7-39+32 & -2-6+8 & -3+21-8 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{1}$$

$= I$ **Hence Proved.**

OR

Given system of equations;

$$x - y + z = 4$$
$$2x + y - 3z = 0$$
$$x + y + z = 2$$

This can be written as $AX = B$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Here, $|A| = 1(1+3) + 1(2+3) + 1(2-1)$
$= 4 + 5 + 1$
$= 10$

Since $|A| \neq 0$

Hence, the system of equations is consistent and has a unique solution given by $X = A^{-1}B$

Now, $A^{-1} = \frac{\text{adj}A}{|A|}$ and adj $A = C^{\text{T}}$

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 1 + 3 = 4$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2+3) = -5$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2-1) = 1$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-1-1) = 2$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (1-1) = 0$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(1+1) = -2$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 3 - 1 = 2$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -(-3-2) = 5$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = (1+2) = 3$$

Hence, the co-factor matrix $c = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$

$$\therefore \quad \text{adj } A = c^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Thus, $A^{-1} = \frac{\text{adj}A}{|A|}$

$$= \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Solution is given by

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Thus, solution is $x = 2$, $y = -1$ and $z = 1$.

33. **(a)** Let the two numbers be x and y.

Then, $x + y = 15$ ...(i)

ATQ,

$P = x^2y^3$

or $P = x^2(15-x)^3$ ...(ii) [from eq (i)] **1**

Now, $\frac{dP}{dx} = 2x(15-x)^3 + 3x^2(15-x)^2(-1)$

or, $\frac{dP}{dx} = 2x(15-x)^3 - 3x^2(15-x)^2$

For maximum or minimum value of P, put $\frac{dP}{dx} = 0$

i.e., $2x(15-x)^3 - 3x^2(15-x)^2 = 0$

$\Rightarrow \quad 2x(15-x) = 3x^2$

$\Rightarrow \quad 30x - 2x^2 = 3x^2$

$\Rightarrow \quad x = 6$ **1**

Form eq (i), we get

$$y = 9$$

Thus, $x = 6$ and $y = 9$ ½

Again, $\frac{d^2P}{dx^2} = 2(15-x)^3 - 6x(15-x)^2$

$$- 6x(15-x)^2 + 6x^2(15-x)$$

$$= 2(15-x)^3 - 12x(15-x)^2 + 6x^2(15-x)$$ **1**

At $x = 6$,

$$\frac{d^2P}{dx^2} = 2(9)^3 - 12 \times 6(9)^2 + 6 \times (6)^2(9)$$

$$= 1458 - 5832 + 1944$$

$$= 3402 - 5832$$

$$= -2430 < 0$$ **1**

Thus, P is maximum when $x = 6$ and $y = 9$.

So, the required two parts into which 15 should be divided are 6 and 9. ½

OR

(b) Given the equation of curve (parabola) is $y^2 = 2x$.

Let (x, y) be the nearest to the point (1, 4).

We have,

$$x = \frac{y^2}{2}$$ ...(i) ½

Using distance formula, we get

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

$$z = d^2 = (x-1)^2 + (y-4)^2$$ **1**

or $$z = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

[from (i)]

or $$z = \frac{y^4}{4} - 8y + 17$$

$\therefore$ $$\frac{dz}{dy} = y^3 - 8$$ **1**

Put $$\frac{dz}{dy} = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y = 2$$ ½

Now, from eq (i), we get

$$x = \frac{(2)^2}{2} = 2$$ ½

Again differentiating w.r.t. y, we get

$$\frac{d^2z}{dy^2} = 3y^2$$

$$\left.\frac{d^2z}{dy^2}\right|_{\text{at } y=2} = 3(2)^2 = 12 > 0 \text{ Minima}$$ **1**

So, the nearest point is (2, 2). ½

34. Let $y_t = a + bx$ be the trend line.

Let $X = x - 2014$

Year (x)	Value (y)	X	X^2	XY
2010	65	– 4	16	– 260
2012	68	– 2	4	– 136
2013	70	– 1	1	– 70
2014	72	0	0	0
2015	75	1	1	75
2016	67	2	4	134
2019	73	5	25	365
	$\Sigma Y = 490$	$\Sigma X = 1$	$\Sigma X^2 = 51$	$\Sigma XY = 108$

2

By the method of least squares, the values of 'a' and 'b' are obtained by solving the normal equations

$$\Sigma y = na + b\Sigma X$$

and $$\Sigma xy = a\Sigma X + b\Sigma X^2$$

$\therefore$ $$490 = 7a + b(1)$$

and $$108 = a(1) + b(51)$$

or, $$7a + b = 490$$ ...(i)

and $$a + 51b = 108$$ ...(ii)

On solving eqs. (i) & (ii), we get

$$a = 69.8933 \text{ and } b = 0.7472$$ **2**

Thus, $$y_t = 69.8933 + (0.7472)x$$

or $$y_t = 69.89 + 0.75x$$

Trend Values,

for 2010, $y_t = 69.89 + 0.75(-4) = 66.89$

for 2012, $y_t = 69.89 + 0.75(-2) = 68.39$

for 2013, $y_t = 69.89 + 0.75(-1) = 69.14$

for 2014, $y_t = 69.89 + 0.75(0) = 69.89$

for 2015, $y_t = 69.89 + 0.75(1) = 70.64$

for 2016, $y_t = 69.89 + 0.75(2) = 71.39$

for 2019, $y_t = 69.89 + 0.75(5) = 73.64$ **1**

35. Compound Annual Growth Rate (CAGR): The compound annual growth rate (CAGR) is annualized average rate of revenue growth between two given years, assuming growth take place at on exponentially compounded rate.

$$\text{CAGR} = \left(\frac{\text{Ending Value}}{\text{Beginning Value}}\right)^{\frac{1}{n}} - 1$$

or $$\text{CAGR} = \left(\frac{EV}{BV}\right)^{\frac{1}{n}} - 1$$ **1**

Here, n is investment period

$$\text{CAGR of Vikas's investment} = \left(\frac{EV}{BV}\right)^{\frac{1}{n}} - 1$$

$$= \left(\frac{14{,}000}{10{,}000}\right)^{\frac{1}{6}} - 1$$

$$= (1.4)^{\frac{1}{6}} - 1$$

$$= 1.058 - 1$$

$$= 0.058$$

$$= 5.8\%$$ **3**

SECTION — E

36. Case Study-1

Given, $n = 100, p = 6\% = 0.06$

$\therefore$ Mean, $np = 100 \times 0.06 = 6\ (\lambda)$

(i) $P(X = r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$

$P(X = 0) = \dfrac{e^{-6}6^0}{0!}$

$= e^{-6}$

$= 0.0024$ (Given) **1**

(ii) $P(X = 2) = \dfrac{e^{-6}6^2}{2!}$

$= \dfrac{e^{-6}.36}{2}$

$= e^{-6} \times 18 = 0.0024 \times 18$

$= 0.0432$

(iii) (a) $P(X = 1) = \dfrac{e^{-6}6^1}{1!}$

$= e^{-6} \times 6$

$= 0.0024 \times 6$

$= 0.0144$ **1**

Now, Probability that sample of 100 bulbs will include not more than one defective bulb

$P(X \le 1) = P(X = 0) + P(X = 1)]$

$= [0.0024 + 0.0144]$

$= 0.016$ **1**

OR

(iii) (b) We know that mean and variance of the Poisson distribution are same. **1**

Thus, mean = 6 and variance = 6. **1**

37. Case Study-2

(i) Here, given profit on tennis racket is ₹ 20 and profit on cricket bat is ₹ 10

Thus, total profit Max. $Z = 10x + 20y$ **1**

(ii) Since, tennis racket requires 1.5 hours and cricket bat requires is 3 hours of machine time. Also, there is maximum 42 hours of machine time available.

$1.5y + 3x \le 42$

$y + 2x \le 28$ ...(i)

Since, tennis racket requires 3 hours and cricket bat requires 1 hour of craftmanship. Also, there is maximum 24 hours of craftmanship time available.

$\therefore$ $3y + x \le 24$ ...(ii)

Thus, required constraints are :

$2x + y \le 28$

$x + 3y \le 24$ **1**

(iii) **(a)** The L.P.P for given problem can be formulated as

Max. $Z = 10x + 20y$

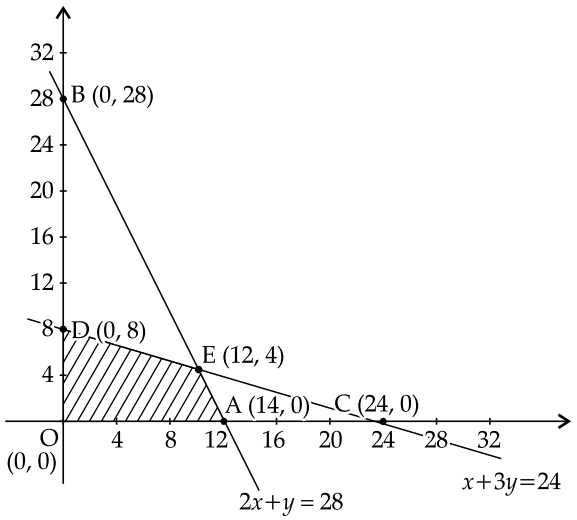
Subject to constraints:

$2x + y \le 28$

$x + 3y \le 24$

$x \ge 0, y \ge 0$

Plotting all constraints, we get the following feasible region.



1

Corner Points	Value of Z = 10x + 20y
A (14, 0)	140
E (12, 4)	200 (Maximum)
D (0, 8)	160

Thus, maximum profit earned by factory is ₹ 200. **1**

OR

(b) From the above solution (*iii – a*), we get maximum profit at E (12, 4).

So, 12 cricket bats and 4 tennis rackets must be made by factory in order to maximize its profit.

38. Case Study-3

(i) Giren, P = ₹ 30,00,000, $i = 7.5\%$ *p.a.*

$= \dfrac{7.5}{1200} = 0.00625$

and $n = 20$ years $= 240$ months

Since, EMI $= \dfrac{Pi(1+i)^n}{(1+i)^n - 1}$

$= \dfrac{30,00,000 \times 0.00625(1+0.00625)^{240}}{(1+0.00625)^{240} - 1}$

$= \dfrac{30,00,000 \times 0.00625(1+0.00625)^{240}}{(1.00625)^{240} - 1}$

$= \dfrac{18750 \times 4.4608}{4.4608 - 1}$

$= \dfrac{83640}{3.4608}$

$= ₹\ 24167.82$

$= ₹\ 24168$ **1**

(ii) Interest paid by Mr. Aggarwal in 150th installment

$= \dfrac{EMI[(1+i)^{n-150+1} - 1]}{(1+i)^{n-150+1}}$

$$= \frac{24167.82[(1.00625)^{240-150+1} - 1]}{(1.00625)^{240-150+1}}$$

$$= \frac{24167.82[(1.00625)^{91} - 1]}{(1.00625)^{91}}$$

$$= \frac{24167.82[1.7629 - 1]}{1.7629}$$

$$= \frac{18437.629}{1.7629}$$

= ₹ 10,458.6929

Principal paid = 24167.82 – 10,458.69

= ₹ 13,709.13 ~ ₹ 13709 **1**

(iii) **(a)** Total interest paid = n(EMI) – P

= 240 × 24167.82 – 30,00,000

= 58,00,276.8 – 30,00,000

= 28,00, 276.8

= 28,00,277 **2**

OR

(b) Total amount paid = n.EMI

= 240 × 24,167.82

= ₹ 58,00,276.8

= ₹ 58,00,277 **2**

❑❑

Sample Question Paper

(Issued by Board dated 16th Sep. 2022)

Applied Mathematics (241)

Class- XII

Session- 2022-23

SOLVED

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions:

(i) This question paper contains five sections A, B, C, D and E. Each section is compulsory.
(ii) Section - A carries 20 marks weightage, Section - B carries 10 marks weightage, Section - C carries 18 marks weightage, Section - D carries 20 marks weightage and Section - E carries 3 case-based with total weightage of 12 marks.
(iii) Section – A: It comprises of 20 MCQs of 1 mark each.
(iv) Section – B: It comprises of 5 VSA type questions of 2 marks each.
(v) Section – C: It comprises of 6 SA type of questions of 3 marks each.
(vi) Section – D: It comprises of 4 LA type of questions of 5 marks each.
(vii) Section – E: It has 3 case studies. Each case study comprises of 3 case-based questions, where 2 VSA type questions are of 1 mark each and 1 SA type question is of 2 marks. Internal choice is provided in 2 marks question in each case-study.
(viii) Internal choice is provided in 2 questions in Section - B, 2 questions in Section – C, 2 questions in Section - D. You have to attempt only one of the alternatives in all such questions.

Section - A

All questions are compulsory. No internal choice is provided in this section

1. What is the least value of 'x' that satisfies $x = 27 (\text{mod } 4)$, when $27 < x \leq 36$?
(A) 27 **(B)** 30 **(C)** 31 **(D)** 35 **1**

2. Let $p > 0$ and $q < 0$ and $p, q \in Z$, then choose the correct inequality from the given below options to complete the statement:
$p + q \boxed{\cdots\cdots} p - q$
(A) $>$ **(B)** $\leq$ **(C)** $\geq$ **(D)** $<$ **1**

3. A machine makes car wheels and in a random sample of 26 wheels, the test statistic is found to be 3.07. As per the t-distribution test (of 5% level of significance), what can you say about the quality of wheels produced by the machine? (Use $t_{25}(0.05) = 2.06$)
(A) Superior quality **(B)** Inferior quality **(C)** Same quality **(D)** Cannot say **1**

4. For the purpose of t-test of significance, a random sample of size (n) 34 is drawn from a normal population, then the degree of freedom (v) is:
(A) $\frac{1}{34}$ **(B)** 33 **(C)** 34 **(D)** 35 **1**

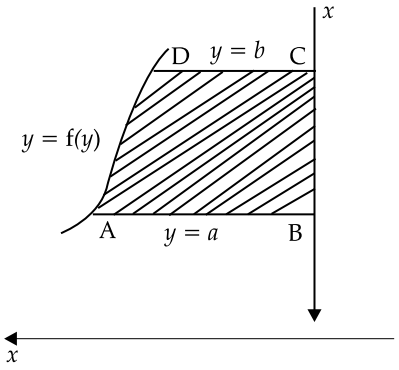
5. A person can row a boat along the stream of the river at 10 km/h and against the stream in 6 km/h. What is the speed of the stream flow?
(A) 1 km/h **(B)** 2 km/h **(C)** 4 km/h **(D)** 5 km/h **1**

6. Standard deviation of a sample from a population is called a:
(A) Standard error **(B)** Parameter **(C)** Statistic **(D)** Central limit **1**

7. Two water supplying trucks – A and B supply water to remote areas. Truck A is carrying 100 litres of water to a village 1.5 km away and truck B is delivering 80 litres of water to another village, 1 km away. Due to bad road conditions, each truck loses 20 ml water while travelling each metre distance. Which truck is able to deliver more water and by how much more?
(A) Truck A, 20 litres **(B)** Truck B, 20 litres **(C)** Truck A, 10 litres **(D)** Truck B, 10 litres **1**

8. What is the face value of a sinking fund that yields a dividend of ₹ 1800 at 10% semi-annually?
(A) ₹ 3600 **(B)** ₹ 18000 **(C)** ₹ 24000 **(D)** ₹ 36000 **1**

9. In the given figure, the area bounded by the curve $x = f(y)$, y-axis and abscissa $y = a$ and $y = b$ is equal to:

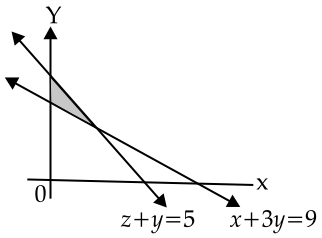


(A) $\int_a^b f(y)dy$ **(B)** $\int_a^b f(x)dx$ **(C)** $\int_a^b |f(y)|dy$ **(D)** $\int_a^b |f(x)|dx$ **1**

10. A factory production is delayed for three weeks due to breakdown of a machine and unavailability of spare parts. Under which trend oscillation does this situation fall under?
(A) Seasonal **(B)** Cyclical **(C)** Secular **(D)** Irregular **1**

11. A newspaper printing machine costs ₹ 4,80,000 and estimated scrap value of ₹ 25,000 at the end of its useful life of 10 years. What is its annual depreciation as per linear method?
(A) ₹ 4,550 **(B)** ₹ 45,500 **(C)** ₹ 50,500 **(D)** ₹ 61,500 **1**

12. In the given figure (I), what is the LPP shaded region known as?



(A) Feasible region **(B)** Feasible solution **(C)** Optimal region **(D)** Objective region **1**

13. General solution of differential equation: $y \log y dx - x\, dy = 0$ is:
(A) $y = \log|Cx|$ **(B)** $y = e^{|Cx|}$ **(C)** $y = e^{-Cx}$ **(D)** $\log y = |C + x|$ **1**

14. An investment of ₹ 10,000 becomes ₹ 60,000 in 4 years, then the CAGR (compound annual growth rate) is given by:
(A) $\frac{\sqrt[4]{6}-1}{100}$ **(B)** $\frac{\sqrt[4]{6}+1}{100}$ **(C)** $[\sqrt[4]{6}-1]\times 100$ **(D)** $[\sqrt[4]{6}+1]\times 100$ **1**

15. In what ratio shall I add water to the liquid detergent costing ₹ 480 per litre to get resulting mixture worth ₹ 300 per litre?
(A) 5 : 3 **(B)** 3 : 8 **(C)** 3 : 5 **(D)** 5 : 8 **1**

16. A grain whole-seller visits the granary market. While going around to make a good purchase, he takes a handful of rice from random sacks of rice, in order to inspect the quality of farmers produce. The handful of rice taken from a sack of rice for quality inspection is a:
(A) statistic **(B)** population **(C)** parameter **(D)** sample **1**

17. For predicting the straight-line trend in the sales of scooters (in thousands) on the basis of 6 consecutive years data, the company makes use of 4-year moving averages method. If the sales of scooters for respective years are a, b, c, d, e and f respectively, then which of the following average will not be computed?
(A) $\frac{a+b+c+d}{4}$ **(B)** $\frac{b+c+d+e}{4}$ **(C)** $\frac{a+c+d+e}{4}$ **(D)** $\frac{c+d+e+f}{4}$ **1**

18. In a school, a random sample of 145 students is taken to check whether a student's average calory intake is 1500 or not. The collected data of average calories intake of sample students is presented in a frequency distribution, which is called a:

(A) Statistics **(B)** Sampling distribution

(C) Parameter **(D)** Population sampling **1**

DIRECTION: For questions 19 and 20, two statements are given – one labelled Assertion(A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (A), (B), (C) and (D) as given below:

(A) Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).
(B) Both assertion (A) and reason (R) are true and reason (R) is not the correct explanation of assertion (A).
(C) Assertion (A) is true but reason (R) is false.
(D) Assertion (A) is false but reason (R) is true. **1**

19. **Assertion (A):** Kuhu and Beena are two equally capable badminton players. Probability that Beena will beat Kuhu in 3 games out of 4 is 25%
Reason (R): The probability of r successes in n trials, denoted by $P(X = r)$ is given by $P(X = r) = {}^nC_r\, p^r q^{n-r}$, $r = 0, 1, ..., n$ where p denotes success and q denotes failure in each trial. **1**

20. **Assertion (A):** If the nominal rate of interest is 12.5% and the inflation is 2%, then the effective rate of interest is 10.5%
Reason (R): If the interest is calculated only at the end of an year, then the effective rate of interest is same as the nominal rate of interest. **1**

Section - B

All questions are compulsory. In case of internal choice, attempt any one question only

21. ₹ 2,50,000 cash is equivalent to a perpetuity of ₹ 7,500 payable at the end of each quarter. What is the rate of interest convertible quarterly? **2**

22. Find value of $2a + 3b - c$, if $A = \begin{bmatrix} 0 & -1 & 28 \\ a-8 & 0 & 3b \\ -c+2 & 2 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

OR

There are two real value(s) of x, for which the value of the determinant $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix}$ is 86. Find the value(s) of x: **2**

23. A book publisher sells a hard cover edition of a book for ₹ 72 and a paperback edition for ₹ 40. In addition to a fixed weekly cost of ₹ 9,600, the cost of printing hardcover and paperback editions are ₹ 56 and ₹ 28 per book respectively. Each edition requires 5 minutes on the printing machine whereas hardcover binding takes 10 minutes and paperback takes 2 minutes on the binding machine. The printing machine and the binding machine are available for 80 hours each week. Formulate the linear programming problem to maximise the publisher's profit. **2**

24. A boatman takes half as much time in rowing his boat for a certain distance downstream than upstream. What is the ratio between his speed of rowing the boat in still water and speed of current? **2**

OR

In a 200-metre race, Anuj can beat Param by 5 metre or 3 seconds. How much time did Anuj take to complete the race?

25. Mitul invested ₹ 3,50,000 in a fund. At the end of the year the value of the fund is ₹ 4,37,500. What is the nominal rate of interest, if the market price is same at the end of the year? **2**

Section - C

All questions are compulsory. In case of internal choice, attempt any one question only

26. Find the interval(s) in which the function $f(x) = \frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x$, is strictly increasing and strictly decreasing. **3**

27. Two badminton teams A and B are staying in the same hotel. Team A has 2 male and 3 female players accompanied by 1 coach. Team B comprises of 1 male, 2 female players and 2 coaches. The daily diet requirement (calories and protein) for each person is as given below:

	Calories	Protein
Male player	2500	65 g
Female player	1900	50 g
Coach	2000	54 g

Use matrix algebra to calculate the total diet requirement of calories and protein for each team. **3**

28. Evaluate $\int \frac{dx}{(1+e^x)(1+e^{-x})}$

OR

Evaluate $\int x \log(1+x^2)dx$ **3**

29. Under the pure market competition scenario, the demand function p_d and the supply function p_s for a certain commodity are given as $p_d = \frac{8}{x+1} - 2$ and $p_s = \frac{x+3}{2}$ respectively, where p is the price and x is the quantity of the commodity. Using integrals, find the producer's surplus.

OR

The demand function p for maximising a profit monopolist is given by $p = 274 - x^2$ while the marginal cost is $4 + 3x$, for x units of the commodity. Using integrals, find the consumer surplus **3**

30. Surjeet purchased a new house, costing ₹ 40,00,000 and made a certain amount of down payment so that he can pay the balance by taking a home loan from XYZ Bank. If his equated monthly instalment is ₹ 30,000, at 9% interest compounded monthly (reducing balance method) and payable for 25 years, then what is the initial down payment made by him? [Use $(1.0075)^{-300} = 0.1062$] **3**

31. 10 years ago, Mr Mehra set up a sinking fund to save for his daughter's higher studies. At the end of 10 years, he has received an amount of ₹ 10,21,760. What amount did he put in the sinking fund at the end of every 6 months for the tenure, which paid him 5% p.a. compounded semi-annually? [Use $(1.025)^{20} = 1.6386$] **3**

Section - D

All questions are compulsory. In case of internal choice, attempt any one question only

32. It is known that 3% of plastic buckets manufactured in a factory are defective. Using the Poisson distribution on a sample of 100 buckets, find the probability of:

(i) Zero defective buckets

(ii) At most one bucket is defective [Use $e^{-3} = 0.049$]

OR

In a math aptitude test, student scores are found to be normally distributed having mean as 45 and standard deviation 5. What percentage of students have scores:

(i) more than the mean score?

(ii) between 30 and 50? **5**

33. An event management company charges ₹ 4,800 per guest, for a bulk booking for 100 guests. In addition, it offers a discount of ₹ 200 for each group of 10 guests over and above 100 guest booking. What is the number of guests that will maximise the amount of money the company receives on a booking? What is the maximum profit on such booking?

OR

To manufacture 'x' number of dolls, a company's total cost function $C(x)$ is given by $C(x) = 100 + 0.025x^2$ and the total revenue function $R(x)$ is described as $R(x) = 5x$. Given that $C(x)$ and $R(x)$ are in thousand rupees, what number of dolls shall be manufactured to maximise the profit of the company? What is the maximum profit? **5**

34. Rahul is at the whole sale market to purchase folding tables and chairs, to later sell them at his furniture shop. He has only ₹ 5,760 to spend and his van has space to carry at the most 20 items. A table costs him ₹ 360 and a chair costs ₹ 240. Back at his shop, he plans to sell a table at a profit of ₹ 22 and a chair at a profit of ₹ 18. Given that he can sell all the items that he purchases, how many tables and chairs shall he purchase in order to maximise his profit? **5**

35. The equilibrium conditions for three competitive markets are described as given below, where p_1,p_2 and p_3 are the equilibrium price for each market respectively.

$$p_1 + 2p_2 + 3p_3 = 85$$
$$3p_1 + 2p_2 + 2p_3 = 105$$

$$2p_1 + 3p_2 + 2p_3 = 110$$

Using matrix method, find the values of respective equilibrium prices. **5**

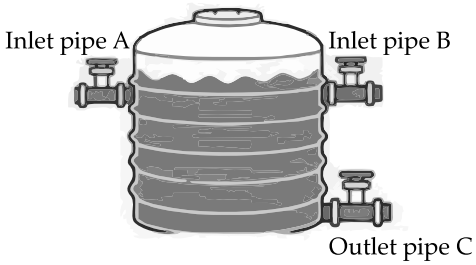
Section - E

All questions are compulsory. In case of internal choice, attempt any one question only

CASE STUDY – I

36. An overhead water tank has three pipes A, B and C attached to it (as shown in figure (II)). The inlet pipes A and B can fill the empty tank independently in 15 hours and 12 hours respectively. The outlet pipe C alone can empty a full tank in 20 hours.

Based on the above information, answer the following questions. Show steps to support your answers.



(a) For a routine cleaning of the tank, the tank needs to be emptied. If pipes A and B are closed at the time when the tank is filled to two-fifth of its total capacity, how long will pipe C take to empty the tank completely? **1**

(b) How long will it take for the empty tank to fill completely, if all the three pipes are opened simultaneously? **1**

(c) On a given day, pipes A, B and C are opened (in order) at 5 am, 8 am and 9 am respectively, to fill the empty tank. In how many hours will the tank be filled completely?

OR

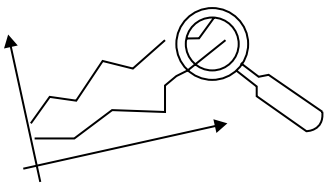
Given that the tank is half-full, only pipe C is opened at 6 AM, to empty the tank. After closing the pipe C and an hour's cleaning time, tank is filled completely by pipe A and B together. What is the total time taken in the whole process? **2**

CASE STUDY – II

37. When observed over a long period of time, a time series data can predict trend that can forecast increase or decrease or stagnation of a variable under consideration. Such analytical studies can benefit a business for forecasting or prediction of future estimated sales or production.

Mathematically, for finding a line of best-fit to represent a trend, many methods are available. Methods like moving-averages and least-squares squares are some of the techniques to predict such trends.

Mrs. Shamita runs a bread factory and the record of her sales of bakery items for the period of 2015 - 2019 is as follows:



Year	2015	2016	2017	2018	2019
Sales (in ₹ thousands)	35	42	46	41	48

Based on the above information, answer the following questions. Show steps to support your answers.

(a) By taking year 2017 as origin, use method of least-squares to find the best-fit trend line equation for Mrs. Shamita's business. Show the steps of your working.

OR

Demonstrate the technique to fit the best-suited straight-line trend by the method of 3-years moving averages. Also draw the trend line. **2**

(b) What are the estimated sales for Mrs. Shamita's business for year 2022? **1**

(c) Mrs Shamita wishes to grow her business to yearly sale of ₹ 67400. In which year will she be able to reach her target? **1**

CASE STUDY – III

38. According to an educational board survey, it was observed that class XII students apply at least one to four weeks ahead of colleges application deadline. Let X represent the week when an average student applies ahead of a college's application deadline and the probability of student to get admission in the college $P(X = x)$ is given as follows:

$$P(X=x)=\begin{cases}\frac{kx}{6} & \text{when } x=0,1 \text{ or } 2\\ \frac{(1-k)x}{6} & \text{when } x=3\\ \frac{kx}{2} & \text{when } x=4\\ 0 & \text{when } x>4\end{cases}$$

Where k is a real number.

Based on the above information, answer the following questions. Show steps to support your answers.

(a) Find the value of k. **1**

(b) What is the probability that Sonali will get admission in the college, given that she applied at least 2 weeks ahead of application deadline? **1**

(c) Calculate the mathematical expectation of number of weeks taken by a student to apply ahead of a college's application deadline.

OR

To promote early admissions, the college is offering scholarships to the students for applying ahead of deadline as follows: ₹ 50000 for applying 4 weeks early, ₹ 20000 for applying 3 weeks early, ₹ 12000 for applying 2 weeks early, and ₹ 9600 for applying 1 week early

What is the expected scholarship offered by the college? **2**

■■■

SOLUTIONS

Sample Question Paper

With CBSE Marking Scheme 2022-23 (Issued by Board)

Applied Mathematics (241)

Section - A

1. **Option (C) is correct.**

Explanation: $x \equiv 27 (\text{mod } 4)$

$\Rightarrow \quad x - 27 = 4k$, for some integer k

$\Rightarrow \quad x = 31$ as $27 < x \leq 36$

2. **Option (D) is correct.**

3. **Option (B) is correct.**

Explanation: $n = 26$

$\Rightarrow \quad |t| = 3.07 > t_{25}(0.05)$

$= 3.07 > 2.06$

4. **Option (B) is correct.**

Explanation: $n = 34$

$\Rightarrow \quad v = 34 - 1 = 33$

5. **Option (B) is correct.**

Explanation:

Speed of boat downstream $= u = 10$ km/h

And, speed of boat upstream $= v = 6$ km/h

$\Rightarrow \quad$ Speed of stream $= \frac{1}{2}(u - v) = 2$ km/h

6. **Option (C) is correct.**

7. **Option (C) is correct.**

Explanation:

Truck A carries water $= 100 - \frac{20 \times 1500}{1000} = 70\ l$

Truck B carries water $= 80 - \frac{20 \times 1000}{1000} = 60\ l$

8. **Option (B) is correct.**

Explanation: Let the face value of the sinking fund $= x$

Then, $\frac{10}{200}x = 1800$

$\Rightarrow \quad x = ₹\ 36000$

9. **Option (C) is correct.**

10. **Option (D) is correct.**

11. **Option (B) is correct.**

Explanation: $D = \frac{C - S}{n}$

$= \frac{480000 - 25000}{10}$

$= ₹\ 45500$

12. **Option (A) is correct.**

13. **Option (B) is correct.**

Explanation: $\int \frac{dy}{y \log y} = \int \frac{dx}{x}$

$\Rightarrow \quad \log(\log y) = \log |x| + \log |C|$

$\Rightarrow \quad \log(\log y) = \log |Cx|$

$\Rightarrow \quad y = e^{|Cx|}$

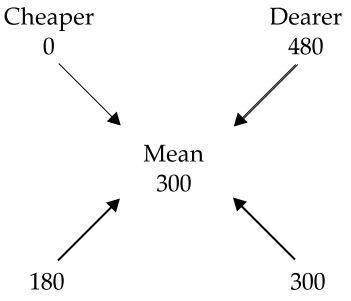
14. **Option (C) is correct.**

Explanation:

$\left[\left(\frac{60000}{10000}\right)^{1/4} - 1\right] \times 100 = [\sqrt[4]{6} - 1] \times 100$

15. **Option (C) is correct.**

Explanation:



$\Rightarrow \quad 180 : 300 = 3 : 5$

16. **Option (D) is correct.**

17. **Option (C) is correct.**

18. **Option (B) is correct.**

19. **Option (A) is correct.**

Explanation:

$$P(\text{Win in one game}) = \text{P(Lose in one game)} = \frac{1}{2}$$

$\Rightarrow P$(Beena to win in 3 out of 4 games)

$$= {}^4C_3 \cdot \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1$$

$$= 4 \times \left(\frac{1}{2}\right)^4$$

$$= \frac{1}{4} = 25\%$$

20. **Option (B) is correct.**

Explanation:

Effective rate of interest = Nominal rate – inflation rate = 12.5 – 2 = 10.5%

Section - B

21. $P = 250000, R = 7500, i = \frac{r}{400}$

$$\Rightarrow \quad 250000 = \frac{7500 \times 400}{r}$$

$$\Rightarrow \quad r = 12\ \%\ \text{p.a.} \qquad \mathbf{2}$$

22.

$$a - 8 = 1$$

$$\Rightarrow \quad a = 9$$

$$3b = -2 \qquad [a_{ij} = -a_{ji}]$$

$$\Rightarrow \quad b = -\frac{2}{3}$$

$$-c + 2 = -28$$

$$\Rightarrow \quad c = 30 \qquad \mathbf{1}$$

$$\Rightarrow \quad 2a + 3b - c = -14 \qquad \mathbf{1}$$

OR

Expanding C_1,

we get $\Delta = 1(2x^2 + 4) - (-2)(4x - 0) + 5(8 - 0)$

$= 86$ **1**

$\Rightarrow x^2 + 4x - 21 = 0$

$\therefore \quad x = 3, -7$ **1**

23. Let the number of hardcopy and paperback copies be x and y respectively

$\Rightarrow$ Maximum profit $Z = (72x + 40y) - (9600 + 56x + 28y)$

$= 16x + 12y - 9600$ **1**

Subject to constraints:

$$x + y \le 960$$

$$5x + y \le 2400$$

$$x, y \ge 0 \qquad \mathbf{1}$$

24. Speed of boat in still waters = x km/h

Speed of stream = y km/h

Distance travelled = d km

Time taken to travel downstream = $\frac{d}{x+y}$

Time taken to travel upstream = $\frac{d}{x-y}$ **1**

Then, $\frac{2d}{x+y} = \frac{d}{x-y}$

$\Rightarrow \quad x : y = 3 : 1$ **1**

OR

Param runs 5 m in 3 seconds

$\Rightarrow$ time taken to run 200 m = $\frac{3}{5} \times 200$

= 120 seconds **1**

Anuj 's time = 120 – 3

= 117 seconds **1**

25. $V_f = 437500, V_i = 350000$

$$\text{Nominal rate} = \frac{V_f - V_i}{V_i} \times 100 \qquad \mathbf{1}$$

$$= \frac{437500 - 350000}{350000} \times 100$$

$$= 25\% \qquad \mathbf{1}$$

Section - C

26. $$f(x) = \frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x$$

$$f'(x) = x^3 - 6x^2 + 11x - 6$$

$$= (x - 1)(x - 2)(x - 3)$$

Strictly increasing in $(1, 2) \cup (3, \infty)$

Strictly increasing in $(-\infty, 1) \cup (2, 3)$

Detailed Answer:

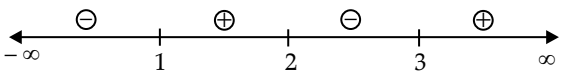
$$f(x) = \frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x$$

$$f'(x) = x^3 - 6x^2 + 11x - 6$$

$$= x^2(x - 1) - 5x(x - 1) + 6(x - 1)$$

$$= (x - 1)(x^2 - 5x + 6)$$

$$= (x - 1)(x - 2)(x - 3)$$



For increasing $f'(x) > 0$

$(1, 2) \cup (3, \infty)$

For decreasing $f'(x) < 0$

$(-\infty, 1) \cup (2, 3)$

27. Daily diet of team $A = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2500 & 65 \\ 1900 & 50 \\ 2000 & 54 \end{bmatrix}$

$$\begin{bmatrix} 5000+5700+2000 \\ 130+150+54 \end{bmatrix} = \begin{bmatrix} 12700 \\ 334 \end{bmatrix}$$

Team A consumes 12700 calories and 334 g protein **1½**

$$\text{Daily diet of team } B = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2500 & 65 \\ 1900 & 50 \\ 2000 & 54 \end{bmatrix}$$

$$\begin{bmatrix} 2500+3800+4000 \\ 65+100+108 \end{bmatrix} = \begin{bmatrix} 10300 \\ 273 \end{bmatrix}$$

Team B consumes 10300 calories and 273 g protein **1½**

28. $$\int \frac{dx}{(1+e^x)(1+e^{-x})} = \int \frac{e^x dx}{(1+e^x)^2}$$

$$= \int \frac{dt}{t^2}$$

where $t = e^x + 1$ and $dt = e^x dx$

$$= \frac{-1}{t} + C$$

$$= \frac{-1}{1+e^x} + C \quad \textbf{3}$$

OR

$\int_{II} x \log(1+x^2)_{I} dx$, Integration by parts

$$= \log(1+x^2).\int x dx - \int \left[\frac{d}{dx}\log(1+x^2).\int x dx \right] dx$$

$$= \frac{x^2}{2}\log(1+x^2) - \int \left[\frac{2x}{1+x^2}.\frac{x^2}{2} \right] dx$$

$$= \frac{x^2}{2}\log(1+x^2) - \int \frac{x^3}{1+x^2} dx$$

$$= \frac{x^2}{2}\log(1+x^2) - \int \left[x - \frac{x}{1+x^2} \right] dx$$

$$= \frac{x^2}{2}\log(1+x^2) - \frac{x^2}{2} + \frac{1}{2}\log(1+x^2) + C$$

$$= \frac{1}{2}[(1+x^2)\log(1+x^2) - x^2] + C \quad \textbf{3}$$

29. Under pure competition, $p_d = p_s$

$$\Rightarrow \quad \frac{8}{x+1} - 2 = \frac{x+3}{2} \quad \textbf{1½}$$

$$\Rightarrow \quad x^2 + 8x - 9 = 0$$

$$\Rightarrow \quad x = -9, 1$$

$$\therefore \quad x = 1$$

When $x_0 = 1$

$$\Rightarrow \quad p_0 = 2$$

$$\therefore \quad \text{Produce surplus} = 2 - \int_0^1 \frac{x+3}{2} dx$$

$$= 2 - \left[\frac{x^2}{4} + \frac{3x}{2} \right]_0^1 = \frac{1}{4} \quad \textbf{1½}$$

OR

$$p = 274 - x^2$$

$$\Rightarrow \quad R = px = 274x - x^3$$

$$\frac{dR}{dx} = 274 - 3x^2$$

Given $MR = 4+3x$

In profit monopolist market,

$$MR = \frac{dR}{dx}$$

$$\Rightarrow \quad 4 + 3x = 274 - 3x^2$$

$$\Rightarrow \quad x^2 + x - 90 = 0 \quad \textbf{1½}$$

$$\Rightarrow \quad x = -10, 9$$

$$\therefore \quad x = 9$$

When $x_0 = 9$

$$\Rightarrow \quad p_0 = 193$$

$$\therefore \quad \text{Consumer surplus} = \int_0^9 (274 - x^2) dx - 193 \times 9$$

$$= \left[274x - \frac{x^3}{3} \right]_0^9$$

$$= 2223 - 1737$$

$$= 486 \quad \textbf{1½}$$

30. Purchase = ₹ 40,00,000

Down payment = ₹ x

Balance = (₹ 40,00,000 – x)

$$i = \frac{9}{1200}$$

$$= 0.0075, n = 25 \times 12$$

$$= 300$$

EMI = ₹ 30,000 **1**

$$\Rightarrow \quad 30000 = \frac{(4000000 - x) \times 0.0075}{1 - (1.0075)^{-300}}$$

$$\Rightarrow \quad 30000 = \frac{(4000000 - x) \times 0.0075}{1 - 0.1062}$$

$$\Rightarrow \quad x = 424800$$

Down payment = ₹ 4,24,800 **2**

31. $n = 10 \times 2 = 20$, $S = 10, 21, 760$, $i = 5200 = 0.025$, $R = ?$

$$S = R\left[\frac{(1+i)^n - 1}{i} \right] \quad \textbf{1½}$$

$$\Rightarrow \quad 1021760 = R\left[\frac{(1+0.025)^{20}-1}{0.025}\right]$$

$$\Rightarrow \quad 1021760 = R\left[\frac{1.6386-1}{0.025}\right]$$

$$\Rightarrow \quad R = \left[\frac{1021760\times 0.025}{0.6386}\right]$$

$$\Rightarrow \quad R = ₹\ 40{,}000$$

Mr. Mehra set aside an amount of ₹ 40,000 at the end of every six months. **1½**

Section - D

32. Probability of defective bucket = 0.03

$$n = 100$$

$$m = np = 100 \times 0.03 = 3$$

Let X = number of defective buckets in a sample of 100

$$P(X = r) = \frac{m^r e^{-m}}{r!}$$

$$r = 0, 1, 2, 3, \ldots. \quad \textbf{1}$$

(i) P(no defective bucket) $= P(r = 0)$

$$= \frac{3^0 e^{-3}}{0!} = 0.049 \quad \textbf{2}$$

(ii) P(at most one defective bucket)

$$= P(r = 0, 1)$$

$$= \frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} \quad \textbf{2}$$

$$= 0.049 + 0.147$$

$$= 0.196$$

OR

X = scores of students, $\mu = 45$, $\sigma = 5$

$$\therefore \quad Z = \frac{X-\mu}{\sigma} = \frac{X-45}{5} \quad \textbf{1}$$

(i) When $X = 45, Z = 0$

$$P(X > 45) = P(Z > 0) = 0.5$$

$\Rightarrow$ 50% students scored more than the mean score **2**

(ii) When $X = 30$, $Z = -3$ and when $X = 50$, $Z = 1$

$P(30 < X < 50) = P(-3 < Z < 1) = P(-3 < Z \le 1)$

$= P(-3 < Z \le 0) + P(0 \le Z < 1)$

$= P(0 \le Z < 3) + P(0 \le Z < 1)$

$= 0.4987 + 0.3413 = 0.84$

$\Rightarrow$ 84% students scored between 30 and 50 marks **2**

33. Let x be the number of guests for the booking

Clearly, $x > 100$ to avail discount

$$\therefore \text{ Profit, } \quad P = \left[4800 - \frac{200}{10}(x-100)\right]x$$

$$= 6800x - 20x^2 \quad \textbf{2}$$

$$\Rightarrow \quad \frac{dP}{dx} = 6800 - 40x$$

$$\Rightarrow \quad x = 170 \quad \textbf{1}$$

$$\text{As} \quad \frac{d^2P}{dx^2} = -40 < 0, \forall\ x \quad \textbf{1}$$

A booking for 170 guests will maximise the profit of the company And, Profit = ₹ 5,78,000 **1**

OR

$$P(x) = R(x) - C(x)$$

$$= 5x - (100 + 0.025x^2) \quad \textbf{2}$$

$$\Rightarrow \quad P'(x) = 5 - 0.05x$$

$$\Rightarrow \quad x = 100 \quad \textbf{1}$$

$$\text{As} \quad P''(x) = -0.05 < 0, \forall\ x \quad \textbf{1}$$

$\therefore$ Manufacturing 100 dolls will maximise the profit of the company And, Profit = ₹ 1, 50, 000 **1**

34. Let the number of tables and chairs be x and y respectively

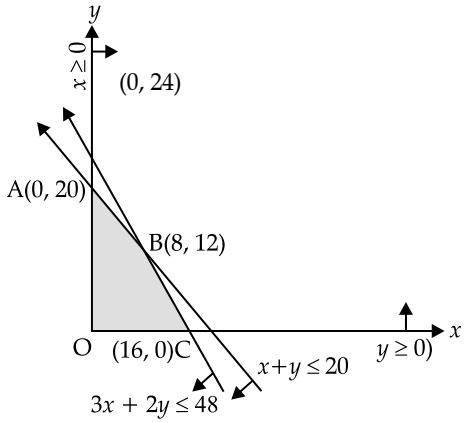
(Max profit) $Z = 22x + 18y$

Subject to constraints:

$$x + y \le 20$$

$$3x + 2y \le 48$$

$$x, y \ge 0 \quad \textbf{1½}$$



2

The feasible region OABCA is closed (bounded)

Corner points	$Z = 22x + 18y$
O (0,0)	0
A (0,20)	360
B (8,12)	392
C (16,0)	352

Buying 8 tables and 12 chairs will maximise the profit **1½**

35. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$

$\Rightarrow \quad |A| = 9 \Rightarrow A^{-1}$ exists

And $\quad A^{-1} = \frac{1}{9}\begin{bmatrix} -2 & 5 & -2 \\ -2 & -4 & 7 \\ 5 & 1 & -4 \end{bmatrix}$ **2**

$AX = B$

$\Rightarrow \quad X = A^{-1}B$

$\Rightarrow \quad X = \frac{1}{9}\begin{bmatrix} -2 & 5 & -2 \\ -2 & -4 & 7 \\ 5 & 1 & -4 \end{bmatrix}\begin{bmatrix} 85 \\ 105 \\ 110 \end{bmatrix}$

$= \begin{bmatrix} 15 \\ 20 \\ 10 \end{bmatrix}$

$\Rightarrow \quad p_1 = 15$

$p_2 = 20, p_3 = 10$ **3**

Detailed Answer:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= -2 - 4 + 15 = 9$$

$\therefore$ A^{-1} is exist.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -2$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -4$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = 7$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4$$

$$A^{-1} = \frac{Adj\ A}{|A|}$$

$$= \frac{1}{9}\begin{bmatrix} -2 & 5 & -2 \\ -2 & -4 & 7 \\ 5 & 1 & -4 \end{bmatrix}$$

$AX = B$

$\Rightarrow \quad X = A^{-1}B$

$$X = \frac{1}{9}\begin{bmatrix} -2 & 5 & -2 \\ -2 & -4 & 7 \\ 5 & 1 & -4 \end{bmatrix}\begin{bmatrix} 85 \\ 105 \\ 110 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -170 + 525 - 220 \\ -170 - 420 + 770 \\ 425 + 105 - 442 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 20 \\ 10 \end{bmatrix}$$

$\therefore\ p_1 = 15, p_2 = 20, p_3 = 10$

Section - E

CASE STUDY - I

36. **(a)** Pipe C empties 1 tank in 20 $h \Rightarrow \frac{2}{5}$ th tank in

$\frac{2}{5} \times 20 = 8$ hours **1**

(b) Part of tank filled in 1 hour $= \frac{1}{15} + \frac{1}{12} - \frac{1}{20}$

$= \frac{1}{10}$th

$\Rightarrow$ time taken to fill tank completely = 10 hours **1**

(c) Let the tank be completely filled in 't' hours

$\Rightarrow$ pipe A is opened for 't' hours

pipe B is opened for '$t - 3$' hours

And, pipe C is opened for '$t - 4$' hours

$\Rightarrow$ In one hour, part of tank filled by pipe $A = \frac{t}{15}$ th

part of tank filled by pipe $B = \frac{t-3}{15}$ th

and, part of tank emptied by pipe $C = \frac{t-4}{15}$ th

Therefore $\frac{t}{15}+\frac{t-3}{12}-\frac{t-4}{20} = 1$

$\Rightarrow \quad t = 10.5$

Total time to fill the tank = 10 hours 30 minutes **2**

OR

6 am, pipe C is opened to empty $\frac{1}{2}$ filled tank

Time to empty = 10 hours

Time for cleaning = 1 hour

Part of tank filled by pipes A and B in 1 hour

$$= \frac{1}{15}+\frac{1}{12}$$

$$= \frac{3}{20} \text{ th tank}$$

$\Rightarrow$ time taken to fill the tank completely $= \frac{20}{3}$ hours

Total time taken in the process

$$= 10 + 1 + \frac{20}{3}$$

$$= 17 \text{ hour } 40 \text{ minutes}$$

CASE STUDY - II

37. (a)

Year	Y	X	X^2	XY
2015	35	– 2	4	– 70
2016	42	– 1	1	– 42
2017	46	0	0	0
2018	41	1	1	41
2019	48	2	4	96
	212		10	25

$$a = \frac{\Sigma Y}{n} = \frac{212}{5}$$

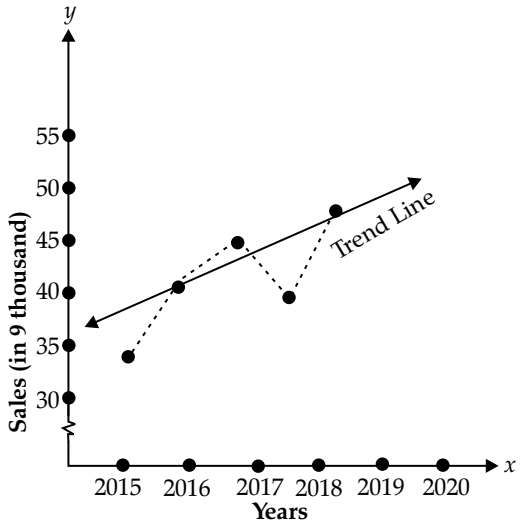
$$= 42.4$$

and $\quad b = \frac{\Sigma XY}{\Sigma X^2} = \frac{25}{10} = 2.5$

$$Y_C = 42.4 + 2.5X$$ **2**

OR

Year	Y	3-years moving average
2015	35	—
2016	42	41
2017	46	43
2018	41	45
2019	48	—



(b) For year 2022,

$$Y_{2022} = 42.4 + 2.5(2022 - 2017)$$
$$= 54.9$$

$\Rightarrow$ the estimated sales for year 2022 = ₹ 54,900 **1**

(c) $\quad Y_C = 42.4 + 2.5X$

$\Rightarrow \quad 67.4 = 42.4 + 2.5X$

$\Rightarrow \quad X = 10$

Sales will be ₹ 67,400 in year (2017 + 10) = year 2027 **1**

CASE STUDY - III

38. (a) $\frac{k}{6}+\frac{2k}{6}+\frac{3(1-k)}{6}+\frac{4k}{2} = 1$

$\Rightarrow \quad k = \frac{1}{4}$ **1**

(b) P(getting admission on applying at least 2 weeks ahead of application deadline)

$$= P(X = 2, 3, 4)$$

$$= \frac{1}{12}+\frac{3}{8}+\frac{1}{2} = \frac{23}{24}$$

[alternate method: $1 - P(X = 1) = 1 - \frac{1}{24} = \frac{23}{24}$]

1

(c) X = week applied ahead of application deadline

X	1	2	3	4
$P(X)$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{3}{8}$	$\frac{1}{2}$
$XP(X)$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{9}{8}$	2

$\therefore$ $E(X) = \frac{80}{24} = 3\frac{1}{3}$ weeks **2**

OR

X = Scholarship money awarded for the week applied in, before the deadline

Week applied in	1	2	3	4
X	9600	12000	20000	50000
P(X)	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{3}{8}$	$\frac{1}{2}$
XP(X)	$\frac{9600}{24}$	$\frac{12000}{12}$	$\frac{60000}{8}$	$\frac{50000}{2}$

$\therefore$ $E(X) = ₹\,33{,}900$

■■■

Solved Paper, 2022
Applied Mathematics
Term-II (Delhi Set-4)

Series : %BAB%

Question Paper Code : 465

Time : 2 Hrs. **Maximum Marks : 40**

General Instructions:

Read the following instructions very carefully and strictly follow them:

*(i) This question paper contains **three** sections – **Section A, B** and **C**.*

*(ii) Each section is **compulsory**.*

*(iii) **Section – A** has **6** short answer type-I questions of **2** marks each.*

*(iv) **Section – B** has **4** short answer type-II questions of **3** marks each.*

*(v) **Section – C** has **4** long answer type questions of **4** marks each.*

(vi) There is an internal choice in some questions.

SECTION - A

Questions number 1 to 6 carry 2 marks each.

1. (a) Evaluate:

$$\int_0^1 \frac{xe^x}{(x+1)^2}\,dx$$ **2**

OR

(b) Solve the following differential equation:

$$\frac{dy}{dx} = e^{x+y} + x^2e^y$$ **2**

2. Find the present value of a perpetuity of ₹ 18,000 payable at the end of 6 months, if the money is worth 8% p.a. compounded semi-annually. **2**

3. (a) Find the effective rate which is equivalent to nominal rate of 10% p.a. compounded monthly. **2**
[Given that: $(1.00833)^{12} = 1.1047$]

(b) Abhay bought a mobile phone for ₹ 30, 000. Th mobile phone is estimated to have a scrap value of ₹ 3,000 after a span of 3 years. Using the linear depreciation method, find the book value of the mobile phone at the end of 2 years.

4. Consider the following hypothesis:

$H_0 : \mu = 36$

$H_1 : \mu \neq 36$

A sample of 81 items is taken whose mean is 37.5 and the standard deviation is 5. Test the hypothesis at 5% level of significance. **2**

[Given: Critical value of Z for a two-tailed test at 5% level of significance is 1.96]

5. The following table shows the annual rainfall (in mm) recorded for Cherrapunji, Meghalaya:

Year	Rainfall (in mm)
2001	1.2
2002	1.9
2003	2
2004	1.4
2005	2.1
2006	1.3
2007	1.8
2008	1.1
2009	1.3

Determine the trend of rainfall by 3-years moving average. **2**

6. Maximize $z = 3x + 4y$, if possible, subject to the constraints:

$$x - y \leq -1$$
$$-x + y \leq 0$$
$$x, y \geq 0$$

SECTION - B

Questions number 7 to 10 carry 3 marks each.

7. **(a)** The supply function of a commodity is $100p = (x + 20)^2$. Find the Producer's Surplus (PS), when the market price is ₹ 25. **3**

OR

(b) Find: **3**

$$\int \frac{2x^2 + 1}{x^2 - 3x + 2} dx$$

8. Fit a straight line trend by the method of least squares and find the trend value for the year 2008 for the following data: **3**

Year	Production (in lakh tonnes)
2001	30
2002	35
2003	36
2004	32
2005	37
2006	40
2007	36

9. Ten cartons are taken at random from an automatic packing machine. The mean net weight of the ten cartons is 11.8 kg and standard deviation is 0.15 kg. Does the sample mean differ significantly from the intended mean of 12 kg ? **3**
[Given that for d.f. = 9, $t_{0.05} = 2.26$]

10. Madhu exchanged her old car valued at ₹ 1,50,000 with a new one priced at ₹ 6,50,000. She paid ₹ x as down payment and the balance in 20 monthly equal instalments of ₹ 21,000 each. The rate of interest offered to her is 9% p.a. Find the value of x. **3**
[Given that: $(1.0075)^{-20} = 0.86118985$]

SECTION - C

Questions number 11 to 14 carry 4 marks each.

11. In a certain culture of bacteria, the rate of increase of bacteria is proportional to the number present. It is found that there are 10,000 bacteria at the end of 3 hours and 40,000 bacteria at the end of 5 hours. Determine the number of bacteria present in the beginning. **4**

12. **(a)** Calculate the EMI under 'Flat Rate System' for a loan of ₹ 5,00,000 with 10% annual interest rate for 5 years. **4**

OR

(b) A machine costing ₹ 2,00,000 has effective life of 7 years and its scrap value is ₹ 30,000. What amount should the company put into a sinking found earning 5% p.a., so that it can replace the machine after its usual life ? Assume that a new machine will cost ₹ 3,00,000 after 7 years. **4**
[Given that: $(1.05)^7 = 1.407$]

13. A start-up company invested ₹ 3,00,000 in shares for 5 years. The value of this investment was ₹ 3,50,000 at the end of second year, ₹ 3,80,000 at the end of third year and on maturity, the final value stood at ₹ 4,50,000. Calculate the compound Annual Growth Rate (CAGR) on the investment. **4**
[Given that: $(1.5)^{1/5} = 1.084$]

14. A dietician wishes to mix two types of food F_1 and F_2 in such a way that the vitamin content of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food F_1 contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C, while Food F_2 contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5 per kg to purchase Food F_1 and ₹ 7 per kg to purchase Food F_2.
Based on the above information, answer the following questions:

(a) To find out the minimum cost of such a mixture, formulate the above problem as a LPP.

(b) Determine the minimum cost of the mixture. **4**

❑❑❑

SOLUTIONS

SECTION - A

1. (a) Let $I = \int_0^1 \frac{xe^x}{(x+1)^2}dx$

$= \int_0^1 \frac{(x+1-1)e^x}{(x+1)^2}dx$

$= \int_0^1 \frac{(x+1)e^x}{(x+1)^2}dx - \int_0^1 \frac{e^x}{(x+1)^2}dx$

$= \int_0^1 \frac{e^x}{(x+1)}dx - \int_0^1 \frac{e^x}{(x+1)^2}dx$

$= I_1 - \int_0^1 \frac{e^x}{(x+1)^2}dx$ ...(i)

Now, $I_1 = \int_0^1 \frac{e^x}{x+1}dx$

$= \left[\frac{e^x}{x+1}\right]_0^1 - \int_0^1 \left(-\frac{1}{(x+1)^2}\right)e^x dx$

$\therefore \quad I_1 = \left(\frac{e}{2}-1\right) + \int_0^1 \frac{e^x}{(x+1)^2}dx$

Now, from eq (i), we get

$I = \left(\frac{e}{2}-1\right) + \int_0^1 \frac{e^x}{(x+1)^2}dx - \int_0^1 \frac{e^x}{(x+1)^2}dx$

$\therefore \quad I = \frac{e}{2}-1$

OR

(b) Given differential equation:

$$\frac{dy}{dx} = e^{x+y} + x^2e^y$$

$$\Rightarrow \quad \frac{dy}{dx} = e^y(e^x + x^2)$$

$$\Rightarrow \quad (e^{-y})dy = (e^x + x^2)dx$$

Integrating both sides, we get

$$\int e^{-y}dy = \int (e^x + x^2)dy$$

$$-e^{-y} = e^x + \frac{x^3}{3} + C$$

or $\quad e^x + e^{-y} + \frac{x^3}{3} + C = 0$

2. We know that

$$\text{Present value of perpetuity} = \frac{\text{Cash flow}}{\text{Interest Rate}}$$

Here, Cash flow = ₹ 18,000

$$\text{interest rate} = \frac{8/2}{100}$$

$$= \frac{4}{100} = 0.04$$

$\therefore$ Present value of perpetuity $= \frac{18,000}{0.04}$

= ₹ 4,50,000

3. (a) We have, $r = \frac{10}{100} = 0.1$, m = 12

$\therefore \quad r_e = \left(1+\frac{r}{m}\right)^m - 1$

$= \left(1+\frac{0.1}{12}\right)^{12} - 1$

$= (1 + 0.00833)^{12} - 1$

$[\because \text{Given } (1.00833)^{12} = 1.1047]$

$= 1.1047 - 1$

$= 0.1047$ or 10.47%

OR

We have,

C = Original cost of mobile = ₹ 30,000

S = Scrap value of mobile = ₹ 3,000

n = Useful life of mobile = ₹ 3 years

Let D be the annual depreciation, then

$$D = \frac{C-S}{n}$$

$$= \frac{30,000-3,000}{3}$$

$$= \frac{27,000}{3}$$

= ₹ 9,000

$\therefore$ The book value at the end of 2 years

= ₹ (30,000 – 2 × 9,000)

= ₹ (30,000 – 18,000)

= ₹ 12,000

4. Given hypothesis:

$H_0 : \mu = 35$

$H_1 : \mu \neq 35$

Also, given sample mean $(\bar{X}) = 37.5$

Sample size $(n) = 81$

Standard deviation $(\sigma) = 5$

Let test statistics be given by

$$Z = \frac{\bar{X}-\mu}{\sigma}$$

$$= \frac{37.5-35}{5}$$

$$= \frac{2.5}{5} = 0.5$$

The level of significance, $\alpha = 5\%$

Here given $Z_{\text{at } 5\%} = 1.96$

and $Z_{\text{calculated at } 5\%} = 0.5$

Since, $Z_{\text{at } 5\%} = Z_{\text{calculated at } 5\%}$

Thus, null hypothesis H_0 is accepted.

5.

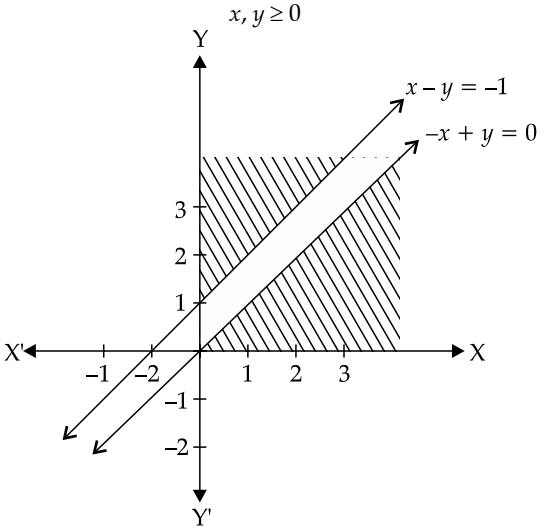
Year	Rain fall (in mm)	Three year moving total	Three year moving average
2001	1.2		—
2002	1.9	5.1	1.7
2003	2	5.3	1.766
2004	1.4	5.5	1.833
2005	2.1	4.8	1.6
2006	1.3	5.2	1.733
2007	1.8	4.2	1.4
2008	1.1	4.2	1.4
2009	1.3		—

6. Given LPP is

Max. $Z = 3x + 4y$

Subject to constraints

$$x - y \le -1$$
$$-x + y \le 0$$
$$x, y \ge 0$$



The region determined by the constraints is shown in above figure.

It is clear, there is no feasible region and thus, Z has no maximum value.

SECTION - B

7. **(a)** The supply function is given by $100p = (x + 20)^2$.

It is given that $p_0 = 25$. Substituting $p_0 = 25$ and $x = x_0$ in the supply function, we have

$$100 \times 25 = (x_0 + 20)^2$$
$$\Rightarrow \quad (x_0 + 20)^2 = 2500$$
$$\Rightarrow \quad x_0 + 20 = 50$$
$$\Rightarrow \quad x_0 = 30$$
$$\therefore \quad x_0 = 30$$

$$PS = p_0 x_0 - \int_0^{x_0} p\,dx$$

$$= 25 \times 30 - \int_0^{30} \frac{(x+20)^2}{100} dx$$

$$= 750 - \frac{1}{100}\left[\left(\frac{x_0 + 20}{3}\right)^3\right]_0^{30}$$

$$= 750 - \frac{1}{300}\{(30 + 20)^3 - 20^3\}$$

$$= 750 - \frac{1}{300}(125000 - 8000)$$

$$= 750 - 390$$

$$= ₹\ 360$$

OR

(b) Here, $\dfrac{2x^2+1}{x^2-3x+2} = 2 + \dfrac{6x-3}{x^2-3x+2}$

(On dividing Nr by Dr)

Now, $\dfrac{6x-3}{x^2-3x+2} = \dfrac{6x-3}{(x-2)(x-1)}$

Let $\dfrac{6x-3}{(x-2)(x-1)} = \dfrac{A}{x-2} + \dfrac{B}{x-1}$

(By Partial fraction)

$$\Rightarrow \quad 6x - 3 = (A + B)\,x + (-A - 2B)$$

on comparing both sides, we get

$A + B = 6$ and $-A - 2B = -3$

On solving, we get

$A = 9$ and $B = -3$

$$\therefore \quad \frac{2x^2+1}{x^2-3x+2} = 2 + \frac{9}{x-2} - \frac{3}{x-1}$$

Let $I = \int \dfrac{2x^2+1}{x^2-3x+2} dx$

$$\therefore \quad I = \int \left(2 + \frac{9}{x-2} - \frac{3}{x-1}\right) dx$$

$$I = 2x + 9\,|x-2| - 3\log|x-1| + C$$

8.

Year	Production (y) (in lakh tonnes)	$x = t - 2004$	xy	x^2
2001	30	– 3	– 90	9
2002	35	– 2	– 70	4
2003	36	– 1	– 36	1

2004	32	0	0	0
2005	37	1	37	1
2006	40	2	80	4
2007	36	3	108	9
$n = 7$	$\Sigma y = 246$	$\Sigma x = 0$	$\Sigma xy = 29$	$\Sigma x^2 = 28$

Now, $a = \frac{\Sigma y}{n}$

$= \frac{246}{7}$

$= 35.14$

$b = \frac{\Sigma xy}{\Sigma x^2}$

$= \frac{29}{28}$

$= 1.035$

$\therefore$ Trend equation is:

$y_t = a + bx$

$\Rightarrow$ $y_t = 35.14 + 1.035x$

Trend value at 2008 is

$y_{2008} = 35.14 + 1.035\ (4)$

$= 35.14 + 4.140$

$= 39.28$

9. Given,

Population mean (μ) = 12

Sample mean ($\bar{X}$) = 11.8

Standrad deviation (σ) = 0.15

Sample size (n) = 10

Let $H_0 : \mu = \bar{X} \neq 11.8$

and $H_1 : \mu \neq \bar{X} = 11.8$

The test statistic is

$$t = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n-1}}}$$

At 0.05 level of significance, for a 2–tail test, for $10 - 1 = 9$ degrees of freedom if the calculated value is greater than 2.262 or less than – 2.262, we reject H_0, else retain it.

Substituting known values of parameters, we get

$$t = \frac{11.8 - 12}{\frac{0.15}{\sqrt{9}}}$$

$$= \frac{-0.2 \times 3}{0.15} = -4$$

Since, $t = -4$ is less than – 2.262, so we reject H_0 at 5% level of significance.

Thus, the sample mean differences significantly from the intended weight of 12 kg.

10. Cash down price = ₹ (6,50,000 – 1,50,000)

= ₹ 5,00,000

Now, Cash down price = down payment (x) + present value (PV) of 21,000 for 20 months at 9% compounded monthly

$$\therefore \quad PV = \frac{21000 \times \left[1 - \left(1 + \frac{0.09}{12}\right)^{-20}\right]}{\frac{0.09}{12}}$$

$$= \frac{21000 \times \left[1 - (1 + 0.0075)^{-20}\right]}{0.0075}$$

$$= \frac{21000 \times \left[1 - (1.0075)^{-20}\right]}{0.0075}$$

$$= \frac{21000 \times (1 - 0.86118985)}{0.0075}$$

$$= \frac{21000 \times 0.13881015}{0.0075}$$

$$= \frac{2,915.01315}{0.0075}$$

= ₹ 388,668.42

$\therefore$ Down payment (x) = Cash down price – PV

= 5,00,000 – 388,668.42

= ₹ 111,331.58

SECTION - C

11. Let N be the number of bacteria in the culture at time t.

$\therefore$ $\frac{dN}{dt} \propto N$

$\Rightarrow$ $\frac{dN}{dt} = KN$

$\Rightarrow$ $\int \frac{dN}{N} = \int K dt$

$\Rightarrow$ $\log N = Kt + C$

$\Rightarrow$ $N = e^{Kt + C}$ ...(i)

Now, at $t = 3$, $N = 10,000 = 10^4$

$\therefore$ From eq (i), we get

$10^4 = e^{3K + C}$ ...(ii)

and at $t = 5$, $N = 40,000 = 4 \times 10^4$

From eq (i), we get

$4 \times 10^4 = e^{5K + C}$ ...(iii)

On dividing eq. (iii) by eq (ii), we get

$$\frac{4 \times 10^4}{10^4} = \frac{e^{5K+C}}{e^{3K+C}}$$

$4 = e^{2K}$

$\Rightarrow$ $\log 4 = 2K \log e$

(on taking log both sides)

$\Rightarrow$ $K = \frac{\log 4}{2}$

$$= \frac{2\log 2}{2} = \log 2 \ (\log e = 1)$$

From eq. (ii), we get

$$10^4 = e^{3\log 2 + C}$$

$$4 \log 10 = 3 \log 2 + C \log e$$

(on taking log both sides)

$$4 \log 10 - 3 \log 2 = C \qquad [\because \log e^e = 1]$$

Now, from eq. (i), at $t = 0$

$$N = e^{K\times 0 + C}$$

$$\Rightarrow \quad N = e^C$$

$$\Rightarrow \quad N = e^{4\log 10 - 3\log 2}$$

$$\Rightarrow \quad N = e^{\log 10^4 - \log 2^3}$$

$$\Rightarrow \quad N = e^{\log\left(\frac{10^4}{2^3}\right)}$$

$$\Rightarrow \quad N = \frac{10^4}{2^3}$$

$$= \frac{10000}{8} = ₹\ 1{,}250$$

12. **(a)** We have, P = Principal = ₹ 5,00,000

i = rate of interest per month

$$= \frac{10}{12\times 100} = \frac{1}{120}$$

n = No. of installments

$= 12 \times 5 = 60$

$$\therefore \quad EMI = ₹\ P\left(i + \frac{1}{n}\right)$$

$$= ₹\ 5{,}00{,}000\left(\frac{1}{120} + \frac{1}{60}\right)$$

$$= ₹\ 5{,}00{,}000\left(\frac{1+2}{120}\right)$$

$$= ₹\ 5{,}00{,}000\left(\frac{3}{120}\right)$$

$$= \frac{₹\ 5{,}00{,}000}{40} = ₹\ 12{,}500$$

OR

Cost of new machine = ₹ 3,00,000

Scrap value of old machine = ₹ 30,000

Hence, money required for new machine after 7 years

$$= ₹\ 3{,}00{,}000 - ₹\ 30{,}000$$

$$= ₹\ 2{,}70{,}000$$

If P is the annual deposit into sinking fund, then we have

Amount of annuity, A = ₹ 2,70,000

No. of periods = 7 years

$i = 5\% = 0.05$

$$A = \frac{P}{i}[(1+i)^n - 1]$$

$$\Rightarrow \quad 2{,}70{,}000 = \frac{P}{0.05}[(1+0.05)^7 - 1]$$

$$\Rightarrow \quad P = \frac{270000\times 0.05}{(1.05)^7 - 1}$$

$$\Rightarrow \quad P = ₹\ 33161.35$$

13. $$\text{CAGR} = \left(\frac{\text{Final value of investment}}{\text{Initial value of investment}}\right)^{\frac{1}{n}} - 1 \qquad ...(i)$$

where n = investment period

Here, given

Final value of investment = ₹ 4,50,000

Initial value of investment = ₹ 3,00,000

investment period = 5

$$\therefore \quad \text{CAGR} = \left(\frac{4{,}50{,}000}{3{,}00{,}000}\right)^{\frac{1}{5}} - 1$$

$$= (1.5)^{1/5} - 1$$

$$= 1.084 - 1$$

$$= 0.084 \text{ or } 8.4\%$$

14. **(a)** let the mixture contain x kg. of food F_1 and y kg of Food F_2.

Clearly $x \geq 0, y \geq 0$. We make the following table from the given data:

Resources	Food		Requirement
	$F_1(x)$	$F_2(y)$	
Vitamin A (unit/kg)	2	1	8
Vitamin C (unit/kg)	1	2	10
cost (₹/kg)	5	7	

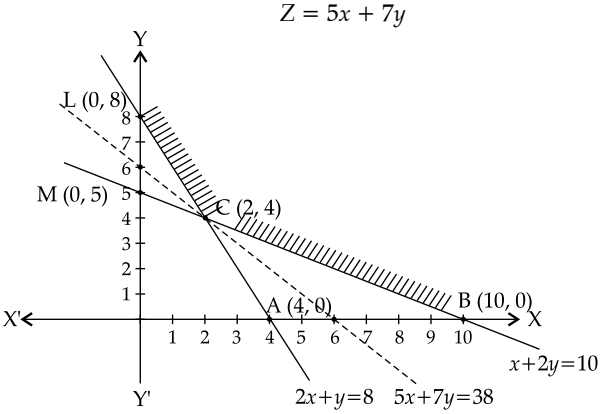
Since the mixture must contain atleast 8 unit of vitamin A and 10 units of vitamin C.

$$\therefore \quad 2x + y \geq 8$$

and $\quad x + 2y \geq 10$

Total cost Z of purchasing x kg of food F_1 and y kg of food F_2 is

$$Z = 5x + 7y$$



Since, feasible region is unbounded.

$\therefore$ We are to check whether this value is minimum For this, we draw the graph of $5x + 7y < 38$ ...(i)

Since (i) has no common point with feasible region.

$\therefore$ Minimum value = 38 at (2, 4)

$\therefore$ Cost is minimum when the dietician mixed 2 kg of food F_1 and 4 kg of Food F_2.

Minimum cost is ₹ 38.

∴ Mathematical formulation of the problem is:

Minimize $Z = 5x + 7y$

Subject to constraints

$$2x + y \geq 8$$
$$x + 2y \geq 10$$
$$x, y \geq 0$$

(b) Consider a set of rectangular cartesian axes OXY in the plane.

It is clear that any point which satisfies $x \geq 0, y \geq 0$ lies in first quadrant.

Let us draw the graph of $2x + y = 8$

For $x = 0$, $y = 8$

For $y = 0$, $x = 4$

∴ Line meets OX at $A(4, 0)$ and OY at $L(0, 8)$

Again, we draw the graph of $x + 2y = 10$

For $x = 10$, $2y = 10$ or $y = 5$

For $y = 0$, $x = 10$

∴ Line meets OX at $B(10, 0)$ and OY at $M(0, 5)$

Since, feasible region satisfies all the constraints.

∴ Shaded region is the feasible region and it is unbounded

corner points are $B(10, 0)$, $C(2, 4)$, $L(0, 8)$

At $B(10, 0)$, $Z = 5 \times 10 + 7 \times 0 = 50$

At $C(2, 4)$, $Z = 5 \times 2 + 7 \times 4 = 38$

At $L(0, 8)$, $Z = 5 \times 0 + 7 \times 8 = 56$

∴ Smallest value of Z is 38 at (2, 4)

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NEW TYPOLOGY INTRODUCED BY CBSE FOR BOARD 2023-24 EXAMINATION

UNIT–I : Numbers, Quantification and Numerical Applications

1. NUMBERS AND QUANTIFICATION

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true, but R is false

(D) A is false, but R is True

Q. 1. Assertion (A): $(186 \times 93) \bmod 7 = 1$

Reason (R): Assertion can be solved using the formula

$$a.b(\text{mod } n) \equiv a(\text{mod } n).b(\text{mod } n)$$

Ans. Option (A) is correct.

Explanation: We know that,

$a.b(\text{mod } n) \equiv a(\text{mod } n).b(\text{mod } n)$

So, $(186 \times 93) \bmod 7 \equiv 186(\text{mod } 7).93(\text{mod } 7)$

$\equiv 4(\text{mod } 7).2(\text{mod } 7)$

$\equiv 4.2(\text{mod } 7)$

$\equiv 8(\text{mod } 7)$

$\equiv 1(\text{mod } 7)$

Q. 2. Assertion (A): It is currently 8.00 a.m. It will be 4.00 a.m. in next 500 hours.

Reason (R): We know that time repeats after every 24 hours.

So, 500(mod 24)

Since, $500 = 24 \times 20 + 20$

Thus, $500 \equiv 20(\text{mod } 24)$

Therefore 500 hours is equivalent to 20 hours.

Now, $(8 + 20) \equiv 4(\text{mod } 24)$

Hence, it will be 4:00 a.m. in next 500 hours.

Ans. Option (A) is correct.

Q. 3. Assertion (A): When 14 is divisible by 5, then remainder is 4. So, we write $14(\text{mod } 5) = 4$.

Reason (R): $37 \equiv 12(\text{mod } 5)$

Here, $37 - 12 \equiv 25$ is an integral multiple of 5.

Ans. Option (B) is correct.

Explanation: Both statements are individually correct. Both the statements are of Modulo arithmetic which is system of arithmetic for integers, which considers the remainder.

Q. 4. Assertion (A): The last two digits of the product 2345×6789 are 05.

Reason (R): To find the last two digits of the product 2345×6789, we find $2345 \times 6789(\text{mod } 100)$.

Ans. Option (A) is correct.

Explanation: Since $2345 = 45(\text{mod } 100)$

and $6789 \equiv 89(\text{mod } 100)$

So, $2345 \times 6789 \equiv 45 \times 89(\text{mod } 100)$

$\equiv 4005(\text{mod } 100)$

$\equiv 05(\text{mod } 100)$

Hence, the last two digits of the product 2345×6789 are 05.

Q. 5. Assertion (A): When $783 \times 657 \times 594 \times 432 \times 346 \times 251$ is divided by 5, then remainder is 3.

Reason (R): To find the remainder, we find $(783 \times 657 \times 594 \times 432 \times 346 \times 251) \equiv 3(\text{mod } 5)$

Ans. Option (A) is correct.

Explanation: Since, $783 \equiv 3(\text{mod } 5)$

$657 \equiv 2(\text{mod } 5)$

$594 \equiv 4(\text{mod } 5)$

$432 \equiv 2(\text{mod } 5)$

$346 \equiv 1(\text{mod } 5)$

and $251 \equiv 1(\text{mod } 5)$

So, $(783 \times 657 \times 594 \times 432 \times 346 \times 251) \bmod 5$

$\equiv (3 \times 2 \times 4 \times 2 \times 1) \bmod 5$

$\equiv 48(\text{mod } 5)$

$\equiv 3(\text{mod } 5)$

Hence, the remainder when $(783 \times 657 \times 594 \times 432 \times 346 \times 251)$ is divided by 5 is 3.

Q. 6. Assertion (A): $3^{16}(\text{mod } 4) \equiv 1$

Reason (R): Use following property to solve the statement given in Assertion.

If $a \equiv b(\text{mod } n)$, then $a^k \equiv b^k(\text{mod } n)$, $\forall\ k \in N$

Ans. Option (A) is correct.

Explanation: As $3^2 = 9 \equiv 1(\text{mod } 4)$

So $(3^2)^8 \equiv 1^8(\text{mod } 4)$

$\Rightarrow$ $3^{16} \equiv 1(\text{mod } 4)$

$\therefore$ $3^{16}(\text{mod } 4) \equiv 1$

Q. 7. Assertion (A): The last three digits of the product 1234×5678 are 6, 5 and 2, respectively.

Reason (R): To find the last two digits of the product 1234×5678, we find $1234 \times 5678 (\text{mod } 1000)$.

Ans. Option (A) is correct.

Explanation: Since,

$1234 \equiv 234 (\text{mod } 1000)$

and $5678 \equiv 678 (\text{mod } 1000)$

So, $1234 \times 5678 \equiv 234 \times 678 (\text{mod } 1000)$

$\equiv 158652 (\text{mod } 1000)$

$\equiv 652 (\text{mod } 1000)$

Hence, the last three digits of the product 1234×5678 are 6, 5 and 2, respectively.

Q. 8. Assertion (A): $[(4 - 9) \times 7] (\text{mod } 4)$ is 1.

Reason (R): $[(4 - 9) \times 5] (\text{mod } 4)$ is –1.

Ans. Option (C) is correct.

Explanation: In case of assertion

$[(4 - 9) \times 7] \text{ mod } 4$

$\equiv (-5 \times 7) (\text{mod } 4)$

$\equiv -35 (\text{mod } 4)$

$\equiv 1 (\text{mod } 4)$

$4\overline{)-35}(-9$, -36, 1

Hence, $[(4 - 9) \times 7] (\text{mod } 4) = 1$

In case of reason:

$[(4 - 9) \times 5] (\text{mod } 4) \equiv -25 (\text{mod } 4)$

Hence, $[(4 - 9) \times 5] (\text{mod } 4) = 3$

Reason is wrong.

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2. NUMERICAL APPLICATIONS

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

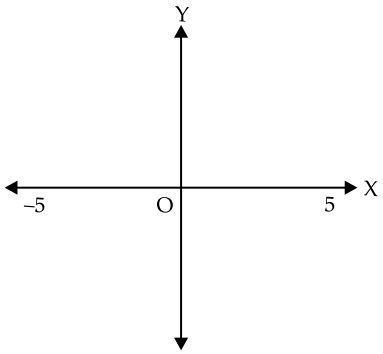
(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false but R is True

Q. 1. Assertion (A): The inequality representing the following graph is $|x| < 5$.

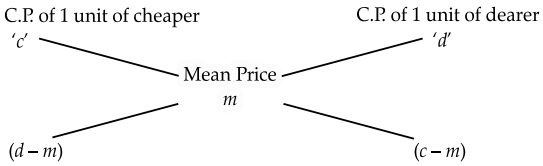


Reason (R): The given graph represents $x > -5$ and $x < 5$ combining the two inequalities $|x| < 5$.

Ans. Option (A) is correct.

Q. 2. Assertion (A): In what ratio water is mixed with milk to gain $16\frac{2}{3}\%$ on selling the mixture at cost price.

Reason (R): To solve the above statement we use the rule of alligation *i.e.*,



(Cheaper Quantity) : (Dearer Quantity)

$= (d - m) : (m - c)$

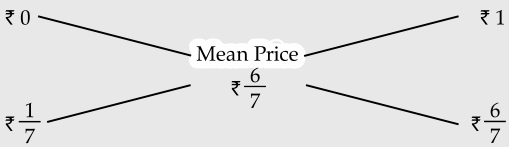
Ans. Option (A) is correct.

Explanation: Let C.P. of 1 litre milk be ₹ 1.

S.P. of 1 litre of mixture = ₹ 1, Gain = $\frac{50}{3}\%$

$\therefore$ C.P. of 1 litre of mixture $= \left(100 \times \frac{3}{350} \times 1\right)$

$= \frac{6}{7}$



By the rule of alligation, we have

$\therefore$ Ratio of water and milk $= \frac{1}{7} : \frac{6}{7} = 1 : 6.$

Q. 3. Assertion (A): A cistern has two taps attached to it. Tap B can empty the cistern in 45 minutes. But Tap A can fill the cistern in just 30 minutes. Rohit started both taps unknowingly but realized his mistake after 30 minutes. He immediately closed Tap B. Now after this, in 20 minutes the cistern will be filled.

Reason (R): In 1 minute, cistern filled by Tap A $= \frac{1}{30}$

In 1 minute, cistern emptied by Tap B $= \frac{1}{45}$

In 1 minute with Taps A and B, cistern filled

$$= \frac{1}{30} - \frac{1}{45} = \frac{1}{90}$$

In 30 minutes, $30 \times \frac{1}{90} = \frac{1}{3}$ part of cistern is filled

$$\text{Remaining cistern} = 1 - \frac{1}{3} = \frac{2}{3}$$

This is filled by Tap A only

In 1 minute, $\frac{1}{30}$ part of cistern is filled

In x minutes, $\frac{2}{3}$ part of cistern is filled

$$\therefore \quad x \text{ minutes} = \frac{\frac{2}{3}}{\frac{1}{30}} = 20 \text{ minutes}$$

= Time to fill after Tap B is closed

Ans. Option (A) is correct.

Q. 4. Assertion (A): If in a 1000 m race, A can beat B by 100 m. In a race of 400 m, B can beat C by 40 m then in a 500 m race A beats C by 95 m.

Reason (R): A beat C by 'x' meters means C is behind A by x meters, when A reached of finishing point.

e.g. To cover a race of 100 meters in this case, A has covered 100 meters while C has covered only $(100 - x)$ meters.

Ans. Option (A) is correct.

Explanation: $A : B = 1000 : 900$,

$B : C = 400 : 360$

$= 100 : 90$

$= 900 : 810$

$\Rightarrow A : B : C = 1000 : 900 : 810$

$\Rightarrow A : C = 1000 : 810$

$\Rightarrow A : C = 500 : 405$

$\Rightarrow$ In a 500 m race,

A beats C by (500 – 405) m = 95 m

Q. 5. Assertion (A): If the boat takes 12 hours to row 48 km upstream and 8 hours to row the same distance downstream, then the boat's speed in still water is 5 km/h and speed of river is 1 km/h.

Reason (R): Let speed of boat and speed of river X km/h and Y km/h, respectively

Down stream speed = X + Y

Up stream speed = X – Y

$$\text{Also, Down stream speed} = \frac{\text{Distance covered}}{\text{Time Taken}}$$

$$\text{Up stream speed} = \frac{\text{Distance covered}}{\text{Time Taken}}$$

Ans. Option (A) is correct.

Explanation: Let speed of a boat is X km/h and speed of river is Y km/h

$$\text{Down stream speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

$$= \frac{48}{8} = 6 \text{ km/h}$$

$$\text{Upstream speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

$$= \frac{48}{12} = 4 \text{ km/h}$$

$X + Y = 6$ km/h

and $X - Y = 4$ km/h

On adding them we get,

$X + Y + X - Y = 10$ km/h

$\therefore \quad X = 5$ km/h

= Speed of Boat

$Y = 6 - 5 = 1$ km/h

= Speed of river

Q. 6. Assertion (A): A began a business with ₹ 85,000. He was joined afterwards by B with ₹ 42,500. If the profits at the end of the year are divided in the ratio of 3 : 1, then B joined for 8 months only.

Reason (R): Suppose B joined for x months.

Then, $\frac{(85000 \times 12)}{(42500 \times x)} = 3$

or, $x = \frac{(85000 \times 12)}{(42500 \times 3)} = 8$

So, B joined for 8 months.

Ans. Option (A) is correct.

Q. 7. Assertion (A): Let x and b are real numbers. If $b > 0$ and $|x| > b$, then $x \in (-\infty, -b) \cup (b, \infty)$.

Reason (R): Let x and b are real numbers. If $b > 0$ and $|x| > b$, then $x \in [-\infty, -b] \cup [b, \infty]$.

Ans. Option (C) is correct.

Explanation:

Given, $|x| > b, b > 0$

$\Rightarrow x < -b$ or $x > b$

$\Rightarrow x \in (-\infty, -b) \cup (b, \infty)$

UNIT–II : Algebra
3. MATRICES

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false but R is True

Q. 1. Assertion (A): If A and B are symmetric matrices, then [BA − 2AB] is not a symmetric matrix.

Reason (R): If A and B are symmetric matrices, then [BA − 2AB] is not a skew-symmetric matrix.

Ans. Option (B) is correct.

Explanation: :

$\therefore (BA-2AB)' = (BA)' - 2(AB)'$

$= A'B' - 2B'A'$

$= AB - 2BA$

[$\because$ A & B both are symmetric matrices *i.e.*, $A' = A$ and $B' = B$]

$= -(2BA - AB)$

So, [BA − 2AB] is neither a symmetric, nor a skew-symmetric matrix.

Q. 2. Assertion (A): Given A and B are symmetric matrices of same order. If AB is symmetric, then $AB = BA$.

Reason (R): Given A and B are symmetric matrices of same order. If $AB = BA$ then AB is symmetric.

Ans. Option (A) is correct.

Explanation: $(AB)' = B'A'$

$= BA$ [$\because$ $A' = A$ and $B' = B$]

$= AB$ [$\because$ $AB = BA$]

Q. 3. Assertion (A): If each of the three matrices of the same order are symmetric, then their sum is a symmetric matrix.

Reason (R): If each of the three matrices of the same order are not symmetric, then their sum is a symmetric matrix.

Ans. Option (C) is correct.

Explanation: Let A, B and C are three matrices of same order.

$\therefore A' = A,\ B' = B$ and $C' = C$

$\therefore (A+B+C)' = A'+B'+C'$

$= (A+B+C)$

Q. 4. Assertion (A): If A and B are two matrices of the same order, then $A - B = B - A$.

Reason (R): If A and B are two matrices of the same order, then $A - B \neq B - A$.

Ans. Option (D) is correct.

Explanation: The addition of two matrices of same order is commutative.

$\therefore A + (-B) = A - B = -[B - A] \neq B - A$

Q. 5. Assertion (A): If $AB = AC = 0$, then it can be possible that B and C are two non-zero matrices such that $B \neq C$.

Reason (R): If $AB = AC = 0$, then it can be possible that B and C are two non-zero matrices such that $B = C$.

Ans. Option (C) is correct.

Explanation: $A \cdot B = 0 = A \cdot C$

Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$

$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

And $\quad AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow AB = AC$ but $B \neq C$

Q. 6. Assertion (A): If A is a symmetric matrix, then A^3 is also a symmetric matrix.

Reason (R): If A is a symmetric matrix, then $(A')^n = (A^n)'$

Ans. Option (A) is correct.

Explanation: $A' = A$

Now, $(A^3)' = (A')^3$ [$\because$ $(A')^n = (A^n)'$]

$= A^3$

Q. 7. Assertion (A): For three matrices A, B and C, we have

$A(B + C) = AB + AC$

Reason (R): Sum of two skew symmetric matrices is always a symmetric matrix.

Ans. Option (C) is correct.

Explanation: Assertion is true due to the distributive property of matrices *i.e.*, multiplication is distributive over addition.
Reason is false because
Let A and B are skew symmetric matrix.
$\therefore \quad A' = -A, B' = -B$
Now, $(A + B)' = A' + B'$
$= -A - B = -(A + B)$
So, sum of two skew symmetric matrices is always skew symmetric matrix.

Q. 8. Assertion (A): If $(AB)' = B'A'$, where A and B are not square matrices, then number of rows in A is equal to number of columns in B and number of columns in A is equal to number of rows in B.
Reason (R): Let, A is of order $m \times n$ and B is of order $p \times q$
Since, $(AB)' = B'A'$
$\therefore A_{m\times n}B_{p\times q}$ is defined.
$\Rightarrow n = p$ ...(i)
And AB is of order $m \times q$.
$\Rightarrow (AB)'$ is of order $q \times m$. ...(ii)
Also, B' is of order $q \times p$ and A' is of order $n \times m$.
$\therefore B'_{q\times p}A'_{m\times n}$ is defined.
$\Rightarrow p = n$
And $B'A'$ is of order $q \times m$. ...(iii)
Also, equality of matrix $(AB)' = B'A'$, we get the given statement as true.
e.g. if A is order (3×1) and B is order of (1×3), we get
Order of $(AB)'$ = Order of $(B'A') = 3 \times 3$

Ans. Option (A) is correct.

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4. DETERMINANTS

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

Q. 1. Assertion (A): If A is a matrix of order 3 × 3, then $|3A| = 27|A|$.
Reason (R): $|kA| = k^n|A|$where A of the order $n \times n$
Ans. Option (A) is correct.

Explanation: :
$|3A| = 3^3|A| = 27|A|$. [$\because$ order of A = 3 × 3]

Ans. Option (C) is correct.

Q. 2. Assertion (A): **A** is a matrix of order 3 × 3, then $(A^2)^{-1} = (A^{-1})^2$
Reason (R): If A is a matrix of order 3 × 3, then number of minors in determinant of A is 6.
Ans. Option (C) is correct.

Explanation: Assertion is one of the property of matrix inverse.
Reason is false because if A is a matrix of order 3 × 3, then number of minors in determinant A is 9 [Since, in a 3 × 3 matrix, there are 9 elements].

Q. 3. Assertion (A): If $x = -9$ is one root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$
then other two roots are 2 and 7.
Reason (R): Since, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$
Expanding along R_1,
$x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) = 0$
$\Rightarrow \quad x^3 - 12x - 6x + 42 + 84 - 49x = 0$
$\Rightarrow \quad x^3 - 67x + 126 = 0$ ...(i)
Here, $126 \times 1 = 9 \times 2 \times 7$
For $x = 2$,
$\Rightarrow \quad 2^3 - 67 \times 2 + 126 = 134 - 134 = 0$
Hence, $x = 2$ is a root.
For $x = 7$,
$\Rightarrow \quad 7^3 - 67 \times 7 + 126 = 469 - 469 = 0$
Hence, $x = 7$ is also a root.

Ans. Option (A) is correct.

Q. 4. Assertion (A): $(aA)^{-1} = \frac{1}{a}A^{-1}$, where a is any real number and A is a square matrix.
Reason (R): If the value of a third-order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.
Ans. Option (D) is correct.

Explanation: Reason (R) is true because
Let A is the determinant.
$\therefore \quad |A| = 12$
We know that, if A is a square matrix of order n, then $|adj\ A| = |A|^{n-1}$
For $n = 3, |adj\ A| = |A|^{3-1} = |A|^2$
$= (12)^2 = 144$
Assertion (A) is wrong because
we know that, if A is a non-singular square matrix, then for any scalar a (non-zero). aA is invertible such that

$(aA)\left(\frac{1}{a}A^{-1}\right)=\left(a,\frac{1}{a}\right)(A.A^{-1})$

i.e., (aA) is inverse of $\left(\frac{1}{a}A^{-1}\right)$ or $(aA)^{-1}=\frac{1}{a}A^{-1}$,

where 'a' is any non-zero scalar.

In the above statement, a is any real number.

Q. 5. Assertion (A): $\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix}$

then value of x is $\pm\sqrt{3}$.

Reason (R):

$$\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix}$$

$\Rightarrow \quad 2\times1-5\times4=2x\times x-6\times4$

$\Rightarrow \quad 2-20=2x^2-24$

$\Rightarrow \quad 2x^2=6$

$\Rightarrow \quad x^2=3$

$\Rightarrow \quad x=\pm\sqrt{3}$

Ans. Option (A) is correct.

Q. 6. Assertion (A): The adjoint of the matrices $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ **is** $\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$.

Reason (R):

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then adj $A=\begin{bmatrix}A_{11} & A_{21}\\ A_{12} & A_{22}\end{bmatrix}$

where $A_{11}=d, A_{12}=-c, A_{21}=-b, A_{22}=a$

Ans. Option (A) is correct.

Explanation:

Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.

We have,

$A_{11}=4,\ A_{12}=-3,\ A_{21}=-2,\ A_{22}=1$

$\therefore$ adj.$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}'=\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}'=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$.

Q. 7. Assertion (A): If $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$ then $|2A|=4|A|$.

Reason (R): $|2A|=-24$ and $4|A|=-24$

Ans. Option (A) is correct.

Explanation: $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$

$\therefore \quad 2A=2\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}=\begin{bmatrix}2 & 4\\ 8 & 4\end{bmatrix}$

Now, $|2A|=\begin{vmatrix}2 & 4\\ 8 & 4\end{vmatrix}$

$=2\times4-4\times8$

$=8-32=-24$

Now, $|A|=\begin{vmatrix}1 & 2\\ 4 & 2\end{vmatrix}$

$=1\times2-2\times4$

$=2-8=-6$

So, $4|A|=4\times(-6)=-24$

Thus, $|2A|=4|A|$

Q. 8. For a matrix A having $|A|=0$, then

Assertion (A): A^{-1} does not exists

Reason (R): A^{-1} exists with $|A^{-1}|=0$

Ans. Option (C) is correct.

Explanation: Inverse does not exist for a singular matrix A singular matrix is the matrix whose determinant is equal to zero.

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UNIT–III : Calculus

5. DIFFERENTIATION AND ITS APPLICATIONS

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false and R is True

Q. 1. Assertion (A): It is given that at $x=1$, the function x^4-62x^2+ax+9 attains its maximum value, on the interval [0, 2]. Then value of a is 120.

Reason (R): To find the value of a put $f'(1)=0$.

Ans. Option (A) is correct.

Explanation:

$$f(x)=x^4-62x^2+ax+9$$

Therefore, $f'(x)=4x^3-124x+a$

f attains its maximum value on the interval [0, 2] at $x=1$.

Therefore, $f'(1) = 0$

$4 - 124 + a = 0$

$a = 120$

Q. 2. Assertion (A): The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 26x + 15$. The marginal revenue, when $x = 15$ is ₹ 116.

Reason (R): Marginal revenue (MR) is the rate of change of total revenue with respect to the number of units sold. Therefore, $(MR)_{\text{at } x = 15} = \left(\frac{dR}{dx}\right)_{\text{at } x=15}$

Ans. Option (A) is correct.

Explanation: Here,

$R(x) = 3x^2 + 26x + 15$

$\frac{dR}{dx} = 6x + 26$

$\Rightarrow \left(\frac{dR}{dx}\right)_{\text{at } x = 15} = 6(15) + 26 = ₹\ 116$

Q. 3. Assertion (A): The function given by $f(x) = 3x + 17$ is strictly increasing on R.

Reason (R): The logarithmic function is strictly increasing on $(0, \infty)$.

Ans. Option (B) is correct.

Explanation: Assertion (A) is true because

Let x_1 and x_2 be any two numbers in R.

Then, we have $x_1 < x_2$

$\Rightarrow 3x_1 < 3x_2$

$\Rightarrow 3x_1 + 17 < 3x_2 + 17$

$\Rightarrow f(x_1) < f(x_2)$

Hence, f is strictly increasing on R.

Reason (R) is correct because

The given function is $f(x) = \log x$.

$\therefore f'(x) = \frac{1}{x}$

It is clear that for $x > 0$, $f'(x) = \frac{1}{x} > 0$

Hence, $f(x) = \log x$ is strictly increasing in interval $(0, \infty)$.

But Reason (R) is not correct explanation of Assertion (A).

Q. 4. Assertion (A): The maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 72x - 18x^2$ is 113.

Reason (R): Let profit function

$p(x) = 41 - 72x - 18x^2$

$\therefore p'(x) = -72 - 36x$

$\Rightarrow x = -\frac{72}{36} = -2$

$p''(x) = -36$

Also, $p''(-2) = -36 < 0$

By second derivative test, $x = -2$ is the point of local maxima of p.

$\therefore$ Maximum profit $= p(-2) = 41 - 72(-2) - 18(-2)2$

$= 41 + 144 - 72 = 113$

Hence, the maximum profit that the company can make is 113 units.

Ans. Option (A) is correct.

Q. 5. Assertion (A): 2 is least value of a function such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2).

Reason (R): – 2 is the least value of a function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2).

Ans. Option (D) is correct.

Explanation:

$f(x) = x^2 + ax + 1$

$\therefore f'(x) = 2x + a$

Now, function f is strictly increasing in (1, 2).

i.e., $f'(x) > 0$

$\therefore 2x + a > 0 \Rightarrow a > -2x$

$\therefore$ We have to find least value of a, such that

$x \in (1, 2)$

$\therefore a > -2$

$\Rightarrow a \in (-2, \infty)$

So, least value of a is – 2.

Q. 6. Assertion (A): The function $g(x) = \log x$ does not has maxima or minima.

Reason (R): The function $g(x) = \log x$ has maxima.

Ans. Option (C) is correct.

Explanation: $g(x) = \log x$

$\therefore g'(x) = \frac{1}{x}$

Since $\log x$ is defined for a positive number x, $g'(x) > 0$ for any x.

Therefore, there does not exist $c \in R$ such that $g'(c) = 0$.

Hence, function g does not have maxima or minima.

Q. 7. Assertion (A): If $y = x^3 \log x$, then $\frac{d^4y}{dx^4} = \frac{6}{x}$.

Reason (R):

If $y = x^3 \log x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 \log x)$

$\Rightarrow \frac{dy}{dx} = x^2 (1 + 3\log x)$

$\frac{d^2y}{dx^2} = x^2 \frac{d}{dx}(1 + 3\log x) + (1 + 3\log x)\frac{d}{dx}x^2$

$\frac{d^2y}{dx^2} = x(5 + 6\log x)$

$\frac{d^3y}{dx^3} = x\frac{d}{dx}(5 + 6\log x) + (5 + 6\log x)\frac{d}{dx}x$

$$\frac{d^3y}{dx^3} = 11 + 6\log x$$

$$\Rightarrow \quad \frac{d^4y}{dx^4} = \frac{6}{x}$$

Ans. Option (A) is correct.

Q. 8. Assertion (A): Given $y = \frac{x}{x^2+1}$, then the value of second derivative at $x = 1$ is $\frac{-1}{2}$.

Reason (R): Since,

$$y'' = \frac{(x^2+1)^2(-2x)-(1-x^2)[2(x^2+1)2x]}{(x^2+1)^4}$$

Ans. Option (A) is correct.

Explanation:

Let $y = \frac{x}{x^2+1}$, then $y' = \frac{1-x^2}{(x^2+1)^2}$

$$y'' = \frac{(x^2+1)^2(-2x)-(1-x^2)[2(x^2+1)2x]}{(x^2+1)^4}$$

At $x = 1$,

$$y'' = \frac{(1^2+1)^2(-2\times1)-(1-1^2)[2(1^2+1)2\times1]}{(1^2+1)^4}$$

$$\Rightarrow \quad y'' = \frac{(2)^2(-2)-(1-1)[2(2)2]}{(2)^4}$$

$$\Rightarrow \quad y'' = \frac{-8+0}{(2)^4}$$

$$\Rightarrow \quad y'' = \frac{-1}{2}$$

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6. INTEGRATION AND ITS APPLICATIONS

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) If Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false but R is True

Q. 1. Assertion (A): $\int \frac{x^3+1}{x^2}dx = \frac{x^2}{2} - \frac{1}{x} + C$

Reason (R): $\int \frac{x^3+1}{x^2}dx = \frac{x^2}{2} + \frac{1}{x} + C$

Ans. Option (C) is correct.

Explanation: Let

$$I = \int \frac{x^3+1}{x^2}dx = \int x\,dx + \int \frac{1}{x^2}dx$$

$$= \frac{x^2}{2} + \frac{x^{-2+1}}{-2+1} + C = \frac{x^2}{2} - \frac{1}{x} + C$$

Hence, only Assertion is true.

Q. 2. Assertion (A): $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx$

$$= \frac{3}{2(a+b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$$

Reason (R): $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx$

$$= \frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$$

Ans. Option (D) is correct.

Explanation: Let

$$I = \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx$$

$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}dx$$

$$= \frac{1}{a-b}\int \sqrt{x+a}\,dx - \int \sqrt{x+b}\,dx$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{3/2}}{3/2} - \frac{(x+b)^{3/2}}{3/2}\right] + C$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$$

Q. 3. Assertion (A): $\int e^x(x^2+2x)dx = e^x x^2 + C$

Reason (R): The integral is solved using the rule

$$\int e^x[f(x)+f'(x)]dx = f(x)e^x + C$$

Ans. Option (A) is correct.

Explanation:

Let $I = \int e^x(x^2+2x)dx$

Here, $f(x) = x^2$ and $f'(x) = 2x$

So, by the rule

$\int e^x[f(x)+f'(x)]dx = f(x)e^x + C$

we get $I = e^x x^2 + C$

Q. 4. Assertion (A): $\int \frac{2x}{1+x^2}dx = \log(1+x^2)+C$

Reason (R): $\int \frac{2x}{1+x^2}dx = \log\left(\frac{1}{1+x^2}\right)+C$

Ans. Option (C) is correct.

Explanation:

Let $1 + x^2 = t$

$\therefore \quad 2x\,dx = dt$

$\Rightarrow \quad \int \frac{2x}{1+x^2}dx = \int \frac{1}{t}dt$

$= \log|t| + C$

$= \log|1 + x^2| + C$

$= \log(1 + x^2) + C$

Q. 5. Assertion (A): Primitive of $\frac{(\log|x|)^2}{x}$ is $\frac{(\log|x|)^3}{2}+C$

Reason (R): Primitive of $\frac{(\log|x|)^2}{x}$ is $\frac{(\log|x|)^3}{2/3}+C$

Ans. Option (C) is correct.

Explanation:

$\int \frac{(\log|x|)^2}{x}dx = \int t^2 dt$

$[\text{let } \log|x| = t \therefore \frac{1}{x}dx = dt]$

$= \frac{t^3}{3}+C = \frac{(\log|x|)^3}{3}+C$

Q. 6. Assertion (A): $\int_4^5 e^x dx = e^4(e-1)$

Reason (R): $\int_5^4 e^x dx = e^4(e-1)$

Ans. Option (C) is correct.

Explanation:

Let $I = \int_4^5 e^x dx$

$\because \quad \int e^x dx = e^x = F(x)$

$I = F(5) - F(4) = e^5 - e^4 = e^4(e-1)$

Q. 7. Assertion (A): $\int \frac{x^2}{(2+3x^3)^3}dx = \frac{-1}{18(2+3x^3)^2}+C$

Reason (R): $\int \frac{x^2}{(2+3x^3)^3}dx = \frac{-1}{18(2+3x^3)}+C$

Ans. Option (C) is correct.

Explanation: $\int \frac{x^2}{(2+3x^3)^3}dx$

To integrate above

Let $2 + 3x^3 = t$

$\therefore \quad 9x^2\,dx = dt$

$\Rightarrow \int \frac{x^2}{(2+3x^3)^3}dx = \frac{1}{9}\int \frac{dt}{(t)^3}$

$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right]+C$

$= \frac{-1}{18}\left(\frac{1}{t^2}\right)+C$

$= \frac{-1}{18(2+3x^3)^2}+C$

Q. 8. Assertion (A): If f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$.

$\int_0^a f(x)g(x)dx = 2\int_{-a}^a f(x)dx$

Reason (R): If f and g are defined as $f(x) = f(a-x)$

$g(x) + g(a-x) = 4$

Then, $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$

Ans. Option (D) is correct.

Explanation:

Let $I = \int_0^a f(x)g(x)dx$ …(i)

$\Rightarrow \quad I = \int_0^a f(a-x)g(a-x)dx$

$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\Rightarrow \quad I = \int_0^a f(x)g(a-x)dx$ …(ii)

Adding equations (i) and (ii), we obtain

$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\times 4dx \quad [\because g(x) + g(a-x) = 4]$

$\Rightarrow \quad I = 2\int_0^a f(x)dx$

Therefore,

$\Rightarrow \int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$

Q. 9. Assertion (A): If $\int_0^1 (3x^2 + 2x + k)dx = 0$, then value of k is 2.

Reason (R): If $\int_0^1 (3x^2 + 2x + k)dx = 0$, then value of k is –2.

Ans. Option (D) is correct.

Explanation:

If $\int_0^1 (3x^2 + 2x + k)dx = 0$

Then, $\left[3.\frac{x^3}{3} + 2.\frac{x^2}{2} + k.x\right]_0^1 = 0$

or, $\left[x^3 + x^2 + k.x\right]_0^1 = 0$

or, $(1 + 1 + k.1) - (0 + 0 + 0) = 0$

or, $2 + k = 0$

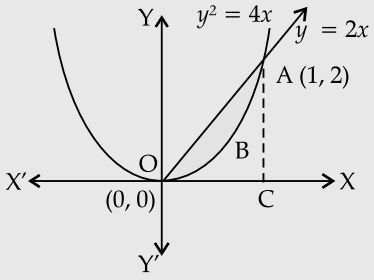
or, $k = -2$

Q. 10. Assertion (A): Area lying between the curve $y^2 = 4x$ and $y = 2x$ is $\frac{1}{3}$ sq. units

Reason (R): Area lying between the curve $y^2 = 4x$ and $y = 2x$ is 3 sq. units.

Ans. Option (C) is correct.

Explanation: The area lying between the curve $y^2 = 4x$ and $y = 2x$ is represented by the shaded area OBAO as



The points of intersection of the curves are O(0, 0) and A(1, 2).

We draw AC perpendicular to x-axis such that coordinate of C is (1, 0).

Area of OBAO = Area of ΔOCA – Area of OCABO

$$A = \int_0^1 2x\,dx - \int_0^1 2\sqrt{x}\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^1$$

$$= \left|\left[1 - \frac{4}{3}\right]\right| = \left|-\frac{1}{3}\right|$$

$$= \frac{1}{3} \text{ sq. units}$$

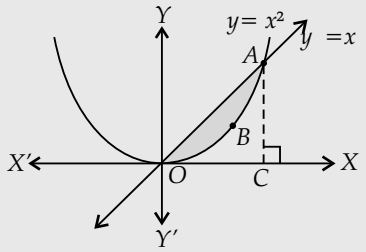
Q. 11. Assertion (A): The area between the curves $y = x$ and $y = x^2$ is $\frac{1}{6}$ sq. units.

Reason (R): The area between curves $y = x$ and $y = x^2$ is $\frac{1}{8}$ sq. units.

Ans. Option (C) is correct.

Explanation:

The area between the curves $y = x$ and $y = x^2$. Area is represented by the shaded area OBAO as



The points of intersection of the curves, $y = x$ and $y = x^2$ are A(1, 1) and O(0, 0).

We draw AC perpendicular to x-axis.

$\therefore$ Area ($OBAO$) = Area (ΔOCA) – Area ($OCABO$)

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ sq. units}$$

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7. DIFFERENTIAL EQUATIONS AND MODELING

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false but R is True

Q. 1. Assertion (A): The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\log\left(\frac{dy}{dx}\right)$ is not defined.

Reason (R): The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\log\left(\frac{dy}{dx}\right)$ is 2.

Ans. Option (C) is correct.

Explanation: : The degree of given differential equation is not defined because when we expand $\log\left(\frac{dy}{dx}\right)$ we get an infinite series in the increasing powers of $\frac{dy}{dx}$. Therefore its degree is not defined.

Q. 2. Assertion (A): The degree of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2} \text{ is 2.}$$

Reason (R): The order of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2} \text{ is 2.}$$

Ans. Option (B) is correct.

Explanation: : Given that,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$$

On squaring both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3} = \left(\frac{d^2y}{dx^2}\right)^2$$

So, the degree and the order of differential equation are 2 and 2.

Q. 3. Assertion (A): The order and degree of the differential equation

$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}+x^{1/5}=0$ are 2 and 2 respectively.

Reason (R): The order and degree of differential equation

$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}+x^{1/5}=0$, are 2 and 4 respectively.

Ans. Option (D) is correct.

Explanation: Given that,

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}+-x^{1/5}=0$$

$$\Rightarrow \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}=-x^{1/5}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^{1/4}=-\left(x^{1/5}+\frac{d^2y}{dx^2}\right)$$

On squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^{1/2}=\left(x^{1/5}+\frac{d^2y}{dx^2}\right)^2$$

Again, on squaring both sides, we get

$$\frac{dy}{dx}=\left(x^{1/5}+\frac{d^2y}{dx^2}\right)^4$$

Order = 2, degree = 4

Q. 4. Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbours to fish on his pond after the fish population reaches 10,000.

Assertion (A): The owner's friends have to wait 26 months approx. to fish in the pond.

Reason (R): The owner's friends have to wait exactly 2 years to fish in the pond.

Ans. Option (C) is correct.

Explanation: We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = \frac{\ln 2}{k}$. Then, $k = \frac{\ln 2}{6}$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. From the passage out when the population reaches 10,000 fish, we must solve the following equation:

$$\begin{aligned} 10{,}000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\text{In } 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right)t \\ t &= \frac{6(\ln 20)}{\ln 2} \approx 25.93 \\ &= 26 \text{ months (approx)} \end{aligned}$$

Q. 5. Consider the population of bacteria grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes.

Assertion (A): There are 80,686 bacteria in the population after 5 hours. (Given $e^6 = 403.4288$)

Reason (R): There are 80,000 bacteria in the population after 5 hours. (Given $e^6 = 403.4288$)

Ans. Option (C) is correct.

Explanation: We have $f(t) = 200e^{0.02t}$.

Then $f(300) = 200e^{0.02t} \approx 80{,}686$

[since, 5 hours = 300 minutes]

Q. 6. Consider the population of bacteria described earlier. This population grows according to the Function $f(t) = 200e^{0.02t}$, where t is measured in minutes.

Assertion (A): The population reaches 100,000 bacteria after 311 (approx.) minutes.

Reason (R): The population reaches 100,000 bacteria after 300 (approx.) minutes.

Ans. Option (C) is correct.

Explanation: To find when the population reaches 100,000 bacteria, we solve the equation:

$$\begin{aligned} 100{,}000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73 \approx 311 \end{aligned}$$

Q. 7. For the given differential equation

$$1-\left(\frac{dy}{dx}\right)^2 = a\left(\frac{d^2y}{dx^2}\right)^{1/3}$$

Assertion (A): The order of differential equation is 2.

Reason (R): The degree of differential equation is 1.

Ans. Option (B) is correct.

Explanation: The given differential equation can be written as

$$\left(1-\left(\frac{dy}{dx}\right)^2\right)^3 = a\frac{d^2y}{dx^2}$$

or, $$1-\left(\frac{dy}{dx}\right)^6-3\left(\frac{dy}{dx}\right)^2\left(1-\left(\frac{dy}{dx}\right)^2\right) = a\frac{d^2y}{dx^2}$$

or, $$a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^6-3\left(\frac{dy}{dx}\right)^4+3\left(\frac{dy}{dx}\right)^2-1=0$$

Here, each term in the derivative is a polynomial, the highest order derivative present is $\frac{d^2y}{dx^2}$ and it's raised to power 1.

So, its degree is 1 and order is 2.

Q. 8. Given differential equation is $\frac{d^3y}{dx^3}+x^2\left(\frac{dy^2}{dx^2}\right)^3=0$

Assertion (A): The order of differential equation is 3.

Reason (R): The degree of differential equation is also 3.

Ans. Option (C) is correct.

Explanation: The highest order derivative present is $\frac{d^3y}{dx^3}$ and it's raised to power 1.

So, its degree is 1 and order is 3.

Q. 9. Given differential equation is $y = x\left(\frac{dy}{dx}\right)^3+\frac{d^2y}{dx^2}$

Assertion (A): The difference of order and degree of the given differential equation is 1.

Reason (R): The difference of order and degree of the given differential equation is 0.

Ans. Option (C) is correct.

Explanation: The highest order derivative present is $\frac{d^2y}{dx^2}$ and its raised to power 1.

So, its order is 2 and degree is 1.

$\therefore$ Difference = 2 – 1 = 1

Hence, only Assertion is true.

Q. 10. Given differential equation is $y\log y dx - xdy = 0$

Assertion (A): Solution of differential equation is $\log|x| = \log|\log y| + C$

Reason (R): Use variable separation method to solve the given differential equation.

Ans. Option (A) is correct.

Explanation: The given differential equation can be written as

$$\frac{dx}{x} = \frac{dy}{y\log y}$$

On integrating both sides, we get

$$\log|x| = \int\frac{dy}{y\log y}dy + C$$

Put $\log y = t$

$$\frac{1}{y}dy = dt$$

$$\therefore \quad \log|x| = \int\frac{1}{t}dt + C$$

$$= \log|t| + C$$

$$= \log|\log y| + C$$

Q. 11. Given differential equation is $\frac{dy}{dx} = e^{x+y}+x^2e^y$

Assertion (A): The solution of differential equation is $e^x+e^{-y}+\frac{x^3}{3}+C=0$

Reason (R): The solution of differential equation is $e^x-e^{-y}+\frac{x^3}{3}+C=0$

Ans. Option (C) is correct.

Explanation: The given differential equation can be written as

$$\frac{dy}{dx} = e^x.e^y+x^2e^y$$

$$\Rightarrow \quad \frac{dy}{dx} = e^y\left(e^x+x^2\right)$$

$$\Rightarrow \quad e^{-y}dy = (e^x+x^2)dx$$

Integrating both sides, we get

$$\Rightarrow \quad -e^{-y} = e^x+\frac{x^3}{3}+C$$

$$\Rightarrow \quad e^x+e^{-y}+\frac{x^3}{3}+C = 0$$

Hence, only Assertion is true.

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8. PROBABILITY

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false but R is True

Q. 1. A random variable X has the following distribution:

X	0	1	2	3	4	5	6
$P(X)$	C	2C	2C	3C	C^2	$2C^2$	$7C^2 + C$

Assertion (A): Value of C is $\frac{1}{10}$.

Reason (R): Using formula $\sum P_i = 1$, we get value of C.

Ans. Option (A) is correct.

Explanation: We know that,

$$\sum P_i = 1$$

$$\therefore C + 2C + 2C + 3C + C^2 + 2C^2 + 7C^2 + C = 1$$

$$\Rightarrow \quad 10C^2 + 9C - 1 = 0$$

$$\Rightarrow \quad (10C - 1)(C + 1) = 0$$

$$\Rightarrow \quad C = \frac{1}{10} \text{ or } C = -1$$

(but C cannot be negative)

$$\Rightarrow \quad C = \frac{1}{10}$$

Q. 2. Assertion (A): Variance of a constant 'a' is a^2.

Reason (R): Variance of a constant 'a' is 0.

Ans. Option (D) is correct.

Explanation: $\because$ Var $(x) = E(X^2) - (E(X))^2$

$\therefore$ Var $(a) = E(a^2) - (E(a))^2$

$= a^2 - a^2 = 0$

Q. 3. Given : If $f(X) = \begin{cases} X - \frac{5}{2}, 0 < X < 1 \\ 2X, \quad 1 < X < 2 \\ 0 \quad , \text{ otherwise} \end{cases}$

Assertion (A): The mean of the random variable (X) is 3.75.

Reason (R): The mean of the random variable (X) is 3.

Ans. Option (C) is correct.

Explanation: Given $f(x) = \begin{cases} X - \frac{5}{2}, 0 < X < 1 \\ 2X, \quad 1 < X < 2 \\ 0 \quad , \text{ otherwise} \end{cases}$

Then $E(X) = \int_0^1 X\left(X - \frac{5}{2}\right)dX + \int_1^2 X(2X)dx$

$$= \left[\frac{X^3}{3} - \frac{5X^2}{4}\right]_0^1 + \left[\frac{2X^3}{3}\right]_1^2$$

$$= \frac{1}{3} - \frac{5}{4} + \frac{16}{3} - \frac{2}{3}$$

$$= 3.75$$

Q. 4. Assertion (A): The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is 2.

Reason (R): Mean is also called average or expected value or expectation and denoted as E(X).

Ans. Option (B) is correct.

Explanation: Assertion (A) is true because Given on three faces of a die 1 is written, on two faces 2 is written and on one face 5 is written. The die is thrown once. The sample space of random experiment = {1, 1, 1, 2, 2, 5}. It has 6 equally likely outcomes

$\therefore P(1) = \frac{3}{6}, P(2) = \frac{2}{6}, P(5) = \frac{1}{6}$

$\therefore$ Probability distribution of the numbers written on face of a die :

x_i	1	2	5
$P_i(x_i)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$\text{Mean } \sum P_i x_i = \frac{1}{2} \times 1 + \frac{1}{3} \times 2 + \frac{1}{6} \times 5$$

$$= \frac{1}{2} + \frac{2}{3} + \frac{5}{6}$$

$$= \frac{12}{6} = 2.$$

Reason is true because expected value and expectation are the other name of mean and mean is represented by $E(X)$.

Q. 5. Assertion (A): In a Binomial Distribution, if p, q and n are probability of success, failure and number of trials respectively then variance is given by npq.

Reason (R): For a discrete probability function, the variance is given by

$$\text{Var.} = \sum_{x=0}^{n} X^2 P(X) - \mu^2,$$

Where μ is the mean,

Substitute $P(X) = {}^nC_x\, p^x\, q^{(n-x)}$ in the above equation. and put $\mu = np$ to obtain Var. $= npq$.

Ans. Option (A) is correct.

Q. 6. Assertion (A): If 'p', 'q' and 'n' are probability of success, failure and number of trials respectively in a Binomial Distribution, then its Standard deviation is $\sqrt{npq}$.

Reason (R): In a Binomial Distribution, the mean and variance are equal.

Ans. Option (C) is correct.

Explanation: Assertion (A) is true because.

The variance for a Binomial Distribution is given by npq.

So, Standard deviation = $\sqrt{\text{Variance}} = \sqrt{npq}$

Reason (R) is false because

Mean of Binomial Distribution = np

and variance of Binomial Distribution = npq

Thus, mean is not equal to variance.

Q. 7. Assertion (A): Poisson distribution is applied for discrete random variable.

Reason (R): In a Poisson distribution, the mean and variance are equal.

Ans. Option (B) is correct.

Explanation: Poisson distribution is a discrete function, means that the variable can only take specific values *i.e.*, 0, 1, 2, ... etc. with no fraction or decimals.

Also, for Poisson distribution

$$\text{mean} = \text{variance} = np$$

Thus, both Assertion and Reason are correct.

Q. 8. Assertion (A): In a Poisson distribution,

if mean, $m = e$, then $P(X)$ is given by $P(x) = \frac{e^{(x-m)}}{x!}$

Reason (R): In a Poisson distribution,

if mean, $m = e$, then $P(X)$ is given by $P(x) = \frac{e^{(x-m)}}{x!}$

Ans. Option (C) is correct.

Explanation: Since, for Poisson distribution

$$P(X) = \frac{e^{-m}m^x}{x!}$$

Put $m = e$, we get

$$P(X) = \frac{e^{-m}e^x}{x!} = \frac{e^{(x-m)}}{x!}$$

Q. 9. Assertion (A): The range of Normal Distribution is $-\infty$ to ∞.

Reason (R): In Normal Distribution,

Mean = Mode = Median

Ans. Option (B) is correct.

Explanation: The area under the standard (*i.e,* mean =0, standard deviation = 1) normal distribution from negative infinity to positive infinity is equal to 1.

The normal distribution is symmetrical, bell-shaped distribution in which the mean, median and mode are all equal.

Q. 10. Assertion (A): For a standard normal variate, the value of mean is 0.

Reason (R): For a normal variate if its mean = 0 and standard deviation = 1, then it is called an standard Normal Variate.

Ans. Option (A) is correct.

Q. 11. Assertion (A): The area under a standard normal curve is 1.

Reason (R): The area under a standard normal curve is 2.

Ans. Option (C) is correct.

Explanation: For any probability distribution, the sum of all probabilities is 1. Area under normal curve refers to sum of all probabilities.

Q. 12. Assertion (A): The standard normal curve is symmetric about the value of mean = 0.

Reason (R): For a standard normal variate, the value of standard deviation is 1.

Ans. Option (B) is correct.

Explanation: If the mean and standard deviation of a normal variate are 0 and 1, respectively it is called as standard normal variate. Thus, both the given statements are individually true.

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UNIT–IV : Inferential Statistics

9. INFERENTIAL STATISTICS

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of *A*

(C) A is true but R is false

(D) A is false but R is True

Q. 1. Assertion (A): A group of objects, events experiments, observations etc is referred to as population.

Reason (R): A statistic is a numerical characteristic of a sample and a parameter is a numerical characteristic of population.

Ans. Option (B) is correct.

Q. 2. Assertion (A): *t*-distribution is also known as student's *t*-distribution.

Reason (R): *t*-distribution is also known as teacher's *t*-distribution.

Ans. Option (C) is correct.

Explanation: *t*-distribution is also known as student's *t*-distribution gets its name from Willam Sealy Gosset who first published it in English in 1908 in scientific Journal Biometrika using his pseudonym 'Student'.

Q. 3. Assertion (A): The assumptison are opposite of what is made in null hypothesis is called alternative hypothesis.

Reason (R): For computing *t*-statistics we use the following formula

$$t = \frac{\overline{X} + \mu}{\sigma / \sqrt{n}}$$

Ans. Option (C) is correct.

> *Explanation:* Correct formula for calculating t-statistic is
>
> $$t = \frac{Z}{SE} = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{\mu}}}$$

Q. 4. Assertion (A): The mean of the t-distribution is zero.

Reason (R): The mean of t-distribution is 1.

Ans. Option (C) is correct.

> *Explanation:* The t-distribution is symmetric like normal distribution.
> So, its mean = 0.

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UNIT–VI : Index Numbers and Time based Data

10. TIME BASED DATA

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false but R is True

Q. 1. Assertion (A): If price index of base year with respect to current year is 125 then it means 125% of prices increased in the current year as compared to base year.

Reason (R): If price index of base year with respect to current year is 125 then it means 25% of prices increased in the current year as compared to base year.

Ans. Option (D) is correct.

> *Explanation:* Since in index number base year is considered as 100. So, it show 25% of prices increased in the current year as compared to base year.

Q. 2. Assertion (A): In time series seasonal variations can occur within a period of one year or shorter.

Reason (R): In time series seasonal variations can occur within 2 or more years.

Ans. Option (C) is correct.

> *Explanation:* Seasonal variations are those short-term fluctuations which occur regularly and periodically within a period of less than one year.

Q. 3. Assertion (A): A fire in a factory is an example of irregular variations.

Reason (R): A fire in a factory is an example of cyclical variations.

Ans. Option (C) is correct.

> *Explanation:* Irregular variations do not have particular pattern and there is no regular period of time of their occurrences where are cyclical variations are due to the ups and downs recurring after a period from time to time.

Q. 4. Assertion (A): The additive model is based on the assumption that the sum of four components is equal to original value

i.e., $Y = T + C + S + I$

Reason (R): In Multiplication model four components have a multiplication relationship

i.e., $Y = T \times C \times S \times I$

Ans. Option (B) is correct.

Q. 5. Assertion (A): For odd number of year, formula to code the values of X by taking origin at centre is:

X = year – average of years

Reason (R): For odd number of year, formula to code the values of X by taking origin at centre is:

X = year – first year

Ans. Option (C) is correct.

Q. 6. Assertion (A): The systematic components of time series which follow regular pattern of variations are called Signal.

Reason (R): The unsystematic sequence which follows irregular pattern of variations is called Noise.

Ans. Option (B) is correct.

> *Explanation:* The sequence which follows some regular patterns of variation and can be completely determined or specified is called the 'systematic sequence' or 'signal'. The sequence following random or irregular patterns of variation is called 'noise'.

Q. 7. Assertion (A): A complete cycle passes through four stages.

Reason (R): A complete cycle passes through three stages.

Ans. Option (C) is correct.

> *Explanation:* A complete cycle passes through four stages *i.e.*, prosperity, recession, depression and recovery.

Q. 8. Assertion (A): Damages due to floods, droughts, strikes, fires and political disturbances are Irregular.

Reason (R): Damages due to floods, droughts, strikes fires and political disturbances are Cyclical.

Ans. Option (C) is correct.

Explanation: Irregular variations which occur on account of random external events. These variations either go very deep downward or too high upward to attain peaks abruptly. These fluctuations are a result of unforeseen and unpredictably forces which operate in absolutely random or erratic manner. These variations are due to floods, wars, famines, earthquakes, strikes, lockouts, epidemics etc.

Q. 9. Assertion (A): When the production of a thing is maximum, this stage is called Boom.
Reason (R): When the production of a thing is minimum, this stage is called Depression.
Ans. Option (B) is correct.

Explanation: When the production of an item is maximum, then this stage is known as Boom whereas when the production of an item is minimum, the stage is known as Depression.

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UNIT–VII : Financial Mathematics

11. FINANCIAL MATHEMATICS

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT the correct explanation of A
(C) A is true but R is false
(D) A is false but R is True

Q. 1. Assertion (A): If the rate of return on an investment is negative, then it indicates profit.
Reason (R): If the rate of return on an investment is negative, then it indicates loss.
Ans. Option (B) is correct.

Explanation: Positive rate of return represents profit where as negative rate of return represents loss.

Q. 2. Assertion (A): Full form of CAGR is Compound Annual Growth Rate.
Reason (R): Full form of EMI is Equally Monthly Instalment.
Ans. Option (C) is correct.

Explanation: Full form of EMI is Equated Monthly Instalment.

Q. 3. Assertion (A): An annuity in which the periodic payment begin on a fixed date and continue forever is called perpetuity.
Reason (R): The amount or future value of perpetuity is always defined.
Ans. Option (C) is correct.

Explanation: The amount of future value of perpetuity is undefined.

Q. 4. Assertion (A): A loan is said to be amortized if each instalment is used to pay interest and part of principal.
Reason (R): If the market value of a share is less than its nominal value, then share is called at premium.
Ans. Option (C) is correct.

Explanation: If the market value of a share is more than its nominal value, then share is called at premium.

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UNIT–VIII : Linear Programming

12. LINEAR PROGRAMMING

Assertion and Reason Based MCQs (1 Mark each)

Directions: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as:

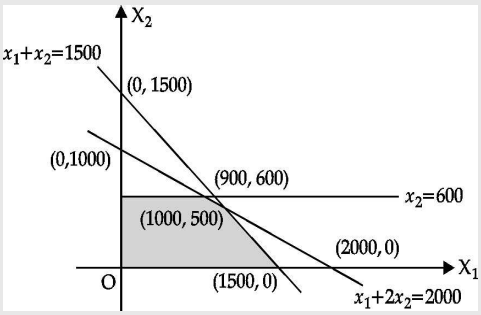
(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is NOT the correct explanation of A
(C) A is true but R is false
(D) A is false but R is True

Q. 1. The constraints of L.P.P. given by

$$x_1 + 2x_2 \le 2000$$
$$x_1 + x_2 \le 1500$$
$$x_2 \le 600 \text{ and } x_1, x_2 \ge 0$$

Assertion (A): The points (1000, 0), (0, 500), (2, 0) lie in the positive bounded region.
Reason (R): The point (2000, 0) does not lies in the positive bounded region.
Ans. Option (B) is correct.

Explanation:



From the graph, it is clear that the point (2000, 0) is outside, where as point (1000, 0), (0, 500), (2, 0) lie in the positive bounded region.

Q. 2. Assertion (A): The feasible solution of a L.P.P. belongs to any quadrant.

Reason (R): The feasible solution of a L.P.P. belongs to only first quadrant.

Ans. Option (D) is correct.

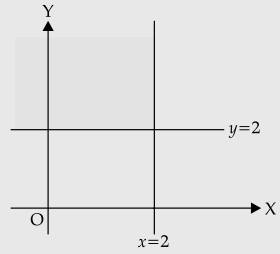
Explanation: The feasible solution of a L.P.P. belongs to only first quadrant because of non-negativity condition of variables associated with L.P.P.

Q. 3. Assertion (A): The graph of $x \leq 2$ and $y \geq 2$ will be situated in the first and second quadrants.

Reason (R): The graph of $x \leq 2$ and $y \geq 2$ will be situated in first and third quadrants.

Ans. Option (C) is correct.

Explanation:



It is clear from the graph that Assertion is true.

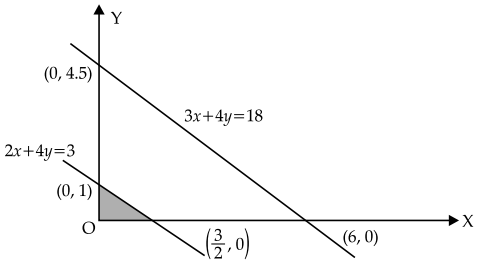
Q. 4. Assertion (A): The vertices of a feasible region by the linear constraints

$3x + 4y \leq 18$

$2x + 3y \geq 3$ and $x, y \geq 0$

are (0, 3), (3, 0) and (4.8, 0).

Reason (R):



Ans. Option (D) is correct.

Explanation: It is clear from the graph given in Reason (R), that the vertices of feasible region are (0, 0), (1, 0) and $(\frac{3}{2}, 0)$.

So, Assertion (A) is wrong.

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CHAPTER 1

NUMBERS AND QUANTIFICATION

Syllabus

Modulo arithmetic, Congruence modulo.

In this chapter you will study

- *Modulo Arithmetic*
- *Congruence Modulo*
- *Properties of Congruence Modulo*

Revision Notes

Modulo Arithmetic

Modulo Arithmetic is a special type of arithmetic that involves only integers and the operations used are addition, subtraction, multiplication and division. The only difference between modulo arithmetic and simple arithmetic that we have learned in our primary classes is that is modulo arithmetic all operations are performed regarding a positive integer *i.e.*, the modulus.

Let's revise the division theorem that tells us that for any two integers a and b where $b \neq 0$, there always exists unique integer a and r such that $a = qb + r$ and $0 \leq r < |b|$.

For example: $a = 39$, $b = 5$, we can find $q = 7$ and $r = 4$.

so, that $39 = 7 \times 5 + 4$.

Here, a is called dividend
b is called divisor
q is called quotient
r is called remainder

If $r = 0$, the we say b divides a or a is divisible by b.

This established a natural **congruence relation** on the integers.

Key Words

Congruence Relation: In abstract algebra, a congruence relation (or simply congruence) is an equivalence relation on an algebraic structure (such as a group, ring, or vector space) that is compatible with the structure in the sense that algebraic operations done with equivalent elements will yield equivalent elements.

Congruence Modulo

Scan to know more about this topic

Congruence Modulo

Let m be a positive integer and I be the set of all integers. The relation "congruence modulo m" is defined on all $a, b \in I$ by $a \equiv b \pmod{m}$ if and only if m divides $(a - b)$.

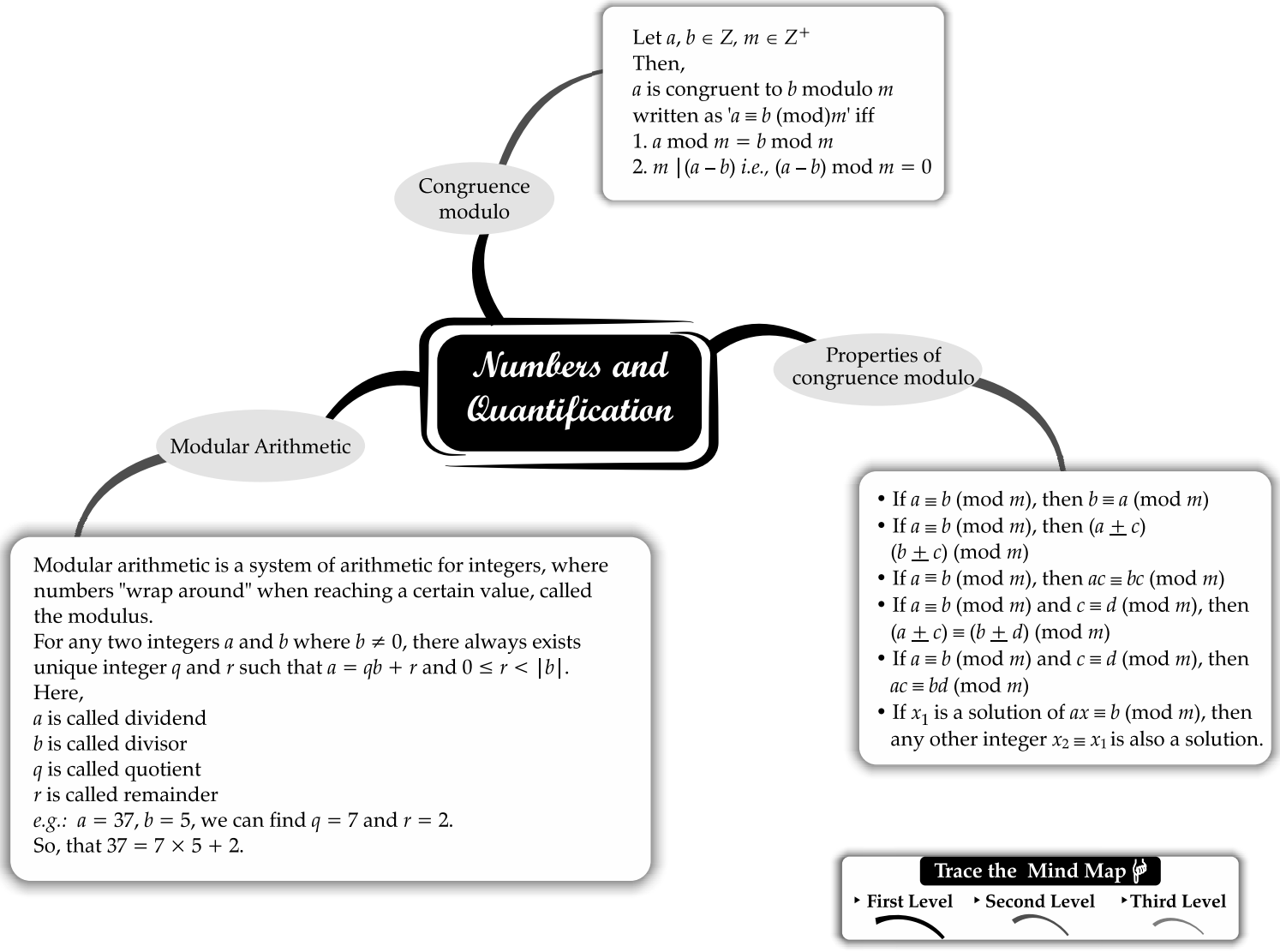
The symbol " $\equiv \pmod{m}$" is read as "congruence modulo m".

"m divides $(a - b)$" or "m is a factor of $(a - b)$" is usually denoted by "$m \mid (a - b)$".

Clearly, $a \equiv b \pmod{m}$ if $(a - b)$ is a multiple of m, *i.e.*, $a = b + k\,m$ for some integer k.

Thus,

- $69 \equiv 25 \pmod{4}$, since $4 \mid (69 - 25)$ or 4 divides $(69 - 25)$ or $(69 - 25) = 44$ is a multiple of 4.
- $8 \not\equiv -8 \pmod{3}$, since $8 - (-8) = 16$, which is not a multiple of 3.


Numbers and Quantification
Congruence modulo
Let $a, b \in Z, m \in Z^+$
Then,
a is congruent to b modulo m
written as '$a \equiv b \pmod{m}$' iff
1. $a \bmod m = b \bmod m$
2. $m \mid (a - b)$ *i.e.*, $(a - b) \bmod m = 0$
Properties of congruence modulo
• If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$
• If $a \equiv b \pmod{m}$, then $(a \pm c)$ $(b \pm c) \pmod{m}$
• If $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{m}$
• If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $(a \pm c) \equiv (b \pm d) \pmod{m}$
• If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$
• If x_1 is a solution of $ax \equiv b \pmod{m}$, then any other integer $x_2 \equiv x_1$ is also a solution.
Modular Arithmetic
Modular arithmetic is a system of arithmetic for integers, where numbers "wrap around" when reaching a certain value, called the modulus.
For any two integers a and b where $b \neq 0$, there always exists unique integer q and r such that $a = qb + r$ and $0 \leq r < |b|$.
Here,
a is called dividend
b is called divisor
q is called quotient
r is called remainder
e.g.: $a = 37, b = 5$, we can find $q = 7$ and $r = 2$.
So, that $37 = 7 \times 5 + 2$.
Trace the Mind Map
First Level
Second Level
Third Level

Properties of Congruence Modulo

Property I: If $a \equiv b \pmod m$, then $b \equiv a \pmod m$

Proof: If $a \equiv b \pmod m$

then, $m \mid (a - b)$

or, $m \mid -(b - a)$

or, $m \mid (b - a)$

or, $b \equiv a \pmod m$

Hence Proved

Scan to know more about this topic

Properties of Congruence Modulo

Property II: If $a \equiv b \pmod m$, then $a \pmod m = b \pmod m$.

Proof: We know that, if $a \equiv b \pmod m$, then

$a - b = m k \quad k \to$ integer constant

Let $k = k_1 - k_2$

so, $a - b = m(k_1 - k_2)$

$a - b = mk_1 - mk_2$

$a - mk_1 = b - mk_2 = r$

or, $r = a - m k_1$	$r = b - m k_2$
or, $a - r = m k_1$	$b - r = m k_2$
or, $a \equiv r \pmod m$	$b \equiv r \pmod m$
or, $r \equiv a \pmod m$	$r \equiv b \pmod m$
(from Property I)	(from Property I)

Thus, $a \pmod m = b \pmod m$ **Hence Proved.**

Property III: If $a \equiv b \pmod m$, then $(a \pm c) = (b \pm c) \pmod m$.

Proof: If $a \equiv b \pmod m$

then $m \mid (a - b)$

or, $m \mid [(a \pm c) - (b \pm c)]$

or, $a \pm c \equiv b \pm c \pmod m$ **Hence Proved.**

Property IV: If $a \equiv b \pmod m$, then $ac \equiv bc \pmod m$.

Proof: If $a \equiv b \pmod m$

then $a - b = m k \quad k \to$ integer constant

or, $ac - bc = m(ck)$

or, $ac - bc = mk_1 \quad k_1 \to$ integer constant

or, $ac \equiv bc \pmod m$ **Hence Proved.**

Property V: If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then, $(a + c) \equiv (b + d) \pmod m$.

Proof: $a \equiv b \pmod m$, $c \equiv d \pmod m$

$\Rightarrow m \mid (a - b), m \mid (c - d)$

$\Rightarrow m \mid [(a - b) + (c - d)]$

$\Rightarrow m \mid [(a + c) - (b + d)]$

$\Rightarrow (a + c) \equiv (b + d) \pmod m$ **Hence Proved.**

Property VI: If $a \equiv b \pmod m$, and $c \equiv d \pmod m$, then, $(a - c) \equiv (b - d) \pmod m$

Proof: $a \equiv b \pmod m$, $c \equiv d \pmod m$

$\Rightarrow m \mid (a - b), m \mid (c - d)$

$\Rightarrow m \mid [(a - b) - (c - d)]$

$\Rightarrow m \mid (a - c) - (b - d)$

$\Rightarrow (a - c) \equiv (b - d) \pmod m$ **Hence Proved.**

Property VII: If $a \equiv b \pmod m$, and $c \equiv d \pmod m$, then, $ac \equiv bd \pmod m$

Proof: $a \equiv b \pmod m$, $c \equiv d \pmod m$

$\Rightarrow m \mid (a - b), m \mid (c - d)$

$\Rightarrow m \mid (ac - bc), m \mid (bc - bd)$

$\Rightarrow m \mid [(ac - bc) + (bc - bd)]$

$\Rightarrow m \mid (ac - bd)$

$\Rightarrow ac \equiv bd \pmod m$ **Hence Proved.**

Property VIII: If x_1 is a solution of $ax \equiv b \pmod m$, then any other integer $x_2 \equiv x_1 \pmod m$ is also a solution.

Proof: Since, x_1 is a solution of $ax \equiv b \pmod m$, we have

$ax_1 \equiv b \pmod m$

But $x_2 \equiv x_1 \pmod m$, therefore $ax_2 \equiv ax_1 \pmod m$, *i.e.*, if m divides $(x_2 - x_1)$, then m also divides a $(x_2 - x_1)$, a being an integer.

Hence, on adding, we get by theorem I,

$ax_2 + ax_1 \equiv ax_1 + b \pmod m$

or, $ax_2 \equiv b \pmod m$

Hence, x_2 is also a solution of $ax \equiv b \pmod m$.

Property IX: The congruence modulo $ax \equiv b \pmod m$ has a solution iff the **greatest common divisor (GCD)** of a and m divides b. If the greatest common divisor d, of a and m, divides b, then congruence modulo has exactly d incongruent solutions.

Key Words

Greatest Common Divisor (GCD):
The greatest common divisor (GCD) of two or more numbers is the greatest common factor number that divides them, exactly. It is also called the highest common factor (HCF).
e.g.: The greatest common factor of 15 and 10 is 5, since both the numbers can be divided by 5.

Example 1. Prove that $35x \equiv 14 \pmod{21}$ has a solution. Also, find the number of incongruent solutions.

Solution: Here, the greatest common divisor (GCD) of 35 and 21 is 7; also 7 divides 14. Hence, the given congruence modulo has a solution and number of incongruent solutions (mod 21) is 7.

⊗	0	1	2	3	4	6	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

Amazing Facts

- In modulo arithmetic, an "even number" is one where it's "0 mod 2" — that is, it has a remainder of 0 when divided by 2. An odd number is "1 mod 2" (has remainder 1).
- Modular arithmetic is also used as a public key in cryptography systems, which are vital for modern commerce.
- In music theory, modulo 12 arithmetic is used to analyze the 12-tone equal temperament system, when notes separated by an octave of 12 semi-tones are treated as equivalent.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. If $100 = x \pmod 7$, then the least positive value of x is:

(A) 2 **(B)** 3
(C) 6 **(D)** 4

[CBSE Board Term-I, 2022]

Ans. Option (A) is correct.

Explanation: We know that,

$$a = b(\text{mod } c)$$

Then b is the remainder, when a is divided by c.

$$100 = x(\text{mod } 7)$$

$$7\overline{)100}(14$$
$$\underline{7}$$
$$30$$
$$\underline{28}$$
$$2 \rightarrow \text{Remainder}$$

Thus, $x = 2$

Q. 2. The remainder when 5^{61} is divided by 7 is:

(A) 1 **(B)** 2
(C) 4 **(D)** 5

[CBSE Board Term-I, 2022]

Ans. Option (D) is correct.

Explanation: $5^{61} = (5^3)^{20} \times 5^1$

$= (126 - 1)^{20} \times 5^1$

$= [126^{20} - {}^{20}C_1\, 126^{19} + {}^{20}C_2\, 126^{18} + \ldots\ldots\ldots {}^{20}C_{19}\, 126^1 + {}^{20}C_{20}\, 126^0\, 1^{20}] \times 5$

$= 126[126^{19} - {}^{20}C_1 126^{18} + {}^{20}C_2 126^{17} + \ldots\ldots\ldots {}^{20}C_{19}] \times 5 + 1 \times 5$

$= 126 \times 5\,[126^{19} - {}^{20}C_1 126^{18} + {}^{20}C_2 126^{17} + \ldots\ldots\ldots {}^{20}C_{19}] + 5$

Here, $126 \times 5 = 630$, which is divisible by 7, thus remainder is 5.

Q. 3. The value of $5 \odot_8 11$, where $\odot$ is multiplication modulo is:

(A) –1 **(B)** 0
(C) 7 **(D)** 9

Ans. Option (C) is correct.

Explanation:

$$5 \odot_8 11 = (5 \times 11) \text{mod } 8$$
$$= 55 \text{ mod } 8$$
$$= 7$$

Q. 4. If $x \equiv -4 \pmod 3$, then a solution for x is:

(A) –2 **(B)** 12
(C) 19 **(D)** 35

Ans. Option (D) is correct.

Explanation: Since $3 | (x + 4)$ is true for

$$x = 35$$

Q. 5. If the present time is 8.40 pm, then the time after $876\frac{1}{2}$ hours will be:

(A) 8.40 am **(B)** 9.10 am
(C) 6.10 pm **(D)** 10.40 pm

Ans. Option (B) is correct.

Explanation: $\because$ $876 \pmod{24} = 12$

$\therefore$ 8.40 pm will change to 8.40 am after 12 hours, further after 30 minutes the time will be 9.10 am.

Q. 6. Let $m \in Z^+$ consider the relation R_m defined as a R_m b iff $a \equiv b \pmod m$, then R_m is:

(A) reflexive but not symmetric
(B) symmetric but not transitive
(C) reflexive, symmetric but not transitive
(D) an equivalence relation

Ans. Option (D) is correct.

Explanation: The relation R_m defined as $a \equiv b \pmod m$ is reflexive, symmetric and transitive.

$\therefore$ R_m is an equivalent relation.

Q. 7. If $a \equiv b \pmod n$, then

(A) $n | a$ and $n | b$ **(B)** $n | b$ only
(C) $n | (a - b)$ **(D)** None of these

Q. 8. For any integers a and b, and positive integer n, then $a \equiv a \pmod n$ = ...

(A) 0 **(B)** n
(C) a **(D)** 1

Q. 9. If $a \equiv b \pmod n$, then

(A) $a \equiv a \pmod n$ **(B)** $b \equiv a \pmod n$
(C) $b \equiv b \pmod n$ **(D)** None of these

Ans. Option (B) is correct.

Explanation: If $a \equiv b \pmod n$

then $n | (a - b)$

or, $n | -(b - a)$

or, $n | (b - a)$

or, $b \equiv a \pmod n$

Q. 10. If $a = b \pmod m$, then

(A) $a \pmod m = b \pmod m$
(B) $a \pmod a = b \pmod m$
(C) $a \equiv a \pmod m$
(D) None of these

Ans. Option (A) is correct.

Q. 11. If $a \equiv b \pmod m$, then

(A) $(a + c) \equiv (b + c) \pmod m$
(B) $(a - c) \equiv (b - c) \pmod m$
(C) $(a \pm c) \equiv (b \pm c) \pmod m$
(D) All of the above

Ans. Option (D) is correct.

Explanation: If $a \equiv b \pmod m$, then

$m | (a - b)$

or, $m | [(a \pm c) - (b \pm c)]$

or, $a \pm c \equiv b \pm c \pmod m$

Q. 12. If $a \equiv b \pmod n$ and $b \equiv c \pmod n$, then

(A) $a \equiv a \pmod n$ **(B)** $b \equiv c \pmod a$
(C) $a \equiv c \pmod n$ **(D)** None of these

Ans. Option (C) is correct.

Explanation: If $a \equiv b \pmod n$ and $b \equiv c \pmod n$, then $n | (b - a)$ and $n | (c - b)$. Using the linear combination, we get

$n | (b - a + c - b)$ or $n | (c - a)$

Thus, $a \equiv c \pmod n$

Q. 13. If $a \equiv b$(mod n), then

(A) $b = a + nq$, for some integer q

(B) $b = a - nq$, for some integer q

(C) $b = nq$, for some integer q

(D) $b = a.nq$, for some integer q

Ans. Option (A) is correct.

Explanation: If $a \equiv b$(mod n), then by definition $n|(b-a)$.

Therefore, $b - a = nq$, for some q.

Thus, $b = a + nq$.

Q. 14. If $a \equiv b$(mod n) and $c \equiv d$(mod n), then

(A) $a + c \equiv b + d$(mod n)

(B) $a.c \equiv b.d$(mod n)

(C) Both A and B

(D) None of these

Ans. Option (C) is correct.

Q. 15. If $a = b$(mod n), then

(A) $a^k = b^k$(mod n) $\forall\ k \in$ N

(B) $ka = kb$(mod n) $\forall\ k \in$ I

(C) Both A and B

(D) Only A

Ans. Option (C) is correct.

Q. 16. 'Modulo arithmetic is the arithmetic of remainders'. If a and b are two integer and r is the remainder, then above statement can be represented Mathematically as:

(A) $a \bmod b = r$ **(B)** $a \bmod r = b$

(C) $r \bmod a = b$ **(D)** None of these

Ans. Option (A) is correct.

Explanation: When we divide integer a by b, we have $a = bq + r$, where $q, r \in$ I and $0 \le |b|$,

a = dividend, b = divisor, q = quotient, r = remainder

In arithmetic modulo, this statement can be represented as:

$$a \bmod b = r$$

Q. 17. Evaluate: 21 mod 2 = ...

(A) 2 **(B)** 19

(C) 10 **(D)** 1

Ans. Option (D) is correct.

Explanation: On dividing 21 by 2, we get remainder = 1

Hence, 21 mod 2 = 1.

Q. 18. Evaluate: (9 + 23) mod 12 = ...

(A) 2 **(B)** 8

(C) 32 **(D)** 12

Ans. Option (B) is correct.

Explanation: (9 + 23) mod 12

or, 32 and 12

On dividing 32 by 12, we get remainder = 8

Hence, (9 + 23) mod 12 = 8

Q. 19. Evaluate: (9 × 8) mod 6 = ...

(A) 2 **(B)** 1

(C) 0 **(D)** 6

Ans. Option (C) is correct.

Explanation: (9 × 8) mod 6

or 72 mod 6

On dividing 72 by 6, we get remainder = 0

Hence, (9 × 8) mod 6 = 0

Q. 20. Assume the current time is 2 : 00 p.m. what time (in a.m. or p.m.) will after 65 hours?

(A) 7 a.m. **(B)** 7 p.m.

(C) 3 a.m. **(D)** 3 p.m.

Ans. Option (A) is correct.

Explanation: We know that time repeats after every 24 hours

So, we find 65 (mod 24)

$\because$ $65 = 24 \times 2 + 17$

So, $65 \equiv 17$ (mod 24)

$\therefore$ 65 hours is equivalent to 17 hours.

Now, 2 : 00 p.m. + 17 hours = 7 : 00 a.m.

Q. 21. Assume that current time is 7:00 p.m. what time (in a.m. or p.m.) will it be in 1000 hours ?

(A) 7 a.m. **(B)** 11 a.m.

(C) 11 p.m. **(D)** 2 a.m.

Ans. Option (B) is correct.

Explanation: We know that time repeats after every 24 hours.

So, we find 1000 (mod 24).

$\because$ $1000 = 24 \times 41 + 16$

So, $1000 \equiv 16$ (mod 24)

$\therefore$ hours is equivalent to 16 hours.

Now, 7 : 00 p.m. + 16 hours = 11 a.m.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. What time is it in 79 hours if it is 4 o'clock now ? U

Sol. We have (4 + 79) mode 12 = 83 mod 12 = 11. Thus 79 hours from now, it is 11 o'clock. **1**

Q. 2. Evaluate: – 97 mod 11 U

Sol. When – 97 divided by 11, we get Quotient = 8 and remainder = – 9.

But since this remainder is negative, we have to increase our quotient by 1 to say – 97 divided by 11 equals – 9 remainder 2, as 11(–9) + 2 = – 97.

Therefore, – 97 mod 11 equals 2. **1**

Q. 3. It's 7:00 (am/pm doesn't matter). Where will the hour hand be in 7 hours? AI

Sol. (7 + 7) mod 12 = (14) mod 12 = 2 mod 12 [2 is the remainder when 14 is divided by 12]

So, its 2 o'clock and hour hand is at 2.

Q. 4. Shamita has a flight arriving at 3 pm. It's getting delayed 14 hours. What time will it land? AI

Sol. We have, 14 mod 12 = 2

So, the flight will land at "3 + 2 = 5 am ". **1**

Q. 5. What is the time 15 hours before 11 p.m. ? A

Sol. $15 \equiv 11 - x \pmod{12}$

[we take (mode 12) in case of clock]

$15 - 11 + x = 12n$ for some integer n

$4 + x = 12n$

$4 + x$ is a multiple of 12.

Therefore, the least positive value of x must be 8.

Since, $4 + 8 = 12$ is a multiple of 12.

Hence, the time 15 hours before 11 p.m. is 8 a.m. **1**

Q. 6. What is the time 100 hours after 7 a.m. ? A

Sol. $7 + 100 \equiv x \pmod{12}$

[we take (mode 12) in case of clock]

$107 - x = 12n$

$107 - x$ is a multiple of 12.

Therefore, the least positive integer of x must be 11.

Since, $107 - 11 = 96$ is the nearest multiple of 11 less than 107.

So, the time 100 hours after 7 a.m. is 11 a.m. **1**

Q. 7. What is 56 mod 7 ? U

Short Answer Type Questions-I (2 or 3 marks each)

Q. 1. Find the least positive value of x such that $71 \equiv x \pmod 8$ U

Q. 2. Find the least positive value of x such that $78 + x \equiv 3 \pmod 5$ A I

Q. 3. Find the least positive value of x such that $5x \equiv 4 \pmod 6$. U

Sol. $5x \equiv 4 \pmod 6$

$5x - 4 = 6n$ for some integer n

$$x = \frac{6n+4}{5}$$ **1**

When we put $n = 1, 6, 11, 16 \ldots$; then $6n + 4$ is divisible by 5.

When $n = 1$, $x = \frac{6\times1+4}{5} = 2$

Therefore, the least positive value of x is 2. **1**

Q. 4. Today is Tuesday. My uncle will come after 45 days. In which day my uncle will be coming ? A

Sol. Number of days in a week = 7

So, $45 \equiv x \pmod 7$ **1**

$45 \equiv 3 \pmod 7$

As $7\overline{)45}$ (6, 42, 3)

The value of x must be 3.

Thus, three days after Tuesday is Friday. So, uncle will come on Friday. **1**

Commonly Made Error

- In such application based questions, sometimes students do not have any idea how to start the solution. Few students used very long method of calculation.

Answering Tip

- The questions based on clock and time can be easily solved using the concept of congruence modulo.

Q. 5. Find $(186 \times 93) \bmod 7$. A I

Sol. We know that,

$a \,.\, b \pmod n \equiv a \pmod n \,.\, b \pmod n$ **1**

So, $(186 \times 93) \bmod 7 \equiv 186 \pmod 7 \,.\, 93 \pmod 7$

$\equiv 4 \pmod 7 \,.\, 2 \pmod 7$

$\equiv 4.2 \pmod 7$

$\equiv 8 \pmod 7$

$\equiv 1 \pmod 7$ **2**

Short Answer Type Questions-II (4 marks each)

Q. 1. (i) Prove that $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n.

(ii) Find the remainder when 2^{81} is divided by 17. A

Sol. (i) $2^n + 6 \times 9^n \equiv \pmod 7$

$2^n + 6 \times 9^n \equiv 7n$ for some integer 7

So, $2^n + 6 \times 9^n$ is a multiple of 7.

When $n = 0$, then $2^0 + 6 \times 9^0 = 1 + 6 = 7$ is a multiple of 7.

When $n = 1$, then $2 + 6 \times 9^1 = 2 + 54 = 56$ is a multiple of 7.

When $n = 2$, then $2^2 + 6 \times 9^2 = 4 + 486 = 490$ is a multiple of 7.

.....

Hence, $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n. **2**

(ii) $2^{81} \equiv x \pmod{17}$

We know that,

$2^3 \equiv 8 \pmod{17}$

$(2^3)^3 \equiv (8)^3 \pmod{17}$

$2^9 \equiv 512 \pmod{17}$

$2^9 \equiv 2 \pmod{17}$

$(2^9)^9 \equiv (2)^9 \pmod{17}$

$2^{81} \equiv 512 \pmod{17}$

$2^{81} \equiv 2 \pmod{17}$

So, when 2^{81} is divided by 17, remainder is 2. **2**

Q. 2. When $783 \times 657 \times 594 \times 432 \times 346 \times 251$ is divided by 5, then find remainder. A I

Sol. Since, $783 \equiv 3 \pmod 5$

$657 \equiv 2 \pmod 5$

$$594 \equiv 4 (\text{mod } 5)$$
$$432 \equiv 2 (\text{mod } 5)$$
$$346 \equiv 1 (\text{mod } 5)$$
and $$251 \equiv 1 (\text{mod } 5)$$ **2**

So, $(783 \times 657 \times 594 \times 432 \times 346 \times 251) \bmod 5$
$$\equiv (3 \times 2 \times 4 \times 2 \times 1) \bmod 5$$ **1**
$$\equiv 48 (\text{mod } 5)$$
$$\equiv 3 (\text{mod } 5)$$ **1**

Hence, the remainder when $(783 \times 657 \times 594 \times 432 \times 346 \times 251)$ is divided by 5 is 3.

Q. 3. Find highest power of 3 in $\lfloor\underline{1000}$. A I U

Sol. Highest power of 3 in $\lfloor\underline{1000}$

$$= \text{Integral part} \left\{\left[\frac{1000}{3}\right]+\left[\frac{1000}{3^2}\right]+\left[\frac{1000}{3^3}\right]+\left[\frac{1000}{3^4}\right]+\left[\frac{1000}{3^5}\right]+\left[\frac{1000}{3^6}\right]+\left[\frac{1000}{3^7}\right]+\ldots\right\}$$

$$= \text{Integral part} \left\{\left[\frac{1000}{3}\right]+\left[\frac{1000}{9}\right]+\left[\frac{1000}{27}\right]+\left[\frac{1000}{81}\right]+\left[\frac{1000}{243}\right]+\left[\frac{1000}{729}\right]+\left[\frac{1000}{2187}\right]+\ldots\right\}$$ **1½**

$$= 333 + 111 + 37 + 12 + 4 + 1 + 0$$ **2½**
$$= 498$$

So, highest power of 3 in $\lfloor\underline{1000}$ is 498.

Long Answer Type Questions-I (5 marks each)

Q. 1. Find last three digits of the product 1234×5678. A I

Sol. To find the last three digits of the product 1234×5678, we find $1234 \times 5678 \pmod{1000}$. **1**

Since,
$$1234 \equiv 234 (\text{mod } 1000)$$ **1**
and $$5678 \equiv 678 (\text{mod } 1000)$$ **1**
So, $$1234 \times 5678 \equiv 234 \times 678 (\text{mod } 1000)$$
$$\equiv 158652 (\text{mod } 1000)$$ **1**
$$\equiv 652 (\text{mod } 1000)$$ **1**

Hence, the last three digits of the product 1234×5678 are 652.

Q. 2. Find last two digits of the product 2345×6789. U

Q. 3. If $n!$ denotes the product of the integers 1 through n, what is the remainder when $(1! + 2! + 3! + 4! + 5! + 6! + \ldots)$ is divided by 9? U

Sol. We know that $k! \equiv 0 \pmod 9$ for all $k \geq 6$. **1**

Thus, we need to find
$(1! + 2! + 3! + 4! + 5!) \pmod 9$. **1**
$$1! \equiv 1 (\text{mod } 9)$$
$$2! \equiv 2 (\text{mod } 9)$$
$$3! \equiv 6 (\text{mod } 9)$$
$$4! \equiv 24 \equiv 6 (\text{mod } 9)$$
$$5! \equiv 5 \cdot 6 \equiv 30 \equiv 3 (\text{mod } 9)$$ **2**

Thus,
$$(1! + 2! + 3! + 4! + 5!) \equiv 1 + 2 + 6 + 6 + 3$$
$$\equiv 18 \equiv 0 (\text{mod } 9)$$

So, the remainder is 0. **1**

Long Answer Type Questions-II (6 marks each)

Q. 1. Show that 'congruence modulo m' is an equivalence relation on the set of all integers. A I R&U

Q. 2. Find the highest power of 6 in $\lfloor\underline{245}$. A

Sol. Since, $6 = 2.3$

Highest power of 2 in $\lfloor\underline{245}$ = Integral part of
$$\left[\frac{245}{2}\right]+\left[\frac{245}{2^2}\right]+\left[\frac{245}{2^3}\right]+\left[\frac{245}{2^4}\right]+\left[\frac{245}{2^5}\right]+\left[\frac{245}{2^6}\right]+\left[\frac{245}{2^7}\right]+\left[\frac{245}{2^8}\right]+\ldots$$

$$= \text{Integral part of } \left[\frac{245}{2}\right]+\left[\frac{245}{4}\right]+\left[\frac{245}{8}\right]+\left[\frac{245}{16}\right]+\left[\frac{245}{32}\right]+\left[\frac{245}{64}\right]+\left[\frac{245}{128}\right]+\left[\frac{245}{256}\right]+\ldots$$

$$= 122 + 61 + 30 + 15 + 7 + 3 + 1 + 0$$
$$= 239$$ **2½**

Highest power of 3 in $\lfloor\underline{245}$ = Integral part of
$$\left[\frac{245}{3}\right]+\left[\frac{245}{3^2}\right]+\left[\frac{245}{3^3}\right]+\left[\frac{245}{3^4}\right]+\left[\frac{245}{3^5}\right]+\left[\frac{245}{3^6}\right]+\ldots$$

$$= \text{Integral part of } \left[\frac{245}{3}\right]+\left[\frac{245}{9}\right]+\left[\frac{245}{27}\right]+\left[\frac{245}{81}\right]+\left[\frac{245}{243}\right]+\left[\frac{245}{729}\right]+\ldots$$

$$= 81 + 27 + 9 + 3 + 1 + 0$$
$$= 121$$ **2½**

Now, highest power of 6 in $\lfloor\underline{245}$ = min (239, 121)
$$= 121$$

Hence, highest power of 6 in 245 is 121. **1**

Q. 3. What is remainder when $2^{20} + 3^{30} + 4^{40} + 5^{50} + 11^7$ is divided by 7?

Sol. Consider, $$2^{7-1} \equiv 1 (\text{mod } 7)$$
$$\Rightarrow 2^6 \equiv 1 (\text{mod } 7)$$
$$\Rightarrow (2^6)^3 \equiv 1 (\text{mod } 7)$$
$$\Rightarrow 2^{18} \equiv 1 (\text{mod } 7)$$

Multiply throughout by 2^2, we get
$$\Rightarrow 2^{18} \cdot 2^2 \equiv 2^2 (\text{mod } 7)$$
$$\Rightarrow 2^{20} \equiv 4 (\text{mod } 7) \quad \ldots(\text{i})$$ **1**

These questions are for practice and their solutions are available at the end of the chapter

Similarly, $3^{7-1} \equiv 1 \pmod 7$

$\Rightarrow 3^6 \equiv 1 \pmod 7$

$\Rightarrow (3^6)^5 \equiv 1 \pmod 7$

$\Rightarrow 3^{30} \equiv 1 \pmod 7$ ...(ii) **1**

and $4^{7-1} \equiv 1 \pmod 7$

$4^6 \equiv 1 \pmod 7$

$\Rightarrow (4^6)^6 \equiv 1 \pmod 7$

$\Rightarrow 4^{36} \equiv 1 \pmod 7$

Multiply throughout by 4^4, we get

$\Rightarrow 4^{36} \cdot 4^4 \equiv 4^4 \pmod 7$

$\Rightarrow 4^{40} \equiv 256 \pmod 7$

$\Rightarrow 4^{40} \equiv 4 \pmod 7$ ...(iii) **1**

Next consider, $5^{7-1} \equiv 1 \pmod 7$

$\Rightarrow 5^6 \equiv 1 \pmod 7$

$\Rightarrow (5^6)^8 \equiv 1 \pmod 7$

$\Rightarrow 5^{48} \equiv 1 \pmod 7$

Multiply throughout by 5^2, we get

$\Rightarrow 5^{48} \cdot 5^2 \equiv 5^2 \pmod 7$

$\Rightarrow 5^{50} \equiv 25 \pmod 7$

$\Rightarrow 5^{50} \equiv 4 \pmod 7$ ...(iv) **1**

Finally consider,

$11^{7-1} \equiv 1 \pmod 7$

$\Rightarrow 11^6 \equiv 1 \pmod 7$

Multiply throughout by 11, we get

$\Rightarrow 11^6 \cdot 11 = 11 \pmod 7$

$\Rightarrow 11^7 \equiv 11 \pmod 7$

$\Rightarrow 11^7 \equiv 4 \pmod 7$ ...(v) **1**

Now adding (i), (ii), (iii), (iv) and (v), we get

$2^{20} + 3^{30} + 4^{40} + 5^{50} + 11^7 \equiv (4 + 1 + 4 + 4 + 4) \pmod 7$

$\Rightarrow 2^{20} + 3^{30} + 4^{40} + 5^{50} + 11^7 \equiv 17 \pmod 7$

$\Rightarrow 2^{20} + 3^{30} + 4^{40} + 5^{50} + 11^7 \equiv 3 \pmod 7$ **1**

COMPETENCY BASED QUESTIONS

Case based MCQs

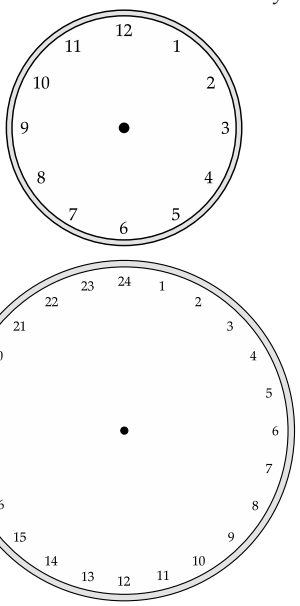
Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

The 12 hour clock is most commonly represented on an analogue clock with number 1-12, when shown on a digital clock, it usually accompanied by a.m. or p.m.

The 24 hour clock is more often shown on digital clocks and is written in a 4 digit form, with the first two digits representing the hours and the last two digits representing the minutes.

There is no need for a.m. or p.m. as each time represents each hour in a 24 hour day.



Q. 1. It is currently 7 : 00 a.m. in 12 hour clock. The time in next 493 hours will

(A) 8 : 00 a.m. **(B)** 8 : 00 p.m.

(C) 20 : 00 a.m. **(D)** 20 : 00 p.m.

Ans. Option (B) is correct.

Explanation: We know that time repeat after every 24 hours.

So, we find 493 (mod 24)

So, $493 \pmod{24} = 13$

$\therefore$ 493 hours is equivalent to 13 hours.

Now, 7 : 00 a.m. + 13 hours = 8 p.m.

Q. 2. If the time after 640 hours from now will be 9 a.m., then the current time is

(A) 5 : 00 a.m. **(B)** 5 : 00 p.m.

(C) 4 : 00 a.m. **(D)** 4 : 00 p.m.

Ans. Option (B) is correct.

Explanation: Here, we find 640 (mod 24)

$640 \pmod{24} \equiv 16$

$\therefore$ 640 hours is equivalent to 16 hours

Now,

9 : 00 a.m.– 16 hours = 5 p.m.

Q. 3. Which of two times are same in 12 hours and 24 hours clocks respectively ?

(A) 1 : 00 p.m., 13 : 00 **(B)** 3 : 00 p.m., 03 : 00

(C) 02 : 00 a.m., 14 : 00 **(D)** 04 : 00 a.m., 16 : 00

Ans. Option (A) is correct.

Explanation: Here,

3 : 00 p.m. in 12 hour clock

= 15 : 00 in 24 hour clock

2 : 00 a.m. in 12 hour clock

= 02 : 00 in 24 hour clock

4 : 00 a.m. in 12 hour clock

= 04 : 00 in 24 hour clock

So, only option (A) is correct *i.e.*, 1 : 00 p.m. in 12 hour clock is same as 13 : 00 in 24 hour clock

Q. 4. a.m. stands for:

(A) ante meridiem **(B)** anti meridiem

(C) after midday **(D)** None of these

Ans. Option (A) is correct.

Explanation: a.m. stands for ante meridiem and p.m. stands for post meridiem.

Q. 5. What time in 24 hour clock is equivalent to 7 a.m. in 12 hour clock ?

(A) 00 : 07 **(B)** 07 : 00
(C) 19 : 00 **(D)** 14 : 00

Ans. Option (B) is correct.

Explanation: 7 : 00 hour in 24 hour clock is equivalent to 07 : 00 a.m. in 12 hour clock.

II. Read the following text and answer the following questions on the basis of the same:

Today Seema learn two properties of Modulo operator in class which are as follows.

Property 1: If the value of X is increased by a multiple of Y (Say, kY), where k is any integer)
Then $X \bmod Y = (X + kY) \bmod Y$;
Where k is any integer.

Property 2: Addition property of modulo arithmetic is if A, B and C are positive integers then,
(A + B) mod C = (A mod C + B mod C) mod C

Q. 1. $(7 + 6 \times 3) \bmod 3 =$

(A) 1 **(B)** 3
(C) 2 **(D)** None of these

Ans. Option (A) is correct.

Explanation: $(7 + 6 \times 3) \bmod 3$
$= 7 \pmod 3$ [from property I]
$= 1$

Q. 2. Express $(18 + 9 \times 6) \bmod 6$ in form of X mod Y.

(A) 18 (mod 9) **(B)** 18 (mod 6)
(C) 9 (mod 6) **(D)** None of these

Ans. Option (B) is correct.

Explanation: Since,
$(X + kY) \bmod Y = X \bmod Y$
Therefore, $(18 + 9 \times 6) \bmod 6 = 18 \bmod 6$

Q. 3. If $k = 12$, $ky = 60$ and $X = 18$, then $X \pmod Y = ...$

(A) 1 **(B)** 2
(C) 3 **(D)** 4

Ans. Option (C) is correct.

Explanation: Here, $k = 12$, $ky = 60$
So, $Y = 5$
$\therefore$ $X \pmod Y = 18 \pmod 5 = 3$

Q. 4. $(12 + 18) \bmod 4 =$ .....

(A) 1 **(B)** 2
(C) 3 **(D)** 0

Ans. Option (B) is correct.

Explanation: $(12 + 18) \bmod 4$
$= (12 \pmod 4 + 18 \pmod 4) \bmod 4$
$= 0 + 2 \pmod 4$
$= 2$

Q. 5. $(171 + 245) \bmod 3 =$ .....

(A) 0 **(B)** 1
(C) 2 **(D)** 3

Ans. Option (C) is correct.

Explanation: $(171 + 245) \bmod 3$
$= (171 \pmod 3 + 245 \pmod 3) \bmod 3$
$= 0 + 2 \pmod 3$
$= 2$

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:

You may never have heard of modular arithmetic, but you use it every day without the slightest difficulty. In this system, numbers wrap around when they reach a certain size called the modulus; it is the arithmetic of remainders. When reckoning hours, we count up to 12 and start again from one. Thus, four hours after 9 o'clock it is 1 o'clock. Numbers that differ by a multiple of the modulus 12 are said to be congruent modulo 12.

A similar situation arises for the days of the week, which are computed modulo seven. Suppose today is Thursday. What weekday will it be 1,000 days from today? We don't have to count through the thousand days, just calculate the remainder when 1,000 is divided by seven, which is six. It is much the same for other time measurements. With 52 weeks in a year, the week number is reset to 1 at the beginning of each year, so it is confined to the range 1 to 52. Likewise, for the months, we use modulo 12 arithmetic.

Q. 1. Assume it is 18:00 hours. What time will it be 59 hours from now ?

Sol. Here, we use the 24 hour clock.
i.e., (18+ 59) (mod 12) **1**
$= 77 \pmod{12}$
$= 5$
So, time will 5:00 hours. **1**

Q. 2. (a) Which month is it 721 months from now if this month is November?

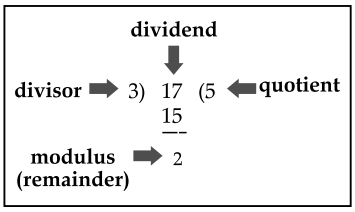
(b) Which day of the week is it in 110 days from today if today is Friday ?

Sol. (a) Months wrap around after 12 months. We have $721 \bmod 12 = 1$.
Since, December is one month after November, months from now, it will be December. **1**

(b) The days of the week wrap around after seven days.
So, $110 \bmod 7 = 5$ days
Thus, 110 days after Friday is the same day of the week as 5 days after Friday, *i.e.*, Wednesday. **1**

II. Read the following text and answer the following questions on the basis of the same:

Modular arithmetic was introduced by Carl Friedrich Gauss on the very first page of his magnum opus Arithmetical Investigations, or *Disquisitiones Arithmeticae*. Modular arithmetic is a system of arithmetic for integers, which considers the remainder. The modulus is another name for the remainder after division. For example, 17 mod 3 = 2, since if we divide 17 by 3, we get 5 with remainder 2. Remainder is simply the remaining part after the arithmetic division between two integer number whereas Modulus is the sum of remainder and divisor when they are oppositely signed and remaining part after the arithmetic division when remainder and divisor both are of same sign.



Q. 1. Find the remainder when the difference between 60002 and 601 is divided by 6.

Sol. We have $60002 = 10000.6 + 2$ and $601 = 100.6 + 1$.

So, $60002 \equiv 2 \pmod 6$

$601 \equiv 1 \pmod 6$

Thus, $60002 - 601 \equiv 2 - 1 \equiv 1 \pmod 6$

so, 1 is the remainder when the difference is divided by 6. **2**

Q. 2. Mayank has 44 boxes of soda in his truck. The cans of soda in each box are packed oddly so that there are 113 cans of soda in each box. Mayank plans to pack the sodas into cases of 12 cans to sell. After making as many complete cases as possible, how many sodas will Mayank have leftover?

Sol. Here, we have to find the remainder of product 44.113 when divided by 12.

So, $44 = 3.12 + 8$

$113 = 9.12 + 5$ **1**

Therefore, $44.113 = (3.12 + 8)(9.12 + 5)$

$= (3.9).12^2 + (3.5 + 8.9).12 + (8.5)$

Here each part of the product is a multiple of 12, except the product of the remainders when each 44 and 113 are divided by 12. That part of the product is $8.5 = 40$, which leaves a remainder of 4 when divided by 12.

So, Mayank has 4 sodas leftover after making as many cases of 12 as possible. **1**

Solutions for Practice Questions

Multiple Choice Questions

Ans. 7: Option (C) is correct.

Explanation: If a and b are integers and $n > 0$, then we write $a \equiv b \pmod n$ to mean $n \mid (a - b)$.

Ans. 8: Option (A) is correct.

Explanation: $a \equiv a \pmod n$ means $n \mid (a - a) = n \mid 0 = 0$.

Very Short Answer Type Questions

Sol. 7: On dividing 56 by 7,

we get, remainder = 0

So, $56 \bmod 7 \equiv 0 \bmod 7$

Short Answer Type Questions-I

Sol. 1: $71 \equiv x \pmod 8$

As, $64 \equiv 0 \pmod 8$ **1**

Now, $64 + 7 \equiv 0 + 7 \pmod 8$

or, $71 \equiv 7 \pmod 8$

So, $x = 7$ **1**

Sol. 2: $78 + x \equiv 3 \pmod 5$

$78 + x - 3 = 5n$ for some integer n

$75 + x = 5n$ **1**

$75 + x$ is a multiple of 5.

Hence, the least positive value of x must be 5, since the nearest value of multiple of 5, which is more than 75 is $75 + 5 = 80$. **1**

Long Answer Type Questions-I

Sol. 2: To find the last two digits of the product 2345×6789, we find $2345 \times 6789 \pmod{100}$. **1**

Since, $2345 = 45 \pmod{100}$ **1**

and $6789 \equiv 89 \pmod{100}$ **1**

So, $2345 \times 6789 \equiv 45 \times 89 \pmod{100}$ **1**

$\equiv 4005 \pmod{100}$

$\equiv 05 \pmod{100}$ **1**

Hence, the last two digits of the product 2345×6789 are 05.

Long Answer Type Questions-II

Sol. 1: Let I be the set of all integers and let R defined on I stand for "congruence modulo m".

Thus $a \, R \, b$ stands for

$a \equiv b \pmod m$, $a, b \in I$ *i.e.*, $m \mid (a - b)$

(i) Reflexivity: Let $a \in I$

Since, $a - a = 0$ and is divisible by m.

$\therefore \quad a \equiv a \pmod m, \forall \, a \in I$

Hence, R is reflexive. **2**

(ii) Symmetry: Let $a, b \in I$

Since, $a \, R \, b \Rightarrow a \equiv b \pmod m$

$\Rightarrow m \mid (a - b)$

$\Rightarrow m \mid (b - a) \quad [\because a - b = -(b - a)]$

$\Rightarrow b \, R \, a$

Hence, R is symmetric **2**

(iii) Transitivity: Let $a, b, c \in$ I. Also, let $a \, R \, b$ and $b \, R \, c$.

Since, $a \, R \, b \Rightarrow m \mid (a - b)$ and $b \, R \, c \Rightarrow m(b - c)$

$\therefore \quad a \, R \, b$ and $b \, R \, c \Rightarrow m \mid [(a - b) + (b - c)]$

$\Rightarrow m \mid (a - c)$

$\Rightarrow a \, R \, c$

$\therefore a \equiv b \pmod m$, $b \equiv c \pmod m \Rightarrow a \equiv c \pmod m$

Hence, R is transitive **2**

Thus, congruence modulo m' is an equivalence relation on I. **Hence Proved ½**

Commonly Made Error

- Sometimes students find difficulty in proving equivalence relation.

Answering Tip

- Always remember that to prove an equivalence relation, first show the given relation is reflexive, then symmetric and finally transitive.

REFLECTIONS

(a) Students can apply the concept of congruence modulo to solve various real time problems.

(b) Students can solve more problems related to remainders based on congruence modulo.

CHAPTER

2

NUMERICAL APPLICATIONS

Syllabus

Allegation and Mixture, Numerical problems on boats and streams, (Upstream and downstream), Pipes and cistern, Races and games, Numerical Inequalities.

In this chapter you will study

- *Understand the rule of alligation to produce a mixture at a given price*
- *Determine the mean price of a mixture*
- *Apply rule of allegation Solve real life problems mathematically Distinguish between upstream and downstream*
- *Express the problem in the form of an equation Determine the time taken by two or more pipes to fill or empty the tank Compare the performance of two players w.r.t. time, distance*
- *Describe the basic concepts of numerical inequalities*
- *Understand and write numerical inequalities*

Revision Notes

Alligation and Mixture:

It is the rule that enables us to find the ratio in which two or more ingredients at the given price must be mixed to produce a **mixture** of a desired price. There are two types of questions asked in **alligation**.

Type 1: If two ingredients are mixed, then

$$\left(\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}}\right) = \frac{\text{(C.P. of dearer)} - \text{(Mean Price)}}{\text{Mean Price} - \text{(C.P. of cheaper)}}$$

Key Words

Mixture: A mixture is a physical combination of two or more substances. The substances are known as ingredients of the mixtures.

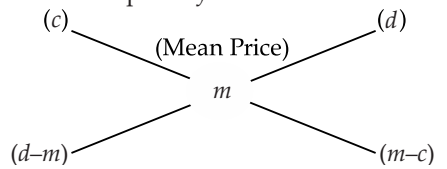
Key Words

Alligation: Alligation is an old and practical method of solving arithmetic problems related to mixtures of two or more ingredients.

We present as under:

C.P. of a unit quantity of cheaper

C.P. of a unit quantity of dearer



$\therefore$ (Cheaper quantity) : (Dearer quantity) = $(d - m)$: $(m - c)$

Here, C.P. = Cost Price

Mean Price (m)= The cost price of a unit quantity of the mixture

Numerical Applications

Alligation or mixture

If ingredients are mixed in a ratio, then

$$\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}} = \frac{\text{C.P. of the dearer- Mean price}}{\text{Mean price - C.P. of the cheaper}}$$

We represent it as writer:

C.P. of unit quantity of cheaper C.P. of unit quantity of dearer

(c) (d) Mean Price (m) (d-m) (m-c)

Races and games

- A race is a contest which can measure the speed in driving running, sailing, rowing, jumping etc. This is calculated over a centre distance.
- In every race end point is known as winning point
- In a dead heat race, every participant reach at the winning point
- The person that comes first wins the race and called winner.
- A game of leo in a race is the game where participants agree that whosoever scores the 100 point first wins the race.

Pipes and cistern

- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $y>x$), then on opening both pipes, the net part filled in 1 hour $= (\frac{1}{x}-\frac{1}{y})$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $x>y$), then on opening both pipes, the net part emptied in 1 hour $= (\frac{1}{y}-\frac{1}{x})$

Boat and streams

Boat (u km/h) → Stream (v km/h) } Down stream $(u+v)$km/h

Boat (u km/h) ← Stream (v km/h) } Up stream $(u-v)$km/h

If the speads of downstream and upstream are a km/h and b km/h, respectively, then

speed in still water $= \frac{1}{2}(a+b)$km/h

Rate of stream $= \frac{1}{2}(a-b)$km/h

Types of Inequalities

Numerical Inequalities : Relation between numbers
e.g.: $3<5, 7>4$

Literal or Variable Ineqalities : Relation between two variables or variable and numbers.
e.g.:$y<8$

Double Inequalities: Relationship from two sides
e.g.:$2<x<5$

Strict Inequalities: An inequality that uses the symbol <or> e.g.: $x<5, 3<5$

Slack Inequality: An inequality that uses the symbols $\leq$ or $\geq$. e.g.: $y\leq 5$

Linear Inequalities in One Variable: An inequality which involves a linear function in one variable
e.g.: $x<5$

Linear Inequalities in two variables: An inequality which involves a linear function in two variables,
e.g.: $3x+2y<5$

Quadratic Inequalities: An inequality which involves a quadratic function, eg.: $x^2+2x \leq 5$.

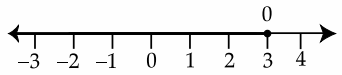
Inequalities

Representation

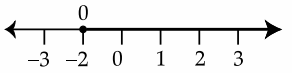
To reprst $x<a$ or $x>-a$ on a number line, put a circle on the number 'a' and dark line to the left (or right) of the number 'a'.

e.g.:

$x<3$



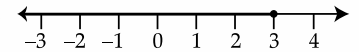
$x > -2$



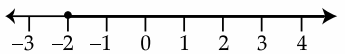
To represent $x\leq a$ (or $x\geq a$) on a number line, put a dark circle on the number 'a' and dark the line to the left (or right) of the number 'a'.

e.g.:

$x \leq 3$



$x \geq -2$



Definition

- An inequality is a relation that holds between two values, when they are different eg.: $x<5$, here, there is a relation between x & 5.
- Two real numbers or algebraic expressions related by symbol '<', '>', '≤' or '≥' form inequality.

e.g.: $y\leq 4, 2x+3y\geq 5, 3>x$.

Rules of inqualities

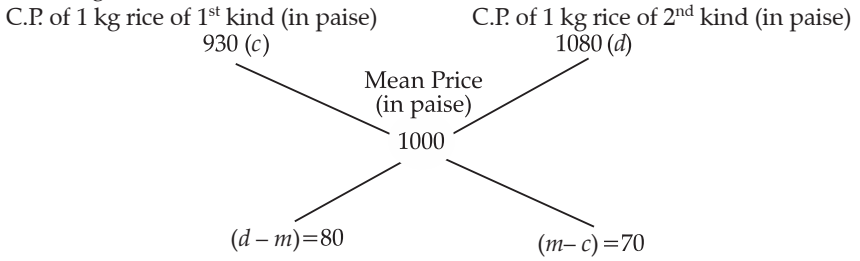
- Equal numbers may be added to (or subtracted from) both the sides of an inequality without affecting the sign of inequality:
  e.g.: $x<7$ is same as $x+2 < 7+2$
- Both sides of an inequality can be multiplied (or divided) by the same positive number without affecting the sign of inequality.
  e.g.: $x+y< 7$ is same as $(x+y)\times 3<7\times 3$
- When both sides are multiplied or divided by a – ve number, then sign of inequality is reversed.
  e.g.: $x+y< 7$ is same as
  $(x+7)\times(-3) > 7\times(-3)$

Trace the Mind Map

First Level · Second Level · Third Level

Example 1: In what ratio must rice at ₹ 9.30 per kg be mixed with rice at ₹ 10.80 per kg so that the mixture be worth ₹ 10 per kg ?

Sol. By the rule of alligation, we have



$\therefore$ Required ratio = 80 : 70 = 8 : 7

Type 2: Suppose a container contains x units of liquids from which y units are taken out and replaced by water. After n operations, the quantity of pure liquid $= \left[x\left(1-\frac{y}{x}\right)^n\right]$ units

Example 2: A container contains 50 litres of milk. From this container, 10 litres of milk was taken out and replaced by water. This process is repeated one more time. How much milk is now left in the container ?

Sol. Applying the method of replacement.

Amount of milk after 2 operations

$$= \left[50\left(1-\frac{10}{50}\right)^2\right]$$

$$= 50\times\frac{4}{5}\times\frac{4}{5} = 32 \text{ litres.}$$

Boats and Streams

Stream is the moving water in the river.

Upstream is the direction against the stream.

Downstream is the direction along the stream.

Still water is the state where water is considered to be stationary and the speed of the water in this case is zero.

- If the speed of a boat in still water is u km/h and the speed of the stream is v km/h, then:

  Speed of boat in downstream = $(u + v)$ km/h

  Speed of boat in upstream = $(u – v)$ km/h

- If the speed of boat in downstream is a km/h and the speed of boat in upstream is b km/h, then:

$$\text{Speed of boat in still water} = \frac{1}{2}(a+b) \text{ km/h}$$

$$\text{Speed of stream} = \frac{1}{2}(a-b) \text{ km/h}$$

Example 3: In a stream running at 2 km/h, a motorboat goes 6 km upstream and back again to the starting point in 33 minutes. Find the speed of the motorboat in still water.

Sol. Let the speed of the motorboat in still water be x km/h.

Then, Speed downstream = $(x + 2)$ km/h

Speed upstream = $(x – 2)$ km/h

$$\therefore \quad \frac{6}{x+2}+\frac{6}{x-2} = \frac{33}{60}=\frac{11}{20}$$

$$\Rightarrow \quad 11x^2 - 240x - 44 = 0$$

$$\Rightarrow \quad 11x^2 - 242x + 2x - 44 = 0$$

$$\Rightarrow \quad (x-22)(11x+2) = 0$$

$$\Rightarrow \quad x = 22$$

Hence, speed of motorboat in still water is 22 km/h.

Pipes and Cistern

Inlet: A pipe connected with a tank or a **cistern** or a **reservoir**, that fills it, is known as inlet.

Outlet: A pipe connected with a tank or cistern or a reservoir, emptying it, is called an outlet.

Key Words

Cistern: A tank for storing water, especially one supplying taps.

Reservoir: A large natural or artificial lake used as a source of supply water.

Scan to know more about this topic

Pipes And Cisterns

- If a pipe can fill a tank in x hours, then part of tank filled in 1 hour $= \frac{1}{x}$
- If a pipe can empty a full tank in y hours, then part of tank emptied in 1 hour $= \frac{1}{y}$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $y > x$), then on opening both pipes, the net part filled in 1 hour $= \left(\frac{1}{x}-\frac{1}{y}\right)$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $x > y$), then on opening both pipes, the net part emptied in 1 hour $= \left(\frac{1}{y}-\frac{1}{x}\right)$

Example 4: A cistern can be filled by a tap in 4 hours while it can be emptied by another tap in 9 hours. If both the taps are opened simultaneously then after how much time will the cistern get filled ?

Sol. Part filled by first cistern in 1 hour $= \frac{1}{4}$

Part emptied by second cistern 1 hour $= \frac{1}{9}$

So, net part filled in 1 hour $= \left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36}$

$\therefore$ The cistern will be filled in $\frac{36}{5}$ hours i.e., 7.2 hours

- **Races and Games**
  - **Races:** A contest of speed in running, riding, driving, sailing or rowing is called a race.
  - **Race Course:** The ground or path on which contests are made is called race course.
  - **Starting Point:** The point from which a race begins is known as a starting point.
  - **Winning Point or Goal:** The point set to bound a race is called a winning point or a goal.
  - **Winner:** The person who first reaches the winning point is called winner.
  - **Dead Heat Race:** If all the persons contesting a race reach the goal exactly at the same time, then the race is said to be a dead heat race.
  - **Start:** Suppose A and B are two contestants in a race. If before the start of the race, A is at the starting point and B is ahead of A by 15 metres, then we say that 'A gives B, a start of 15 metres'.

    So, in case to cover a race of 100 metres, A will have to cover 100 metres while B will have to cover only (100 – 15) metres = 85 metres.

    In other words, we can say that to cover a race of 100 metres 'A can give B 15 metres' or 'A can give B a start of 15 metres' or 'A beats B by 15 metres'.
  - **Games:** A game of 100, means that the person among the contestants who scores 100 points first is the winner.

    If A scores 100 points while B scores only 80 points, then we say that 'A can give B 20 points'.

    Example 5: Aman can run 1 km in 3 min 10 s and Binay can cover the same distance in 3 min 20 s. By what distance can A beat B ?

    Sol. Clearly, Aman beats Binay by 10 s.

    Distance covered by Binay in 10 s

    $$= \left(\frac{1000}{200} \times 10\right) \text{ m} = 50 \text{ m}$$

    [$\because$ 1 km = 1000 m and 3 min 20 s = 200 s]

    Thus, A beats B by 50 metres.

    Example 6: In a game of 80 points, Ajay can give Beena 5 points and Seema 15 points. Then how many points Beena can give Seema in a game of 60 ?

    Sol. Ajay : Beena = 80 : 75, Ajay : Seema = 80 : 65

    $$\frac{\text{Beena}}{\text{Seema}} = \left(\frac{\text{Beena}}{\text{Ajay}} \times \frac{\text{Ajay}}{\text{Seema}}\right)$$

    $$= \left(\frac{75}{80} \times \frac{80}{65}\right)$$

    $$= \frac{15}{13} = \frac{60}{52} = 60 : 52 = 15 : 13$$

    $\therefore$ In a game of 15, Beena can give Seema 13 points.
- **Inequalities:** Two real numbers or two algebraic expression related by the symbol '$<$', '$>$', '$\leq$', or '$\geq$', form an inequality.

Key Facts

- Inequalities are arguably used more than often in 'real life' than equalities. Businesses use inequalities to control inventory, plan production lines, produce pricing models and for shipping warehousing goods and materials.
- Roads have speed limits, certain movies have age restrictions and the time it takes you to walk to the pack are all examples of inequalities. Inequalities do not represent an exact amount, but instead represent a limit of what is allowed or what is possible.
- Alligation is one of the simple and illustrative methods in pharmacy to calculate the proportion of any two solutions to be mixed to prepare final solution of required concentration.

Types of inequalities

- **Numerical inequality:** An inequality which does involve any variable is called a numerical inequality.

  e.g.: $4 > 2, 8 < 21$
- **Literal inequality:** An inequality which have variable is called a literal inequality.

  e.g.: $x < 7, y \geq 11, x \leq 4$
- **Strick inequality:** An inequality which have only $<$ or $>$ is called strick inequality.

  e.g.: $3x + y < 0, x > 7$
- **Slack inequality:** An inequality which have only $\geq$ or $\leq$ is called slack inequality.

  e.g.: $3x + 2y \leq 0, y \leq 4$
- **Linear inequality:** An inequality is said to be linear, if each variable occurs in first degree only and there is no term involving the product of the variables.

  e.g.: $ax + b \leq 0, ax + by + c > 0, ax \leq 4$

  An inequality in one variable in which degree of variable is 2, is called quadratic inequality in one variable.

  e.g.: $ax^2 + bx + c \geq 0, 3x^2 + 2x + 4 \leq 0$

  ❖ **Linear inequality in one variable**

  A linear inequality which has only one variable, is called linear inequality in one variable.

e.g.**:** $ax + b < 0$, where $a \neq 0$

$4c + 7 \geq 0$

Replacement Set: The set from which values of the variables (involved in the inequality) are chosen is called the 'replacement set'.

Solution Set: A solution to an inequality is a number (chosen from replacement set) which, when substituted for the variable, make the inequality true. The set of all solutions of an inequality is called the 'solution set' of the inequality.

e.g.: Consider the inequality $x < 5$

	Replacement Set	Solution Set
(i)	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}	{1, 2, 3, 4}
(ii)	{– 1, 0, 1, 2, 3, 5, 8}	{– 1, 0, 1, 2, 3}
(iii)	{– 5, 1, 9, 10}	{– 5, 1}
(iv)	{6, 7, 8, 9, 10}	ϕ

Fundamental Facts

- Solution set always depends upon replacement set.
- If the replacement set is not given, then we shall take it as R(set of real numbers).

Rules for solving inequalities in one variable: The rules for solving inequalities are similar to those for solving equations except for multiplying or dividing by a negative number.

(i) If $a \geq b$, then $a \pm k \geq b \pm k$, where k is any real number.

(ii) If $a \geq b$, then ka is not always $\geq kb$

(iii) If $k > 0$ (i.e., positive), then

$a \geq b \Rightarrow ka \geq kb$

(iv) If $k < 0$ (i.e., negative), then

$a \geq b \Rightarrow ka \leq kb$

Thus, always reverse the sign of inequality while multiplying or dividing both sides of an inequality by a negative number.

Procedure to solve a linear inequality in one variable:

(i) Simplify both sides by collecting like terms.

(ii) Remove fraction (or decimals) by multiplying both sides by appropriate factor (L.C.M. of denominator or a power of 10 in case of decimals)

(iii) Isolate the variables on one side and all constant on the other side.

(iv) Make the coefficient of the variable equal to 1.

(v) Choose the solution set from the replacement set.

Example 7: Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Sol. We have, $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$

or, $\frac{3x-4}{2} \geq \frac{x-3}{4}$

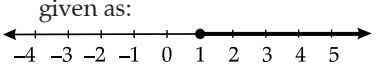
or, $2(3x-4) \geq (x-3)$

or, $6x - 8 \geq x - 3$

or, $5x \geq 5$

or, $x \geq 1$

The graphical representation of solution is given as:



❖ **Linear inequality in two variable**

The inequality of form $ax + by + c > 0$, $ax + by + c = 0$, or $ax + by + c < 0$ etc. where $a \neq 0$, $b \neq 0$ is called a linear equality in two variables x and y.

Graphical solution of Linear Inequalities in Two Variables

(i) The graph of the inequality $ax + by > c$ is one of the half planes and is called the solution region.

(ii) When the inequality involves the sign $\leq$ or $\geq$ then the points on the line are included in the solution region but if it has the sign $<$ or $>$ then the points on the line are not included in the solution region and it has to be drawn as a dotted line.

(iii) The common values of the variable form the required solution of the given system of linear inequalities in one variable.

(iv) The common part of coordinate plane is the required solution of the system of linear inequations in two variables when solved by graphical method.

Example 8: Solve the following system of inequalities graphically

Sol. $3x + 4y \leq 60$, $x + 3y \leq 30$, $x \geq 0$, $y \geq 0$.

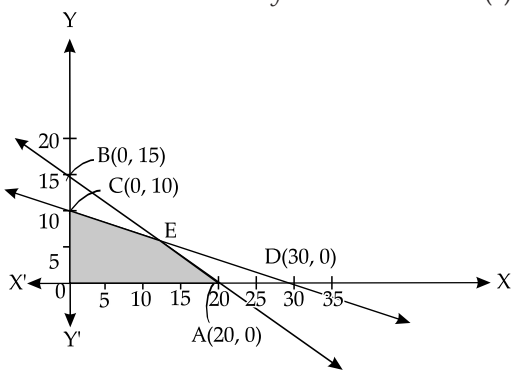
The inequalities are:

$3x + 4y \leq 60$ ...(1)

$x + 3y \leq 30$ ...(2)

$x \geq 0$ ...(3)

$y \geq 0$ ...(4)



We first draw the graph of line $3x + 4y = 60$, $x + 3y = 30$, $x = 0$ and $y = 0$

(a) The line $3x + 4y = 60$ passes through the points (20, 0) and (0, 15) and represented by the line AB. Putting $x = 0, y = 0$ in (1), we have $0 < 60$, which is true.

So, origin lies in this region.

$\therefore$ $3x + 4y \leq 60$ represents the region below the line AB and all the points on it.

(b) Further, $x + 3y = 30$ passes through (0, 10) and (30, 0). CD represents this line.

Put $x = 0, y = 0$ in (2), we have $0 < 30$, which is true. So, origin lies in this region $x + 3x \leq 30$. The inequality represent the region below the line CD and the line itself.

The solution region is OAEC as shown in figure.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. If a man rows 32 km downstream and 14 km upstream in 6 hours each, then the speed of the stream is: [CBSE Board 2021]

(A) 2 km/h **(B)** 1.5 km/h

(C) 2.5 km/h **(D)** 2.25 km/h

Ans. Option (B) is correct.

Explanation: Upstream speed $= \frac{14}{6}$ km/h

Downstream speed $= \frac{32}{6}$ km/h

$$\text{Speed of the stream} = \frac{\frac{32}{6} - \frac{14}{6}}{2} = \frac{\frac{18}{6}}{2}$$

$$= \frac{18}{6 \times 2} = \frac{3}{2}$$

$= 1.5$ km/h

Q. 2. In a 2 km race, P can give Q a start of 200 m and R a start of 560 m. Then in the same race, Q can give R a start of: [CBSE Board 2021]

(A) 360 m **(B)** 380 m

(C) 400 m **(D)** 430 m

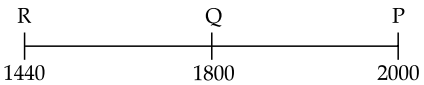
Ans. Option (C) is correct.

Explanation: Let P is complete the race 2000 m

and Q is at 1800 m

and R is at 1440 m

$\Rightarrow$ In a race of 1800 m



Q gives R a start at $= 1800 - 1440$

$= 360$

$\Rightarrow$ 1 m race $= \frac{360}{1800}$

$\Rightarrow$ 2000 m race $= \frac{360}{1800} \times 2000$

$= 400$ m

Q. 3. Pipe A and B can fill a tank in 5 hours and 6 hours respectively. Pipe C can empty it in 12 hours. If all the three pipes are opened together, then the time taken to fill the tank is: [CBSE Board 2021]

(A) 2 hours **(B)** $2\frac{3}{4}$ hours

(C) 3 hours **(D)** $3\frac{9}{17}$ hours

Ans. Option (D) is correct.

Explanation: Here, LCM of 5, 6 and 12 is 60.

i.e., LCM(5, 6, 12) = 60

Suppose capacity of the tank is 60 *l*.

Then, quantity filled by pipe A in 1 hr $= \frac{60}{5} = 12\,l$

Quantity filled by pipe B in 1 hr $= \frac{60}{6} = 10\,l$

Quantity emptied by pipe C in 1 hr $= \frac{60}{12} = 5\,l$

So,quantity filled by all three pipes in 1 hr

$= (12 + 10 - 5)l$

$= 17\,l$

Required time $= \frac{60}{17} = 3\frac{9}{17}$ hours

Q. 4. The solution of $\frac{x-3}{x+5} > 0, x \neq -5, x \in \mathbb{R}$ is: [CBSE Board 2021]

(A) $x > 3$ **(B)** $x < -5$

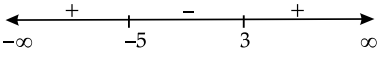
(C) $x < -5$ or $x > 3$ **(D)** no solution

Ans. Option (C) is correct.

Explanation: Given, $\frac{x-3}{x+5} > 0, x \neq -5$

Multiply by $(x + 5)^2$

$(x + 5)(x - 3) > 0$



$\therefore$ Solution $(-\infty, -5) \cup (3, \infty)$

or $x < -5$

$x > 3$

Q. 5. 20 litres of a mixture contains milk and water in the ratio 3 : 1. The amount of milk, in litres, to be added to the mixture so as to have milk and water in the ratio 4 : 1, is: [CBSE Board 2021]

(A) 7 **(B)** 4
(C) 5 **(D)** 6

Ans. Option (C) is correct.

Explanation: In 20 l of mixture,

$$\text{Quantity of milk} = \frac{3}{4}\times 20 = 15\ l$$

$$\text{Quantity of water} = \frac{1}{4}\times 20 = 5\ l$$

Let the quantity of milk added be x l.

According to be question,

$$\frac{15+x}{5} = \frac{4}{1}$$

$$\Rightarrow \quad 15 + x = 20$$

$$\Rightarrow \quad x = 5\ l$$

Q. 6. Pipe A can fill a tank 6 times faster than a pipe B. If B can fill a tank in 21 minutes, then the time taken by both the pipes together to fill the tank is: [CBSE Board 2021]

(A) 3 minutes **(B)** $4\frac{1}{2}$ minutes
(C) 7 minutes **(D)** 9 minutes

Ans. Option (A) is correct.

Explanation: Given, slower pipe filled the tank in 21 minutes.

Then faster pipe filled the tank in $\frac{21}{6}$ minutes

Thus, tank filled by both the pipes in 1 minute

$$= \frac{1}{21}+\frac{6}{21} = \frac{7}{21} = \frac{1}{3}$$

Hence, time taken by both pipes together to fill the tank is 3 minutes.

Q. 7. The solution set of the inequation $|x + 2| \le 5$ is: [CBSE Board 2021]

(A) $(-7, 5)$ **(B)** $[-7, 3]$
(C) $[-5, 5]$ **(D)** $(-7, 3)$

Ans. Option (B) is correct.

Explanation: $|x + 2| \le 5$

$$\Rightarrow \quad -5 \le x + 2 \le 5$$

$$[\because\ |x| \le a, \text{ then } -a \le x \le a]$$

$$\Rightarrow \quad -5 - 2 \le x \le 5 - 2$$

$$\Rightarrow \quad -7 \le x \le 3$$

$$\Rightarrow \quad x \in [-7, 3]$$

Q. 8. For two distinct positive numbers x and y

(A) $x + y > 2\sqrt{xy}$ **(B)** $\frac{x+y}{2} > xy$

(C) $\sqrt{xy} > \frac{x+y}{2}$ **(D)** $\frac{2xy}{x+y} > \sqrt{xy}$

Ans. Option (A) is correct.

Explanation: For distinct $x, y > 0$;

A.M. > G.M.

$$\Rightarrow \quad \frac{x+y}{2} > \sqrt{xy}$$

$$\Rightarrow \quad x + y > 2\sqrt{xy}$$

Q 9. A person can row in still water at the rate of 8 km/h. If it takes him thrice as long to row upstream as to row downstream then the speed of the stream is:

(A) 2 km/h **(B)** 3 km/h
(C) 4 km/h **(D)** 6 km/h

Ans. Option (C) is correct.

Explanation: Let x be the speed of the stream

$$\frac{V_d}{t} = 3\frac{V_u}{t}$$

$$\therefore \quad 8 + x = 3(8 - x)$$

$$\Rightarrow \quad 4x = 16$$

$$\Rightarrow \quad x = 4 \text{ km/h}$$

Q. 10. Two athletes Vijay and Samuel finish 100 meters race in 12 seconds and 16 seconds respectively. By how many meters does Vijay defeat Samuel?

(A) 10.2 meters **(B)** 15 meters
(C) 25 meters **(D)** 33.3 meters

Ans. Option (C) is correct.

Explanation: Since Vijay is faster by 4 seconds.

$$\therefore \text{ He beats Samuel by} = \frac{100}{16}\times 4$$

$$= 25 \text{ meters}$$

Q. 11. A retailer buys 250 kg of rice, a part of which he sells at 10% profit and the remaining at 5% loss. If the net profit made by the retailer in the whole transaction is 7%, then the quantity of rice sold at 10% profit is

(A) 200 kg **(B)** 150 kg
(C) 100 kg **(D)** 50 kg

Ans. Option (A) is correct.

Explanation: $C = -5\ \%\ d = 10\%, m = 7\%$

$$(d - m) : (m - c) = 1 : 4$$

Quantity sold at 10 % profit

$$= \frac{4}{5}\times 250$$

$$= 200 \text{ kg}$$

Q 12. Two pipes A and B can fill a cistern in 8 hours and 12 hours respectively. The pipes when opened simultaneously takes 12 minutes more to fill the cistern due to leakage. Once the cistern is full, it will get emptied due to leakage in

(A) 5 hours **(B)** 20 hours
(C) 60 hours **(D)** 120 hours

Ans. Option (D) is correct.

Explanation: Portion of cistern filled by both pipes in 1 hour

$$= \frac{1}{8}+\frac{1}{12}$$

$$= \frac{5}{24}.$$

Time taken by both pipes to fill the cistern

$$= 4 \text{ h } 48 \text{ minutes}$$

Time taken to fill tank due to leakage

$$= 5 \text{ h}$$

Cistern empty by leakage in 1 h

$$= \frac{5}{24}-\frac{1}{5}$$

$$= \frac{1}{120}$$

Time taken by leakage to empty the cistern
$= 120$ h

Q. 13. If B > A, then which expression will have the highest value, given that A and B are positive integers. [CBSE Text Book 2022]

(A) A – B **(B)** A × B
(C) A + B **(D)** Can't say

Ans. Option (D) is correct.

Explanation: As, $B > A$
$\Rightarrow A < B$
$\Rightarrow A - B < 0$
$A + B > 0$ and $AB > 0$
If $A = 1, B = 4$ then, $AB < A + B$
If $A = 2, B = 4$ then, $AB > A + B$
Thus, we can't say which one of A + B and AB has higher value.

Q. 14. If $0 < x < 1$, then which of the following is greatest [CBSE Text Book 2022]

(A) x **(B)** x^2
(C) $\frac{1}{x}$ **(D)** $\frac{1}{x^2}$

Ans. Option (D) is correct.

Explanation: $0 < x < 1$
between 0 and 1, $x^2 < x < 1$

$\frac{1}{x^2} > \frac{1}{x} > 1$, also, $1 > x > 0$ (given)

Therefore, $\frac{1}{x^2} > \frac{1}{x} > 1 > x > 0$

Thus, $\frac{1}{x^2}$ is greatest.

Q. 15. If $p > q$ and $r < 0$, then which of the following is true ?

(A) $pr < qr$ **(B)** $p - r < q - r$
(C) $p + r < q + r$ **(D)** None of these

Ans. Option (A) is correct.

Explanation: Since, given $p > q$ and r < 0
then $pr < qr$
(inequality sign is reversed when multiplied by a negative integer)

Q. 16. If a and b are positive integers and $\frac{a-b}{6.25} = \frac{8}{2.5}$, then

(A) $b > a$ **(B)** $b < a$
(C) $b = a$ **(D)** $b \geq a$

Ans. Option (B) is correct.

Explanation: Given,

$$\frac{a-b}{6.25} = \frac{8}{2.5}$$

$$a - b = \frac{6.25 \times 8}{2.5}$$

$\Rightarrow a - b = 20$
$\Rightarrow a = 20 + b$
Hence, $a > b$ or $b < a$

Q. 17. If $x < 5$, then

(A) $-x < -5$ **(B)** $-x \leq -5$
(C) $-x > -5$ **(D)** $-x \leq -5$

Ans. Option (C) is correct.

Explanation: Since $x < 5$, therefore $-x > -5$.

Q. 18. If $-3x + 17 < -13$, then

(A) $x \in (10, \infty)$ **(B)** $x \in [10, \infty)$
(C) $x \in (-\infty, 10)$ **(D)** $x \in [-10, 10)$

Ans. Option (A) is correct.

Explanation: Given,

$-3x + 17 < -13$
$\Rightarrow -3x < -17 - 13$
$\Rightarrow -3x < -30$
$\Rightarrow 3x > 30$
$\Rightarrow x > 10$
$\Rightarrow x \in (10, \infty)$

Q. 19. If x is a real number and $|x| < 3$, then

(A) $x \geq 3$ **(B)** $-3 < x < 3$
(C) $x \leq -3$ **(D)** $-3 \leq x \leq 3$

Ans. Option (B) is correct.

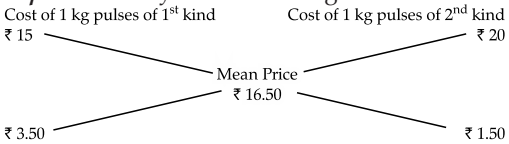
Explanation: Given, $|x| < 3 \Rightarrow -3 < x < 3$
[Since, $|x| < a \Rightarrow -a < x < a$]

Q. 20. In what ratio must a grocer mix two varieties of pulses costing ₹ 15 and ₹ 20 per kg respectively so as to get a mixture worth ₹ 16.50 kg?

(A) 3 : 7 **(B)** 5 : 7
(C) 7 : 3 **(D)** 7 : 5

Ans. Option (C) is correct.

Explanation: By the rule of alligation:



$\therefore$ Required rate = 3.50 : 1.50 = 7 : 3.

Q. 21. A dishonest milkman professes to sell his milk at cost price but he mixes it with water and thereby gains 25%. The percentage of water in the mixture is:

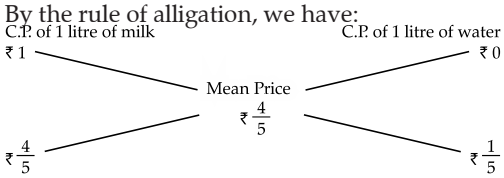
(A) 4% **(B)** 6%
(C) 20% **(D)** 25%

Ans. Option (C) is correct.

Explanation: Let C.P. of 1 litre milk be ₹ 1
Then, S.P. of 1 litre of mixture = ₹ 1,
Gain = 25%

C.P. of 1 litre mixture $= ₹\left(\frac{100}{125} \times 1\right) = \frac{4}{5}$

By the rule of alligation, we have:



$\therefore$ Ratio of milk to water $= \frac{4}{5} : \frac{1}{5} = 4 : 1$

Hence, percentage of water in the mixture

$$= \left(\frac{1}{5} \times 100\right)\% = 20\%.$$

Q. 22. There is a leak at the bottom of a cistern. Due to this it takes 8 hours to fill the cistern. Had there not been a leak, it would take one hour less to fill the cistern. How much time does it take for the leak to completely empty the cistern?

(A) 48 hours **(B)** $55\frac{1}{3}$ hours

(C) 56 hours **(D)** 15 hours

Ans. Option (C) is correct.

Explanation: Normally, cistern gets filled in one hour less than 8 hours that means 7 hours.

So in 1 hour it fills = $\frac{1}{7}$ part

Due to leak, it takes 8 hours. So in 1 hour it actually fills = $\frac{1}{8}$ part

$\therefore$ Water removed by the leak in 1 hour

$$= \frac{1}{7} - \frac{1}{8} = \frac{1}{56}$$

$\therefore$ Leak empties the tank in 56 hours.

Q. 23. A team played 40 games in a season and won in 24 of them. What percent of games played did the team win ?

(A) 70% **(B)** 40%

(C) 60% **(D)** 35%

Ans. Option (C) is correct.

Explanation: Number of games played = 40

Number of games won = 24

Percentage of games played

$$= \left(\frac{24}{40} \times 100\right)$$

$$= 60\%$$

Q. 24. In a one-kilometre race, A, B and C are three participants. A can give B start of 50 m and C a start of 69 m. The start, which B can allow C, is:

(A) 17 m **(B)** 18 m

(C) 19 m **(D)** 20 m

Ans. Option (D) is correct.

Explanation: $A : B : C$

$$= 1000 : (1000 - 50) : (1000 - 69)$$

$$= 1000 : 950 : 931$$

In a 950 m race, B can give C a start of (950 – 931) m

$$= 19 \text{ m}$$

In a 1000 m race. B can give C a start of $\left(\frac{19}{950} \times 1000\right)$

$$= 20 \text{ m}$$

Q. 25. A and B can cover a 200 m race in 22 seconds and 25 seconds respectively. When A finished the race, the B is at what distance from the finishing line ?

(A) 24 m **(B)** 30 m

(C) 48 m **(D)** 54 m

Ans. Option (A) is correct.

Explanation: Distance covered by B in 25 s

$$= 200 \text{ m}$$

Distance covered by B in 22 s

$$= \left(\frac{200}{25} \times 22\right) \text{m}$$

$$= 176 \text{ m}$$

$\therefore$ B was at a distance of (200 – 176) m = 24 m from the finishing line.

Q. 26. Speed of river is 6 km/hr. Speed of a motorboat in still water is 30 km/hr. How much distance can it cover downstream in 24 minutes?

(A) 9.8 km **(B)** 12 km

(C) 12.8 km **(D)** 14.4 km

Ans. Option (D) is correct.

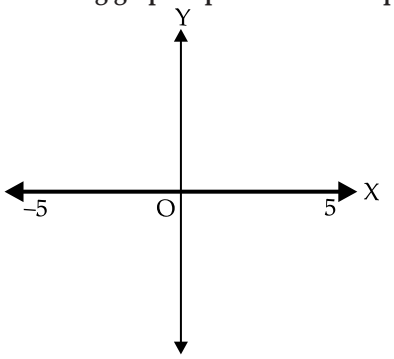
Explanation: Distance = Speed × Time

$X + Y = 30 + 6 = 36$ km/hr

$$24 \text{ minutes} = \frac{24}{60} \text{ hours}$$

$$\text{Distance} = 36 \times \frac{24}{60} = 14.4 \text{ km}$$

Q. 27. The following graph represents the inequality.



(A) $x > -5$ **(B)** $x < 5$

(C) $|x| < 5$ **(D)** $|x| < -5$

Ans. Option (C) is correct.

Explanation: The given graph represents $x > -5$ and $x < 5$ combining the two inequalities $|x| < 5$.

Q. 28. In what ratio water is mixed with milk to gain $16\frac{2}{3}\%$ on selling the mixture at cost price?

(A) 1 : 6 **(B)** 6 : 1

(C) 2 : 3 **(D)** 3 : 2

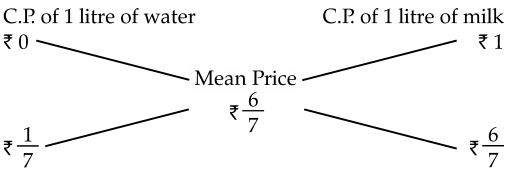
Ans. Option (A) is correct.

Explanation: Let C.P. of 1 litre milk be ₹ 1.

S.P. of 1 litre of mixture = ₹ 1, Gain = $\frac{50}{3}\%$

$\therefore$ C.P. of 1 litre of mixture $= \left(100 \times \frac{3}{350} \times 1\right)$

$$= \frac{6}{7}$$



By the rule of alligation, we have

$\therefore$ Ratio of water and milk = $\frac{1}{7} : \frac{6}{7} = 1 : 6$.

Q. 29. In a 1000 m race, A can beat B by 100 m. In a race of 400 m, B can beat C by 40 m. In a race of 500 m by what margin will A beat C.

(A) 95 m **(B)** 85 m

(C) 75 m **(D)** 55 m

Ans. Option (A) is correct.

Explanation: $A : B = 1000 : 900$,
$B : C = 400 : 360$
$= 100 : 90$
$= 900 : 810$

$\Rightarrow A : B : C = 1000 : 900 : 810$
$\Rightarrow A : C = 1000 : 810$
$\Rightarrow A : C = 500 : 405$
$\Rightarrow$ In a 500 m race,
A beats C by (500 – 405) m = 95

SUBJECTIVE TYPE QUESTIONS

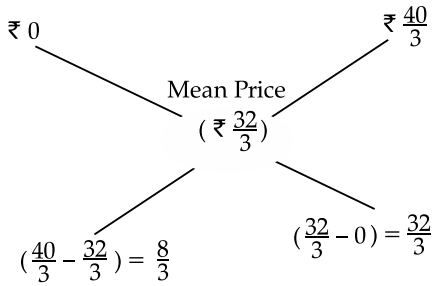
Very Short Answer Type Questions (1 mark each)

Q. 1. How much water must be added to 60 litres of milk at $1\frac{1}{2}$ litres for ₹ 20 so as to have a mixture worth ₹ $10\frac{2}{3}$ a litre ? U

Sol. C.P. of 1 litre of milk

$$= ₹\left(20 \times \frac{2}{3}\right)$$
$$= ₹\frac{40}{3}$$

C.P. of 1 litre of water — C.P. of 1 litre of milk



$\therefore$ Ratio of water and milk $= \frac{8}{3} : \frac{32}{3} = 8 : 32 = 1 : 4$

$\therefore$ Quantity of water to be added to 60 litres of milk

$$= \left(\frac{1}{4} \times 60\right) \text{ litres } = 15 \text{ litres.} \quad \mathbf{1}$$

Q. 2. A man can row upstream at 7 km/h and downstream at 10 km/h. Find man's rate in still water and rate of current. U

Q. 3. An electric pump can fill a tank in 3 hours. Because of a leak in the tank it took $3\frac{1}{2}$ hours to fill the tank. If the tank is full, how much time will the leak take to empty it ? U

Sol. Work done by the leak in 1 hour $= \left[\frac{1}{3} - \frac{1}{\left(\frac{7}{2}\right)}\right]$

$$= \left(\frac{1}{3} - \frac{2}{7}\right) = \frac{1}{21}$$

$\therefore$ The leak will empty the tank in 21 hours. **1**

Q. 4. In a game of 80 points, A can give 5 points to B and 15 points to C. Calculate the number of points that B can give to C in a game of 60 points ? U

Q. 5. Solve for x(x is real number), $4x + 3 < 5x + 7$. U

Sol. Given, $4x + 3 < 5x + 7$

Transposing $5x$ to LHS and 3 to RHS, we get

$$4x - 5x < 7 - 3$$
or, $$-x < 4$$
or, $$x > -4$$

$\therefore$ When x is real, the solution is $(-4, \infty)$. **1**

Q. 6. Let x and b are real numbers. If $b > 0$ and $|x| > b$, then find interval for x.

Sol. Given, $|x| > b, b > 0$
$\Rightarrow x < -b$ or $x > b$
$\Rightarrow x \in (-\infty, -b) \cup (b, -\infty)$ **1**

Q. 7. Tea worth ₹ 126 per kg and ₹ 135 per kg are mixed with a third variety in the ratio 1 : 1 : 2. If the mixture is worth ₹ 153 per kg, then find the price of the third variety in per kg.

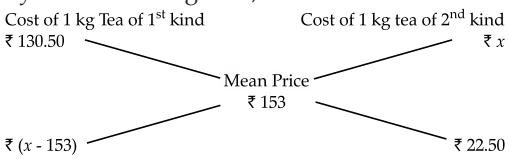
Sol. Since first and second varieties are mixed in equal proportions.

So, their average price $= ₹\left(\frac{126 + 135}{2}\right)$

$= ₹ 130.50$

So, the mixture is formed by mixing two varieties, one at ₹ 130.50 per kg and the other at say ₹ x per kg in the ratio 2 : 2, i.e., 1 : 1. We have to find x.

By the rule of alligation, we have:



$\therefore \quad \frac{x - 153}{22.50} = 1$

$\Rightarrow x - 153 = 22.50$
$\Rightarrow x = 175.50 \therefore$ ₹ 175.50/kg **1**

Q. 8. Tap P alone fills a cistern in 2 hours; while tap Q alone fills the same cistern in 3 hours. A new tap R is attached to the bottom of the cistern which can empty the completely filled cistern in 6 hours. Sunny started all three taps together at 9 a.m. When will the tank be full?

Sol. Cistern filled or work done by Tap P in 1 hour $= \frac{1}{2}$

Cistern filled or work done by Tap Q in 1 hour $= \frac{1}{3}$

These questions are for practice and their solutions are available at the end of the chapter

Tank emptied or work done by Tap R in 1 hour

$$= \frac{1}{6}$$

Tank filed or work done by all three pipes in 1 hour

$$= \frac{1}{2}+\frac{1}{3}-\frac{1}{6} = \frac{2}{3}$$

So Tank gets completely filled in $\frac{3}{2}$ hours

$$= 1.5 \text{ hours} = 1 \text{ hr } 30 \text{ min} \quad \mathbf{1}$$

So time will be 9.00 a.m. + 1 hour 30 min = 10.30 a.m.

Short Answer Type Questions-I (2 or 3 marks each)

Q. 1. How many kg of wheat costing ₹ 8 per kg must be mixed with 36 kg wheat costing ₹ 5.40 per kg so that 20% gain may be obtained by selling the mixture at ₹ 7.20 per kg ? U

Q. 2. There is a road beside a river. Two friends started from a place A, moved to a temple situated at another place B and then returned to A again. One of them moves on a cycle at a speed of 12 km/h, while the other sails on a boat at a speed of 10 km/h. If the river flows at the speed of 4 km/h, which of the two friends will return to place A first ? A I

Sol. Here, cyclist moves both ways at a speed of 12 km/h.
So, average speed of the cyclist = 12 km/h ½
The boat sailor moves at rate of (10 ± 4) i.e., 6 km/h
So, average speed of the boat sailor

$$= \left(\frac{2\times 14\times 6}{14+6}\right) \text{ km/h}$$

$$= \frac{42}{5} \text{ km/h} = 8.4 \text{ km/h} \quad \mathbf{1}$$

Since, the average speed of the cyclist is greater, he will return to A first. ½

Q. 3. Two pipes A and B can fill a tank in 24 minutes and 32 minutes respectively. If both the pipes are opened simultaneously, after how much time B should be closed so that the tank is full in 18 minute ? A I

Sol. Let B be closed after x minute.
Then, part filled by (A + B) in x min + part filled by A in $(18 - x)$ min = 1

$$\therefore\ x\left(\frac{1}{24}+\frac{1}{32}\right)+(18-x)\times\frac{1}{24} = 1 \quad \mathbf{1}$$

$$\Rightarrow \quad \frac{7x}{96}+\frac{18-x}{24} = 1$$

$$\Rightarrow \quad 7x + 4(18 - x) = 96$$

$$\Rightarrow \quad x = 8$$

Hence, B must be closed after 8 minutes. **1**

Q. 4. In a 100 m race, A runs at 8 km/h. If A gives B a start of 4 m and still beats him by 15 seconds, what is the speed of B ? U

Sol. Time taken by A to cover 100 m

$$= \left(\frac{60\times 60}{8000}\times 100\right) \text{ s}$$

$$= 45 \text{ s}$$

$\therefore$ B covers (100 – 4) m = 96 m in (45 + 15) s
= 60 s **1**

$$\therefore \text{ B's speed} = \left(\frac{96\times 60\times 60}{60\times 1000}\right) \text{ km/h}$$

$$= 5.76 \text{ km/h} \quad \mathbf{1}$$

Q. 5. If the boat takes 12 hours to row 48 km upstream and 8 hours to row the same distance downstream, then find the boat's speed in still water and speed of river.

Sol. Let speed of a boat is X km/hour and speed of river is Y km/hour.

$$\text{Downstream speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

$$= \frac{48}{8} = 6 \text{ km/h}$$

$$\text{Upstream speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

$$= \frac{48}{12} = 4 \text{ km/h}$$

$$X + Y = 6 \text{ km/h}$$

and $X - Y = 4$ km/h **1**

Adding them we get,

$$X + Y + X - Y = 10 \text{ km/h}$$

$$\therefore \quad X = 5 \text{ km/h} \quad ½$$

5 km/h = Speed of Boat

$$Y = 6 - 5 = 1 \text{ km/h}$$

1 km/h = Speed of river ½

Q. 6. Draw the graph of the solution set of $x < y$. A

Q. 7. Solve the linear inequality $|x - 2| \geq 6$ U

Sol. We know that, $|x - a| \geq k$

$$\Rightarrow \quad -6 \leq x - 2 \leq 6 \quad \mathbf{1}$$

$$\Rightarrow \quad -4 \leq x < 8$$

$$\Rightarrow \quad x \in (-\infty, -4] \cup [8, \infty)$$

Hence, the solution set of the inequation is $(-\infty, 4] \cup [8, \infty)$. **1**

Commonly Made Error

- In absolute inequality $|x - a| \geq k$, mostly students expand either $x \leq a - k$ or $x \geq a + k$. But actually we have to expand both inequality at a time.

Answering Tip

- While expanding the absolute type of inequality carefully use the concept and practice more problems related to this concept.

These questions are for practice and their solutions are available at the end of the chapter

Q. 8. Guddi speed of swimming in still water to that of river water is 7 : 1. She swims 4.2 km up the river in just 14 min. How much time will Guddi take to swim 18.4 km down the river?

Sol. $X : Y = 7 : 1$

$\therefore \quad X = 7Y$

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$\therefore$ Upstream speed $= \frac{4.2 \text{ km}}{14 \text{ min}}$ ½

$= 0.3$ km/min

Upstream speed $= X - Y$

$= 7Y - Y = 0.3$ km/min

$\therefore \quad Y = 0.05$ km/min ½

Downstream speed $= X + Y$

$= (7Y + Y) = 8Y$

$= 8 \times 0.05$

$= 0.4$ km/min ½

$$\text{Downstream time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{18.4 \text{ km}}{0.4 \text{ km/min}}$$

$$\text{Downstream time} = \frac{184}{4}$$

$= 46$ minutes ½

Q. 9. Find the ratio of swimming speed of Raj in still water to speed of river, if ratio of time taken to go 10 km upstream to time taken to go 10 km downstream is 11 : 5?

Ans. $$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\frac{\text{Upstream time}}{\text{Downstream time}} = \frac{11}{5}$$ ½

$$\therefore \frac{\text{Upstream Speed}}{\text{Downstream Speed}} = \frac{5}{11}$$ ½

$$\therefore \quad \frac{X - Y}{X + Y} = \frac{5}{11}$$

$\therefore \quad 11X - 11Y = 5X + 5Y$

$\therefore \quad 6X = 16Y$

$\therefore \quad \frac{X}{Y} = \frac{8}{3}$

$$= \frac{\text{Swimming speed of Raj}}{\text{Speed of river}}$$

$= 8 : 3$ **1**

Q. 10. A cistern has two taps attached to it. Tap B can empty the cistern in 45 minutes. But Tap A can fill the cistern in just 30 minutes. Rohit started both taps unknowingly but realized his mistake after 30 minutes. He immediately closed Tap B. Now, in how many minutes the cistern will be filled ?

Sol. In 1 minute, cistern filled by Tap $A = \frac{1}{30}$

In 1 minute, cistern emptied by Tap $B = \frac{1}{45}$

In 1 minute with Taps A and B, cistern filled

$$= \frac{1}{30} - \frac{1}{45} = \frac{1}{90}$$

In 30 minutes, $30 \times \frac{1}{90} = \frac{1}{3}$ amount of cistern is filled

$$\text{Remaining cistern} = 1 - \frac{1}{3} = \frac{2}{3}$$

This is filled by Tap A only

In 1 minute, $\frac{1}{30}$ part of cistern is filled

In x minutes, $\frac{2}{3}$ amount of cistern is filled

$$\therefore \quad x \text{ minutes} = \frac{\frac{2}{3}}{\frac{1}{30}} = 20 \text{ minutes}$$

20 minutes = Time to fill after Tap B is closed

Short Answer Type Questions-II (4 marks each)

Q. 1. (i) The milk and water in two vessels A and B are in the ratio 4 : 3 and 2 : 3 respectively. In what ratio, the liquids in both the vessels be mixed to obtain a new mixture in vessel C containing half milk and half water ?

(ii) A man can row $7\frac{1}{2}$ km/h in still water. If in a river running at 1.5 km in an hour, it takes him 50 minutes to row to place and back, how far off is the place ? U

Q. 2. Two pipes can fill a cistern in 14 hours and 16 hours respectively. The pipes are opened simultaneously and it is found that due to leakage in the bottom it took 32 minutes more to fill the cistern. When the cistern is full, in what time will the leak empty it ?

Sol. Fill the cistern by the two pipes in 1 hour $= \left(\frac{1}{14} + \frac{1}{16}\right)$

$$= \frac{15}{112}$$

$\therefore$ Time taken by these pipes to fill the cistern

$$= \frac{112}{15} \text{ hours}$$

$= 7$ h 28 min

Due to leakage, time taken = 7 h 28 min + 32 min = 8h

$\therefore$ Fill the cistern by (two pipes + leak) in 1 hour $= \frac{1}{8}$

These questions are for practice and their solutions are available at the end of the chapter

empty the cistern by the leak in 1 hour $= \left(\frac{15}{112} - \frac{1}{8}\right)$

$= \frac{1}{112}$

$\therefore$ Leak will empty the full cistern in 112 hours. **4**

Q. 3. (i) A runs $1\frac{3}{4}$ times as fast as B. If A gives B a start of 84 m, how far must winning post be so that A and B might reach it at the same time ?

(ii) 8 litres are drawn from a cask full of wine and is then filled with water. This operation is performed three more times. The ratio of the quantity of wine now left in cask to that of the water is 16 : 65. How much wine did the cask hold originally ?

(iii) Show that graph of the solution $2x - 3 > x - 5$ on number line. U

Sol. (i) Ratio of the rates of A and B $= \frac{7}{4} : 1 = 7 : 4$

So, in a race of 7 m, A gains 3 m over B.

$\therefore$ 3 m are gained by A in a race of 7 m.

$\therefore$ 84 m are gained by A in a race of $\left(\frac{7}{3} \times 84\right)$ m

$= 196$ m

$\therefore$ Winning post must be 196 m away from the starting point. **1½**

(ii) Let the quantity of the wine in the cask originally be x litres.

Then, quantity of wine left in cask after 4 operations

$= \left[x\left(1 - \frac{8}{x}\right)^4\right]$ litres

$\therefore \quad \frac{x\left(1 - \frac{8}{x}\right)^4}{x} = \frac{16}{81}$

$\Rightarrow \quad \left(1 - \frac{8}{x}\right)^4 = \left(\frac{2}{3}\right)^4$

$\Rightarrow \quad \frac{x - 8}{x} = \frac{2}{3}$

$\Rightarrow \quad 3x - 24 = 2x$

$\Rightarrow \quad x = 24$

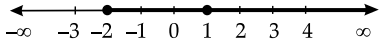
Hence, cask originally hold 24 litres of wine. **1½**

(iii) We have, $2x - 3 > x - 5$

Transposing x to LHS and -3 to RHS, we get

$2x - x > -5 + 3$

$\Rightarrow \quad x > -2 \Rightarrow (-2, \infty)$



1

Commonly Made Error

- Errors are made while making circle on the number line.

Answering Tip

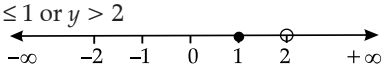
- Take precaution while making circle (○) |dark circle (•) to avoid mistakes.

Q. 4. Solve $\frac{|x| - 1}{|x| - 2} \geq 0$ $x \in \mathbb{R}$, $x \neq \pm 2$

Sol. We have, $\frac{|x| - 1}{|x| - 2} \geq 0$

$\Rightarrow \frac{y - 1}{y - 2} \geq 0$, where $y = |x|$ **1**

$\Rightarrow y \leq 1$ or $y > 2$ **1**



$\Rightarrow |x| \leq 1$ or $|x| > 2$

$\Rightarrow (-1 \leq x \leq 1)$ or $(x < -2$ or $x > 2)$ **1**

$\Rightarrow x \in [-1, 1]$ or $x \in [-\infty, -2) \cup (2, \infty)$

$\Rightarrow x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$ **1**

Hence, solution set of the given inequation is $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$.

Commonly Made Error

- Errors are found while calculating inequations such as incorrect sign-convention, inequalities symbols, open and closed brackets on solution set.

Answering Tip

- Equal numbers may be added to (or subtracted from) both sides of an inequality.

Long Answer Type Questions (5 marks each)

Q. 1. (i) A can contains a mixture of two liquids A and B in the ratio 7 : 5. When 9 litres of mixture are drawn off and can is filled with B, the ratio of A and B becomes 7 : 9. How many litres A was contained by the can initially ?

(ii) A man can row 40 km upstream and 55 km downstream in 13 hours. Also, he can row 30 km upstream and 44 km downstream in 10 hours. Find the speed of the man in still water and speed of the current. U

Sol. (i) Suppose the can initially contains $7x$ and $5x$ litres of mixtures A and B, respectively.

Quantity of A in mixture left $= \left(7x - \frac{7}{12} \times 9\right)$ litres

$$= \left(7x - \frac{21}{4}\right) \text{litres}$$

Quantity of B in mixture left $= \left(5x - \frac{5}{12} \times 9\right)$ litres

$$= \left(5x - \frac{15}{4}\right) \text{litres}$$

$$\therefore \quad \frac{\left(7x - \frac{21}{4}\right)}{\left(5x - \frac{15}{4}\right) + 9} = \frac{7}{9}$$

$$\Rightarrow \quad \frac{28x - 21}{20x + 21} = \frac{7}{9}$$

$$\Rightarrow \quad 252x - 189 = 140x + 147$$

$$\Rightarrow \quad 112x = 336$$

$$\Rightarrow \quad x = 3$$

So, the can contained 21 litres of A. **2½**

(ii) Let rate upstream = x km/h and rate downstream = y km/h

Then, $\frac{40}{x} + \frac{55}{y} = 13$ ...(i)

and $\frac{30}{x} + \frac{44}{y} = 10$ ...(ii)

Multiplying (ii) by 4 and (i) by 3 and subtracting, we get

$$\frac{11}{y} = 1 \text{ or } y = 11$$

Substituting $y = 11$ in (i), we get $x = 5$

$\therefore$ Rate in still water $= \frac{1}{2}(11 + 5)$ km/h $= 8$ km/h

Rate of current $= \frac{1}{2}(11 - 5)$ km/h $= 3$ km/h **2½**

Q. 2. (i) If two pipes function simultaneously, the reservoir will be filled in 12 hours. One pipe fills the reservoir 10 hours faster than the other. How many hours will the second pipe take to fill the reservoir ?

(ii) The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shorter side.

Sol. (i) Let the reservoir be filled by first pipe in x hours. Then, second pipe will fill it in $(x + 10)$ hours.

$$\therefore \quad \frac{1}{x} + \frac{1}{(x+10)} = \frac{1}{12} \quad \textbf{1}$$

$$\Rightarrow \quad x^2 - 14x - 120 = 0$$

$$\Rightarrow \quad (x - 20)(x + 6) = 0$$

$$\Rightarrow \quad x = 20$$

[neglecting the –ve value of x]

So, the second pipe will take $(20 + 10)$ hours, *i.e.*, 30 hours to fill the reservoir. **1**

(ii) Let shorter side be x cm.

Then, according to question,

Longest side $= 3x$ cm and third side $= (3x - 2)$ cm

Also, perimeter of triangle ≥ 61 **1**

i.e., Sum of all sides ≥ 61

$$\Rightarrow \quad x + 3x + (3x - 2) \geq 61$$

$$\Rightarrow \quad 7x - 2 \geq 61$$

$$\Rightarrow \quad 2 + 7x - 2 \geq 61 + 2$$

$$\Rightarrow \quad 7x \geq 63$$

$$\Rightarrow \quad \frac{7x}{7} \geq \frac{63}{7}$$

$$\Rightarrow \quad x \geq 9$$

$\therefore$ Minimum length of the shortest side = 9 cm. **2**

Q. 3. Solve the following system of inequalities graphically:

$$2x + y \leq 24,$$

$$x + y \geq 11,$$

$$2x + 5y \leq 40, \quad x, y \geq 0.$$ A

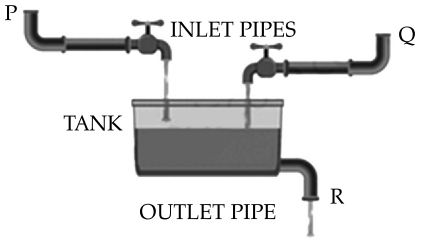
COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

Sohil is good in Mathematics. His friend gave him a question from the topic Pipes and Cisterns *i.e.*, "Pipe R can empty a full tank in 30 hours. But two pipes P and Q can fill a tank in 15 hours and 10 hours respectively. Ram unknowingly opened all three taps. After 2 hours Shyam realized it and closed Pipe R."



Q. 1. In 1 hour tank filled by pipe P is

(A) $\frac{1}{15}$ **(B)** $\frac{1}{30}$

(C) $\frac{1}{10}$ **(D)** $\frac{5}{30}$

These questions are for practice and their solutions are available at the end of the chapter

Ans. Option (A) is correct.

Explanation: Pipe P can fill the tank in 15 hours.

So, in 1 hour tank filled by pipe P = $\frac{1}{15}$

Q. 2. In 1 hour tank emptied by pipe R is

(A) $\frac{1}{15}$ **(B)** $\frac{1}{30}$

(C) $\frac{1}{10}$ **(D)** $\frac{5}{30}$

Ans. Option (B) is correct.

Explanation: Pipe R can empty the tank in 30 hours.

So, in 1 hour tank emptied by pipe R = $\frac{1}{30}$

Q. 3. When P, Q and R all are open, then tank filled in 1 hour is

(A) $\frac{1}{15}$ **(B)** $\frac{1}{30}$

(C) $\frac{4}{30}$ **(D)** $\frac{5}{30}$

Ans. Option (C) is correct.

Explanation: When P, Q and R all are open, then tank filled in 1 hour = $\frac{1}{15}+\frac{1}{10}-\frac{1}{30} = \frac{4}{30}$

Q. 4. If R is closed, then entire tank get filled in how much time?

(A) 6 hours **(B)** 4 hours

(C) 8 hours **(D)** 2 hours

Ans. Option (A) is correct.

Explanation: If pipe R is closed, then

In one hour tank filled = $\frac{1}{15}+\frac{1}{10} = \frac{5}{30}$

So entire tank gets filled in $\frac{30}{5}$ = 6 hours

Q. 5. When P, Q and R all are open, then tank filled in 2 hours is

(A) $\frac{1}{15}$ **(B)** $\frac{8}{30}$

(C) $\frac{4}{30}$ **(D)** $\frac{5}{30}$

Ans. Option (B) is correct.

Explanation: When P, Q and R all are open, then

tank filled in 1 hour = $\frac{1}{15}+\frac{1}{10}-\frac{1}{30} = \frac{4}{30}$

So, in 2 hours = $2\times\frac{4}{30} = \frac{8}{30}$

II. Read the following text and answer the following questions on the basis of the same:

Riya is doing her Mathematics homework given in the class. Her teacher gave a problem related to the topic 'Boats and Streams'. She is very confused in this topic, the question given by the teacher is

"A man can row 5 km/h in still water. If in a river running at 2 km an hour, it takes him 40 minutes to row to a place and return back".

Q. 1. What is the downstream speed ?

(A) 2 km/h **(B)** 5 km/h

(C) 7 km/h **(D)** 3 km/h

Ans. Option (C) is correct.

Explanation: Speed of the man in still water = 5 km/h

and speed of the river = 2 km/h

Downstream speed = Speed of man in still water + Speed of the river

= 5 + 2 = 7 km/h

Q. 2. What is the upstream speed ?

(A) 2 km/h **(B)** 5 km/h

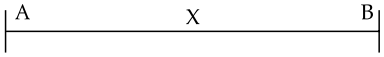
(C) 7 km/h **(D)** 3 km/h

Ans. Option (D) is correct.

Explanation: Upstream speed = Speed of man in still water – Speed of the river

= 5 – 2 = 3 km/h

Q. 3. Man starts from A, travels to B and comes back. Let distance between A and *B* = *x*.



Total time to travel from A to B and come back to *A* =

(A) Distance from A to B/downstream speed + Distance from B to A/upstream speed

(B) Distance from B to A/downstream speed + Distance from A to B/upstream speed

(C) Distance from A to B + Distance from B to A

(D) None of these

Ans. Option (A) is correct.

Q. 4. A man rows downstream 30 km and upstream 12 km. If he takes 4 hours to cover each distance, then the speed of the stream is:

(A) 2.25 km/h **(B)** 25.5 km/h

(C) 22.5 km/h **(D)** 52.2 km/h

Ans. Option (A) is correct.

Explanation: Downstream speed = Distance travelled downstream / Time taken

= $\frac{30}{4}$ Km/h

Upstream speed = Distance travelled in upstream / Time taken

= $\frac{12}{4}$

$$\text{Speed of the stream} = \frac{1}{2}\text{ (Downstream speed} - \text{upstream speed)}$$
$$= \frac{1}{2}\left(\frac{30}{4} - \frac{12}{4}\right)$$
$$= \frac{1}{2}\left(\frac{18}{4}\right)$$
$$= 2.25 \text{ km/h}$$

Q. 5. A boat travels at 9 km/h along the stream and 6 km/h against the stream. Find the speed of the boat in still water.

(A) 7.5 km/h **(B)** 5.5 km/h
(C) 7 km/h **(D)** 8.7 km/h

Ans. Option (A) is correct.

Explanation: Downstream speed of the boat = 9 km/h
Upstream speed of the boat = 6 km/h
Speed of the boat in still water

$$= \frac{1}{2}\text{ (Downstream speed + Upstream speed)}$$
$$= \frac{1}{2}(9 + 6)$$
$$= 7.5 \text{ km/h}$$

III. Read the following text and answer the following questions on the basis of the same:

While making the notes on linear inequalities, Riya note down the following points in her notebook.

Inequality: Two real numbers or two algebraic expressions related by the symbol '<', '>', '≤', form an inequality.

Linear Inequality: An inequality is said to be linear, if each variable occurs in first degree only and there is no term involving the product of the variables.

e.g., $ax + b \le 0$, $ax + by + c > 0$, $ax \le 4$.

An inequality in one variable in which degree of variable is 2, is called quadratic inequality in one variable,

e.g., $ax^2 + bx + c \ge 0$, $3x^2 + 2x + 4 \le 0$.

Linear equality In one Variable: A linear inequality which has only one variable, is called linear inequality in one variable.

e.g., $ax + b < 0$, where $a \ne 0$, $4c + 7 \ge 0$.

(i) Rules of solving inequalities:

- If $a \ge b$ then $a \pm k \ge b \pm k$ where k is any number.
- If $a \ge b$ then ka is not always $\ge kb$

If $k > 0$ (*i.e.*, positive) then $a \ge b \Rightarrow ka \ge kb$
If $k > 0$ (*i.e.*, negative) then $a \ge b \Rightarrow ka \le kb$

Thus, always reverse the sign of inequality while multiplying or dividing both sides of an inequality by a negative number.

(ii) Procedure to solve a linear inequality in one variable:

- Simplify both sides by collecting like terms.
- Remove fractions (or decimals) by multiplying both sides by appropriate factor (L.C.M. of denominator or a power of 10 in case of decimals.)
- Isolate the variable on one side and all constants on the other side. Collect like terms whenever possible.
- Make the coefficient of the variable equal to 1.
- Choose the solution set from the replacement set.

Solution set: A solution to an inequality is a number which when substituted for the variable, makes the inequality true. The set of all solutions of an inequality is called the solution set of the inequality.

Q. 1. The solution set for the following figure is:

(A) $x \in (-\infty, 5)$ **(B)** $x \in (-\infty, 5]$
(C) $x \in [5, \infty)$ **(D)** $x \in (5, \infty)$

Ans. Option (D) is correct.

Explanation: The given figure represents all value of x greater than 5 excluding 5 on the real number line.

So, $x \in (5, \infty)$

Q. 2. If $\frac{-3}{4}x \le -3$, then x ... 4.

(A) $x < 4$ **(B)** $x \ge 4$
(C) $x > 4$ **(D)** $x = 4$

Ans. Option (B) is correct.

Explanation: $\frac{-3}{4}x \le -3$

$$x \ge 4 \quad \Rightarrow \quad x \ge -3 \times \frac{-4}{3}$$

Q. 3. The solution set for the given inequality is: $4x + 3 \ge 2x + 17$, $3x - 5 < -2$

(A) $x < 1$ **(B)** $x \ge 7$
(C) No Solution **(D)** $x > 1$

Ans. Option (C) is correct.

Explanation: We have,

$$4x + 3 \ge 2x + 17$$
$$\Rightarrow \quad 4x - 2x \ge 17 - 3$$
$$\Rightarrow \quad 2x \ge 14$$
$$\Rightarrow \quad x \ge \frac{14}{2}$$
$$\Rightarrow \quad x \ge 7$$

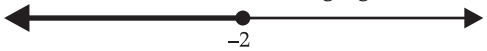
Also, we have $3x - 5 < -2$

$$\Rightarrow \quad 3x < -2 + 5$$
$$\Rightarrow \quad 3x < 3$$
$$\Rightarrow \quad x < 1$$

On combining Eqs. (i) and (ii), we see that solution is not possible because nothing is common between these two solutions,

(i.e., $x < 1$, $x \ge 7$).

Q. 4. The solution set for the following figure is:



(A) $x \in (-\infty, -2]$ **(B)** $x \in (\infty, 2)$
(C) $x \in (-2, \infty]$ **(D)** $x \in [-2, \infty)$

Ans. Option (A) is correct.

Explanation: The given figure represent all real values of x less than and equal to – 2.

So $x \in (-\infty, -2]$.

Q. 5. If $|x + 2| \le 9$, then

(A) $x \in (-7, 11)$
(B) $x \in [-11, 7]$
(C) $x \in (-\infty, -7) \cup (11, \infty)$
(D) $x \in (-\infty, -7) \cup (11, \infty)$

Ans. Option (B) is correct.

Explanation: Given, $|x + 2| \le 9$

$\Rightarrow \quad -9 \le x + 2 \le 9$

$\Rightarrow \quad -9 - 2 \le x \le 9 - 2$

$\Rightarrow \quad -11 \le x \le 7$

$\Rightarrow \quad x \in [-11, 7]$

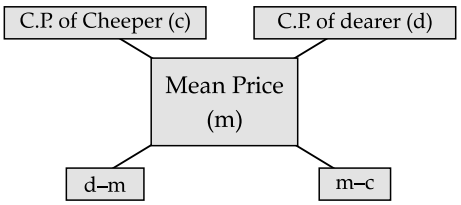
Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:

Akshay and Ritesh are the neighbours. They are studying in the same school. Both are taking tuitions from same teacher. One day they have combined tution class in which teacher explain about the topic Alligation and Mixtures.

Teacher said, according to alligation rule" if two ingredients at a given price are mixed to produce a mixture at given price, the ratio of quantity of cheaper ingredient and quantity of dearer ingredient is given by

$$\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}} = \frac{\text{Cost price of dearer} - \text{Mean price}}{\text{Mean price} - \text{Cost price of cheaper}}$$



Q. 1. Five litres of water is added to a certain quantity of pure milk costing ₹ 60 per litre. If by selling the mixture at same price as before, a profit of 20% is made, what is the amount of pure milk in the mixture ?

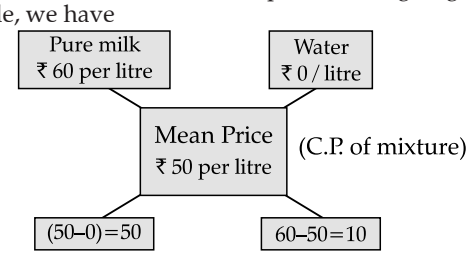
Sol. Let quantity of pure milk be x litres. It is given that the mixture is sold at a profit of 20% and the selling price of the mixture is ₹ 60 per litre.

$$\therefore \quad \text{S.P.} = \text{C.P.}\left(1 + \frac{\text{Profit}}{100}\right)$$

$$\Rightarrow \quad 60 = \text{C.P.}\left(1 + \frac{20}{100}\right)$$

$$\Rightarrow \quad \text{C.P.} = 60 \times \frac{5}{6} = 50$$

So, the mixture is sold at ₹ 50 per litre using alligation rule, we have



1

Thus, $\dfrac{\text{Quantity of pure milk}}{\text{Quantity of water}} = \dfrac{50}{10}$

$$\Rightarrow \quad \frac{x}{5} = \frac{5}{1}$$

$$\Rightarrow \quad x = 25$$ **1**

Hence, the amount of pure milk in the mixture is 25 litres.

Q. 2. Two liquids one mixed in the proportion 3 : 2 and the mixture is sold at ₹ 110 per litre at 10% profit. If first liquid costs ₹ 20 more per litre than the second, what does it cost per litre ?

Sol. We have,

S.P. of mixture = ₹ 110, profit = 10%

$$\therefore \quad \text{S.P.} = \text{C.P.}\left(1 + \frac{\text{Profit}}{100}\right)$$

$$\Rightarrow \quad 110 = \text{C.P.}\left(1 + \frac{10}{100}\right)$$

$$\Rightarrow \quad 110 = \text{C.P.}\frac{110}{100}$$

$$\Rightarrow \quad \text{C.P.} = ₹\ 100$$

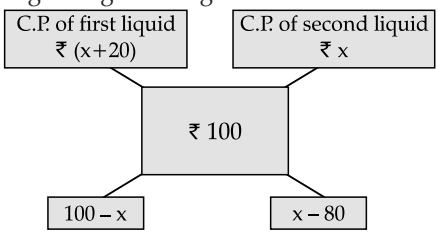
Let the C.P. of second liquid be ₹ x. then, C.P. of first liquid = ₹ $(x + 20)$

Thus, we have

C.P. of first liquid = ₹ $(x + 20)$

C.P. of mixture = ₹ 100 **1**

The alligation grid is as given below:



Using alligation rule, we get

$$\frac{\text{Quantity of first liquid}}{\text{Quantity of second liquid}} = \frac{100 - x}{x - 80}$$

But it is given that two liquids are mixed in the proportion 3 : 2.

$$\therefore \quad \frac{100 - x}{x - 80} = \frac{3}{2}$$

$$\Rightarrow \quad 200 - 2x = 3x - 240$$

$$\Rightarrow \quad 5x = 440$$

$$\Rightarrow \quad x = 88$$ **1**

hence, cost price of second liquid is ₹ 88.

II. Read the following text and answer the following questions on the basis of the same:

Sheena is preparing for a competitive exam, in which questions are asked from the topic 'pipe and cisterns'. While searching this topic online, she found the following matter about pipe and cisterns.

'The problems often asked on pipes and cisterns one similar to the problems on time and work. In problems of pipe and cisterns, we are usually come across two types of pipes.

(i) Inlet Pipe: A pipe connected with a tank or a reservoir or a cistern, that fills it, is called an inlet pipe.

(ii) Outlet pipe: A pipe connected with a tank or a reservoir or a cistern, that empties it, is called on outlet pipe.

Also,

- If a pipe can fill a tank in n hours, then in 1 hour, it will fill $\left(\frac{1}{n}\right)^{th}$ part of the tank.
- If a pipe can empty a tank in n hours, then in 1 hour, it will empty $\left(\frac{1}{n}\right)^{th}$ part of the tank.

Q. 1. A pump can fill a tank with water in 2 hours. Because of a leak, it took $2\frac{1}{3}$ hours to fill the tank. In how many hours the leak can drain all the water of the tank.

Sol. Suppose the leak can drain all the water of the tank in n hours.

Here, $n_A = 2$, $n_B = n$ and $n_{AB} = 2\frac{1}{3}$ hours

$= \frac{7}{3}$ hours ½

We know that, if an inlet pipe A can fill a tank in n_A hours and an outlet pipe B can empty the full tank in n_B hours ($n_B > n_A$). If on opening both the pipes together, the tank can be filled in n_{AB} hours, then

$$\frac{1}{n_{AB}} = \frac{1}{n_A} \quad \textbf{1}$$

$$\therefore \quad \frac{1}{\frac{7}{3}} = \frac{1}{2} - \frac{1}{n}$$

$$\Rightarrow \quad \frac{3}{7} = \frac{1}{2} - \frac{1}{n}$$

$$\Rightarrow \quad \frac{1}{n} = \frac{1}{2} - \frac{3}{7}$$

$$\Rightarrow \quad \frac{1}{n} = \frac{1}{14}$$

$$\Rightarrow \quad n = 14 \text{ hours.} \quad ½$$

Hence, the leak can drain all the water of the tank in 14 hours.

Q. 2. One pipe can fill a tank three lines as fast as another pipe. If together the two pipes can fill the tank in 36 minutes, how much time slower pipe alone

Sol. Let slower pipe (A) alone fill the tank in n minutes.

Then the faster pipe (B) alone will fill the tank in $\frac{n}{3}$ minutes. It is given that two pipes together can fill the tank in 36 minutes.

Using formula, $\frac{1}{n_{AB}} = \frac{1}{n_A} + \frac{1}{n_B}$

we get $\frac{1}{36} = \frac{1}{n} + \frac{1}{\frac{n}{3}}$

[Putting $n_{AB} = 36$, $n_A - n$ and $n_B = \frac{n}{3}$]

$$\Rightarrow \quad \frac{1}{36} = \frac{1}{n} + \frac{3}{n}$$

$$\Rightarrow \quad \frac{1}{36} = \frac{4}{n}$$

$$\Rightarrow \quad n = 144 \text{ minutes}$$

Hence, the slower pipe (A) alone fill the tank in 144 minutes.

Solutions for Practice Questions

Very Short Answer Type Questions

Sol. 2: Rate in still water $= \frac{1}{2}(10 + 7)$ km/h $= 8.5$ km/h ½

Rate of current $= \frac{1}{2}(10 - 7)$ km/h $= 1.5$ km/h ½

Sol. 4: $A : B = 80 : 75$, $A : C = 80 : 65$

$$\frac{B}{C} = \left(\frac{B}{A} \times \frac{A}{C}\right)$$

$$= \left(\frac{75}{80} \times \frac{80}{65}\right) = \frac{15}{13}$$

$$= \frac{60}{52} = 60 : 52$$

$\therefore$ In a game of 60, B can give C 8 points. **1**

Short Answer Type Questions-I

Sol. 1: S.P. of 1 kg mixture = ₹ 7.20, Gain = 20%

$\therefore$ C.P. of 1 kg mixture = $₹\left(\frac{100}{120} \times 7.20\right)$ = ₹ 6 ½

By the rule of alligation, we have:

C.P. of 1 kg wheat of 1st kind (800 p) — C.P. of 1 kg wheat of 2nd kind (540 p)

Mean Price (600 p)

60 — 200

Wheat of 1st kind : Wheat of 2nd kind = 60 : 200 = 3 : 10 **1**

Let x kg of wheat of 1st kind be mixed with 36 kg of wheat of 2nd kind.

Then, $3 : 10 = x : 36$ or $10x = 3 \times 36$ or $x = 10.8$ kg ½

Sol. 6: The given inequality is $x < y$ ...(i)

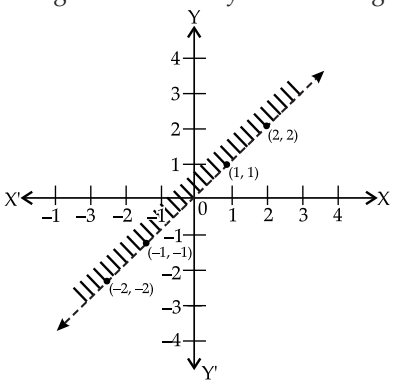
Step 1: Consider inequation as strict equation i.e., $x = y$

Step 2: Find the points on x-axis and y-axis i.e.,

x	1	2	– 1	– 2
y	1	2	– 1	– 2

Step 3: Plot the graph using the above table.

Step 4: Take a point (1, 1) and put it in the given inequation (i), we get $1 > 1$, which is false, so shaded region will be away from the origin. **1**



2

Here, shaded region shows the inequality $x < y$.

Commonly Made Error

- Steps are skipped by students which leads to errors in graph.

Answering Tip

- Students are advised to practice more and more questions of graphical method.

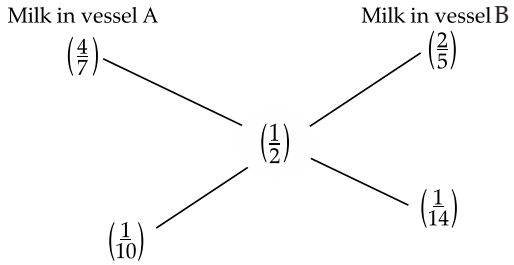
Short Answer Type Questions-II

Sol. 1: (i) Let the C.P. of milk be ₹ 1 per litre

$$\text{Milk in 1 litre mixture of } A = \frac{4}{7} \text{ litre}$$

$$\text{Milk in 1 litre mixture of } B = \frac{2}{5} \text{ litre}$$

$$\text{Milk in 1 litre mixture of } C = \frac{1}{2} \text{ litre}$$



$$\therefore \quad \text{Required ratio} = \frac{1}{10} : \frac{1}{14} = 7 : 5 \qquad \mathbf{1}$$

(ii) Speed downstream $= (7.5 + 1.5)$ km/h $= 9$ km/h

Speed upstream $= (7.5 - 1.5)$ km/h $= 6$ km/h **1**

Let the required distance be x km. Then,

$$\frac{x}{9} + \frac{x}{6} = \frac{50}{60}$$

$$\Rightarrow \quad 2x + 3x = \left(\frac{5}{6} \times 18\right)$$

$$\Rightarrow \quad 5x = 15 \Rightarrow x = 3$$

Hence, the required distance is 3 km. **1**

Long Answer Type Questions

Sol. 3: Inequality $(x \geq 0)$ represents the region on the right of y-axis and y-axis itself. Inequality $(y \geq 0)$ represents the region above x-axis and x-axis itself.

$$2x + y \leq 24 \qquad (1)$$

$$x + y \geq 11 \qquad (2)$$

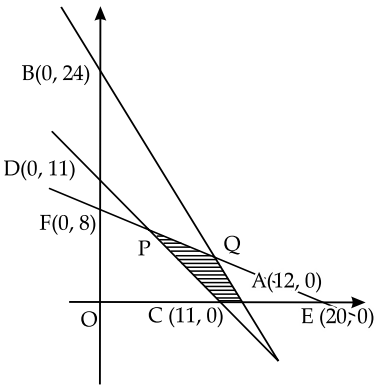
$$2x + 5y \leq 40 \qquad (3)$$

We first draw the graph of lines

$2x + y = 24$, $x + y = 11$ and $2x + 5y = 40$ **1**

Now, $2x + y = 24$, passes through A(12, 0) and B(0, 24) **1**

Again, $x + y = 11$, passes through C(11, 0) and D(0, 11) **1**



1

Further $2x + 5y = 40$, passes through $E(20, 0)$ and F(0, 8) **1**

Shaded area PQAC is the solution area.

REFLECTIONS

(a) Will you be able to solve the complex problems of boats and streams?

(b) Can you find the ratio of water and milk in a certain mixture of water and milk?

(c) Will you be able to handle the questions based on race and games easily?

SELF ASSESSMENT PAPER – 01

Maximum Mark: 30 **Max. Time: 1 Hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. The cost of type I rice is ₹ 15 per kg and type II rice is ₹ 20 per kg. If type 1 and type 2 are mixed is the ratio 2 : 3. The price per kg of mixed variety is:

(A) ₹ 18 per kg **(B)** ₹ 16 per kg

(C) ₹ 20 per kg **(D)** ₹ 22 per kg

Q. 2. Navisha can row upstream a boat at 10 km/h and downstream at 18 km/h. The rate at which she rows the boat in still water is:

(A) 12 km/h **(B)** 14 km/h

(C) 16 km/h **(D)** None of these

Q. 3. Two pipes A and B can fill a tank in 15 minutes and 20 minutes. Both the pipes are opened together but after 4 minutes, pipe A is turned off. What is the total time required to fill the tank.

(A) $\frac{22}{3}$ minutes **(B)** $\frac{44}{3}$ minutes

(C) $\frac{23}{3}$ minutes **(D)** $\frac{42}{3}$ minutes

Q. 4. If $5x - 3 < 3x + 1$, then find x, when x is a real number.

(A) $(-\infty, 2)$ **(B)** $(2, \infty)$

(C) $(-5, -3)$ **(D)** $(3, \infty)$

Q. 5. A can run 25 m which B can run 30 m. By what distance B boat A is kilometre race ?

(A) 150 m **(B)** 166.67 m

(C) 169 m **(D)** 175.78 m

Q. 6. Solve for x: $7x \equiv 3 \pmod 5$

(A) 3 **(B)** 4

(C) 5 **(D)** 7

Case-based

Q. 7. Attempt any 4 sub-parts from each questions. Each question carries 1 mark.

Read the following text and answer the following questions on the basis of the same:

A linear inequation in one variable is of the form $ax + b < 0$ or, $ax + b \leq 0$ or $ax + b > 0$ or, $ax + b \geq 0$

We follow the following algorithm to solve a linear inequation in one variable.

Algorithm:

Step I: Obtain the linear inequation

Step II: Collect all terms involving the variable on one side of the inequation and the constant term on the other side.

Step III: Simplify both sides of inequality in their simplest forms to reduce the inequation in the form $ax < b$, or $ax \leq b$, or $ax > b$, or $ax \geq b$

Step IV: Solve the inequation obtained in step III by dividing both sides of the inequation by the coefficient of the variable.

Step V: Write the solution set obtained in step IV in the form of an interval on the real line.

(i) Solve the linear inequation $2x - 8 \leq 0$

(A) $x < 4$ **(B)** $x < 2$

(C) $x \leq 2$ **(D)** $x \leq 4$

(ii) Solve the linear inequality $-3x + 15 < 0$

(A) $x < 5$ **(B)** $x > 5$

(C) $x \leq 5$ **(D)** $x \geq 5$

(iii) Solve the linear inequality: $8x - 40 \geq 0$

(A) $[5, \infty)$ **(B)** $(5, \infty)$

(C) $[-5, \infty)$ **(D)** $(-5, \infty)$

(iv) Solve the linear inequality: $7x + 9 > 30$

(A) $(-3, \infty)$ **(B)** $[3, \infty)$

(C) $(-3, 3)$ **(D)** $(0, \infty)$

(v) Solve the linear inequations $11 - 5x > -4$

(A) $x > -4$ **(B)** $x < 4$

(C) $x < 3$ **(D)** $x > 7$

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. A grocer wishes to sell a mixture of two variety of pulses worth ₹ 16 per kg. In what ratio must he mix the pulses to reach this selling price, when cost of one variety of pulses is ₹ 14 per kg and the other is ₹ 24 per kg?

Q. 9. In one hour, a boat goes 13 km/hr in the direction of the stream and 7 km/hr against the direction of the stream. What will be the speed of the boat in still water?

Q. 10. Solve $5x < 24$, when $x \in$ N.

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. Find all solutions to the congruence $55x \equiv 35 \pmod{75}$.

Q. 12. Three pipes A, B and C can fill a tank from empty to full in 30 minutes, 20 minutes, and 10 minutes respectively. When the tank is empty, all the three pipes are opened. A, B and C discharge chemical solutions P, Q and R respectively. What is the proportion of the solution R in the liquid in the tank after 3 minutes?

Q. 13. Show the graph of the solution of $2x - 3 > x - 5$ on the number line.

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. (a) Compute the last digit in the decimal expansion of 4^{100}.

(b) Is 4^{100} divisible by 3?

Q. 15. A man rows a boat 30 km upstream and 44 km down-stream in 10 hours. If he rows 40 km upstream and 55 km downstream in 13 hours, then find the rate at which he rows the boat in still water.

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 16. Solve the following system of inequalities for all $x \in R$.

$$2(2x + 3) - 10 < 6(x - 2),\ \frac{(2x-3)}{4} + 6 \geq 4 + \frac{4x}{3}$$

CHAPTER

3 MATRICES

Syllabus

Matrices and types of matrices
Equality of matrices, Transpose of a matrix, Symmetric and Skew symmetric matrix
Algebra of Matrices

In this chapter you will study

- *Define matrix*
- *Identify different kinds of matrices*
- *Find the size / order of matrices Determine equality of two matrices*
- *Write transpose of given matrix*
- *Define symmetric and skew symmetric matrix*
- *Perform operations like addition & subtraction on matrices of same order*
- *Perform multiplication of two matrices of appropriate order*
- *Perform multiplication of a scalar with matrix*

Revision Notes

► **Matrix:**

A matrix is an ordered rectangular **array** of numbers (real or complex) or functions which are known as elements or the entries of the matrix. It is denoted by the uppercase letters i.e., *A, B, C* etc.

Scan to know more about this topic

Basic of Matrices

Key Words

Array: An array is a rectangular arrangement of objects in equal rows (horizontal) and equal columns (vertical). Everyday example of arrays include a muffin tray and an egg carton.

Consider a matrix A given as,

$$A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$$

Here in matrix A depicted above, the horizontal lines of elements are said to constitute rows of the matrix A and vertical lines of elements are said to constitute columns of the matrix. Thus, matrix A has m **rows** and n **columns**.

The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars ‖ ‖.

- A matrix having m rows and n columns is called a matrix of order $m \times n$ (read as 'm by n' matrix). A matrix 'A' of order $m \times n$ is depicted as $A=[a_{ij}]_{m\times n}; i, j \in \mathbb{N}$.
- Also, in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.
- No. of elements in the matrix $A=[a_{ij}]_{m\times n}$ is given as (m) (n).

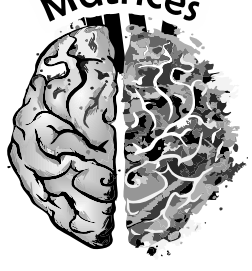
► **Types of Matrices:**

(i) Column matrix: A matrix having only one column is called a **column matrix** or **column vector**.

Scan to know more about this topic

Types of matrices

Matrices



Definition and its types

A matrix of order $m \times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A=[a_{ij}]_{m\times n}$, $1 \le i \le m$, $1 \le j \le n$; $i, j \in N$ is given by

$$A=\begin{bmatrix} a_{11} & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22} & \ldots a_{2n} \\ a_{m_1} & a_{m_2} & \ldots a_{mn} \end{bmatrix}_{m\times n}$$

- **Column matrix :** It is of the form $\begin{bmatrix} a_{ij} \end{bmatrix}_{m\times 1}$
- **Row matrix :** It is of the form $\begin{bmatrix} a_{ij} \end{bmatrix}_{1\times n}$
- **Square matrix :** Here, $m = n$ (no. of rows = no. of columns)
- **Diagonal matrix :** All non-diagonal entries are zero *i.e* $a_{ij} = 0 \forall i \ne j$
- **Scalar matrix :** $a_{ij} = 0, i \ne j$ and $a_{ij} = k$ (Scalar), $i = j$, for some constant k.
- **Identity matrix :** $a_{ij} = 0, i \ne j$ and $a_{ij} = 1, i = j$
- **Zero matrix :** All elements are zero.

Equality of two matrices

$A=\begin{bmatrix} a_{ij} \end{bmatrix}=\begin{bmatrix} b_{ij} \end{bmatrix} = B$ if, A and B are of same order and $a_{ij}=b_{ij} \forall$ i and j; $i,j \in$ N

Transpose of a Matrix

If $A=[a_{ij}]_{m\times n}$, then its transpose $A'= (A^T)=[a_{ji}]_{n\times m}$ *i.e.* if $A=\begin{pmatrix} 2 & 1 \end{pmatrix}$ then $A^T=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Also, $(A')' = A$, $(kA)' = kA'$, $(A+B)' = A'+B'$, $(AB)' = B'A'$.

- A is symmetric matrix if $A=A'$ *i.e.* $A'=A$.
- A is skew – symmetric if $A= -A'$ *i.e.* $A'=-A$.
- A is any square matrix, then–

$A=\frac{1}{2}\{(A+A')+(A-A')\}$ = sum of a symmetric and a skew-symmetric matrix.
(S.M. under $A+A'$; Skew S.M. under $A-A'$)

For example if $A=\begin{pmatrix} 2 & 8 \\ 6 & 4 \end{pmatrix}$, then $A=\frac{1}{2}\left\{\begin{pmatrix} 4 & 14 \\ 14 & 8 \end{pmatrix}+\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right\}$.

Operations on matrices

Multiplication

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then $AB = C =[C_{ik}]_{m\times p}$, $[C_{jk}] =\sum_{j=1}^{n} a_{ij} b_{jk}$.

Also, $A(BC) = (AB)C$, $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$, but $AB \ne BA$ (always).

Addition

If A, B are two matrices of same order, then $A+B=[a_{ij}+b_{ij}]$. The addition of A and B follows: $A+B=B+A$, $(A+B)+C = A+(B+C)$, $-A = (-1)A$, $k(A+B) = kA + kB$, k is scalar and $(k+I)A = kA + IA$, k and I are constants.

- If $A=\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, $B=\begin{bmatrix} -3 & 2 \\ -4 & 5 \end{bmatrix}$ then $A+B=\begin{bmatrix} -1 & 5 \\ -2 & 9 \end{bmatrix}$
- If $A=\begin{bmatrix} 2 & 3 \end{bmatrix}_{1\times 2}$, $B=\begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$, then $AB=[2\times 4+3\times 5]=[23]_{1\times 1}$

Properties for applying the operations

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

$R_i \to kR_i$ or $C_i \to kC_i$

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Trace the Mind Map

First Level · Second Level · Third Level

e.g., $A = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times1}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times1}$

General notation: $A = [a_{ij}]_{m\times1}$

(ii) Row matrix: A matrix having only one row is called a **row matrix** or **row vector**.

e.g., $A = [2 \quad 5 \quad -4]_{1\times3}$, $B = [\sqrt{2} \quad 4]_{1\times2}$

General notation: $A = [a_{ij}]_{1\times n}$

(iii) Square matrix: It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $n\times n$ matrix is said to constitute a square matrix of order $n \times n$ and is known as a **square matrix of order** 'n'.

e.g., $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times3}$ is a square matrix of order 3.

General notation: $A = [a_{ij}]_{n\times n}$

(iv) Diagonal matrix: A square matrix $A = [a_{ij}]_{m\times n}$ is said to be diagonal matrix if all the elements, except those in the leading diagonal are zero i.e., $a_{ij} = 0$ for all $i \neq j$.

e.g., $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$ is a diagonal matrix of order 3.

- Also, there are more notation specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can also be written as diag (2 5 4) or diag [2, 5, 4]
- Note that the elements a_{11}, a_{22}, a_{33}, ..., a_{mm} of a square matrix $A = [a_{ij}]_{m\times n}$ of order m × m are said to constitute the principal diagonal or simply **the diagonal of the square matrix** A. These elements are known as **diagonal elements of matrix** A.

(v) Scalar matrix: A diagonal matrix $A = [a_{ij}]_{m\times n}$ is said to be a scalar matrix if its diagonal elements are equal.

i.e., $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$

e.g., $A = \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}_{3\times3}$ is a scalar matrix of order 3.

(vi) Unit or Identity matrix: A square matrix $A = [a_{ij}]_{m\times n}$ is said to be an identity matrix if

$a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$.

A unit matrix can also be defined as the scalar matrix each of whose diagonal elements is unity. We denote the identity matrix of order m by I_m or I.

e.g., $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

(vii) Zero matrix or Null matrix: A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is '0'.

e.g.,

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$, $C = [0 \quad 0]_{1\times2}$

(viii) Triangular matrix:

(a) Lower triangular matrix: A square matrix is called a lower triangular matrix if all the entries above the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 3 & 5 \end{bmatrix}$

(b) Upper triangular matrix: A square matrix is called a **upper triangular matrix** if all the entries below the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & -8 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

► **Equality of Matrices:**

Two matrices A and B are said to be equal and written as $A = B$, if they are of the same order and their corresponding elements are identical i.e., $a_{ij} = b_{ij}$ i.e., $a_{11} = b_{11}$, $a_{22} = b_{22}$, $a_{32} = b_{32}$ etc.

► **Algebra of Matrices:**

- **Addition of Matrix:** If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

Thus, if $A = [a_{ij}]$, $B = [b_{ij}] \Rightarrow A + B = [a_{ij} + b_{ij}]$.

Scan to know more about this topic

Addition and subtraction of matrices

Properties of matrix addition:

- Commutative property: $A + B = B + A$
- Associativity property: $A + (B + C) = (A+B) + C$
- Cancellation law:
  (i) Left cancellation: $A + B = A + C \Rightarrow B=C$
  (ii) Right cancellation: $B + A = C + A \Rightarrow B = C$
- **Subtraction of Matrices:**
  If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by subtracting the corresponding elements of the matrices A and B is called the subtraction of the matrices A and B and is denoted by '$A - B$'.
  Thus if $A = [a_{ij}]$, $B = [b_{ij}]$, or $A - B = [a_{ij} - b_{ij}]$.

- **Multiplication of Matrices:**

Scan to know more about this topic

Matrix multiplication

We can only multiply two matrices if their dimensions are compatible , which means the number of columns in the first matrix is the same as the number of rows in the second matrix.

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the product AB is an $m \times p$ matrix.

$AB = [c_{ij}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$.

(The entry in the i^{th} row and j^{th} column is denoted by the double subscript notation a_{ij}, b_{ij}, and c_{ij}. For instance, the entry a_{23} is the entry in the second row and third column.)

The definition of matrix multiplication indicates a row-by-column multiplication, where the entries in the i^{th} row of A are multiplied by the corresponding entries in the j^{th} column of B and then adding the results.

Let's take the following example, multiplying a 2×3 matrix with a 3×2 matrix, to get a 2×2 matrix as the product. The entries of the product matrix are called e_{ij} when they're in the i^{th} row and j^{th} column.

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}3 & 5\\ -1 & 0\\ 2 & -1\end{bmatrix} = \begin{bmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{bmatrix}$$

Key Facts

- The term matrix was introduced by the 19th century English Mathematician James Sylvester, but it was his friend the Mathematics Arthur Cayley who developed the algebraic aspect of matrices in two papers in the 1850s.
- The English Mathematician Cuthbert Edmund Cullis was the first to use modern bracket notation for matrices in 1913.

To get e_{11}, multiply Row 1 of the first matrix by Column 1 of the second matrix.

$$e_{11} = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\ -1\\ 2\end{bmatrix} = 1(3) + 0(-1) + 1(2) = 5$$

To get e_{12}, multiply Row 1 of the first matrix by Column 2 of the second matrix.

$$e_{12} = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\ 0\\ -1\end{bmatrix} = 1(5) + 0(0) + 1(-1) = 4$$

To get e_{21}, multiply Row 2 of the first matrix by Column 1 of the second matrix.

$$e_{21} = \begin{bmatrix}0 & 1 & 2\end{bmatrix}\begin{bmatrix}3\\ -1\\ 2\end{bmatrix} = 0(3) + 1(-1) + 2(2) = 3$$

To get e_{22}, multiply Row 2 of the first matrix by Column 2 of the second matrix.

$$e_{21} = \begin{bmatrix}0 & 1 & 2\end{bmatrix}\begin{bmatrix}5\\ 0\\ -1\end{bmatrix} = 0(5) + 1(0) + 2(-1) = -2$$

Writing the product matrix, we get

$$\begin{bmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{bmatrix} = \begin{bmatrix}5 & 4\\ 3 & -2\end{bmatrix}$$

Therefore, $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}3 & 5\\ -1 & 0\\ 2 & -1\end{bmatrix} = \begin{bmatrix}5 & 4\\ 3 & -2\end{bmatrix}$

Properties of matrix multiplication :

- In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
- If two matrices A and B are such that AB is possible then it is not necessary that the product BA is also possible.
- If A is an $m \times n$ matrix and both AB as well as BA are defined, then B will be an $n \times m$ matrix.
- If A is an $n \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$.
- Matrix multiplication is **associative** i.e., $A(BC) = (AB)C$.
- Matrix multiplication is **distributive** over the **addition** i.e., $A.(B + C) = AB + AC$.
- Matrix multiplication is not commutative.

- **Multiplication of a Matrix by a Scalar:**

If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$, and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, i.e., $A = [a_{ij}]$.

$\Rightarrow kA = [ka_{ij}]$

e.g., $A = \begin{bmatrix}2 & -4\\ 5 & 6\end{bmatrix} \Rightarrow 3A = \begin{bmatrix}6 & -12\\ 15 & 18\end{bmatrix}$

Properties of scalar multiplication:

If A, B are two matrices of the same order and k, l are any scalars (numbers), then

(i) $(k + l)A = kA + lA$

(ii) $k(A + B) = kA + kB$

(iii) $(kl)A = k(lA) + l(kA)$

(iv) $IA = A$

(v) $(-1)A = -A$

- **Existence of non-zero matrices whose product is zero.**

The product of two matrices can be zero without either factor being a zero matrix.

e.g., Let $A = \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}$ and $B = \begin{bmatrix}3 & 4\\ 0 & 0\end{bmatrix}$

Here, $A \neq 0$ and $B \neq 0$.

Also, $AB = \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}3 & 4\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

- **Transpose of a Matrix:**

If $A=[a_{ij}]_{m\times n}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a transpose of matrix A. The transpose of A is denoted by A' or A^T i.e., if $A^T = [a_{ji}]$ $n\times m$.

For example, if $A=\begin{bmatrix}5 & -4 & 1\\ 0 & \sqrt{5} & 3\end{bmatrix}$ then $A^T=\begin{bmatrix}5 & 0\\ -4 & \sqrt{5}\\ 1 & 3\end{bmatrix}$

Properties of Transpose of Matrices:

- $(A+B)^T = A^T+B^T$
- $(A^T)^T = A$
- $(kA)^T = kA^T$, where k is any constant
- $(AB)^T = B^TA^T$
- $(ABC)^T = C^TB^TA^T$

Scan to know more about this topic

Transpose of a matrix

- **Symmetric matrix:** A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A^T = A$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = [a_{ij}]$.

For example,

$$A=\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$$

$$B=\begin{bmatrix}2+i & 1 & 3\\ 1 & 2 & 3+2i\\ 3 & 3+2i & 4\end{bmatrix}$$

Scan to know more about this topic

Symmetric and skew symmetric matrices

Note

- For any matrices AA^T and A^TA are symmetric matrices
- If A and B are two symmetric matrices of same order, then
  (i) AB is symmetric if and only if $AB = BA$.
  (ii) $A \pm B$, $AB + BA$ are also symmetric matrices.

- **Skew Symmetric Matrix:**

A square matrix $A = [a_{ij}]$ is said to be **a skew symmetric matrix** if $A^T = -A$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = -[a_{ij}]$.

For example, $A=\begin{bmatrix}0 & 1 & -5\\ -1 & 0 & 5\\ 5 & -5 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$

Note

- All the diagonal elements in a skew-symmetric matrix are zero.
- If A and B are two symmetric matrices, then $AB - BA$ is a skew symmetric matrix.

Mnemonics

Concept: Types of Matrices

Mnemonics: Remember **C**hrist **S**ubah **D**oophar **S**hyam

Interpretations:

Remember	Christ	Subah	Doophar	Shyam
R	**C**	**S**	**D**	**S**
↓	↓	↓	↓	↓
Row Matrix	Column Matrix	Square Matrix	Diagonal Matrix	Scalar Matrix

Mnemonics: NITE

Interpretations:

N	**I**	**T**	**E**
↓	↓	↓	↓
Null Matrix	Identity Matrix	Triangular Matrix	Equal Matrix

Key Formulae

- For any square matrix A, the matrix $A + A^T$ is a symmetric and $A - A^T$ is always a skew symmetric matrix.
- A square matrix can be expressed as the sum of a symmetric and skew symmetric matrix *i.e.*, $A = \frac{1}{2}(P)+\frac{1}{2}(Q)$, where $P = A + A^T$ is a symmetric matrix and $Q = A - A^T$ is a skew symmetric matrix.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. If matrix $A=\begin{pmatrix}a & b & -5\\ c & d & 0\\ 5 & 0 & 0\end{pmatrix}$ **is skew symmetric, then value of** $2a + b + c - 3d$ **is:**

(A) 1 **(B)** –1
(C) 0 **(D)** 2

Ans. Option (C) is correct.

Explanation:

$$2a + b + c - 3d = b + c \qquad (\because a = d = 0)$$
$$= b + (-b) \qquad (\because c = -b)$$
$$= 0$$

Q. 2. If matrix A is given by $A = [a_{ij}]_{2\times 2}$, where $a_{ij} = i + j$, then A is equal to:

(A) $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ **(B)** $\begin{bmatrix}2 & 3\\ 3 & 4\end{bmatrix}$

(C) $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$ **(D)** $\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$

[Board Term-I, 2021]

Ans. Option (B) is correct.

Explanation: Here, $a_{11} = 1 + 1 = 2$

$a_{12} = 1 + 2 = 3$

$a_{21} = 2 + 1 = 3$

and $a_{22} = 2 + 2 = 4$

$\therefore \quad A = \begin{bmatrix}2 & 3\\ 3 & 4\end{bmatrix}$

Q. 3. If A is a square matrix such that $A^2 = A$, then $(I + A)^2 - 3A$ is equal to:

(A) I **(B)** $2A$

(C) $3I$ **(D)** A

Ans. Option (A) is correct.

Explanation: We have, $(I + A)^2 - 3A$

$= I^2 + 2IA + A^2 - 3A$

$= I + 2A + A - 3A$ [Since given $A^2 = A$]

$= I$

Q. 4. If $A = \begin{bmatrix}1 & 2 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & -2 & y\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $AB = I_3$, then $(x + y)$ equals:

(A) 0 **(B)** –1

(C) 2 **(D)** –2

Ans. Option (A) is correct.

Explanation: $AB = \begin{bmatrix}1 & 2 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -2 & y\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 0 & y+x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Since given $AB = I_3$

$\therefore \quad \begin{bmatrix}1 & 0 & y+x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

On comparing, we get

$y + x = 0$ or $x + y = 0$

Q. 5. If $A = \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ a & b & 2\end{bmatrix}$, then $aI + bA + 2A^2$ equals:

(A) A **(B)** $-A$

(C) abA **(D)** None of these

Ans. Option (D) is correct.

Explanation: Given, $A = \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ a & b & 2\end{bmatrix}$

$\therefore \quad A^2 = \begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ a & b & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ a & b & 2\end{bmatrix}$

$= \begin{bmatrix}1+a & b & 3\\ a & b & 2\\ 3a & 2b & a+b+4\end{bmatrix}$

Now, $aI + bA + 2A^2$

$= a\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + b\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ a & b & 2\end{bmatrix} + 2\begin{bmatrix}1+a & b & 3\\ a & b & 2\\ 3a & 2b & a+b+4\end{bmatrix}$

$= \begin{bmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{bmatrix} + \begin{bmatrix}b & 0 & b\\ 0 & 0 & b\\ ab & b^2 & 2b\end{bmatrix} + \begin{bmatrix}2+2a & 2b & 6\\ 2a & 2b & 4\\ 6a & 4b & 2a+2b+8\end{bmatrix}$

$= \begin{bmatrix}3a+b+2 & 2b & b+6\\ 2a & a+2b & b+4\\ 6a+ab & 4b+b^2 & 3a+4b+8\end{bmatrix}$

Q. 6. $A = [a_{if}]_{m\times n}$ is a square matrix, if

(A) $m < n$ **(B)** $m > n$

(C) $m = n$ **(D)** None of these

Ans. Option (C) is correct.

Explanation: It is known that a given matrix is said to be a square matrix if the number of rows is equal to the number of columns.

Therefore,

$A = [a_{if}]_{m\times n}$ is a square matrix, if $m = n$.

Q. 7. Which of the given values of x and y make the following pair of matrices equal?

$$\begin{bmatrix}3x+7 & 5\\ y+1 & 2-3x\end{bmatrix}, \begin{bmatrix}0 & y-2\\ 8 & 4\end{bmatrix}$$

(A) $x = \frac{-1}{3}, y = 7$ **(B)** Not possible to find

(C) $y = 7, x = \frac{-2}{3}$ **(D)** $x = \frac{-1}{3}, y = \frac{-2}{3}$

Ans. Option (B) is correct.

Explanation: It is given that

$\begin{bmatrix}3x+7 & 5\\ y+1 & 2-3x\end{bmatrix} = \begin{bmatrix}0 & y-2\\ 8 & 4\end{bmatrix}$

Equating the corresponding elements, we get:

$3x + 7 = 0$

$\Rightarrow \quad x = -\frac{7}{3}$

$5 = y - 2$

$\Rightarrow \quad y = 7$

$y + 1 = 8$

$\Rightarrow \quad y = 7$

$2 - 3x = 4$

$\Rightarrow \quad x = -\frac{2}{3}$

We find that on comparing the corresponding elements of the two matrices, we get two different values of x, which is not possible.

Hence, it is not possible to find the values of x and y for which the given matrices are equal.

Q. 8. The number of all possible matrices of order 3×3 with each entry 0 or 1 is:

(A) 27 **(B)** 18

(C) 81 **(D)** 512

Ans. Option (D) is correct.

Explanation: The given matrix of the order 3×3 has 9 elements and each of these elements can be either 0 or 1.

Now, each of the 9 elements can be filled in two possible ways.

Therefore, by the multiplication principle, the required number of possible matrices is $2^9 = 512$.

Q. 9. Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$

(B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$

(D) $k = 2, p = 3$

Ans. Option (A) is correct.

Explanation: Matrices P and Y are of the orders $p \times k$ and $3 \times k$, respectively. Therefore, matrix PY will be defined if $k = 3$. Consequently, PY will be of the order $p \times k$. Matrices W and Y are of the orders $n \times 3$ and $3 \times k$ respectively.

Since the number of columns in W is equal to the number of rows in Y, matrix WY is well-defined and is of the order $n \times k$. Matrices PY and WY can be added only when their orders are the same.

However, PY is of the order $p \times k$ and WY is of the order $n \times k$. Therefore, we must have $p = n$. Thus, $k = 3$ and $p = n$ are the restrictions on n, k, and p so that PY + WY will be defined.

Q. 10. Assume X, Y, Z W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. If $n = p$, then the order of the matrix 7X – 5Z is:

(A) $p \times 2$ **(B)** $2 \times n$

(C) $n \times 3$ **(D)** $p \times n$

Ans. Option (B) is correct.

Explanation: Matrix X is of the order $2 \times n$.

Therefore, matrix 7X is also of the same order.

Matrix Z is of the order $2 \times p$, i.e., $2 \times n$ [Since $n = p$]

Therefore, matrix 5Z is also of the same order.

Now, both the matrices 7X and 5Z are of the order $2 \times n$.

Thus, matrix 7X – 5Z is well-defined and is of the order $2 \times n$.

Q. 11. If A and B are symmetric matrices of same order, AB – BA is a:

(A) Skew-symmetric matrix

(B) Symmetric matrix

(C) Zero matrix

(D) Identity matrix

Ans. Option (A) is correct.

Explanation: Given that,

A and B are symmetric matrices.

$\Rightarrow$ $A = A'$ and $B = B'$

Now, $(AB - BA)' = (AB)' - (BA)'$ ...(i)

$\Rightarrow$ $(AB - BA)' = B'A' - A'B'$ [By reversal law]

$\Rightarrow$ $(AB - BA)' = BA - AB$ [From Eq. (i)]

$\Rightarrow$ $(AB - BA)' = -(AB - BA)$

Q. 12. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$

(B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$

(D) $1 + \alpha^2 - \beta\gamma = 0$

Ans. Option (C) is correct.

Explanation:

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$$

$$\therefore A^2 = A \cdot A$$

$$= \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 + \beta\gamma & \alpha\beta - \alpha\beta \\ \alpha\gamma - \alpha\gamma & \beta\gamma + \alpha^2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \beta\gamma + \alpha^2 \end{bmatrix}$$

Now,

$$A^2 = I$$

$$\begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \beta\gamma + \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing the corresponding elements, we have:

$$\alpha^2 + \beta\gamma = 1$$

$$\Rightarrow \alpha^2 + \beta\gamma - 1 = 0$$

$$\Rightarrow 1 - \alpha^2 - \beta\gamma = 0$$

Q. 13. If the matrix A is both symmetric and skew symmetric, then

(A) A is a diagonal matrix

(B) A is a zero matrix

(C) A is a square matrix

(D) None of these

Ans. Option (B) is correct.

Explanation: If A is both symmetric and skew-symmetric matrices, then we should have,

$$A' = A \text{ and } A' = -A$$

$$\Rightarrow A = -A$$

$$\Rightarrow A + A = 0$$

$$\Rightarrow 2A = 0$$

$$\Rightarrow A = 0$$

Therefore, A is a zero matrix.

Q. 14. If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(A) A **(B)** $I - A$

(C) I **(D)** $3A$

Ans. Option (C) is correct.

Explanation:

$$(I+A)^3 - 7A = I^3 + A^3 + 3I^2A + 3A^2I - 7A$$
$$= I + A^3 + 3A + 3A^2 - 7A$$
$$= I + A^2 \cdot A + 3A + 3A - 7A \quad [\because A^2 = A]$$
$$= I + A \cdot A - A$$
$$= I + A^2 - A$$
$$= I + A - A$$
$$= I$$
$$\therefore (I+A)^3 - 7A = I$$

Q. 15. The matrix $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a:

(A) square matrix **(B)** diagonal matrix
(C) unit matrix **(D)** None of these

Ans. Option (A) is correct.

Explanation: We know that, in a square matrix number of rows is equal to the number of columns.

So, the matrix $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a square matrix.

Q. 16. Total number of possible matrices of order 3 × 3 with each entry 2 or 0 is:

(A) 9 **(B)** 27
(C) 81 **(D)** 512

Ans. Option (D) is correct.

Explanation: Total number of possible matrices of order 3 × 3 with each entry 2 or 0 is 2^9 i.e., 512.

Q. 17. If $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of $x + y$ is:

(A) 4 **(B)** 5
(C) 6 **(D)** 8

Ans. Option (B) is correct.

Explanation: We have,

$$4x = x + 6$$
$$\Rightarrow \quad x = 2$$
And $$4x = 7y - 13$$
$$\Rightarrow \quad 8 = 7y - 13$$
$$\Rightarrow \quad 7y = 21$$
$$\Rightarrow \quad y = 3$$
$$\therefore \quad x + y = 2 + 3 = 5$$

Q. 18. If A and B are two matrices of the order 3 × *m* and 3 × *n*, respectively, and *m* = *n*, then the order of matrix (5A – 2B) is:

(A) $m \times 3$ **(B)** 3×3
(C) $m \times n$ **(D)** $3 \times n$

Ans. Option (D) is correct.

Explanation: $A_{3 \times m}$ and $B_{3 \times n}$ are two matrices. If $m = n$, then A and B have same orders $3 \times n$ as each, so the order of (5A − 2B) should be same as $3 \times n$.

Q. 19. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then A^2 is equal to

(A) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(B)** $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
(C) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ **(D)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ans. Option (D) is correct.

Explanation:

$$A^2 = A \cdot A$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q. 20. If matrix $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = 1$ if $i \neq j$, and 0 if $i = j$ then A^2 is equal to

(A) I **(B)** A
(C) 0 **(D)** None of these

Ans. Option (A) is correct.

Explanation: We have,

$$A = [a_{ij}]_{2 \times 2},$$

where $a_{ij} = 1$ if $i \neq j$ and 0 if $i = j$.

$$\therefore A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

And

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= I$$

Q. 21. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a:

(A) identity matrix
(B) symmetric matrix
(C) skew-symmetric matrix
(D) None of these

Ans. Option (B) is correct.

Explanation:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\therefore A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = A$$

So, the given matrix is a symmetric matrix.

[Since, in a square matrix A, if $A' = A$, then A is called symmetric matrix.]

Q. 22. The matrix $\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$ is a:

(A) diagonal matrix
(B) symmetric matrix

(C) skew symmetric matrix
(D) scalar matrix

Ans. Option (C) is correct.

Explanation: We know that, in a square matrix, if b_{ij}, when $i \neq j$ then it is said to be a diagonal matrix. Here, b_{12}, b_{13}.... $\neq 0$ so the given matrix is not a diagonal matrix.

Now, $B = \begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$

$$B' = \begin{bmatrix} 0 & 5 & -8 \\ -5 & 0 & -12 \\ -8 & 12 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$$

$$= -B$$

So, the given matrix is a skew-symmetric matrix, since we know that in a square matrix B, if $B' = -B$, then it is called skew-symmetric matrix.

Q. 23. If A is matrix of order $m \times n$ and B is a matrix such that AB′ and B′A are both defined, then order of matrix B is:

(A) $m \times m$ **(B)** $n \times n$

(C) $n \times m$ **(D)** $m \times n$

Ans. Option (D) is correct.

Explanation: Let, $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{p\times q}$

$$B' = \left[b_{ji}\right]_{q\times p}$$

Now, AB′ is defined, so $n = q$

and $B'A$ is also defined, so $p = m$

$\therefore$ Order of $B' = \left[b_{ji}\right]_{n\times m}$

And order of $B = \left[b_{ij}\right]_{m\times n}$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. If $A = \begin{pmatrix} 0 & 3 \\ 2 & -5 \end{pmatrix}$ and $kA = \begin{pmatrix} 0 & 4a \\ -8 & 5b \end{pmatrix}$, find the values of k and a. R&U

Sol. $A = \begin{bmatrix} 0 & 3 \\ 2 & -5 \end{bmatrix}$

$$kA = \begin{bmatrix} 0 & 3k \\ 2k & -5k \end{bmatrix}$$

But given $kA = \begin{bmatrix} 0 & 4a \\ -8 & 5b \end{bmatrix}$

$$\therefore \begin{bmatrix} 0 & 3k \\ 2k & -5k \end{bmatrix} = \begin{bmatrix} 0 & 4a \\ -8 & 5b \end{bmatrix} \quad ½$$

Equating individual terms,

$$2k = -8 \Rightarrow k = -4$$

$$3k = 4a$$

$$3 \times (-4) = 4a \Rightarrow a = -3 \quad ½$$

Q. 2. Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3. A I

Q. 3. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$. U

Sol. Given $A^2 = I$

$\therefore (A - I)^3 + (A + I)^3 - 7A$

$= A^3 - I^3 - 3A^2 + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$ ½

$= AI - I - 3I + 3A + AI + I + 3I + 3A - 7A$

$= A + 3A + A + 3A - 7A = A$ ½

Q. 4. If $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$, write the value of $x + y + z$. R&U

Sol. $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$

By equating $z + 6 = 0$ and $x + y = 6$ ½

or $z = -6$, $x + y = 6$

$x + y + z = 6 - 6$

$\therefore x + y + z = 0$ ½

Q. 5. If $(2x \;\; 4)\begin{pmatrix} x \\ -8 \end{pmatrix} = 0$, find the positive value of x. U

Q. 6. If $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$, write the value of $a - 2b$. U

Sol. $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$

By equating, $a + 4 = 2a + 2$ or $a = 2$ ½

$3b = b + 2$ or $b = 1$

$\therefore a - 2b = 2 - 2(1)$

$= 2 - 2$

$= 0$ ½

Q. 7. If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the matrix A. A I

Sol. Given $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

or $3A - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$

These questions are for practice and their solutions are available at the end of the chapter

or $3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ ½

or $3A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$

or $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ ½

Q. 8. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ -1 \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, **find** $x + y + z$. AI

Sol. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$x = 1, y = -2, z = 1$

or $x + y + z = 0$

Commonly Made Error

- Mostly candidates commit errors while calculation as they confuse in multiplication of two matrices.

Answering Tip

- Give ample practice on problems based on multiplication of two matrices.

Q. 9. Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ **is given to be symmetric, find values of** a **and** b. U

Q. 10. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ **is skew symmetric. Find the values of 'a' and 'b'.** AI

Sol. A is a skew symmetric matrix

$\Rightarrow \quad A' = -A$

$$A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$$

$A' = -A$

Now, $\begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}$$

On comparing we get,

$a = -2$

$b = 3$ **1**

Q. 11. Write a 2 × 2 matrix which is both symmetric and skew symmetric. U

Sol. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 2 × 2 symmetric as well as skew symmetric matrix. **1**

Q. 12. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{pmatrix}$ **is a matrix satisfying** $AA' = 9I$, **find** x. AI

Sol. Given, $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{pmatrix}$

$$A' = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & x & -1 \end{pmatrix}$$

Since, $AA' = 9I$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & x & -1 \end{pmatrix} = 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+4+4 & 2+2+2x & -2+4-2 \\ 2+2+2x & 4+1+x^2 & -4+2-x \\ -2+4-2 & -4+2-x & 4+4+1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 2x+4 & 0 \\ 2x+4 & x^2+5 & -x-2 \\ 0 & -x-2 & 9 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$ ½

Now, $x^2 + 5 = 9$

$x^2 = 9 - 5$

$x^2 = 4$

$x = \sqrt{4}$

$x = \pm 2$

Also, $2x + 4 = 0$

$2x = -4$

$x = -\frac{4}{2}$

So, $x = -2$ ½

Q. 13. If $(2 \quad 1 \quad 3)\begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = A$, **then write the order of matrix** A. U

Sol. We have,

$$A = (2 \quad 1 \quad 3)\begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= (-2-1 \quad 1+3 \quad -2+3)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = (-3 \quad 4 \quad 1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$= (-3-1) = (-4)_{1\times 1}$

$\therefore$ Order of matrix A is 1 × 1. **1**

OR

Let $P = (2\ 1\ 3)$, $Q = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

order of $PQ = 1 \times 3$

order of $PQR(A) = 1 \times 1$

Short Answer Type Questions-I (2 marks each)

Q. 1. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = 5A + kI$. R&U

Sol. $A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} \quad ½$$

$$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix},$$

$$kI = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \quad ½$$

$\because \quad A^2 - 5A = kI$

$$\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \Rightarrow k = -7 \quad 1$$

Q. 2. Find a matrix A such that $2A - 3B + 5C = 0$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$. R

Q. 3. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find $(A^2 - 5A)$. R

Sol. Given, $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad ½$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} \quad ½$$

Now, $A^2 - 5A = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ ½

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix} \quad ½$$

Q. 4. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, show that $(A - 2I)(A - 3I) = 0$. R

Sol. Given, $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$$\therefore A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2 & 2+0 \\ -1+0 & 1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \quad ½$$

and $A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 2+0 \\ -1+0 & 1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \quad ½$$

Now, LHS $= (A - 2I)(A - 3I)$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 4-4 \\ -1+1 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad ½$$

$= 0 =$ RHS ½

Q. 5. Show that $A'A$ and AA' are both symmetric matrices for any matrix A. A I

Q. 6. Prove that the diagonal elements of a skew symmetric matrix are all zero. U

Sol. Let A be a skew-symmetric matrix. Then by definition $A' = -A$ ½

or the $(i, j)^{th}$ element of A' = the $(i, j)^{th}$ element of $(-A)$ ½

or the $(j, i)^{th}$ element of A = – the $(i, j)^{th}$ element of A ½

For the diagonal elements $i = j$ or the $(i, j)$the element of A = – the $(i, j)^{th}$ element of A

or the $(i, i)^{th}$ element of $A = 0$

Hence the diagonal elements are all zero. ½

These questions are for practice and their solutions are available at the end of the chapter

Short Answer Type Questions-II (3 marks each)

Q. 1. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$. R&U

Sol. Getting $A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} \quad \textbf{1}$$

$$\therefore\ A^2 - 5A + 4I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ½$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -3 \\ -1 & -3 & -10 \\ -5 & 4 & 2 \end{bmatrix} \quad \textbf{1}$$

$\therefore\ X = -(A^2 - 5A + 4I)$

$$\text{or } X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix} \quad ½$$

Commonly Made Error

- Most of the students attempt this question incorrectly. They make errors while calculating A^2. Sometimes they add two matrices to solve A^2.

Answering Tip

- Stress upon developing logical and reasoning skills to apply the correct property of matrix.

Q. 2. Find matrix X so that

$$X\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$$ A I U

Q. 3. A trust fund has ₹ 35,000 is to be invested in two different types of bonds. The first bond pays 8% interest per annum which will be given to orphanage and second bond pays 10% interest per annum which will be given to an N.G.O. (Cancer Aid Society). Use matrix multiplication, determine how to divide ₹ 35,000 among two types of bonds if the trust fund obtains an annual total interest of ₹ 3,200. U

Sol. Let investment in first type of bond be ₹ x.

$\therefore$ The investment in second type of bond $= ₹\,(35{,}000 - x)$ **1**

$$\therefore\quad \begin{bmatrix} x & 35{,}000 - x \end{bmatrix}\begin{bmatrix} \frac{8}{100} \\ \frac{10}{100} \end{bmatrix} = [3{,}200] \quad \textbf{1½}$$

$$\text{or } \frac{8}{100}x + (35{,}000 - x)\frac{10}{100} = 3{,}200$$

$$\text{or } x = ₹\,15{,}000 \quad \textbf{1}$$

$\therefore$ Investment in first bond = ₹ 15,000

and Investment in second bond $= ₹(35{,}000 - 15{,}000)$

$= ₹\,20{,}000$ ½

[CBSE Marking Scheme 2015]

Commonly Made Error

- Students get wrong with the order while forming matrices.

Answering Tip

- Take the first matrix as a row matrix and the second one as a column matrix.

Q. 4. Show that the matrix $B^T AB$ is symmetric or skew-symmetric accordingly when A is symmetric or skew-symmetric. U

Sol. Case I: Let A be a symmetric matrix. Then $A^T = A$.

Now, $(B^T AB)^T = B^T A^T (B^T)^T$ [By reversal law]

$= B^T A^T B$ $[\because (B^T)^T = B]$

or $(B^T AB)^T = B^T AB$ $[\because A^T = A]$

$\therefore\ B^T AB$ is a symmetric matrix. **1½**

Case II: Let A be a skew-symmetric matrix. Then, $A^T = -A$.

Now, $(B^T AB)^T = B^T A^T (B^T)^T$ [By reversal law]

or $(B^T AB)^T = B^T A^T B$ $[\because (B^T)^T = B]$

or $(B^T AB)^T = B^T(-A)B$ $[\because A^T = -A]$

or $(B^T AB)^T = -B^T AB$ **1½**

$\therefore\ B^T AB$ is a skew-symmetric matrix.

These questions are for practice and their solutions are available at the end of the chapter

Long Answer Type Questions (5 marks each)

Q. 1. Express the matrix $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix. A I

Sol. $A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$

then $A' = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$ **1**

Let $P = \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix}$

$= \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix}$ **1**

Since $P' = P$

$\therefore$ P is a symmetric matrix. ½

Let $Q = \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$

$Q' = -\begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix} = -Q$ **1**

Since $Q' = -Q$

$\therefore$ Q is a skew symmetric matrix. ½

Also $P + Q = \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} = A$ **1**

Q. 2. Express the following matrix as a sum of a symmetric and skew symmetric matrices and verify your result:

$\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$ A

Q. 3. If $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$, then show that $A^3 - 4A^2 - 3A + 11I = 0$. R&U

Sol. $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ **1**

$= \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$

$A^3 = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$ **1**

$\therefore A^3 - 4A^2 - 3A + 11A$

$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$+ \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ **1**

$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - \begin{bmatrix} 36 & 28 & 20 \\ 4 & 16 & 4 \\ 32 & 36 & 36 \end{bmatrix} - \begin{bmatrix} 3 & 9 & 6 \\ 6 & 0 & -3 \\ 3 & 6 & 9 \end{bmatrix}$

$+ \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ **1**

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= 0$ **Hence Proved. 1**

Q. 4. Find matrix X if : $X\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \\ 11 & 10 & 9 \end{pmatrix}$ R&U

Sol. Clearly order of X is 3×2

Let $X = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$

These questions are for practice and their solutions are available at the end of the chapter

So $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \\ 11 & 10 & 9 \end{pmatrix}$ **1**

$\left.\begin{matrix} a+4b=-7 & c+4d=2 & e+4f=11 \\ 2a+5b=-8 & 2c+5d=4 & 2e+5f=10 \end{matrix}\right\}$ **1**

$3a + 6b = -9, 3c + 6d = 6, 3e + 6f = 9$

Solving we get

$a = 1, b = -2, c = 2, d = 0, e = -5, f = 4$ **2**

Thus, $X = \begin{pmatrix} 1 & -2 \\ 2 & 0 \\ -5 & 4 \end{pmatrix}$ **1**

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

Three schools A, B and C organised a ***mela*** for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycle material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below: A

School Articles	A	B	C
Fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Q. 1. Which approach we used to solve the given problem:

(A) Matrix addition

(B) Matrix multiplication

(C) Transpose of matrix

(D) None of these

Ans. Option (B) is correct.

Explanation:

$$\begin{matrix} & HF & M & P \\ A \\ B \\ C \end{matrix}\begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

Q. 2. Fund collected by School A is:

(A) ₹ 7,000 **(B)** ₹ 6,000

(C) ₹ 6,125 **(D)** ₹ 2,000

Ans. Option (A) is correct.

Explanation:

$$\text{Since, } \begin{matrix} & HF & M & P \\ A \\ B \\ C \end{matrix}\begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

So, Fund collected by School A

$$= \begin{bmatrix} 40 & 50 & 20 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

$= 40(25) + 50(100) + 20(50)$

$= 1000 + 5000 + 1000$

$= ₹ 7000$

Q. 3. Fund collected by School B is:

(A) ₹ 7,000 **(B)** ₹ 6,000

(C) ₹ 6,125 **(D)** ₹ 2,000

Ans. Option (C) is correct.

Explanation:

$$\text{Since, } \begin{matrix} & HF & M & P \\ A \\ B \\ C \end{matrix}\begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

So, Fund collected by School B

$$= \begin{bmatrix} 25 & 40 & 30 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

$= 25(25) + 40(100) + 30(50)$

$= 625 + 4000 + 1500$

$= ₹ 6125$

Q. 4. Fund collected by School C is:

(A) ₹ 7,000 **(B)** ₹ 7,875

(C) ₹ 6,125 **(D)** ₹ 21,000

Ans. Option (B) is correct.

Explanation:

$$\text{Since, } \begin{matrix} & HF & M & P \\ A \\ B \\ C \end{matrix}\begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

So, Fund collected by School C

$$= \begin{bmatrix} 35 & 50 & 40 \end{bmatrix}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

$= 35(25) + 50(100) + 40(50)$
$= 875 + 5000 + 2000$
$= ₹\,7875$

Q. 5. Total fund collected by all three school is:
(A) ₹ 7,000 **(B)** ₹ 7,875
(C) ₹ 6,125 **(D)** ₹ 21,000

Ans. Option (D) is correct.

Explanation:

Total collection: ₹ 7,000 + ₹ 6,125 + ₹ 7,875
= ₹ 21,000

II. Read the following text and answer the following questions on the basis of the same:

A manufacture produces three stationery products Pencil, Eraser and Sharpener which he sells in two markets. Annual sales are indicated below.

Market	Products (in numbers)		
	Pencil	Eraser	Sharpener
A	10,000	2,000	18,000
B	6,000	20,000	8,000

If the unit Sale price of Pencil, Eraser and Sharpener are ₹2.50, ₹1.50 and ₹1.00 respectively, and unit cost of the above three commodities are ₹2.00, ₹1.00 and ₹ 0.50 respectively, then, **[CBSE QB 2021]**

Q. 1. Total revenue of market *A*:
(A) ₹ 64,000 **(B)** ₹ 60,400
(C) ₹ 46,000 **(D)** ₹ 40600

Ans. Option (C) is correct.

Explanation: Total revenue of

$$= [10,000 \quad 2,000 \quad 18,000]\begin{bmatrix}2.50\\1.50\\1.00\end{bmatrix}$$
$$= 2.50\times 10,000 + 1.50\times 2,000 + 1.00\times 18,000$$
$$= ₹46,000$$

Q. 2. Total revenue of market *B* is:
(A) ₹ 35,000 **(B)** ₹ 53,000
(C) ₹ 50,300 **(D)** ₹ 30,500

Ans. Option (B) is correct.

Explanation: Total revenue of market B

$$= [6,000 \quad 20,000 \quad 8,000]\begin{bmatrix}2.50\\1.50\\1.00\end{bmatrix}$$
$$= 2.50\times 6,000 + 1.50\times 20,000 + 1.00\times 8,000$$
$$= ₹53,000$$

Q. 3. Cost incurred in market *A*:
(A) ₹ 13,000 **(B)** ₹ 30,100
(C) ₹ 10,300 **(D)** ₹ 31,000

Ans. Option (D) is correct.

Explanation: Cost incurred in market A

$$= [10,000 \quad 2,000 \quad 18,000]\begin{bmatrix}2.00\\1.00\\0.50\end{bmatrix}$$
$$= 2.00\times 10,000 + 1.00\times 2,000 + 0.50\times 18,000$$
$$= ₹31,000$$

Q. 4. Profits in market A and B respectively are:
(A) (₹15,000, ₹17,000) **(B)** (₹17,000, ₹15,000)
(C) (₹51,000, ₹71,000) **(D)** (₹10,000, ₹20,000)

Ans. Option (A) is correct.

Explanation: Cost incurred in market B

$$= [6,000 \quad 20,000 \quad 8,000]\begin{bmatrix}2.00\\1.00\\0.50\end{bmatrix}$$
$$= [2.00\times 6,000 + 1.00\times 20,000 + 0.50\times 8,000]$$
$$= [36,000]$$

Profit of market A & B = total revenue of A and B – Cost increased in market A and B

$$\begin{bmatrix}A\\B\end{bmatrix} = \begin{bmatrix}46,000\\50,000\end{bmatrix} - \begin{bmatrix}31,000\\36,000\end{bmatrix}$$
$$= \begin{bmatrix}15,000\\17,000\end{bmatrix}$$

i.e., (₹15,000, ₹17,000)

Q. 5. Gross profit in both markets is:
(A) ₹23,000 **(B)** ₹20,300
(C) ₹32,000 **(D)** ₹30,200

Ans. Option (C) is correct.

Explanation:

Gross profit in both markets = Profit in A + Profit in B
= 15,000 + 17,000 = ₹32,000

III. Read the following text and answer the following questions on the basis of the same:

Amit, Biraj and Chirag were given the task of creating a square matrix of order 2.

Below are the matrices created by them. A, B, C are the matrices created by Amit, Biraj and Chirag respectively.

$$A = \begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix}\ B = \begin{bmatrix}4 & 0\\1 & 5\end{bmatrix}\ C = \begin{bmatrix}2 & 0\\1 & -2\end{bmatrix}$$

If $a = 4$ and $b = -2$

Q. 1. Sum of the matrices *A*, *B* and *C*, *A* + (*B* + *C*) is

(A) $\begin{bmatrix}1 & 6\\2 & 7\end{bmatrix}$ **(B)** $\begin{bmatrix}6 & 1\\7 & 2\end{bmatrix}$

(C) $\begin{bmatrix}7 & 2\\1 & 6\end{bmatrix}$ **(D)** $\begin{bmatrix}2 & 1\\7 & 6\end{bmatrix}$

Ans. Option (C) is correct.

Explanation:

$$A + (B + C) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \left(\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix}$$

Q. 2. $(A^T)^T$ is equal to

(A) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ **(B)** $\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **(D)** $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

Ans. Option (A) is correct.

Explanation: $(A^T) = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$(A^T)^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

Q. 3. $(bA)^T$ is equal to

(A) $\begin{bmatrix} -2 & -4 \\ 2 & -6 \end{bmatrix}$ **(B)** $\begin{bmatrix} -2 & 2 \\ -4 & -6 \end{bmatrix}$

(C) $\begin{bmatrix} -2 & 2 \\ -6 & -4 \end{bmatrix}$ **(D)** $\begin{bmatrix} -6 & -2 \\ 2 & 4 \end{bmatrix}$

Ans. Option (B) is correct.

Explanation: $bA = -2A = \begin{bmatrix} -2 & -4 \\ 2 & -6 \end{bmatrix}$

$$(bA)^T = \begin{bmatrix} -2 & 2 \\ -4 & -6 \end{bmatrix}$$

Q. 4. $AC - BC$ is equal to

(A) $\begin{bmatrix} -4 & -6 \\ -4 & 4 \end{bmatrix}$ **(B)** $\begin{bmatrix} -4 & -4 \\ 4 & -6 \end{bmatrix}$

(C) $\begin{bmatrix} -4 & -4 \\ -6 & 4 \end{bmatrix}$ **(D)** $\begin{bmatrix} -6 & 4 \\ -4 & -4 \end{bmatrix}$

Ans. Option (C) is correct.

Explanation:

$$AC = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ 1 & -6 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$$

$$AC - BC = \begin{bmatrix} 4 & -4 \\ 1 & -6 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -4 \\ -6 & 4 \end{bmatrix}$$

Q. 5. $(a + b)B$ is equal to

(A) $\begin{bmatrix} 0 & 8 \\ 10 & 2 \end{bmatrix}$ **(B)** $\begin{bmatrix} 2 & 10 \\ 8 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$ **(D)** $\begin{bmatrix} 2 & 0 \\ 8 & 10 \end{bmatrix}$

Ans. Option (C) is correct.

Explanation: $(a + b)B = (4 - 2)\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$$

IV. Read the following text and answer the following questions on the basis of the same:

Two farmers Ramakishan and Gurucharan Singh cultivate only three varieties of rice namely Basmati, Permal and Naura. The sale (in ₹) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B **[CBSE QB 2021]**

September sales (in Rupees).

$$A = \begin{bmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} \begin{matrix} \text{Ramakrishan} \\ \text{Gurcharan} \end{matrix}$$

October sales (in Rupees)

$$B = \begin{bmatrix} 5,000 & 10,000 & 6,000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \begin{matrix} \text{Ramakrishan} \\ \text{Gurcharan} \end{matrix}$$

Q. 1. The total sales in September and October for each farmer in each variety can be represented as _____.

(A) A + B **(B)** A – B

(C) A > B **(D)** A < B

Ans. Option (A) is correct.

Explanation: Combined sales in September and October for each farmer in each variety is given by $A + B =$

$$\begin{matrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ \begin{bmatrix} 15,000 & 30,000 & 36,000 \\ 70,000 & 40,000 & 20,000 \end{bmatrix} & \begin{matrix} \text{Ramkrishan} \\ \text{Gurcharan singh} \end{matrix} \end{matrix}$$

Q. 2. What is the value of A_{23}?

(A) 10,000 **(B)** 20,000

(C) 30,000 **(D)** 40,000

Ans. Option (A) is correct.

Explanation: $A_{23} = 10,000$

Q. 3. The decrease in sales from September to October is given by _____ .

(A) A + B **(B)** A – B

(C) A > B **(D)** A < B

Ans. Option (B) is correct.

Explanation: Change in sales from September to October is given by

$A - B =$

$$\begin{matrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ \begin{bmatrix} 5000 & 10,000 & 24,000 \\ 30,000 & 20,000 & 0 \end{bmatrix} & & \begin{matrix} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{matrix} \end{matrix}$$

Q. 4. If Ramkishan receives 2% profit on gross sales, compute his profit for each variety sold in October.

(A) ₹100, ₹200 and ₹120

(B) ₹100, ₹200 and ₹130

(C) ₹100, ₹220 and ₹120

(D) ₹110, ₹200 and ₹120

Ans. Option (A) is correct.

Explanation:

$$2\% \text{ of } B = \frac{2}{100} \times B$$

$$= 0.02 \times B$$

$$= 0.02$$

$$\begin{matrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ \begin{bmatrix} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} & & \begin{matrix} \text{Ramkishan} \\ \text{Gurcharn Singh} \end{matrix} \end{matrix}$$

Thus, in October Ramkishan receives ₹ 100, ₹ 200 and ₹ 120 as profit in the sale of each variety of rice, respectively.

Q. 5. If Gurucharan receives 2% profit on gross sales, compute his profit for each variety sold in September:

(A) ₹100, ₹200, ₹120 **(B)** ₹1000, ₹600, ₹200

(C) ₹400, ₹200, ₹120 **(D)** ₹1200, ₹200, ₹120

Ans. Option (B) is correct.

Explanation:

$$2\% \text{ of } A = \frac{2}{100} \times A$$

$$= 0.02 \times A$$

$$= 0.02$$

$$\begin{matrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ \begin{bmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} & & \begin{matrix} \text{Ramkishan} \\ \text{Gurcharn Singh} \end{matrix} \end{matrix}$$

$$= \begin{matrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ \begin{bmatrix} 200 & 400 & 600 \\ 1000 & 600 & 200 \end{bmatrix} & & \begin{matrix} \text{Ramkishan} \\ \text{Gurcharn Singh} \end{matrix} \end{matrix}$$

Thus, in September Gurucharan receives ₹ 1000, ₹ 600 and ₹ 200 as Profit in the sale of each variety of rice, respectively.

V. Read the following text and answer the following questions on the basis of the same:

There are three families *A*, *B* and *C*. The number of members in these families are given in the table below.

	Men	Women	Children
Family *A*	3	2	1
Family *B*	2	4	2
Family *C*	4	3	2

The daily expenses of each man, woman and child are respectively ₹200, ₹100 and ₹50.

Q. 1. The total daily expense of family A is _____.

(A) 850 **(B)** 900

(C) 1,200 **(D)** 2,950

Ans. Option (A) is correct.

Explanation: Expense of family *A*

$$= [3 \quad 2 \quad 1] \begin{bmatrix} 200 \\ 100 \\ 50 \end{bmatrix} = [850]$$

Q. 2. The total daily expense of family C is _____.

(A) 850 **(B)** 900

(C) 1,200 **(D)** 2,950

Ans. Option (C) is correct.

Explanation: Expense of family *C*

$$= [4 \quad 3 \quad 2] \begin{bmatrix} 200 \\ 100 \\ 50 \end{bmatrix} = [1200]$$

Q. 3. The combined daily expense of all the women is _____.

(A) 850 **(B)** 900

(C) 1,200 **(D)** 2,950

Ans. Option (B) is correct.

Explanation: Combined expense of women

$$= [2 \quad 4 \quad 3] \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} = [900]$$

Q. 4. The family with highest expense is _____.

(A) *A* **(B)** *B*

(C) *C*

(D) All have same expense

Ans. Option (C) is correct.

Explanation: Most expensive family is *C* with an expense of ₹1200.

Q. 5. The combined expense of men in family A and children in family C is _____.

(A) 600 **(B)** 700

(C) 800 **(D)** 900

Ans. Option (B) is correct.

Explanation: $[3 \quad 2]\begin{bmatrix}200\\50\end{bmatrix} = [700]$

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:

(Each Sub-part carries 2 marks)

In a city there are two factories A and B. Each factory produces sports clothes for boys and girls. There are three types of clothes produced in both the factories type I, type II and type III. For boys the number of units of types I, II and III respectively are 80, 70 and 65 in factory A and 85, 65 and 72 are in factory B. For girls the number of units of types I, II and III respectively are 80, 75, 90 in factory A and 50, 55, 80 are in factory B.

Q. 1. Write the matrices P and Q, if P represents the matrix of number of units of each type produced by factory A for both boys and girls; and Q represents the matrix of number of units of each type produced by factory B for both boys are girls.

Sol. In factory A, number of units of type I, II and III for boys are 80, 70, 65 respectively and for girls number of units of type I, II and III are 80, 75, 90 respectively.

$$\therefore \quad P = \begin{matrix} \\ \text{I} \\ \text{II} \\ \text{III} \end{matrix}\begin{matrix}\begin{matrix}\text{Boys} & \text{Girls}\end{matrix}\\\begin{bmatrix}80 & 80\\70 & 75\\65 & 90\end{bmatrix}\end{matrix} \qquad \textbf{1}$$

In factory B, number of units of type I, II and III for boys are 85, 65, 72 respectively and for girls number of units of type I, II and III are 50, 55, 80 respectively.

$$\therefore \quad Q = \begin{matrix} \\ \text{I} \\ \text{II} \\ \text{III} \end{matrix}\begin{matrix}\begin{matrix}\text{Boys} & \text{Girls}\end{matrix}\\\begin{bmatrix}85 & 50\\65 & 55\\72 & 80\end{bmatrix}\end{matrix} \qquad \textbf{1}$$

Q. 2. Find the total production of sports clothes of each type for boys and girls.

Sol. Let matrix X represent the number of units of each type produced by factory A for boys and matrix Y represents the number of units of each type produced by factory B for boys.

$$\therefore \quad X = \begin{matrix}\text{I} & \text{II} & \text{III}\\\begin{bmatrix}80 & 70 & 65\end{bmatrix}\end{matrix}$$

$$Y = \begin{matrix}\text{I} & \text{II} & \text{III}\\\begin{bmatrix}85 & 65 & 72\end{bmatrix}\end{matrix} \qquad \textbf{1}$$

Now, total production of sports clothes of each type for boys = X + Y

= [80 70 65] + [85 65 72]

= [165 135 137]

Similarly, for girls, let matrix S represents the number of units of each type produced by factory A and matrix T represents the number of units of each type produced by factory B.

$$\therefore \quad S = \begin{matrix}\text{I} & \text{II} & \text{III}\\\begin{bmatrix}80 & 75 & 90\end{bmatrix}\end{matrix}$$

$$T = \begin{matrix}\text{I} & \text{II} & \text{III}\\\begin{bmatrix}50 & 55 & 80\end{bmatrix}\end{matrix}$$

Now, required matrix = S + T

= [80 75 90] + [50 55 80]

= [130 130 170] **1**

II. Read the following text and answer the following questions on the basis of the same:

Three car dealers, say A, B and C deals in three cars, Sedan, cars Hatchback cars and SUV cars. The sales figure of 220 and 2021 showed that dealer A sold 120 Hatch back, 50 Sedan and 10 SUV cars in 2020 and 300 Hatchback, 150 Sedan and 20 SUV in 2021. Dealer B sold 100 Hatchback, 30 Sedan, 5 SUV cars in 2020 and 200 Hatchback, 50 Sedan, 6 SUV cars in 2021. Dealer C sold 90 Hatchback, 40 Sedan and 2 SUV cars in 2020 and 100 Hatchback, 60 Sedan and 5 SUV cars in 2021

Q. 1. Find the total number of cars sold in two given years, by each dealer.

Sol. Let matrix P denotes that dealer A sold 120 Hatchback, 50 sedan and 10 SUV; dealer B sold 100 Hatchback, 30 sedan and 5 SUV cars and dealer C sold 90 Hatchback, 40 sedan and 2 SUV cars in year 2020.

$$\therefore \quad P = \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{matrix}\begin{matrix}\text{Hatchback} & \text{Sedan} & \text{SUV}\end{matrix}\\\begin{bmatrix}120 & 50 & 10\\100 & 30 & 5\\90 & 40 & 2\end{bmatrix}\end{matrix} \qquad \tfrac{1}{2}$$

Similarly, matrix Q denotes the data for the year 2021.

$$\therefore \quad Q = \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{matrix}\begin{matrix}\text{Hatchback} & \text{Sedan} & \text{SUV}\end{matrix}\\ \begin{bmatrix} 300 & 150 & 20 \\ 200 & 50 & 6 \\ 100 & 60 & 5 \end{bmatrix}\end{matrix} \qquad ½$$

Now, total number of cars sold in two given years, by each dealer, is given by

$$P + Q = \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{matrix}\begin{matrix}\text{Hatchback} & \text{Sedan} & \text{SUV}\end{matrix}\\ \begin{bmatrix} 120+300 & 50+150 & 10+20 \\ 100+200 & 30+50 & 5+6 \\ 90+100 & 40+60 & 2+5 \end{bmatrix}\end{matrix}$$

$$= \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{matrix}\begin{matrix}\text{Hatchback} & \text{Sedan} & \text{SUV}\end{matrix}\\ \begin{bmatrix} 420 & 200 & 30 \\ 300 & 80 & 11 \\ 190 & 100 & 7 \end{bmatrix}\end{matrix} \qquad \mathbf{1}$$

Q. 2. If each dealer receive profit of ₹ 50,000 on sale of a Hatch car, ₹ 1,00,000 on sale of a sedan car and ₹ 1,50,000 on a sale of a SUV car, then find the amount of profit received in year 2021 by each dealer.

Sol. The amount of profit in year 2021 received by each dealer is given by the matrix.

$$\begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{matrix}\begin{matrix}\text{Hatchback} & \text{Sedan} & \text{SUV}\end{matrix}\\ \begin{bmatrix} 300 & 150 & 20 \\ 200 & 50 & 6 \\ 100 & 60 & 5 \end{bmatrix}\end{matrix}\begin{bmatrix} 50000 \\ 100000 \\ 150000 \end{bmatrix}$$

$$= \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{bmatrix} 15000000+15000000+3000000 \\ 10000000+5000000+900000 \\ 5000000+6000000+750000 \end{bmatrix}$$

$$= \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{bmatrix} 33000000 \\ 15900000 \\ 11750000 \end{bmatrix}$$

Q. 3. Show that A'A' and AA' are both symmetric matrices for any matrix A.

Sol. Let, $P = A'A$

$\therefore \quad P' = (A'A)'$

$= A'(A')' \qquad [\because (AB')' = B'A']$

$= A'A = P$

So, A'A is symmetric matrix for any matrix A.

Similarly,

Let $Q = AA'$

$Q' = (AA')' = (A')'A'$

$= AA' = A$

So, AA' is symmetric matrix for any matrix A.

Solutions for Practice Questions

Very Short Answer Type Questions

Sol. 2: Number of elements of 2×2 matrix = 4 ½

$\therefore$ Number of ways to write 1, 2 or 3 at 4 places

$= 3^4$

$= 81$

$\therefore$ 81 matrices of order 2×2 are possible with each entry 1, 2 or 3. ½

Sol. 5: $$(2x \quad 4)\begin{pmatrix} x \\ -8 \end{pmatrix} = 0$$

or $(2x^2 - 32) = 0$ ½

By equating,

$2x^2 - 32 = 0$

or $2x^2 = 32$

or $x^2 = 16$

$x = \pm 4$

or $x = +4$ ½

$\therefore$ Positive value of $x = 4$

Sol. 9: As A is a symmetric matrix,

or $A' = A$ ½

$$\therefore \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$$

$\therefore$ By equality of matrices, $a = \dfrac{-2}{3}$ and $b = \dfrac{3}{2}$. ½

Commonly Made Error

- Few students commit error in solving the problems based on symmetric matrix.

Answering Tip

- Learn the difference between symmetric and skew symmetric matrices.

Short Answer Type Questions-I

Sol. 2: Given, $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$

Also, given $2A - 3B + 5C = 0$

$\Rightarrow 2A = 3B - 5C$ ½

$$\Rightarrow 2A = 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix} \qquad ½$$

$$\Rightarrow 2A = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} + \begin{bmatrix} -10 & 0 & 10 \\ -35 & -5 & -30 \end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix} \qquad ½$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix} \qquad ½$$

Sol. 5: Let $P = A'A$

$\therefore \quad P' = (A'A)'$

$= A'(A')' \qquad [\because (AB)' = B'A']$

$= A'A = P$

So, $A'A$ is symmetric matrix for any matrix A. **1**

Similarly,

Let $Q = AA'$

$Q' = (AA')'$

$= (A')'A'$

$= AA'$

$= A$

So, AA' is symmetric matrix for any matrix A. **1**

Short Answer Type Questions-II

Sol. 2: Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ½

then, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$ **1**

$a + 4b = -7$ $\quad 3a + 6b = -9$ $\quad c + 4d = 2$

$2a + 5b = -8$ $\quad 3c + 6d = 6$ $\quad 2c + 5d = 4$

Equating and solving to get $a = 1, b = -2, c = 2, d = 0$ **1**

$X = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$ ½

Commonly Made Error

- Some students take $X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ but this is wrong find the order of X using the order of A and B.

$XA = B$

$(X_{p\times q})\,(A)_{2\times 3} = B_{2\times 3}$

$p = 2, q = 2$

Answering Tip

- Learn the properties of matrix multiplication.

Long Answer Type Questions

Sol. 2: We know that

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Here, $\frac{1}{2}(A + A')$ is symmetric matrix and $\frac{1}{2}(A - A')$ is skew symmetric matrix. **1**

Now, $A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$\Rightarrow A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ **1**

$\therefore \quad \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 3+3 & -2+3 & -4-1 \\ 3-2 & -2-2 & -5+1 \\ -1-4 & 1-5 & 2+2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$

which is symmetric. **1**

$\frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 3-3 & -2-3 & -4+1 \\ 3+2 & -2+2 & -5-1 \\ -1+4 & 1+5 & 2-2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$

which is skew-symmetric. **1**

$\therefore \quad A = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$ **1**

REFLECTIONS

(a) Do you know matrices are used in Cryptography for the process of encryption?

(b) Can you apply matrices in study of electrical circuit ?

(c) Will you be able to solve complex problem of matrix multiplication ?

CHAPTER

4

DETERMINANTS

Syllabus

Determinant, Inverse of Matrix, Solving system of Simultaneous equations using matrix method, Cramer's rule.

In this chapter you will study

- *Define the inverse of a square matrix*
- *Apply properties of inverse of matrices*
- *Solve the system of simultaneous equations using i) Cramer's Rule ii) Inverse of coefficient matrix*
- *Formulate real life problems into a system of simultaneous linear equations and solve it using these methods*

Revision Notes

Determinant: A unique number (real or complex) can be associated to every square matrix $A = [a_{ij}]$ of order m. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

The determinant of matrix A is denoted by det A or $|A|$.

- **Determinant of a square matrix of order 2**

  If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2,

  then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$

  It follows that the value of a determinant of order 2 is equal to the product of the elements along the principal diagonal *minus* the product of the off diagonal elements.

Example 1: Let $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$, then

$$|A| = 5 \times 3 - (-2) \times 4 = 15 + 8 = 23.$$

- **Determinant of a square matrix of order 3**

  If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3,

  then $|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

  or $|A| = a_{11}(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}(a_{33}\,a_{21} - a_{23}\,a_{31}) + a_{13}(a_{21}\,a_{32} - a_{22}\,a_{31})$

  It follows that the value of a determinant of order 3 is the sum of the product of elements a_{ij} in first row with $(-1)^{i+j}$ times the determinant of a 2×2 sub matrix obtained by leaving the first row and column passing through elements.

Elementary properties of Determinants

Property I: The value of a determinant is not altered by inter changing its rows into columns and columns into rows, i.e.,

Determinants

Cramer's Rule

The solution of the system $a_1x + b_1y + c_1z + d$, $a_2x + b_2y + c_2z + d_2$ and $a_3x + b_3y + c_3z + d_3$ is given by $x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$ and $z=\frac{D_z}{D}$, where

$$\text{If } D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Provided that $D \neq 0$

Determinant of Square Matrix

A unique number (real or complex) can be associated, to every square matrix $A=[a_{ij}]$ of order m, is called determinant of matrix A. It is denoted by $|A|$ or det (A)

Expansion of a Determinant

of Order 2

$$|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}.a_{22}-a_{21}.a_{12}$$

of Order 3

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}(a_{22}.a_{33}-a_{32}.a_{23})-a_{12}$$
$$(a_{21}.a_{33}-a_{31}.a_{23})+a_{13}(a_{21}.a_{32}-a_{31}.a_{22})$$

Minors and Cofactors

Minors

If $|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of $a_{11}=M_{11}$

$=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, Minor of $a_{12}=M_{12}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Cofactors

If the minor M_{ij} is multiplied by $(-1)^{i+j}$, then we obtain the factor of a_{ij}. It is denoted by A_{ij} or C_{ij}, i.e., $A_{ij}=(-1)^{i+j}$ M_{ij}, cofactor of $a_{11}=(-1)^{1+1}M_{11}$ or $A_{11}=\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

Inverse of Square Matrix

Statement

Let A be a non-singular (*i.e.* $|A| \neq 0$) square matrix. Then, a square matrix B, such that $AB = BA = I$ is called of inverse of A and it is denoted by A^{-1}

i.e. $A^{-1}=\frac{1}{|A|}\text{adj}(A)$.

Properties

Let A and B be square matrices of same order, then

(i) A square matrix is invertible if and only if it is non-singular

(ii) $(A^{-1})^{-1}=A$

(iii) $(A')^{-1}=(A^{-1})'$

(iv) $(AB)^{-1}=B^{-1}A^{-1}$

(v) $|A^{-1}|=|A|^{-1}$

(vi) $AA^{-1}=A^{-1}A=I$

(vii) $(kA)^{-1}=\frac{1}{k}A^{-1}$, if $k \neq 0$

Properties of Determinants

If each element of a row or column is zero, then determinant becomes zero. *e.g.* $A=\begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{bmatrix}$ $\therefore |A|=0$

The value of a determinant remains unchanged if its row and columns are interchanged *e.g.* $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

If any two adjacent rows or columns are interchanged then the determinant retains its absolute value, but its sign is changed. *e.g.* $\Delta=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $\Delta_1=\begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $\Rightarrow \Delta_1=-1\Delta$

If each element of a row or a column of a determinant is multiplied by a constant k, then the value of new determinant are k times the value of the original determinant *e.g.* $\Delta=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $\Delta_1=\begin{bmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $=k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \Rightarrow \Delta_1=k\Delta$

If each element of a row (or column of determinant is the sum of the two more terms, then the determinants can be expressed as the sum of the two or more determinants

e.g. $\begin{bmatrix} a_1+a_2 & b & c \\ p_1+p_2 & q & r \\ u_1+u_2 & v & w \end{bmatrix}=\begin{bmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{bmatrix}+\begin{bmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{bmatrix}$

Singular and Non-Singular Matrix

Non-Singular

If A is a square matrix of order $n \times n$, then

(i) $A(\text{adj } A)=(\text{adj } A)A=|A|I_n$

(ii) $\text{adj}(A^T)=(\text{adj } A)^T$

(iii) $|\text{adj } A|=|A|^{n-1}$, provided $|A| \neq 0$

Singular

If A is a square matrix and $|A| \neq 0$,

then A is known as non - singular matrix.

Statement

If A is a square matrix, and $|A|=0$,

then A is known as singular matrix.

Adjoint of a Matrix

Example

Let $A=[a_{ij}]_{n \times n}$ be a square matrix of order n and C_{ij} (or A_{ij}) be the cofactor of a_{ij} in the determinant $|A|$. Then, the adjoint of A is defined as the transpose of the cofactor matrix and it is denoted by adj (A).

If $A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then adj $A=\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$,

where A_{ij}'s are coafactors of A.

Trace the Mind Map

First Level · Second Level · Third Level

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & b_3 \end{vmatrix}$$

Scan to know more about this topic

Properties of Determinants

Property II: If any two adjacent rows or columns of a determinant are interchanged, the sign of the determinant get changed but its numerical value remains unaltered, *i.e.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property III: If any two rows or columns of a determinant are same, the value of determinant is zero, i.e.,

$$\text{Let, } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ or, } \begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = 0$$

Property IV : If every element in a row or a column of a determinant is multiplied by the same non-zero constant k, then the value of the determinant gets multiplied by k, i.e.,

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property V : A determinant can be expressed as the sum of several determinants of the same order, i.e.,

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Property VI : The value of the determinant is not affected if the elements of a row or column are increased or diminished by the same multiple of the corresponding elements of any other row or column, i.e.,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1 & c_1 \\ a_2+pb_2+qc_2 & b_2 & c_2 \\ a_3+pb_3+qc_3 & b_3 & c_3 \end{vmatrix}$$

Property VII : If each element of a row or column of a determinant is zero, its value is zero, i.e.,

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

- **Singular Matrix & Non-Singular Matrix:**

(a) Singular matrix: A square matrix A is said to be singular if $|A| = 0$ i.e., its determinant is zero.

e.g., $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{vmatrix}$

$= 1(15-12) - 2(12-12) + 3(4-5)$

$= 3 - 0 - 3 = 0$

$\therefore A$ is singular matrix.

(b) Non-singular matrix : A square matrix A is said to be non-singular if $|A| \neq 0$.

Scan to know more about this topic

Minors and Co-factors

e.g., $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 0(0-1) - 1(0-1) + 1(1-0)$

$= 0 + 1 + 1 = 2 \neq 0$

$\therefore$ A is non-singular matrix.

- A square matrix A is **invertible** if and only if A is **non-singular**.

- **Minors:** Minors of an element a_{ij} of a determinant *(or a determinant corresponding to matrix* A*)* is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (i.e., 3×3) determinant.
- **Co-factors:** Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.
- **Adjoint of a Square Matrix:**

Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by "*adj* **(A)".**

To find adjoint of a 2 × 2 matrix: If the adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of off-diagonal elements i.e. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 2: consider a square matrix of order 3 as $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then in order to find the adjoint matrix A, we find a matrix B (formed by the co-factors of elements of matrix A as mentioned above in the definition)

i.e., $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj\ A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$

- **Algorithm to find A^{-1} by Determinant Method:**

Scan to know more about this topic

How to find Determinant

Step 1: Find $|A|$.

Step 2: If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is a non-singular matrix and hence invertible".

Step 3: Calculate the co-factors of elements of matrix A.

Step 4: Write the matrix of co-factors of elements of A and then obtain its transpose to get *adjA* (i.e., adjoint A).

Step 5: Find the inverse of A by using the relation:

$$A^{-1} = \frac{1}{|A|}(adj\, A).$$

Note

For a 2×2 matrix, the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } ad - bc \neq 0$$

Just swap the 'a' and 'd', negate the 'b' and 'c', then divide all by the determinant $ad - bc$.

- **Properties associated with various Operation of Matrices & Determinants:**

(a) $AB = I = BA$
(b) $AA^{-1} = I$ or $A^{-1}I = A^{-1}$
(c) $(AB)^{-1} = B^{-1}A^{-1}$
(d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
(e) $(A^{-1})^{-1} = A$
(f) $(A^T)^{-1} = (A^{-1})^T$
(g) $A(adjA) = (adj\, A)A = |A|\, I$
(h) $adj(AB) = adj(B)\, adj\, (A)$
(i) $adj(A^T) = (adj\, A)^T$
(j) $(adj\, A)^{-1} = (adj\, A^{-1})$
(k) $|adj\, A| = |A|^{n-1}$, if $|A| \neq 0$, where n is of the order of A.
(l) $|AB| = |A||B|$
(m) $|A\, adj\, A| = |A|^n$, where n is of the order of A.
(n) $|A^{-1}| = \frac{1}{|A|}$, if matrix A is invertible.
(o) $|A| = |A^T|$

- $|kA| = k^n |A|$, where n is of the order of square matrix A and k is any scalar.
- If A is a non-singular matrix of order n, then $adj\,(adj\, A) = |A|^{n-2} A$.

- **Uniqueness Theorem:** Let A be an invertible square matrix of order n. Suppose B and C are the two inverse of A.

Then $AB = BA = I_n$
(by definition of inverse matrix)
$AC = CA = I_n$
Now, $B = BI_n = B(AC)$
[$\because$ Matrix multiplication is associative]
$= (BA)C$
$= I_nC$
$= C$

$\therefore B = C$, i.e., any two inverse of A are equal matrices. Hence, the inverse of A is unique.

- **Cramer's Rule:**

Let the three nonhomogeneous linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The solution of the system of linear equations is given by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

Where,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

Provided that $D \neq 0$

- **Condition for consistency:**

(a) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

(b) If $D = 0$ and $D_x = D_y = D_z = 0$, the given system of equations is consistent has infinitely many solutions.

(c) If $D = 0$ and atleast one of the determinants D_x, D_y, D_z is non-zero, then the given system of equations is inconsistent.

Key Facts

- In mathematics, the determinant is a scalar value that is a function of the entries of a square matrix.
- There are 10 main properties of determinants which include reflection property, all-zero property, proportionality or repetition property, switching property, scalar multiple property, sum property, invariance property, factor property, triangle property and determinant of factor property.

Mnemonics

Concept: Cramer's Rule

Mnemonics:

For X	**For Y**
Interpretations:	
Every **B**oy **F**inds **D**ogs **(E*D — B*F)** divided by (A*D — B*C) for X.	**A**ll **E**lephants **C**an't **F**orget **(A*F — E*C)** divided by (A*D — B*C) for Y.

$$2x - 3y = 9$$
$$x + 5y = -2$$

Comparing the above set of equations by

$$Ax + By = E$$
$$Cx + Dy = F$$

We, get $A = 2, B = -3, C = 1, D = 5, E = 9$ and $F = -2$

So, $$x = \frac{9\times5-(-3)\times(-2)}{2\times5-(-3)\times1}$$

$$= \frac{45-6}{10+3} = \frac{39}{13} = 3$$

and $$y = \frac{2\times(-2)-9\times1}{2\times5-(-3)\times1}$$

$$= \frac{-4-9}{10+3} = \frac{-13}{13} = -1$$

So, the solution set is $(3, -1)$.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. If A is a square matrix of order 3 and $|A| = -2$, then $|adj(A)|$ is equal to

(A) –8 **(B)** –2
(C) 0 **(D)** 4

Ans. Option (D) is correct.

Explanation:

$$|\text{adj}(A)| = |A|^{n-1}$$
$$\Rightarrow |\text{adj}(A)| = (-2)^2$$
$$= 4$$

Q. 2. In a 3 × 3 matrix A, value of $a_{12}c_{13} + a_{22}c_{23} + a_{32}c_{33}$, where c_{ij} is the cofactor of a_{ij} is:

(A) 0 **(B)** –1
(C) 1 **(D)** $|A|$

Ans. Option (A) is correct.

Explanation: The summation of product of a_{ij} of 2nd column with corresponding c_{ij} of 3 column =0

Q. 3. If two square matrices A and B are such that $|AB| = 12$ and $|B| = -4$, then value of $|A|$ is:

(A) 8 **(B)** –8
(C) –3 **(D)** 16

Ans. Option (C) is correct.

Explanation:

$$|AB| = 12$$
$$\Rightarrow |A|\,|B| = 12$$
$$\Rightarrow -4|A| = 12$$
$$\Rightarrow |A| = -3$$

Q. 4. If solving a system of linear equations in 3 variables by Cramer's rule, we get $\Delta = 0$ and at least one of $\Delta_x, \Delta_y, \Delta_z$ is non-zero then the system of linear equations has

(A) no solution
(B) unique solution
(C) infinitely many solutions
(D) trivial solution

Ans. Option (A) is correct.

Explanation: If $\Delta = 0$ and at least (one of $\Delta_x, \Delta_y, \Delta_z$) $\neq 0$

The system of linear equations has no solution.

Q. 5. If A is a square matrix of order 3 × 3 such that $|A| = 4$, then $|3A|$ is equal to:

(A) 27 **(B)** 81
(C) 108 **(D)** 256

Ans. Option (C) is correct.

Explanation:

$$|3A| = (3)^3|A| \quad [\because \text{order of matrix} = 3]$$
$$= 27 \times 4 \quad [\text{Given } |A| = 4]$$
$$= 108$$

Q. 6. If $A^2 - A + I = 0$, then the inverse of matrix A is:

(A) A^2 **(B)** $A + I$
(C) $I - A$ **(D)** $A - I$

Ans. Option (C) is correct.

Explanation: We have,

$$A^2 - A + I = 0$$
$$\Rightarrow A^{-1}(A^2 - A + I) = A^{-1}(0)$$
$$\Rightarrow A^{-1}A^2 - A^{-1}A + A^{-1}I = 0$$
$$\Rightarrow A - I + A^{-1} = 0$$
$$\Rightarrow A^{-1} = I - A$$

Q. 7. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is:

(A) 3 **(B)** ±3
(C) ±6 **(D)** 6

Ans. Option (C) is correct.

Explanation: Given that

$$\therefore \begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix},$$
$$\Rightarrow 2x^2 - 40 = 18 + 14$$
$$\Rightarrow 2x^2 = 32 + 40$$
$$\Rightarrow x^2 = \frac{72}{2} = 36$$
$$\therefore x = \pm 6$$

Q. 8. If $A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$, then A^{-1} exist, if

(A) $\lambda = 2$ **(B)** $\lambda \neq 2$
(C) $\lambda \neq -2$ **(D)** None of these

Ans. Option (D) is correct.

Explanation: Given that,

$$A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$$

Expanding along R_1,

$$|A| = 2(6-5) - \lambda(-5) - 3(-2)$$
$$= 2 + 5\lambda + 6$$

We know that A^{-1} exists, if A is non-singular matrix, *i.e.*, $|A| \neq 0$

$$\therefore 2 + 5\lambda + 6 \neq 0$$
$$\Rightarrow 5\lambda \neq -8$$
$$\therefore \lambda \neq \frac{-8}{5}$$

So, A^{-1} exists if and only if $\lambda \neq \frac{-8}{5}$.

Q. 9. If A and B are invertible matrices, then which of the following is not correct?

(A) $adj.\ A = |A|.A^{-1}$

(B) $\det\left(A^{-1}\right) = \left[\det(A)\right]^{-1}$

(C) $(AB)^{-1} = B^{-1}A^{-1}$

(D) $(A+B)^{-1} \neq B^{-1} + A^{-1}$

Ans. Option (D) is correct.

Explanation: Since, A and B are invertible matrices, so, we can say that

$$(AB)^{-1} = B^{-1}A^{-1} \quad ...(i)$$

Also, $$A^{-1} = \frac{1}{|A|}(adj\ A)$$

$$\Rightarrow \quad adj\ A = A^{-1}.|A| \quad ...(ii)$$

Also, $$\det(A)^{-1} = [\det(A)]^{-1}$$

$$\Rightarrow \quad \det(A)^{-1} = \frac{1}{[\det(A)]}$$

$$\Rightarrow \det(A).\det(A)^{-1} = 1 \quad ...(iii)$$

From equation (iii), we conclude that it is true.

Again, $$(A+B)^{-1} = \frac{1}{|(A+B)|}adj\ (A+B)$$

$$\Rightarrow \quad (A+B)^{-1} \neq B^{-1} + A^{-1} \quad ...(iv)$$

Q. 10. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{5} \\ \mathbf{2} & \boldsymbol{a} & \mathbf{-1} \\ \mathbf{0} & \mathbf{4} & \mathbf{2a} \end{vmatrix} = \mathbf{86}$. Then sum of these numbers is:

(A) 4 **(B)** 5

(C) –4 **(D)** 9

Ans. Option (C) is correct.

Explanation: We have,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$$

Expanding along first column

$$\Rightarrow 1(2a^2+4) + 2(4a+0) + 5(8-0) = 86$$

$$\Rightarrow \quad 2a^2 + 4 + 8a + 40 = 86$$

$$\Rightarrow \quad 2a^2 + 8a + 44 - 86 = 0$$

$$\Rightarrow \quad a^2 + 4a - 21 = 0$$

$$\Rightarrow \quad a^2 + 7a - 3a - 21 = 0$$

$$\Rightarrow \quad (a+7)(a-3) = 0$$

$$\therefore a = -7,\ 3$$

Required sum $= -7 + 3 = -4$

Q. 11. If $\begin{vmatrix} \boldsymbol{x} & \mathbf{2} \\ \mathbf{18} & \boldsymbol{x} \end{vmatrix} = \begin{vmatrix} \mathbf{6} & \mathbf{2} \\ \mathbf{18} & \mathbf{6} \end{vmatrix}$, then x is equal to

(A) 6 **(B)** ± 6

(C) -6 **(D)** 0

Ans. Option (B) is correct.

Explanation: Given that,

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

$$\Rightarrow \quad x^2 - 36 = 36 - 36$$

$$\Rightarrow \quad x^2 \quad 36 = 0$$

$$\Rightarrow \quad x^2 = 36$$

$$\Rightarrow \quad x = \pm 6$$

Q. 12. Let A be a square matrix of order 3 × 3, then $|kA|$ is equal to

(A) $k|A|$ **(B)** $k^2|A|$

(C) $k^3|A|$ **(D)** $3k|A|$

Ans. Option (C) is correct.

Explanation: We know that, A be a square matrix of order 3 × 3

i.e., $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then, $$kA = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$$

$$\therefore \quad |kA| = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$$

Taking out common factor k from each row

$$= k^3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k^3 |A|$$

$$\therefore \quad |kA| = k^3 |A|$$

Q. 13. Which of the following is correct?

(A) Determinant is a square matrix.

(B) Determinant is a number associated to a matrix.

(C) Determinant is a number associated to a square matrix.

(D) None of these.

Ans. Option (C) is correct.

Explanation: We know that every square matrix, $A = [a_{ij}]$ of order n. We can associate a number called the determinant of a square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

Thus, the determinant is a number associated to a square matrix.

Q. 14. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is cofactor of a_{ij}, then value of Δ is given by

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Ans. Option (D) is correct.

Explanation: We know that:

Δ = Sum of the product of the elements of a column (or a row) with their corresponding co-factors

$\therefore \Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Q. 15. Let A be a non-singular square matrix of order 3×3. Then $|adj\ A|$ is equal to

(A) $|A|$ **(B)** $|A|^2$

(C) $|A|^3$ **(D)** $3|A|$

Ans. Option (B) is correct.

Explanation: We know that,

$$(adj\ A)A = |A|$$

$$I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |(adj\ A)A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$\Rightarrow |adj\ A||A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A|^3\ (I)$$

$$\therefore \quad |adj\ A| = |A|^2$$

Q. 16. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

(A) det(A) **(B)** $\dfrac{1}{\det(A)}$

(C) 1 **(D)** 0

Ans. Option (B) is correct.

Explanation: Given that A is an invertible matrix,

A^{-1} exists and $A^{-1} = \dfrac{1}{|A|} adj.A.$

As matrix A of order 2,

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $|A| = ad - bc$

and $adj.A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now,

$$A^{-1} = \frac{1}{|A|} adj.A = \begin{bmatrix} \dfrac{d}{|A|} & \dfrac{-b}{|A|} \\ \dfrac{-c}{|A|} & \dfrac{a}{|A|} \end{bmatrix}$$

$$\therefore \quad |A^{-1}| = \begin{vmatrix} \dfrac{d}{|A|} & \dfrac{-b}{|A|} \\ \dfrac{-c}{|A|} & \dfrac{a}{|A|} \end{vmatrix}$$

$$= \frac{1}{|A|^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

$$= \frac{1}{|A|^2}(ad - bc)$$

$$= \frac{1}{|A|^2} \cdot |A|$$

$$= \frac{1}{|A|}$$

$$\therefore \det(A^{-1}) = \frac{1}{\det(A)}$$

Q. 17. If $|A| = 0$, then

(A) Singular Matrix **(B)** Non-Singular Matrix

(C) Zero Matrix **(D)** None of these

Ans. Option (A) is correct.

Explanation: For singular matrix, $|A| = 0$.

Q. 18. The inverse of a matrix is defined for

(A) Only square matrices

(B) Diagonal matrices

(C) All matrices

(D) Rectangular matrices

Ans. Option (A) is correct.

Explanation: If a given matrix is a square matrix, then only we can find its inverse.

Q. 19. Cramer's rule fails for _____

(A) Determinant > 0

(B) Determinant < 0

(C) Determinant $= 0$

(D) Determinant = non-real

Ans. Option (C) is correct.

Explanation: Cramer's rule is applicable only when the determinant value is not equal to zero.

Q. 20. Cramer's rule is not suitable for which type of problems?

(A) Small systems with 4 unknowns

(B) Systems with 2 unknowns

(C) Large systems

(D) Systems with 3 unknowns

Ans. Option (C) is correct.

Explanation: Generally, in large systems, excessive multiplicative operations are required which becomes very cumbersome to solve.

Q. 21. If $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$, then x = ... $(x \in I)$

(A) 3 **(B)** -3

(C) 2 **(D)** -2

Ans. Option (B) is correct.

Explanation: Given $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$

$$\Rightarrow (x-2).2x - 3x(-3) = 3$$

$$\Rightarrow 2x^2 - 4x + 9x = 3$$

$$\Rightarrow 2x^2 + 5x - 3 = 0$$

$$\Rightarrow (2x-1)(x+3) = 0$$

$$\Rightarrow x = \frac{1}{2} \text{ or } x = -3$$

$$\Rightarrow x = -3 \quad (\text{since } x \in I)$$

Q. 22. $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ = ...

(A) 0 **(B)** x
(C) $3x$ **(D)** None of these

Ans. Option (A) is correct.

Explanation: $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

(Taking $3x$ common from R_3)

$= 3x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix}$

($\because$ R_1 and R_3 are same)

$= 3x \times 0$

$= 0$

Q. 23. For a determinant containing 4 elements a_1, b_1, a_2 and b_2; what will the elements of leading diagonals ?

(A) a_1, b_1 **(B)** a_1, b_2
(C) a_2, b_1 **(D)** a_1, b_2

Ans. Option (B) is correct.

Explanation:

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Leading diagonal

Q. 24. If A and B are two square matrices, then $|A.B|$ is same as which of the following ?

(A) $|A|.|B|$ **(B)** $|B|.|A|$
(C) $|B - A|$ **(D)** $|A - B|$

Ans. Option (A) is correct.

Explanation: Determinant of product of two square matrices is equal to the product of the individual determinant of the square matrices.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. If A and B are matrices of order 3 and $|A| = 5$, $|B| = 3$, then find $|3AB|$. A I

Sol. We know that,

$|AB| = |A|.|B|$

$|3AB| = (3)^3|AB|$

$= 27|AB|$

$= 27|A|.|B|$

$= 27 \times 5 \times 3$

$= 405$ **1**

Q. 2. If $x \in \mathbb{N}$ and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x. R&U

Sol. Given, $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$

or $2x(x + 3) - (-2)(-3x) = 8$

or $2x^2 + 6x - 6x = 8$

or $x^2 = \frac{8}{2}$ ½

or $x^2 = 4$

or $x = \pm 2$ ½

Since $x \in \mathbb{N}$, so, $x = 2$

Q. 3. Write the value of the determinant $\begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix}$. R&U

Sol. $\begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix} = p^2 - (p^2 - 1)$

$= p^2 - p^2 + 1$ ½

$= 1$ ½

Q. 4. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then find the value of x. R&U

Sol. $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$2x^2 - 40 = 18 + 14$

or $2x^2 = 32 + 40$ ½

or $2x^2 = 72$

or $x^2 = 36$

or $x = \pm 6$ ½

Q. 5. If A is a 3×3 matrix, $|A| \neq 0$ and $|3A| = k|A|$, then find the value of k. A

Q. 6. If A and B are invertible matrices of order 3, $|A| = 2$ and $|(AB)^{-1}| = -\frac{1}{6}$, then find $|B|$. A

Sol. $\frac{1}{|AB|} = -\frac{1}{6}$

$\Rightarrow \frac{1}{|A||B|} = -\frac{1}{6}$

$\Rightarrow |B| = -3$. **1**

Q. 7. Given $A = \begin{pmatrix} 4 & 2 & 5 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}$, write the value of det. $(2AA^{-1})$. A

Sol. $|2AA^{-1}| = (2)^3$ [$\because AA^{-1} = I$]

$= 8$ **1**

Q. 8. If A is square matrix of order 3 such that $|adj\, A| = 64$, then find $|A|$. A

Sol. $|adj\, A| = |A|^{n-1}$, where n is the order of the matrix ½

$|A|^2 = 64$

or, $|A| = \pm 8$ ½

These questions are for practice and their solutions are available at the end of the chapter

Q. 9. For what value of x, the given matrix $A = \begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix}$ is a singular matrix ?

Q. 10. Find $|AB|$, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$. A

Sol. Given, $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

$\therefore \quad AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

or $\quad AB = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix}$

or $\quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence, $|AB| = 0$ **1**

Q. 11. If A and B are square matrices of the same order 3, such that $|A| = 2$ and $AB = 2I$, write the value of $|B|$. U

Sol. Given, $AB = 2I$

or $AB = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\therefore \quad |AB| = 2(4-0) - 0 + 0 = 8$

Since, $|AB| = |A||B|$

$\therefore \quad |B| = \dfrac{|AB|}{|A|} = \dfrac{8}{2} = 4$

[given, $|A| = 2$] **1**

Q. 12. If A is a square matrix satisfying $A'A = I$, write the value of $|A|$. A I U

Sol. Let the value of $|A| = x$

Since, $|A| = |A'|$ and $|I| = 1$

Given, $AA' = I$

$\therefore \quad |AA'| = |I|$

$\Rightarrow \quad |A||A'| = |I| \qquad [|AA'| = |A||A'|]$

$\Rightarrow \quad x.x = 1$

$\Rightarrow \quad (x^2 - 1) = 0$

$\Rightarrow \quad (x-1)(x+1) = 0$

$\Rightarrow \quad x = \pm 1$

$\therefore$ Value of $|A| = \pm 1$ **1**

Q. 13. Check whether $(l + m + n)$ is a factor of the determinant $\begin{vmatrix} l+m & m+n & n+l \\ n & l & m \\ 2 & 2 & 2 \end{vmatrix}$ or not. Give reason. A

Sol. Apply $R_1 \to R_1 + R_2$

$$\begin{vmatrix} l+m+n & m+n+l & n+l+m \\ n & l & m \\ 2 & 2 & 2 \end{vmatrix}$$

$$= 2(l+m+n)\begin{vmatrix} 1 & 1 & 1 \\ n & l & m \\ 1 & 1 & 1 \end{vmatrix}$$; yes $(l + m + n)$ is a factor. **1**

Q. 14. For what values of k, the system of linear equations

$$x + y + z = 2$$
$$2x + y - z = 3$$
$$3x + 2y + kz = 4$$

has a unique solution ? A

Sol. For a unique solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$k + 2 - 2k - 3 + 4 - 3 \neq 0$

$\Rightarrow k \neq 0$ **1**

Q. 15. If $A = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}.

Sol. Given, $A = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$\therefore$ Minor of $a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$= 10 - 3 = 7$ **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then find the other two roots. R

Sol. Since $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Expanding along R_1,

$\Rightarrow x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) = 0$

$\Rightarrow \quad x^3 - 12x - 6x + 42 + 84 - 49x = 0$

$\Rightarrow \quad x^3 - 67x + 126 = 0$ ...(i)

Here, $126 \times 1 = 9 \times 2 \times 7$

For $x = 2$,

$\Rightarrow 2^3 - 67 \times 2 + 126 = 134 - 134 = 0$

Hence, $x = 2$ is a root.

For $x = 7$,

$\Rightarrow 7^3 - 67 \times 7 + 126 = 469 - 469 = 0$

Hence, $x = 7$ is also a root. **2**

Q. 2. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$. A

These questions are for practice and their solutions are available at the end of the chapter

Sol. $|A| = 2$,

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad \textbf{1}$$

$$\text{LHS} = 2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix},$$

$$\text{RHS} = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{LHS} = \text{RHS} \quad \textbf{1}$$

Q. 3. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, be such that $A^{-1} = kA$, then find the value of k. A

Q. 4. Find the inverse of the matrix $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

Hence, find the matrix P satisfying the matrix equation $P\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. R&U

Sol. $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1} = \frac{1}{9-10}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ **1 + ½**

$$\therefore\ P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 1 & 1 \end{bmatrix} \quad \textbf{½}$$

Q. 5. If A is a skew-symmetric matrix of order 3, then prove that det $A = 0$. R&U

Sol. Since A is a skew-symmetric matrix

$\therefore\ A^T = -A$

$\therefore\ |A^T| = |-A| = (-1)^3.|A|$ **1**

or $|A^T| = -|A|$

or $2|A| = 0$ or $|A| = 0$. **1**

Q. 6. If $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ and $|A^3| = 125$, then find the value(s) of p. R

Sol. Given, $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ and $|A^3| = 125$

Since, $|A^n| = |A|^n$

$\therefore\ |A^3| = |A|^3$

Now, $|A| = \begin{vmatrix} p & 2 \\ 2 & p \end{vmatrix} = p^2 - 4$ **1**

According to given condition,

$|A^3| = 125$

$\Rightarrow\ |A|^3 = 125$

$\Rightarrow\ (p^2-4)^3 = 125$

$\Rightarrow\ (p^2-4)^3 = 5^3$

$\Rightarrow\ p^2 - 4 = 5$

$\Rightarrow\ p^2 = 9$

$\Rightarrow\ p = \pm 3$

Hence, values of $p = \pm 3$. **1**

Commonly Made Error

- Some students find A^3 first and then take its determinant which is time consuming.

Answering Tip

- Learn all the properties of determinants thoroughly.

Q. 7. Without expanding at any stage, find the value of $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$. R&U

Sol. Let, $\Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

Applying $R_1 \Rightarrow R_1 + 2R_3$, we get

$$\Delta = \begin{vmatrix} a+2x & b+2y & c+2z \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$$

Since, R_1 and R_2 are identical, therefore value of Δ is zero. **2**

Commonly Made Error

- Many candidates expand the determinant to solve it, though it was clearly mentioned in the question *find the value of determinant without expansion at any stage.*

Answering Tip

- Do extensive practice of different types of questions based on properties of determinants.

Q. 8. Without expanding at any stage, find the value of the determinant:

$$\Delta = \begin{vmatrix} 2 & x & y+z \\ 2 & y & z+x \\ 2 & z & x+y \end{vmatrix}$$ R&U

Q. 9. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32}A_{32}$. A I

Sol. $a_{32}A_{32} = -5\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$

$= -5(8 - 30)$

$= -5(-22) = 110$ **2**

Q. 10. Write the value of the determinant:

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$ R&U

Sol. Let, $\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

$= \begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} \qquad \left[R_1 \to \frac{1}{6}R_1\right]$

$= 0 \quad (\because R_1 \text{ and } R_3 \text{ are identical})$ **2**

Q. 11. Write the value of $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$. R&U

Short Answer Type Questions-II (3 marks each)

Q. 1. Using properties of determinants, solve for x:

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$ A I R&U

Sol. $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$

$\begin{vmatrix} 3a-x & 3a-x & 3a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ **1**

Applying $C_2 \to C_2 - C_1$

$\begin{vmatrix} 3a-x & 0 & 3a-x \\ a-x & 2x & a-x \\ a-x & 0 & a+x \end{vmatrix} = 0$ ½

Applying $C_3 \to C_3 - C_1$

$\begin{vmatrix} 3a-x & 0 & 0 \\ a-x & 2x & 0 \\ a-x & 0 & 2x \end{vmatrix} = 0$ ½

Expanding along C_3

$4x^2(3a - x) = 0$

$\therefore x = 0, 3a$ **1**

Q. 2. Using properties of determinants, prove that:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 3 & 4+3p & 2+4p+3q \\ 4 & 7+4p & 2+7p+4q \end{vmatrix} = 1$$

Sol. Taking L.H.S. $= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 3 & 4+3p & 2+4p+3q \\ 4 & 7+4p & 2+7p+4q \end{vmatrix}$

Applying $R_2 \to R_2 - 3R_1$

$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 4 & 7+4p & 2+7p+4q \end{vmatrix}$ **1**

Applying $R_3 \to R_3 - 4R_1$

$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 3 & -2+3p \end{vmatrix}$ ½

Applying $R_3 \to R_3 - 3R_2$

$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 0 & 1 \end{vmatrix}$ ½

Expanding along $R_1 = \begin{vmatrix} 1 & -1+p \\ 0 & 1 \end{vmatrix}$

$= 1 =$ R.H.S. **Hence Proved. 1**

Q. 3. Using properties of determinants, prove the following:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$$

A

Q. 4. Using properties of determinants, prove that:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab.$$ R&U

Sol. $R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2, R_3 \to \frac{1}{c}R_3$

$\therefore \quad \text{LHS} = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ ½

$R_1 \to R_1 + R_2 + R_3$

$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ ½

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ ½

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$ **1**

Expand along R_1

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1)$

$= abc + bc + ca + ab$ = RHS ½

Commonly Made Error

- Most of the candidates make errors while applying the properties of determinants in the correct order.

Answering Tip

- Elucidate all properties of determinants and their applications.

Q. 5. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$

R&U

Sol. $\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Using $C_1 \to C_1 - bC_3$, $C_2 \to C_2 + aC_3$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$ 1

Taking $(1 + a^2 + b^2)$ common from C_1 and C_2

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$ ½

Using $R_3 \to R_3 + aR_2 - bR_1$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$ ½

Taking $(1 + a^2 + b^2)$ common from R_3

$$= (1+a^2+b^2)^3 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{vmatrix}$$ ½

Expanding with respect to R_3

$\Delta = (1 + a^2 + b^2)^3(1)$

or $\Delta = (1 + a^2 + b^2)^3$ ½

Commonly Made Error

- Sometimes candidates fail to take common factor $(1 + a^2 + b^2)^3$ and could not simplify the determinant.

Answering Tip

- If a row or a column contains a common factor, then it can be taken out.

Q. 6. If $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ then using A^{-1}, solve the following system of equations : $x - 2y = -1$, $2x + y = 2$.

R&U

Q. 7. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 \\ -2 & 4 \end{bmatrix}$. Then compute AB. Hence, solve the following system of equations: $2x + y = 4$, $3x + 2y = 1$. A

Sol. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

and $B = \begin{bmatrix} 4 & -6 \\ -2 & 4 \end{bmatrix}$

then $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ -2 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$$ **1**

$\Rightarrow \quad A\left(\frac{1}{2}B\right) = I$

On multiplying by A^{-1}

$$A^{-1} = \frac{1}{2}B$$

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$ **1**

The given system of equations are equivalent to $A'X = C$,

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$

and $C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $A' = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$X = (A')^{-1}C = (A^{-1})'C$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \end{bmatrix}$

$\therefore \quad x = 7$

and $\quad y = -10$ **1**

Q. 8. Ishan wants to donate a rectangular plot of land for a school in his village. When he was asked to give dimensions of the plot, he told that if its length is decreased by 50 m and breadth is increased by 50 m, then its area will remain the same. But if length is decreased by 10 m and breadth is decreased by 20 m, then its area will decrease by 5300 m^2. Using matrices, find the dimensions of the plot.

AI R&U

Sol. Let length be x m and breadth be y m.

$\therefore \quad (x - 50)(y + 50) = xy$

$\Rightarrow \quad 50x - 50y = 2500$ or $x - y = 50$ ½

and $(x - 10)(y - 20) = xy - 5300$

$\Rightarrow \quad 2x + y = 550$ ½

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50 \\ 550 \end{pmatrix}$$ ½

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 50 \\ 550 \end{pmatrix}$$ 1

$\Rightarrow \quad x = \frac{1}{3}(600) = 200$ m,

$y = \frac{1}{3}(450) = 150$ m ½

Q. 9. On her birthday Seema decided to donate some money to children of an orphanage home. If there were 8 children less, everyone would have got ₹10 more. However if there were 16 children more, everyone would have got ₹10 less. Using matrix method, find the number of children and the amount distributed by Seema. A I R&U

Long Answer Type Questions (5 marks each)

Q. 1. Prove that $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$ **is divisible by $(x + y + z)$, and hence find the quotient.** R&U

Sol. L.H.S. $= \begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ 1

$$= \begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ (x-y)(x+y+z) & (y-z)(x+y+z) & (z-x)(x+y+z) \\ (x-z)(x+y+z) & (y-x)(x+y+z) & (z-y)(x+y+z) \end{vmatrix}$$

Taking $(x + y + z)$ common from R_2 and R_3

$$= (x+y+z)^2 \begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ x - y & y - z & z - x \\ x - z & y - x & z - y \end{vmatrix}$$ ½

Applying $C_1 \to C_1 + C_2 + C_3$ 1

$= (x + y + z)^2$

$$\begin{vmatrix} (xy + yz + zx) - (x^2 + y^2 + z^2) & zx - y^2 & xy - z^2 \\ 0 & y - z & z - x \\ 0 & y - x & z - y \end{vmatrix}$$

Expanding along C_1

$= (x + y + z)^2[xy + yz + zx - (x^2 + y^2 + z^2)]$

$[(y - z)(z - y) - (z - x)(y - x)]$

$= (x + y + z)^2[xy + yz + zx - (x^2 + y^2 + z^2)]$

$[yz - y^2 - z^2 + yz - yz + xz + xy - x^2]$

$= (x + y + z)^2[xy + yz + zx - (x^2 + y^2 + z^2)]^2$

Hence, it is divisible by $(x + y + z)$ and the quotient is $(x + y + z)[xy + yz + zx - (x^2 + y^2 + z^2)]^2$. 2½

Q. 2. Using properties of determinants, prove that:

$$\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3.$$ A I

Sol. LHS $= \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix}$

Apply $(R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3)$ 1

$$= \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & zx^2 \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ 1

$$= \begin{vmatrix} (y+z)^2 & x^2 - (y+z)^2 & x^2 - (y+z)^2 \\ y^2 & (x+z)^2 - y^2 & 0 \\ z^2 & 0 & (x+y)^2 - z^2 \end{vmatrix}$$

Taking $(x + y + z)$ common from C_2 and C_3.

$$= (x+y+z)^2 \begin{vmatrix} (y+z)^2 & x - y - z & x - y - z \\ y^2 & x + z - y & 0 \\ z^2 & 0 & x + y - z \end{vmatrix}$$

Apply, $R_1 \to R_1 - R_2 - R_3$ ½

$$= (x+y+z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x + z - y & 0 \\ z^2 & 0 & x + y - z \end{vmatrix}$$

Apply, $C_2 \to yC_2, C_3 \to zC_3$ ½

$$= \frac{(x+y+z)^2}{yz} \begin{vmatrix} 2yz & -2yz & -2yz \\ y^2 & xy + zy - y^2 & 0 \\ z^2 & 0 & zx + yz - z^2 \end{vmatrix}$$

Apply, $C_2 \to C_2 + C_1$ & $C_3 \to C_3 + C_1$ 1

$$= \frac{(x+y+z)^2}{yz} \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & xy + zy & y^2 \\ z^2 & z^2 & zx + yz \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{yz} \cdot |(2yz) \cdot (x^2yz + xy^2z + xyz^2 + y^2z^2 - y^2z^2)|$$

$= 2xyz(x + y + z)^3 =$ RHS 1

Commonly Made Error

- Sometimes candidates make errors while applying the properties of determinant in the correct order.

Answering Tip

- Practice the problem based an properties clearly mentioning the property used in every step.

These questions are for practice and their solutions are available at the end of the chapter

Q. 3. Using properties of determinants, prove that

$$\begin{vmatrix} \frac{(a+b)^2}{c} & c & c \\ a & \frac{(b+c)^2}{a} & a \\ b & b & \frac{(c+a)^2}{b} \end{vmatrix} = 2(a+b+c)^3.$$ R&U

Q. 4. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & -3 \\ 2 & 0 & -1 \end{bmatrix}$**, Find A^{-1}.**

Hence, solve the system of equations :

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$2x - 3y - z = 5$$ A

Sol. $A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & -3 \\ 2 & 0 & -1 \end{bmatrix}$

$|A| = 3(-2) - 1(3) + 2(-4)$
$= -6 - 3 - 8 = -17 \neq 0$

$\therefore A^{-1}$ exists. **1**

Cofactor matrix of $A = \begin{bmatrix} -2 & -3 & -4 \\ 1 & -7 & 2 \\ -7 & 15 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\text{Adj}A$$

$$= \frac{1}{-17}\begin{bmatrix} -2 & 1 & -7 \\ -3 & -7 & 15 \\ -4 & 2 & 3 \end{bmatrix}$$ **2½**

Now for given system of equations.

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$(A^T)X = B$

$\Rightarrow X = (A^T)^{-1}B$

$\Rightarrow X = (A^{-1})^T B \quad [\because (A^T)^{-1} = (A^{-1})^T]$

$$X = \frac{1}{-17}\begin{bmatrix} -2 & -3 & -4 \\ 1 & -7 & 2 \\ -7 & 15 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$X = \frac{1}{-17}\begin{bmatrix} -34 \\ -17 \\ 68 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

$x = 2, y = 1, z = -4$ **1½**

Q. 5. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$**, Find A^{-1}. Use it to solve the system of equations.**

Hence, solve the system of equations:

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$ A I R&U

Sol. $|A| = -1 \neq 0 \therefore A^{-1}$ exists

Co-factors of A are: **1**

$A_{11} = 0; \quad A_{12} = 2; \quad A_{13} = 1$
$A_{21} = -1; \quad A_{22} = -9; \quad A_{23} = -5$ **2**
$A_{31} = 2; \quad A_{32} = 23; \quad A_{33} = 13$

$$adj(A) = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}.adj(A)$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$ ½

For, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$, the system of equation is $A \cdot X = B$

$$\therefore X = A^{-1} \cdot B = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore x = 1, y = 2, z = 3$ **1½**

Q. 6. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$**, find A^{-1}. Hence, solve the system of equations $x + y + z = 6, x + 2z = 7, 3x + y + z = 12$.** A I R&U

Q. 7. Given $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$**, compute $(AB)^{-1}$.** R&U

Sol. $|A| = 5(-1) + 4(1) = -1$ **1**

$C_{11} = -1 \quad C_{21} = 8 \quad C_{31} = -12$
$C_{12} = 0 \quad C_{22} = 1 \quad C_{32} = -2$ **2**
$C_{13} = 1 \quad C_{23} = -10 \quad C_{33} = 15$

$$A^{-1} = \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1}$ **1**

$$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$$ **1**

Q. 8. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **to solve the system of equations** $x + 3z = 9, -x + 2y - 2z = 4, 2x - 3y + 4z = -3$ U&A

Sol. $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **1**

$$AB = I$$

or $A^{-1} = B = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **2**

Given equations in matrix form are:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$AX = C \quad ½$$

or $X = (A)^{-1} C = (A^{-1}) C$ ½

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$

or $x = 0, y = 5, z = 3$ **1**

Q. 9. In $A = \begin{pmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{pmatrix}$ **find** A^{-1}**. Using** A^{-1} **solve the system of equations** $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2; \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5;$ $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4$ U&A

Sol. Here, $|A| = 1200$ **1**

Co-factors are

$$\left.\begin{array}{lll} C_{11} = 75, & C_{21} = 150, & C_{31} = 75 \\ C_{12} = 110, & C_{22} = -100, & C_{32} = 30 \\ C_{13} = 72, & C_{23} = 0, & C_{33} = -24 \end{array}\right\} \quad \textbf{2}$$

$$A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \quad ½$$

Given equation in matrix from is :

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} \quad \textbf{1}$$

or $AX = B$

or $X = A^{-1}B$

or $\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{3} \\ \frac{1}{5} \end{bmatrix}$ **1**

or $x = 2, y = -3, z = 5$ ½

Q. 10. For what values of a **and** b**, the system of equations**

$$2x + ay + 6z = 8$$
$$x + 2y + bz = 5$$
$$x + y + 3z = 4,$$

has (i) unique solution (ii) no solution? R&U

Sol. For the given system of equations, we have

$$D = \begin{vmatrix} 2 & a & 6 \\ 1 & 2 & b \\ 1 & 1 & 3 \end{vmatrix} = 2(6 - b) - a(3 - b) + 6(1 - 2)$$

$$= 12 - 2b - 3a + ab - 6$$
$$= 6 - 3a - 2b + ab$$
$$= (b - 3)(a - 2) \quad \textbf{1}$$

$$D_x = \begin{vmatrix} 8 & a & 6 \\ 5 & 2 & b \\ 4 & 1 & 3 \end{vmatrix} = 8(6 - b) - a(15 - 4b) + 6(5 - 8)$$

$$= 48 - 8b - 15a + 4ab - 18$$
$$= 30 - 15a - 8b + 4ab$$
$$= (a - 2)(4b - 15) \quad \textbf{1}$$

$$D_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & b \\ 1 & 4 & 3 \end{vmatrix} = 2(15 - 4b) - 8(3 - b) + 6(4 - 5)$$

$$= 30 - 8b - 24 + 8b - 6$$
$$= 0 \quad \textbf{1}$$

$$D_z = \begin{vmatrix} 2 & a & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 2(8 - 5) - a(4 - 5) + 8(1 - 2)$$

$$= 6 + a - 8 = a - 2 \quad \textbf{1}$$

(i) For unique solution, we must have
$D \neq 0 \Rightarrow (a - 2)(b - 3) \neq 0 \Rightarrow a \neq 2$ or $b \neq 3$

(ii) For no solution, we must have $D = 0$ and at least one of D_x, D_y and D_z is non-zero.
Clearly, $b = 3$, we have
$D = 0$ and $D_z \neq 0$
Hence, the system has no solution for $b = 3$. **1**

Q. 11. Solve the following system of equations by Cramer's rule :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1 \text{ and } \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

A I

Sol. Let $\frac{1}{x} = p$, $\frac{1}{y} = q$ and $\frac{1}{z} = r$, then the above system of equations can be written as

$$2p + 3q + 10r = 4$$
$$4p - 6q + 5r = 1$$
$$6p + 9q - 20r = 2 \quad \textbf{1}$$

Here, $D = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$

$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$
$= 150 + 330 + 720 = 1200$

$D_1 = \begin{vmatrix} 4 & 3 & 10 \\ 1 & -6 & 5 \\ 2 & 9 & -20 \end{vmatrix}$

$= 4(120 - 45) - 3(-20 - 10) + 10(9 + 12)$
$= 300 + 90 + 210 = 600$

$D_2 = \begin{vmatrix} 2 & 4 & 10 \\ 4 & 1 & 5 \\ 6 & 2 & -20 \end{vmatrix}$

$= 2(-20 - 120) - 4(-80 - 30) + 10(8 - 6)$
$= -60 + 440 + 20 = 400$

and $D_3 = \begin{vmatrix} 2 & 3 & 4 \\ 4 & -6 & 1 \\ 6 & 9 & 2 \end{vmatrix}$

$= 2(-12 - 9) - 3(8 - 6) + 4(36 + 36)$
$= -42 - 6 + 288 = 240$ **2**

$\therefore \quad p = \frac{D_1}{D} = \frac{600}{1200} = \frac{1}{2} \Rightarrow \frac{1}{x} = \frac{1}{2} \Rightarrow x = 2,$

$q = \frac{D_2}{D} = \frac{400}{1200} = \frac{1}{3} \Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3,$

$r = \frac{D_3}{D} = \frac{240}{1200} = \frac{1}{5} \Rightarrow \frac{1}{z} = \frac{1}{5} \Rightarrow z = 5$

Hence, $x = 2$, $y = 3$ and $z = 5$. **2**

Q. 12. Using determinants, show that the following system of linear equation is inconsistent:

$$x - 3y + 5z = 4$$
$$2x - 6y + 10z = 11$$
$$3x - 9y + 15z = 12$$

R&U

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same: A I

Two schools P and Q want to award their selected students on the values of the Discipline, Politeness and Punctuality. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students with the total award money of ₹ 1,000. School Q wants spend ₹ 1,500 to award its 4, 1 and 3 students on their respective values (by giving the same award money for the three values as before). The total amount of awards for one prize on each value is ₹ 600.

Q. 1. For the given information, the system of equations can be written as

(A) $3x + 2y + z = 1000$
$4x + y + 3z = 1500$
$x + y + z = 600$

(B) $3x + 2y + z = 1500$
$4x + y + 3z = 1000$
$x + y + z = 600$

(C) $3x + 2y + z = 1000$
$x + y + z = 15000$
$4x + y + 3z = 600$

(D) $x + 2y + z = 1000$
$4x + y + 3z = 1500$
$3x + 2y + z = 600$

Ans. Option (A) is correct.

Explanation: According to the given conditions of the questions, we have

$$3x + 2y + z = 1000$$
$$4x + y + 3z = 1500$$
$$x + y + z = 600$$

Q. 2. If the given system of equations can be written as $AX = B$, then find $|A|$:

(A) 10 **(B)** 5
(C) -5 **(D)** -10

Ans. Option (C) is correct.

Explanation: The given system of the questions can be written as $AX = B$, i.e.

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

Now, $|A| = 3(1 - 3) - 2(4 - 3) + 1(4 - 1)$
$= -6 - 2 + 3 = -5$

Q. 3. The award money for Discipline is:

(A) ₹ 100 **(B)** ₹ 200
(C) ₹ 300 **(D)** ₹ 400

Ans. Option (A) is correct.

Explanation: Since $|A| \neq 0$, thus A^{-1} exists, so the given system of equation has unique solution $X = A^{-1}B$

Hence, $A_{11} = -2$, $A_{12} = -1$, $A_{13} = 3$
$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$
And $A_{31} = 5$, $A_{32} = -5$, $A_{33} = -5$

These questions are for practice and their solutions are available at the end of the chapter

$$\therefore \quad adj\ A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad X = A^{-1}B$$

$$= \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} -500 \\ -1000 \\ -1500 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

Hence, the award money for Discipline is ₹ 100.

Q. 4. The award money for Politeness is:

(A) ₹ 100 **(B)** ₹ 200

(C) ₹ 300 **(D)** ₹ 400

Ans. Option (B) is correct.

Explanation: From the above, it clear that the award money for Politeness is ₹ 200.

Q. 5. The award money for Punctuality is:

(A) ₹ 100 **(B)** ₹ 200

(C) ₹ 300 **(D)** ₹ 400

Ans. Option (C) is correct.

Explanation: From the above, it clear that the award money for Punctuality is ₹ 300.

II. Read the following text and answer the following questions on the basis of the same:

A shopkeeper has 3 varieties of pen 'A', 'B' and 'C'. Meenu purchase 1 pen of each variety for a total of ₹ 21. Jeevan purchase 4 pens of 'A' variety, 3 pens say 'B' variety and 2 pens of 'C' variety for ₹ 60. While Shikha purchased 6 pens of 'A' variety, 2 pens of 'B' variety and 3 pens of 'C' variety for ₹ 70.

R&U

Q. 1. For the given information, the system of equations can be written as:

(A) $x + y + z = 21$
$4x + 3y + 2z = 60$
$6x + 2y + 3z = 70$

(B) $x + y + z = 60$
$4x + 3y + 2z = 21$
$6x + 2y + 3z = 70$

(C) $4x + 3y + 2z = 21$
$x + y + z = 70$
$6x + 2y + 3z = 60$

(D) $x + y + z = 70$
$4x + 3y + 2z = 60$
$6x + 2y + 3z = 21$

Ans. Option (A) is correct.

Explanation: Let the cost of 'A' variety pen be x, the cost of 'B' variety pen be y, the cost of 'C' variety pen be z.

According to the question,

$$x + y + z = 21$$
$$4x + 3y + 2z = 60$$
$$6x + 2y + 3z = 70$$

Q. 2. If the given system of equations can be written as $AX = B$, then find $|A|$.

(A) 10 **(B)** 5

(C) – 5 **(D)** – 10

Ans. Option (C) is correct.

Explanation: The given system of the questions can be written as $AX = B$, *i.e.*

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

Now, $$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 1(9 - 4) - 1(12 - 12) + 1(8 - 18)$$

$$= 5 - 10 = -5 \neq 0$$

Q. 3. The cost of 'A' variety pen is

(A) ₹ 4 **(B)** ₹ 5

(C) ₹ 8 **(D)** ₹ 2

Ans. Option (B) is correct.

Explanation: Since $|A| \neq 0$, thus A^{-1} exists, so the given system of equation has unique solution $X = A^{-1}B$

$$A_{11} = (9 - 4) = 5$$
$$A_{12} = -(12 - 12) = 0$$
$$A_{13} = (8 - 18) = -10$$
$$A_{21} = -(3 - 2) = -1$$
$$A_{22} = (3 - 6) = -3$$
$$A_{23} = -(2 - 6) = 4$$
$$A_{31} = (2 - 3) = -1$$
$$A_{32} = -(2 - 4) = 2$$
$$A_{33} = 3 - 4 = -1$$

$$adj\ A = \begin{bmatrix} 5 & 0 & -10 \\ -1 & -3 & 4 \\ -1 & 2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ A)$$

Also, $AX = B$

$\Rightarrow$ $X = A^{-1}B$

$\Rightarrow$ $A^{-1} = \frac{1}{|A|}(adj\ A)B$

$$= \frac{1}{|A|}\begin{bmatrix} 5 & -1 & -1 \\ 0 & -3 & 2 \\ -10 & 4 & -1 \end{bmatrix}\begin{bmatrix} 21 \\ 60 \\ 70 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} 105-60-70 \\ 0-180+140 \\ -210+240-70 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} -25 \\ -40 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Hence, cost of 'A' variety pen is ₹ 5.

Q. 4. The cost of 'B' variety pen is

(A) ₹ 4 **(B)** ₹ 5

(C) ₹ 8 **(D)** ₹ 2

Ans. Option (C) is correct.

Explanation: From the above, it clear that the cost of 'B' variety pen is ₹ 8.

Q. 5. The cost of 'C' variety pen is

(A) ₹ 4 **(B)** ₹ 5

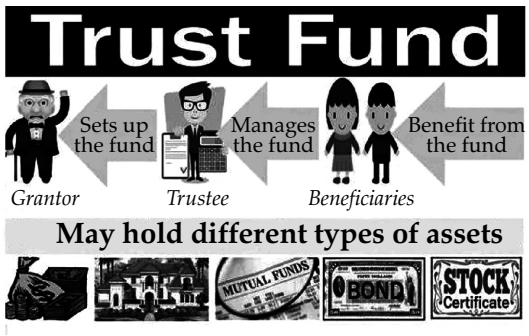
(C) ₹ 8 **(D)** ₹ 2

Ans. Option (C) is correct.

Explanation: From the above, it clear that the cost of 'C' variety pen is ₹ 8.

III. Read the following text and answer the following questions on the basis of the same:

A trust invested some money in two types of bonds. The first bond pays 10% interest and second bond pays 12% interest the trust received ₹ 2800 as interest. However if trust had interchanged money in bonds, they would have got ₹ 100 less as interest.



R&U

Q. 1. For the given information, the system of equations can be written as

(A) $5x + 6y = 135000$
$6x + 5y = 140000$

(B) $5x + 6y = 140000$
$6x + 5y = 135000$

(C) $5x + 5y = 140000$
$6x + 6y = 135000$

(D) $x + y = 140000$
$6x + 5y = 135000$

Ans. Option (B) is correct.

Explanation: According to the questions:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} \frac{10}{100} \\ \frac{12}{100} \end{bmatrix} = [2800]$$

i.e., $\frac{10x}{100} + \frac{12y}{100} = 2800$

$\Rightarrow$ $10x + 12y = 280000$

$\Rightarrow$ $5x + 6y = 140000$ ...(i)

Also, $\begin{bmatrix} y & x \end{bmatrix}\begin{bmatrix} \frac{10}{100} \\ \frac{12}{100} \end{bmatrix} = [2700]$

i.e., $\frac{10y}{100} + \frac{12x}{100} = 2700$

$\Rightarrow$ $10y + 12x = 270000$

$\Rightarrow$ $5y + 6x = 135000$ ...(ii)

Q. 2. If the given system of equations can be written as $AX = B$, then find $|A|$.

(A) -11 **(B)** 11

(C) -5 **(D)** -10

Ans. Option (A) is correct.

Explanation: By matrix method,

$$\begin{bmatrix} 5 & 6 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 140000 \\ 135000 \end{bmatrix}$$

where, $A = \begin{bmatrix} 5 & 6 \\ 6 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 140000 \\ 135000 \end{bmatrix}$

It is of the form $AX = B$

$\Rightarrow$ $X = A^{-1}B$

$$|A| = \begin{vmatrix} 5 & 6 \\ 6 & 5 \end{vmatrix}$$

$$= 25 - 36 = -11 \neq 0$$

Q. 3. The amount invested in first type of bond is

(A) ₹ 9000 **(B)** ₹ 12000

(C) ₹ 15000 **(D)** ₹ 10000

Ans. Option (D) is correct.

Explanation: Since $|A| \neq 0$, thus A^{-1} exists, so the given system of equation has unique solution

$$X = A^{-1}B$$

Co-factors of $|A|$ are

$A_{11} = 5, \quad A_{12} = -6,$

$A_{21} = -6, \; A_{22} = 5$

$\therefore \quad adj\, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{vmatrix} 5 & -6 \\ -6 & 5 \end{vmatrix}$

$A^{-1} = \frac{1}{|A|}(adj\, A)$ and

$X = A^{-1}B$

$= \frac{1}{-11}\begin{bmatrix} 5 & -6 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 140000 \\ 135000 \end{bmatrix}$

$= \frac{1}{-11}\begin{bmatrix} 700000 - 810000 \\ -840000 + 675000 \end{bmatrix}$

$= \frac{1}{-11}\begin{bmatrix} -110000 \\ -165000 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10000 \\ 15000 \end{bmatrix}$

Thus, the amount invested in first type of bond is ₹ x = ₹ 10000

Q. 4. The amount invested in first type of bond is

(A) ₹ 9000 **(B)** ₹ 12000

(C) ₹ 15000 **(D)** ₹ 10000

Ans. Option (C) is correct.

Explanation: From the above, it clear that the amount invested in first type of bond is ₹ y = ₹ 15000

Q. 5. The total amount invested is

(A) ₹ 25000 **(B)** ₹ 18000

(C) ₹ 15000 **(D)** ₹ 10000

Ans. Option (A) is correct.

Explanation: Total amount invested is

₹ x + ₹ y = ₹ 10000 + ₹ 15000

= ₹ 25,000

IV. Read the following text and answer the following questions on the basis of the same:

The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to kept the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. The sum of the number of awardees for honesty and supervision is twice the number of awardees for helping.

Q. 1. $x + y + z =$ _____.

(A) 3 **(B)** 5

(C) 7 **(D)** 12

Ans. Option (D) is correct.

Explanation:

$x + y + z = 12$ ...(i)

$2x + 3y + 3z = 33$ ...(ii)

$x - 2y + z = 0$ ...(iii)

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$|A| = 1(3 + 6) - 1(2 - 3) + 1(-4 - 3)$

$= 9 + 1 - 7$

$= 3$

$A^{-1} = \frac{1}{|A|}(adj\, A)$

$= \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$

$X = A^{-1}B$

$= \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$\Rightarrow x = 3, y = 4, z = 5$

$x + y + z = 12$ [from (i)]

Q. 2. $x - 2y =$ _____.

(A) z **(B)** $-z$

(C) $2z$ **(D)** $-2z$

Ans. Option (B) is correct.

Explanation: $x - 2y = -z$ [from (iii)]

Q. 3. The value of z is _____.

(A) 3 **(B)** 4

(C) 5 **(D)** 6

Ans. Option (C) is correct.

Explanation: $z = 5$

Q. 4. The value of $x + 2y$ = _____.

(A) 9 **(B)** 10

(C) 11 **(D)** 12

Ans. Option (C) is correct.

Explanation: $x + 2y = 3 + 8 = 11$

Q. 5. The value of $2x + 3y + 5z$ = _____.

(A) 40 **(B)** 43

(C) 50 **(D)** 53

Ans. Option (B) is correct.

Explanation:

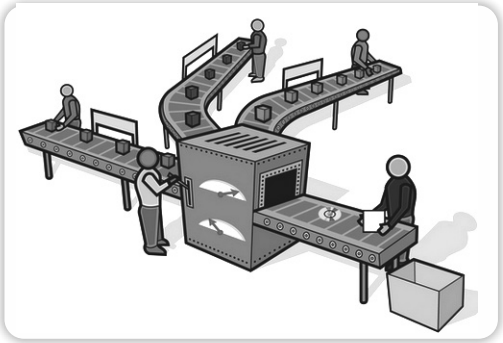
$2x + 3y + 5z = 6 + 12 + 25$

$= 43$

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:

A factory produces three products every day. Their production on a particular day is 45 tons. It is found that production of third product exceeds the production of first product by 8 tons while the total production of first and third product is twice the production of second product.



Q. 1. It x, y and z respectively denotes the quantity (in tons) of first, second and third product produced, then construct the system of equations and write it in matrix form.

Sol. According to given information, the system of linear equations is:

$$x + y + z = 45$$
$$z = x + 8$$
$$x + z = 2y$$

or,
$$x + y + z = 45$$
$$-x + z = 8$$
$$x - 2y + z = 0 \qquad \textbf{1½}$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \\ 0 \end{bmatrix} \qquad ½$$

Q. 2. If $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$, **then find the inverse of** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

Then $A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$

We know that,

$$(A')^{-1} = (A^{-1})'$$

$$\therefore \quad (A^{-1})' = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{bmatrix} \qquad \textbf{1}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} \end{bmatrix} \qquad \textbf{1}$$

or, $(A')^{-1} = (A^{-1})'$

Solutions for Practice Questions

Very Short Answer Type Questions

Sol. 5: Since $|kA| = k^n|A|$, where n is the order of matrix. ½

$$\therefore |3A| = 3^3|A| = 27|A|$$

or $k = 27$ ½

Sol. 9: $A = \begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix}$

Since A is a singular matrix. *i.e.*, $|A| = 0$

$$\begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix} = 0 \qquad ½$$

or, $4(3-2x) - 2(x+1) = 0$

or, $12 - 8x - 2x - 2 = 0$

or, $-10x + 10 = 0$ or, $x = 1$ ½

Short Answer Type Questions-I

Sol. 3: Finding $A^{-1} = \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$ **1**

$$\Rightarrow \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2k & 3k \\ 5k & -2k \end{bmatrix} \qquad ½$$

$$\Rightarrow k = \frac{1}{19} \qquad ½$$

Sol. 8: Given, $\Delta = \begin{vmatrix} 2 & x & y+z \\ 2 & y & z+x \\ 2 & z & x+y \end{vmatrix}$

By applying, $C_3 \Rightarrow C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2 & x & x+y+z \\ 2 & y & x+y+z \\ 2 & z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$$

$$= 2 \times 0 = 0$$

[Since, $C_1 = C_2 \therefore \Delta = 0$] **2**

Sol. 11: Let $\Delta = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

$$= \begin{vmatrix} 2 & 7 & 0 \\ 3 & 8 & 0 \\ 5 & 9 & 0 \end{vmatrix} \quad [C_3 \to C_3 - 9C_2 - C_1]$$

$= 0$ **2**

[Since all values of C_3 is zero so determinant value is zero]

Short Answer Type Questions-II

Sol. 3: LHS $= \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$

$C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_1$

$$= \begin{vmatrix} a+b+c & a+b & a+c \\ -c & a+b & -(a+c) \\ -b & -(a+b) & (a+b) \end{vmatrix}$$ **1½**

$$= (a+b)(a+c)\begin{vmatrix} a+b+c & 1 & 1 \\ -c & 1 & -1 \\ -b & -1 & 1 \end{vmatrix}$$ ½

$C_3 \to C_3 + C_2$

$$= (a+b)(a+c)\begin{vmatrix} a+b+c & 1 & 2 \\ -c & 1 & 0 \\ -b & -1 & 0 \end{vmatrix}$$

$= 2(a+b)(a+c)(c+b) = \text{RHS}.$ **1**

Sol. 6: $|A| = 5$

$$adj\ A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$ **1**

Given system of equations is $AX = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ ½

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1+4 \\ 2+2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$ **1**

$$x = \frac{3}{5} \text{ and } y = \frac{4}{5}$$ ½

Sol. 9: Let the number of children be x and the amount distributed by Seema for one student be ₹y.

So, $(x-8)(y+10) = xy$

$\Rightarrow$ $5x - 4y = 40$ ...(i) ½

and $(x+16)(y-10) = xy$

$\Rightarrow$ $5x - 8y = -80$ ...(ii) ½

Here $A = \begin{bmatrix} 5 & -4 \\ 5 & -8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 40 \\ -80 \end{bmatrix}$

$$AX = B \Rightarrow X = A^{-1}B$$ **1**

$$A^{-1} = -\frac{1}{20}\begin{bmatrix} -8 & 4 \\ -5 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 32 \\ 30 \end{bmatrix}$$

$\Rightarrow$ $x = 32, y = 30$ **1**

No. of students = 32

Amount given to each student = ₹ 30.

Long Answer Type Questions

Sol. 3: LHS $= \frac{1}{abc}\begin{vmatrix}(a+b)^2 & c^2 & c^2\\ a^2 & (b+c)^2 & a^2\\ b^2 & b^2 & (c+a)^2\end{vmatrix}$

$C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$ ½

$= \frac{1}{abc}$

$\begin{vmatrix}(a+b+c)(a+b-c) & 0 & c^2\\ 0 & (b+c+a)(b+c-a) & a^2\\ (b+c+a)(b-c-a) & (b+c+a)(b-c-a) & (c+a)^2\end{vmatrix}$ **1**

$= \frac{(a+b+c)^2}{abc}\begin{vmatrix}(a+b-c) & 0 & c^2\\ 0 & (b+c-a) & a^2\\ (b-c-a) & (b-c-a) & (c+a)^2\end{vmatrix}$ ½

$R_3 \to R_3 - R_1 - R_2$

$= \frac{(a+b+c)^2}{abc}\begin{vmatrix}(a+b-c) & 0 & c^2\\ 0 & (b+c-a) & a^2\\ (-2a) & (-2c) & 2ca\end{vmatrix}$ ½

Taking common $2ac$ from R_3

$= \frac{(2ac)(a+b+c)^2}{ba^2c^2}\begin{vmatrix}ac+bc-c^2 & 0 & c^2\\ 0 & ab+ca-a^2 & a^2\\ -1 & -1 & 1\end{vmatrix}$

$(C_1 \to C_1 + C_3, C_2 \to C_2 + C_3)$

$= \frac{2ac(a+b+c)^2}{ba^2c^2}\begin{vmatrix}(ac+bc) & c^2 & c^2\\ a^2 & (ba+ca) & a^2\\ 0 & 0 & 2ca\end{vmatrix}$ **1**

Taking c from R_1 & a from R_2

$= \frac{2ac(a+b+c)^2}{(ac)b}\begin{vmatrix}a+b & c & c\\ a & b+c & a\\ 0 & 0 & 1\end{vmatrix}$ **1**

Expanding along R_3

$= \frac{2ac(a+b+c)^2}{abc}\left[ab+ac+b^2+bc-ac\right]$

$= \frac{2abc(a+b+c)^3}{abc}$

$= 2(a+b+c)^3$ ½

Sol. 6: $|A| = 4 \neq 0 \Rightarrow A^{-1}$ exists. **1**

$$\text{Adj } A = \begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$$ **2**

$$\therefore A^{-1} = \frac{1}{|A|}\cdot adj\ A = \frac{1}{4}\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$$ ½

Given system of equations can be written as $AX = B$

where, $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}, B = \begin{bmatrix}6\\ 7\\ 12\end{bmatrix}$

$\therefore X = A^{-1}\cdot B$

$$= \frac{1}{4}\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}6\\ 7\\ 12\end{bmatrix} = \begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$$ **1**

$\therefore x = 3, y = 1, z = 2$ ½

Sol. 12: For the given system of equations, we have

$$D = \begin{vmatrix}1 & -3 & 5\\ 2 & -6 & 10\\ 3 & -9 & 15\end{vmatrix}$$

Taking (– 3) common from C_2, we get

$$D = \begin{vmatrix}1 & 1 & 5\\ 2 & 2 & 10\\ 3 & 3 & 15\end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are identical] **1**

Now, $D_x = \begin{vmatrix}4 & -3 & 5\\ 11 & -6 & 10\\ 12 & -9 & 15\end{vmatrix}$

Taking common (– 3) and 5 from C_2 and C_3, we get

$$D_x = -15\begin{vmatrix}4 & 1 & 1\\ 11 & 2 & 2\\ 12 & 3 & 3\end{vmatrix} = 0$$

[$\therefore$ C_2 and C_3 are identical] **1**

$$D_y = \begin{vmatrix}1 & 4 & 5\\ 2 & 11 & 10\\ 3 & 12 & 15\end{vmatrix}$$

Taking common (5) from C_3, we get

$$D_y = 5\begin{vmatrix}1 & 4 & 1\\ 2 & 11 & 2\\ 3 & 12 & 3\end{vmatrix} = 0$$

[$\because$ C_1 and C_3 are identical] **1**

$$D_z = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -6 & 11 \\ 3 & -9 & 12 \end{vmatrix}$$

Taking Common (– 3) from C_2, we get

$$D_z = (-3)\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 11 \\ 3 & 3 & 12 \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are identical]

Thus, $D = D_x = D_y = D_z = 0$

So, given system of equations may or many not be consistent. **1**

If we put $z = k$ in any two of three equations, we find that the two equations obtained are inconsistent as they represent a pair of parallel lines.

i.e., $x - 3y = 4 - 5k$

$2x - 6y = 11 - 10k$

[Taking first two equations]

or $x - 3y = 4 - 5k$

$$x - 3y = \frac{11 - 10k}{2},$$

which represent pair of parallel lines.

Hence, the given system of equations is inconsistent. **1**

REFLECTIONS

(a) Here you practiced of questions based on properties of determinants?

(b) Are you able to easily apply Cramer's rule for the solution of system of equations?

(c) Will you be able to decide the given system of equations is consistent or inconsistent?

(d) Will you be able to find the volume of parallelepiped using determinants?

SELF ASSESSMENT PAPER - 02

Maximum Mark: 30 **Max. Time: 1 Hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. A 2×2 matrix whose elements a_{ij} are given by $a_{ij} = \frac{(i-j)^2}{2}$ is:

(A) $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ **(B)** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ **(C)** $\begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix}$ **(D)** $\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

Q. 2. If $A = [a_{ij}] = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 4 & 9 \\ 0 & 7 & -2 \end{bmatrix}$ and $B = [b_{ij}] = \begin{bmatrix} 2 & -1 \\ -3 & 4 \\ 1 & 2 \end{bmatrix}$, then value of $a_{11}b_{11} + a_{22}b_{22}$ is:

(A) 10 **(B)** 15 **(C)** 20 **(D)** 25

Q. 3. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, then values of x and y are:

(A) 2, 3 **(B)** 2, 1 **(C)** 1, 2 **(D)** 3, 2

Q. 4. Value of $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$ is:

(A) 1 **(B)** 0 **(C)** –1 **(D)** None of these

Q. 5. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, then $|A^2 - 2A|$ = ...

(A) 15 **(B)** 20 **(C)** 25 **(D)** –15

Q. 6. Find the value of $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$.

(A) 0 **(B)** 1 **(C)** –1 **(D)** 2

Case-Based MCQs

Q. 7. Attempt any 4 sub-parts from each questions. Each question carries 1 mark.

Read the following text and answer the questions on the basis of the same.

A bolt manufacturing company produces three types of bolts x, y and z which he sells in two markets. Annual sales (in ₹) are indicated in the following table.

Markets	Products		
	x	y	z
I	10000	2000	18000
II	6000	20000	8000

It unit sales price of x, y and z are ₹2.50, ₹1.50 and ₹1.00 respectively, then answer the following questions using the concept of matrices.

(i) The total revenue collected from market I is:

(A) ₹44,000 **(B)** ₹48,000 **(C)** ₹46,000 **(D)** ₹53,000

(ii) Total revenue collected from market II is:

(A) ₹46,000 **(B)** ₹51,000 **(C)** ₹53,000 **(D)** ₹49,000

(iii) If the unit costs of the above three types of bolts are ₹2, ₹1 and ₹0.50 respectively, then the gross profit from the market is:

(A) ₹46,000 **(B)** ₹25,000 **(C)** ₹32,000 **(D)** ₹20,000

(iv) If matrix $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = 1$ if $i \neq j$ and $a_{ij} = 0$ if $i = j$, then A^2 is:

(A) I **(B)** A **(C)** 0 **(D)** none of these

(v) If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, then AB is:

(A) $\begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$ **(B)** $\begin{bmatrix} 0 & 3 \\ 22 & 43 \end{bmatrix}$ **(C)** $\begin{bmatrix} 43 & 22 \\ 0 & 3 \end{bmatrix}$ **(D)** $\begin{bmatrix} 22 & 43 \\ 3 & 0 \end{bmatrix}$

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then find $A + A^T$.

Q. 9. Find the trace of matrix $A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}$.

Q. 10. If A is a skew symmetric matrix of odd order n, then $|A| = 0$.

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. Find values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

Q. 12. For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y so that $A^2 + xI = yA$.

Q. 13. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then show that $A^{-1} = \frac{1}{19} A$.

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. Evaluate: $\Delta = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$.

Q. 15. Solve linear equations $x + y = 2$ and $2x + 3y = 4$ using Cramer's Rule Method.

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 16. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equation :

$x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$

CHAPTER 5

DIFFERENTIATION AND ITS APPLICATIONS

Syllabus

Higher Order Derivatives, Applications of Derivatives, Marginal cost and marginal revenue using derivatives, Increasing/Decreasing functions, Maxima and Minima.

In this chapter you will study

- *Determine second and higher order derivatives*
- *Understand differentiation of parametric functions and implicit functions*
- *Determine the rate of change of various quantities*
- *Understand the gradient of tangent and normal to a curve at a given point*
- *Write the equation of tangents and normal to a curve at a given point*
- *Determine whether a function is increasing or decreasing*
- *Determine the conditions for a function to be increasing or decreasing*
- *Determine critical points of the function*
- *Find the point(s) of local maxima and local minima and corresponding local maximum and local minimum values*
- *Find the absolute maximum and absolute minimum value of a function*
- *Solve applied problems*

Revision Notes

Higher Order Derivatives of an Explicit Function

Let the function $y = f(x)$ have a finite derivative $f'(x)$ in a certain interval (a, b), i.e., the derivative $f'(x)$ is also a function in this interval. If this function is differentiable, we can find the second derivative of the original function $y = f(x)$, which is denoted by various notations as

$$f'' = (f')' = \left(\frac{dy}{dx}\right)'$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d^2y}{dx^2}$$

Similarly, if f'' exists and is differentiable, we can calculate the third derivative of the function $f(x)$:

$$f''' = \frac{d^3y}{dx^3} = y'''$$

The result of taking the derivative n times is called the n^{th} derivative of $f(x)$ with respect to x and is denoted as

$$\frac{d^ny}{dx^n} = \frac{d^ny}{dx^n} \text{ (in Leibniz's notation)}$$

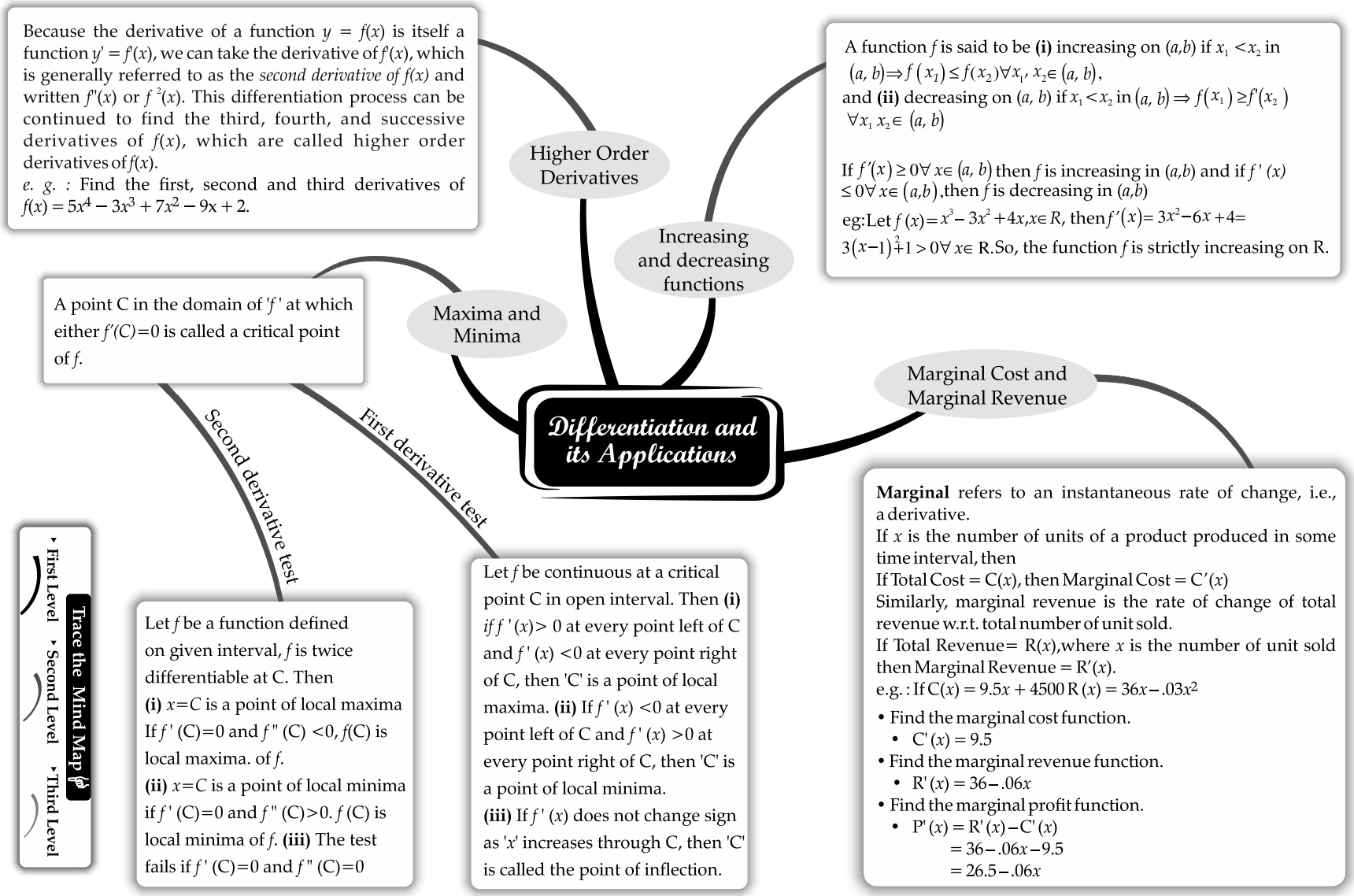
$$f^{(n)}(x) = y^{(n)}(x) \text{ (in Lagrange's notation)}$$

Thus, the notation of the nth order derivative in introduced inductively by sequential calculation of n derivatives starting from the first order derivative. Transition to the next higher-order derivative is performed using the recurrence formula

$$y^{(n)} = (y^{(n-1)})'$$

Higher Order Derivatives of an Implicit Function

The nth order derivative of an implicit function can be found by sequential (n times) differentiation of the equation $F(x, y) = 0$. At each step, after appropriate substitutions and transformations, we can obtain an explicit expression for the derivative, which depends only on the variables x and y, *i.e.*, the derivatives have the form


Differentiation and its Applications
Higher Order Derivatives
Because the derivative of a function $y = f(x)$ is itself a function $y' = f'(x)$, we can take the derivative of $f'(x)$, which is generally referred to as the *second derivative of f(x)* and written $f''(x)$ or $f^2(x)$. This differentiation process can be continued to find the third, fourth, and successive derivatives of $f(x)$, which are called higher order derivatives of $f(x)$.
e. g. : Find the first, second and third derivatives of $f(x) = 5x^4 - 3x^3 + 7x^2 - 9x + 2$.
Increasing and decreasing functions
A function f is said to be **(i)** increasing on (a,b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \le f(x_2) \forall x_1, x_2 \in (a, b)$,
and **(ii)** decreasing on (a, b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \ge f'(x_2)$ $\forall x_1\, x_2 \in (a, b)$
If $f'(x) \ge 0 \forall x \in (a, b)$ then f is increasing in (a,b) and if $f'(x) \le 0 \forall x \in (a,b)$, then f is decreasing in (a,b)
eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall x \in$ R. So, the function f is strictly increasing on R.
Maxima and Minima
A point C in the domain of 'f' at which either $f'(C) = 0$ is called a critical point of f.
First derivative test
Let f be continuous at a critical point C in open interval. Then **(i)** *if* $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. **(ii)** If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.
(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.
Second derivative test
Let f be a function defined on given interval, f is twice differentiable at C. Then
(i) $x = C$ is a point of local maxima If $f'(C) = 0$ and $f''(C) < 0$, $f(C)$ is local maxima. of f.
(ii) $x = C$ is a point of local minima if $f'(C) = 0$ and $f''(C) > 0$. $f(C)$ is local minima of f. **(iii)** The test fails if $f'(C) = 0$ and $f''(C) = 0$
Marginal Cost and Marginal Revenue
Marginal refers to an instantaneous rate of change, i.e., a derivative.
If x is the number of units of a product produced in some time interval, then
If Total Cost = $C(x)$, then Marginal Cost = $C'(x)$
Similarly, marginal revenue is the rate of change of total revenue w.r.t. total number of unit sold.
If Total Revenue = $R(x)$, where x is the number of unit sold then Marginal Revenue = $R'(x)$.
e.g. : If $C(x) = 9.5x + 4500$ $R(x) = 36x - .03x^2$
• Find the marginal cost function.
  • $C'(x) = 9.5$
• Find the marginal revenue function.
  • $R'(x) = 36 - .06x$
• Find the marginal profit function.
  • $P'(x) = R'(x) - C'(x)$
    $= 36 - .06x - 9.5$
    $= 26.5 - .06x$
Trace the Mind Map
First Level
Second Level
Third Level

$$y' = f_1(x, y)$$
$$y_n = f_2(x, y), \ldots$$
$$y(n) = f_n(x, y)$$

► **Higher Order Derivatives of Parametric Function**

Consider a function $y = f(x)$ given parametrically by the equations

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

The first derivative of this function is given by

$$y' = y'_x = \frac{y'_t}{x'_t}$$

Differentiating once more with respect to x, we find the second derivative:

$$y'' = y''_{xx} = \frac{(y'_x)'_t}{x'_t}$$

Example 1: Given the function $y = (2x + 1)^3 (x - 1)$. Find all derivatives of the n^{th} order from $n = 1$ to $n = 5$.

Sol. First we convert the given function into a polynomial:

$$y = (2x + 1)^3$$
$$= (8x^3 + 12x^2 + 6x + 1).(x - 1)$$
$$= 8x^4 + 12x^3 + 6x^2 + x - 8x^3 - 12x^2 - 6x - 1$$
$$= 8x^4 + 4x^3 - 6x^2 - 5x - 1.$$

Now we successively calculate the derivatives from 1^{st} to 5^{th} order:

$$y' = (8x^4 + 4x^3 - 6x^2 - 5x - 1)'$$
$$= 32x^3 + 12x^2 - 12x - 5$$
$$y'' = (y')' = (32x^3 + 12x^2 - 12x - 5)'$$
$$= 96x^2 + 24x - 12$$
$$y''' = (y'')' = (96x^2 + 24x - 12)'$$
$$= 192x + 24$$
$$y^{(4)} = (y''')' = (192x + 24)' = 192$$
$$y^{(5)} = (y''')' = (192)' = 0.$$

► **Increasing/Decreasing Functions**

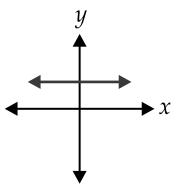
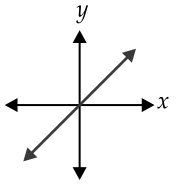
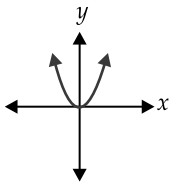
Scan to know more about this topic

Increasing and Decreasing Functions

- A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases, i.e., if $\alpha, \beta \Leftrightarrow [a, b]$ and $\alpha > \beta$, $f(\alpha) > f(\beta)$.
  If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.
- A function $f(x)$ is said to be a decreasing function in $[a, b]$, if as x increases, $f(x)$ decreases, i.e., if $\alpha, \beta \Leftrightarrow [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.
  If $f'(x) = 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.
  - ♦ A function $f(x)$ is a constant function in $[a, b]$, if $f'(x) = 0$ for each $x \Leftrightarrow (a, b)$.
  - ♦ By monotonic function $f(x)$ in interval I, we mean that f is either only increasing in I or only decreasing in I.
- Finding the intervals of increasing and/or decreasing of a function:

  Algorithm

  Step 1: Consider the function $y = f(x)$.
  Step 2: Find $f'(x)$.
  Step 3: Put $f'(x) = 0$ and solve to get the critical point(s).
  Step 4: The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Function	Increasing/ Decreasing	Graph
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Example 2: Find the intervals in which the function $f(x)$ is strictly increasing where, $f(x) = 10 - 6x - 2x^2$.

Sol. Given, $f(x) = 10 - 6x - 2x^2$

$\therefore \quad f'(x) = -6 - 4x$

$f'(x) = -6 - 4x = 0$

$\Rightarrow \quad x = \frac{-6}{4} = \frac{-3}{2}$

$\therefore$ The intervals are $\left(-\infty, \frac{-3}{2}\right)$ and $\left(\frac{-3}{2}, \infty\right)$

$f'(x) > 0$ in $\left(-\infty, \frac{-3}{2}\right)$

$\therefore f(x)$ is strictly increasing in $\left(-\infty, \frac{-3}{2}\right)$

► **Maxima and Minima**

Scan to know more about this topic

Local Maxima & Minima

- **Understanding maxima and minima:**

  Consider $y = f(x)$ be a well defined function on an **interval** I, then

Key Words

Interval: In mathematics, an interval is a set of real numbers between two given numbers called end points of the interval.

Scan to know more about this topic

Maxima & Minima

(a) f is said to have a maximum value in I, if there exists a point c in I such that

$$f(c) > f(x), \text{ for all } x \Leftrightarrow I$$

The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the point of maximum value of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \Leftrightarrow I$.

The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of f in I**.

(c) f is said to have an extreme value in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.

- Let f be a real valued function and also take a point c from its **domain**. Then

Key Words

Domain: The domain refers to the set of possible input values, the domain of a graph consists of all input values shown on the x-axis.

(i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local maximum value of f**.

(ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local minimum value** of f.

- **Critical points:**

It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes, i.e., $f'(c) = 0$ or f is not differentiable.

- **First Derivative Test:**

Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm:

Step 1: Find $\frac{dy}{dx}$.

Step 2: Find the critical point(s) by putting $\frac{dy}{dx} = 0$. Suppose $c \Leftrightarrow I$ (where I is the interval) be any critical point and f be continuous at this point c. Then we may have following situations:

- $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.
- $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.
- $\frac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**. Rather in this case, the point $x = c$ is called the **point of inflection**.

- **Second Derivative Test:**

Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that:

- $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

  The value $f(c)$ is called the local maximum value of f.
- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

  The value $f(c)$ is called the local minimum value of f.

  This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed in the above.

- **Absolute maxima and absolute minima:**

If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

Algorithm

Step 1: Find all the critical points of f in the given interval, i.e., find all the points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the given interval.

Step 3: At all these points (i.e., the points found in Step 1 and Step 2) calculate the values of f.

Step 4: Identify the maximum and minimum value of f out of the values calculated in Step 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly, absolute minimum value is called as **global minimum value** or the **least value**.

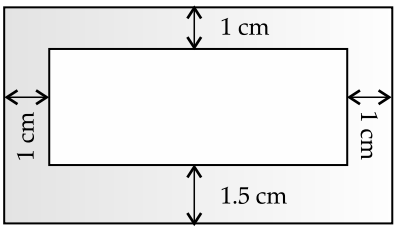
Second Derivative Test:

Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that:

Example 3: A printed page is to have total area of 80 sq. cm with a margin of 1 cm at the top and on each side and a margin of 1.5 cm at the bottom. What should be the dimensions of the page so that the printed area will be maximum?

Sol. Let x cm and y cm be the dimensions of the printed page.

$$xy = 80 \quad \ldots(i)$$



Key Facts

- Maxima and minima is used to solve optimization problems such as maximizing profit, minimizing the amount of material used in manufacturing or finding the maximum height a rocket can reach.
- A function is called increasing if it increases as the input x moves from left to right, and is called decreasing if it decreases as x moves from left to right.
- In Computer Science, Calculus is used for machine learning, data mining, scientific computing, image processing, and creating the graphics and physics engines for video games, including the 3D visuals for simulations.

Let A (x sq. cm) be the printed area then,

$$A = (x-2)(y-5/2)$$

$$\Rightarrow \quad A = xy - \frac{5x}{2} - 2y + 5$$

$$\Rightarrow \quad A = 80 - \frac{5x}{2} - \frac{2\times 80}{x} + 5 \qquad \text{[using (i)]}$$

$$\Rightarrow \quad A = 85 - \frac{5x}{2} - \frac{160}{x}$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = -\frac{5}{2} + \frac{160}{x^2}$$

Differentiating once again w.r.t. x

$$\frac{d^2A}{dx^2} = -\frac{320}{x^3}$$

For maximum and minimum Area,

$$\Rightarrow \quad \frac{dA}{dx} = 0$$

$$-\frac{5}{2} + \frac{160}{x^2} = 0$$

$$\Rightarrow \quad \frac{160}{x^2} = \frac{5}{2}$$

$$\Rightarrow \quad x^2 = 64$$

$$\Rightarrow \quad x = -8$$

Also, $\left(\frac{d^2A}{dx^2}\right)_{x=8} = -\frac{320}{512} < 0$

$\Rightarrow A$ is maximum at $x = 8$

From (i), $\quad xy = 80$

$$\Rightarrow \quad 8 \times y = 80$$

$$\Rightarrow \quad y = 10$$

Therefore, the dimensions of the printed page are 10 cm and 8 cm.

Marginal Cost and Marginal Revenue

Derivatives are useful in analysing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Scan to know more about this topic
Marginal Cost

Scan to know more about this topic
Marginal Revenue

- The Fixed Cost (FC) is the amount of money we have to spend regardless of how many items we produce. FC can include things like rent, purchase costs of machinery, and salaries for office staff. We have to pay the fixed costs even if we don't produce anything.
- The Total Variable Cost (TVC) for q items is the amount of money we spend to actually produce them. TVC includes things like the materials we use, the electricity to run the machinery, gasoline for our delivery vans, maybe the wages of our production workers. These costs will vary according to how many items we produce.
- The Total Cost (TC) for x items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing q items.
- The Average Cost (AC) for x items is the total cost divided by x i.e., $\frac{TC}{x}$.

  Also, Average Fixed Cost = $\frac{FC}{x}$ and Average Variable Cost = $\frac{TVC}{x}$.
- If $C(x)$ is the cost of producing x items, then the marginal cost $MC(x)$ is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the marginal profit MP(x) is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.
- Break even point (BEP) is the point where the profit from the transaction is zero and the total sales is equal to total costs. Break even point is the inflection point where the revenue sales are same as the costs. At the break even point, there is zero profit or zero loss for the company.

Break even point (*BEP*)

= Total Sales – Total Costs

- Break even point is important for companies to understand the minimum business required to sustain any product or service. The units break even i.e. number of units to be sold at the break even pricing helps in evaluating the break even point. This method of evaluation is known as break even analysis.

Example 4: The total cost associated with provision of free mid-day meals to x students of a school in primary classes is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$. If the marginal cost is given by rate of change $\frac{dC}{dx}$ of total cost, write the marginal cost of food for 300 students.

Sol. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

$$\Rightarrow \quad \frac{dC}{dx} = C'(x) = 0.015x^2 - 0.04x + 30 + 0$$

At $x = 300$,

$C'(300) = 0.015(300)^2 - 0.04(300) + 30 = 1368$

So, the marginal cost of food for 300 students is ₹ 1,368.

Mnemonics

Concept: Increasing and Decreasing Function

Mnemonics: **D**ead **Z**ombies **C**onsider **G**reen **L**emons

Interpretations:

Dead	**Z**ombies	**C**onsider	**G**reen	**L**emons
↓	↓	↓	↓	↓
Find **Derivative** of function $f(x)$ *i.e.*, $f'(x)$	Put Derivative equal to **zero** *i.e.*, $f'(x) = 0$	Get **Critical** points *i.e.*, values of x	If $f'(x)$ is **Greater** than zero *i.e.*, $f'(x) > 0$, then $f(x)$ is increasing	If $f'(x)$ is **Less** than zero i.e., $f'(x) < 0$, then $f(x)$ is decreasing

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. The function $f(x) = a^x$ is increasing on R, if:

(A) $a > 0$ **(B)** $a < 0$

(C) $0 < a < 1$ **(D)** $a > 1$

[CBSE Term-I 2021]

Ans. Option (D) is correct.

Explanation: Given, $f(x) = a^x$

$\therefore \quad f'(x) = a^x \log a$

for increasing function, $f'(x) > 0$

$\therefore \quad a^x \log a > 0$

$\Rightarrow a^x > 0$ and $\log a > 0$ or $a^x < 0$ and $\log a < 0$

But log function is always positive

$\Rightarrow \quad \log a > 0$

$\Rightarrow \quad a > 1$

Q. 2. If $C(x)$ and $R(x)$ are respectively Cost function and Revenue function, then the Profit function P(x) is given by:

(A) $P(x) = R(x)$ **(B)** $P(x) = C(x) + R(x)$

(C) $P(x) = R(x) - C(x)$ **(D)** $P(x) = R(x).C(x)$

[CBSE Term-I 2021]

Ans. Option (C) is correct.

Explanation: Profit is revenue minus cost

i.e., $P(x) = R(x) - C(x)$

Q. 3. If $y = Ae^{5x} + Be^{-5x}$, then $\frac{d^2y}{dx^2}$ is:

(A) $25y$ **(B)** $5y$

(C) $-25y$ **(D)** $15y$

[CBSE Term-I 2021]

Ans. Option (A) is correct.

Explanation: Given, $y = Ae^{5x} + Be^{-5x}$

$\therefore \quad y' = 5Ae^{5x} - 5Be^{-5x}$

[Differentiating w.r.t. x]

$\Rightarrow \quad y' = 5(Ae^{5x} - Be^{-5x})$

$\Rightarrow \quad y' = 5(5Ae^{5x} + 5Be^{-5x})$

$\Rightarrow \quad y'' = 5(5Ae^{5x} + 5Be^{-5x})$

[Differentiating again w.r.t. x]

$\Rightarrow \quad y'' = 25(Ae^{5x} + Be^{-5x})$

$\Rightarrow \quad y'' = 25y$

or $\quad \frac{d^2y}{dx^2} = 25y$

Q. 4. If the total revenue (₹) received from the sale of x units of a product is given by:

$$R(x) = 3x^2 + 36x + 5,$$

then the marginal revenue,when $x = 15$, is

(A) ₹ 116 **(B)** ₹ 96

(C) ₹ 90 **(D)** ₹ 126

[CBSE Term-I 2021]

Ans. Option (D) is correct.

Explanation: $R(x) = 3x^2 + 36x + 5$

Marginal Revenue, $R'(x) = \frac{dR}{dx} = 6x + 36$

$\therefore$ Marginal Revenue at $x = 15$ is $\left.\frac{dR}{dx}\right|_{x=15}$

$= 6(15) + 36 = ₹126$

Q. 5. The least value of 'a' such that the function $f(x) = x^2 + ax + 1$ is increasing on (1, 2) is:

(A) 0 **(B)** –1

(C) –2 **(D)** –4

[CBSE Term-I 2021]

Ans. Option (C) is correct.

Explanation: We have, $f(x) = x^2 + ax + 1$

$\therefore \quad f'(x) = 2x + a$

Now, function f will be increasing in (1, 2),

if $f'(x) > 0$ in (1, 2)

$f'(x) > 0$

$\Rightarrow \quad 2x + a > 0$

$\Rightarrow \quad 2x > -a$

$\Rightarrow \quad x > \frac{-a}{2}$

Therefore, we have to find the least value of a such that

$x > \frac{-a}{2}$, when $x \in (1, 2)$

$\Rightarrow \quad x > \frac{-a}{2}$, when $1 < x < 2$

Thus, the least value of a for f to be increasing on (1, 2) is given by,

$\frac{-a}{2} = 1$

$\Rightarrow \quad a = -2$

Hence, the required value of a is –2.

Q. 6. The demand function of a commodity is given by:

$$x = 82 - p$$

and its total cost function is given by:

$$T.C. = 100 + 60x$$

For maximum profit, the value of x is:

(A) 15 units **(B)** 14 units

(C) 13 units **(D)** 11 units

[CBSE Term-I 2021]

Ans. Option (D) is correct.

Explanation: Here, demand function.

$p(x) = 82 - x$

and cost function $c(x) = 100 + 60x$

$\therefore$ Profit function, $P(x) = xp(x) - c(x)$

$\Rightarrow \quad P(x) = x(82 - x) - (100 + 60x)$

$\Rightarrow \quad P(x) = 82x - x^2 - 100 - 60x$

$\Rightarrow \quad P(x) = -x^2 + 22x - 100$

Now, put $P'(x) = 0 \quad P'(x) = -2x + 22$

i.e., $-2x + 22 = 0$

$\Rightarrow \quad x = 11 \quad P''(x) = -2$

Also, $P''(x) = -2 < 0$ (Hence, maximum)

Thus, value of x is 11 units.

Q. 7. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is:

(A) $(2\sqrt{2}, 4)$ **(B)** $(2\sqrt{2}, 0)$

(C) (0, 0) **(D)** (2, 2)

[CBSE Term-I 2021]

Ans. Option (A) is correct.

Explanation: Let a point on the curve be (h, k)

Then, $h^2 = 2k$ ...(i)

Distance, $D = \left|\sqrt{(h-0)^2 + (k-5)^2}\right|$

$= \left|\sqrt{2k + k^2 + 25 - 10k}\right|$ [from (i)]

$= \left|\sqrt{k^2 - 8k + 25}\right|$

or, Let $z = D^2 = k^2 - 8k + 25$

$\Rightarrow \quad z = k^2 - 8k + 25$

$\Rightarrow \quad \frac{dz}{dk} = 2k - 8$ ...(ii)

for maxima & minima, $\frac{dz}{dk} = 0$

$\therefore \quad 2k - 8 = 0$

$\Rightarrow \quad k = 4$

On differentiating again eq. (i), we get

$\frac{d^2z}{dk^2} = 2 > 0$

$\Rightarrow z$ has a minima at $k = 4$

from (i), when $k = 4, h^2 = 8$

$\Rightarrow \quad h = \pm 2\sqrt{2}$

$\therefore$ Required points are $(\pm 2\sqrt{2}, 4)$.

Q. 8. The total cost function is given by $C(x) = x^2 + 30x + 1500$. The marginal cost when 10 units are produced is:

(A) ₹ 20 **(B)** ₹ 30
(C) ₹ 50 **(D)** ₹ 70

Ans. Option (C) is correct.

Explanation:

$$C(x) = x^2 + 30x + 1500$$
$$\text{M.C.} = C'(x)$$
$$= 2x + 30$$

M.C. when 10 units are produced

$$= C'(10)$$
$$= ₹\ 50$$

Q. 9. The function $y = \frac{1}{x}$ is strictly decreasing in the interval(s)

(A) $(0, \infty)$
(B) $(-\infty, 0)$
(C) $(-\infty, 0)$ as well as $(0, \infty)$
(D) R

Ans. Option (C) is correct.

Explanation:

$$y = \frac{1}{x}$$
$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2} < 0$$

for $(-\infty, 0)$ and $(0, \infty)$

Q. 10. Given that $x = at^2$ and $y = 2at$, then value of $\frac{d^2y}{dx^2}$ is

(A) $-\frac{1}{2at^3}$ **(B)** $-\frac{1}{2at^2}$
(C) $\frac{1}{t^2}$ **(D)** $\frac{-2a}{t}$

Ans. Option (A) is correct.

Explanation:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$= \frac{2a}{2at}$$
$$= \frac{1}{t}$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{t^2} \times \frac{dt}{dx}$$
$$= -\frac{1}{2at^3}$$

Q 11. The variable cost of producing x units is $V(x) = x^2 + 2x$. If the company incurs a fixed cost of ₹ 10,000, then the level of output where the average cost is minimum is

(A) 10 units **(B)** 50 units
(C) 100 units **(D)** 200 units

Ans. Option (C) is correct.

Explanation:

$$TC = VC + FC$$
$$= x^2 + 2x + 10000$$
$$AC = x + 2 + \frac{10000}{x}$$
$$\frac{d(AC)}{dx} = 1 - \frac{10000}{x^2}$$
$$= 0$$
$$\Rightarrow \quad x = 100$$

Q. 12. The demand function of a toy is, $x = 75 - 3p$ and its total cost function is $TC = 100 + 3x$. For maximum profit the value of x is

(A) 33 **(B)** 31
(C) 29 **(D)** 24

Ans. Option (A) is correct.

Explanation:

$$TR = px$$
$$= \frac{75x - x^2}{3}$$
$$P = TR - TC$$
$$= \frac{75x - x^2}{3} - (3x + 100)$$
$$\frac{dP}{dx} = 22 - \frac{2}{3}x$$
$$= 0$$
$$\Rightarrow \quad x = 33$$
$$\Rightarrow \quad \frac{d^2P}{dx^2} = \frac{-2}{3} < 0$$

$\therefore$ Max. at 33 unit

Q. 13. Let $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$ for every real number x. Then

(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is increasing
(D) Nothing can be said in general

Ans. Option (A) is correct.

Explanation: For the given relation h is increasing whenever f is increasing as:

$$h(x) = f(x) - (f(x))^2 + (f(x))^3$$

Q. 14. The interval on which the function

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$

is decreasing is:

(A) $[-1, \infty)$ **(B)** $[-2, -1]$
(C) $(-\infty, -2]$ **(D)** $[-1, 1]$

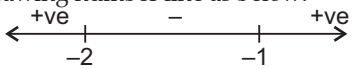
Ans. Option (B) is correct.

Explanation: Given that,

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$
$$f'(x) = 6x^2 + 18x + 12$$
$$= 6(x^2 + 3x + 2)$$
$$= 6(x + 2)(x + 1)$$

So, $f'(x) \le 0$, for decreasing.

On drawing number line as below:



We see that $f(x)$ is decreasing in $[-2, -1]$

Q. 15. $y = x(x-3)^2$ **decreases for the values of** x **given by:**

(A) $1 < x < 3$ **(B)** $x < 0$

(C) $x > 0$ **(D)** $0 < x < \frac{3}{2}$

Ans. Option (A) is correct.

Explanation: Given that,

$$y = x(x-3)^2$$

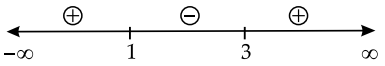
$$\therefore \quad \frac{dy}{dx} = x.2(x-3).1 + (x-3)^2.1$$

$$= 2x^2 - 6x + x^2 + 9 - 6x$$

$$= 3x^2 - 12x + 9$$

$$= 3(x^2 - 3x - x + 3)$$

$$= 3(x-3)(x-1)$$



So, $y = x(x-3)^2$ decreases for (1, 3).

[Since, $y' < 0$ for all $x \in (1, 3)$, hence y is decreasing on (1, 3)].

Q. 16. If x **is real, the minimum value of** $x^2 - 8x + 17$ **is**

(A) –1 **(B)** 0

(C) 1 **(D)** 2

Ans. Option (C) is correct.

Explanation: Let,

$$f(x) = x^2 - 8x + 17$$

On differentiating with respect to x, we get

$$f'(x) = 2x - 8$$

$$\text{So, } f'(x) = 0$$

$$\Rightarrow 2x - 8 = 0$$

$$\Rightarrow \quad 2x = 8$$

$$\therefore \quad x = 4$$

Now, Again on differentiating w.r.t. x, we get

$$f''(x) = 2 > 0, \forall x$$

So, $x = 4$ is the point of local minima.

Minimum value of $f(x)$ at $x = 4$

$$f(4) = 4 \times 4 - 8 \times 4 + 17 = 1$$

Q. 17. The smallest value of the polynomial $x^3 - 18x^3 + 96x$ **in [0, 9] is**

(A) 126 **(B)** 0

(C) 135 **(D)** 160

Ans. Option (B) is correct.

Explanation: Given that, the smallest value of polynomial $x^3 - 18x^2 + 96x$ is

On differentiating with respect to x, we get

$$f'(x) = 3x^2 - 36x + 96$$

So,

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2 - 36x + 96 = 0$$

$$\Rightarrow 3(x^2 - 12x + 32) = 0$$

$$\Rightarrow \quad (x-8)(x-4) = 0$$

$$\Rightarrow \quad x = 8, 4 \in [0, 9]$$

We shall now calculate the value of f at these points and at the end points of the interval [0, 9], i.e., at $x = 4$ and $x = 8$ and at $x = 0$ and at $x = 9$.

$$f(4) = 4^3 - 18 \times 4^2 + 96 \times 4$$

$$= 64 - 288 + 384$$

$$= 160$$

$$f(8) = 8^3 - 18 \times 8^2 + 96 \times 8$$

$$= 128$$

$$f(9) = 9^3 - 18 \times 9^2 + 96 \times 9$$

$$= 729 - 1458 + 864$$

$$= 135$$

$$\text{and } f(0) = 0^3 - 18 \times 0^2 + 96 \times 0$$

$$= 0$$

Thus, we conclude that absolute minimum value of f on [0, 9] is 0 occurring at $x = 0$.

Q. 18. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$ **, has**

(A) two points of local maximum

(B) two points of local minimum

(C) one maxima and one minima

(D) no maxima or minima

Ans. Option (C) is correct.

Explanation: We have,

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

$$f'(x) = 6x^2 - 6x - 12$$

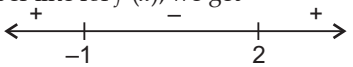
Now, $f'(x) = 0$

$\Rightarrow \quad 6(x^2 - x - 2) = 0$

$\Rightarrow \quad 6(x + 1)(x - 2) = 0$

$\Rightarrow \quad x = -1$ and $x = +2$

On number line for $f'(x)$, we get



Hence, $x = -1$ is point of local maxima and $x = 2$ is point of local minima.

So, $f(x)$ has one maxima and one minima.

Q. 19. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$.

(A) $\sin 2x$ **(B)** $\tan x$

(C) $\cos x$ **(D)** $\cos 3x$

Ans. Option (C) is correct.

Explanation: In the given interval $\left(0, \frac{\pi}{2}\right)$

$$f(x) = \cos x$$

On differentiating w.r.t. x, we get

$$f'(x) = -\sin x$$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

Q. 20. The function $f(x) = \tan x - x$

(A) always increases

(B) always decreases

(C) never increases

(D) sometimes increases and sometimes decreases

Ans. Option (A) is correct.

Explanation: We have,

$$f(x) = \tan x - x$$

On differentiating with respect to x, we get

$$f'(x) = \sec^2 x - 1$$

$$\Rightarrow f'(x) > 0, \forall x \in R$$

So, $f(x)$ always increases.

Q. 21. The interval in which $y = x^2e^{-x}$ is increasing in

(A) $(-\infty, \infty)$ **(B)** $(-2, 0)$

(C) $(2, \infty)$ **(D)** $(0, 2)$

Ans. Option (D) is correct.

Explanation: Given that,

$$y = x^2e^{-x}$$

$$\therefore \quad \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x}$$

$$= xe^{-x}(2 - x)$$

$$\text{Now, } \frac{dy}{dx} = 0$$

$$\Rightarrow \quad x = 0 \text{ and } x = 2$$

The points $x = 0$ and $x = 2$ divide the real line into three disjoint intervals *i.e.*, $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$. In intervals $(-\infty, 0)$ and $(2, \infty)$, $f'(x) < 0$ as e^{-x} is always positive.

$\therefore f$ is decreasing on $(-\infty, 0)$ and $(2, \infty)$

In interval $(0, 2)$, $f'(x) > 0$

$\therefore f$ is strictly increasing on $(0, 2)$.

Hence, f is strictly increasing in interval $(0, 2)$.

Q. 22. What is the nature of function $f(x) = 7x - 4$ on R?

(A) Increasing

(B) Decreasing

(C) Strictly Increasing

(D) Increasing and Decreasing

Ans. Option (C) is correct.

Explanation:

Let x_1 and x_2 be any two numbers in R.

Then $x_1 < x_2$

$\Rightarrow \quad 7 \times 1 < 7 \times 2.$

$\Rightarrow \quad 7 \times 1 - 4 < 7 \times 2 - 4.$

As $f(x_1) < f(x_2)$, thus the function f is strictly increasing on R.

Q. 23. What is the nature of function $f(x) = x^3 - 3x^2 + 4x$ on R ?

(A) Increasing

(B) Decreasing

(C) Constant

(D) Increasing and Decreasing

Ans. Option (A) is correct.

Explanation:

$$f(x) = x^3 - 3x^2 + 4x$$

$$f'(x) = 3x^2 - 6x + 4.$$

$$f'(x) = 3(x^2 - 2x + 1) + 1.$$

$\Rightarrow 3(x-1)^2 + 1 > 0$, in every interval of R. Therefore the function f is increasing on R.

Q. 24. Find the interval in which function $f(x) = x^2 - 4x + 5$ is increasing.

(A) $(2, \infty)$ **(B)** $(-\infty, 2)$

(C) $(3, \infty)$ **(D)** $(-\infty, \infty)$

Ans. Option (A) is correct.

Explanation: $f(x) = x^2 - 4x + 5.$

$f'(x) = 2x - 4.$

Therefore $f'(x) = 0$ gives $x = 2$.

Now this point $x = 2$ divides the line into two disjoint intervals and the interval namely $(2, \infty)$ is increasing on $f(x)$.

Q. 25. Find the intervals in which $f(x) = 2x^2 - 3x$ is increasing.

(A) $\left(-\frac{1}{4}, \infty\right)$ **(B)** $\left(-\frac{3}{4}, \infty\right)$

(C) $\left(\frac{1}{4}, \infty\right)$ **(D)** $\left(\frac{3}{4}, \infty\right)$

Ans. Option (D) is correct.

Explanation: $f(x) = 2x^2 - 3x.$

$f'(x) = 4x - 3.$

As we know $f'(x) = 0$, $x = \frac{3}{4}$. This shows that function f is increasing in interval $\left(\frac{3}{4}, \infty\right)$ for all $x \in$ R.

Q. 26. Nature of the function $f(x) = e^{2x}$ is _____

(A) Increasing

(B) Decreasing

(C) Constant

(D) Increasing and Decreasing

Ans. Option (A) is correct.

Explanation:

$$f(x) = e^{2x}$$

$$f'(x) = 2e^{2x}$$

As we know $2e^{2x} > 0$, so it always has a value greater than zero.

Which shows that function f is increasing for all $x \in$ R.

Q. 27. Find the second derivative of $x^3 - 5x^2 + x = 0$

(A) $10x - 5$ **(B)** $6x - 10$

(C) $3x + 10$ **(D)** $3x^2 - 5x$

Ans. Option (B) is correct.

Explanation:

Let $f(x) = x^3 - 5x^2 + x$

So, $f'(x) = 3x^2 - 10x + 1$

And $f''(x) = 6x - 10$

Q. 28. Find the tangent to the curve $y = 3x^2 + x + 4$ at $x = 3$.

(A) 19 **(B)** 1.9

(C) 18 **(D)** 17

Ans. Option (A) is correct.

Explanation: The slope of tangent at $x = 3$ is given by

$$\left.\frac{dy}{dx}\right|_{x=3} = 6x + 1|_{x=3}$$

$$= 18 + 1$$

$$= 19$$

Q. 29. Find the slope of normal to the curve $y = 4x^2 - 14x + 5$ at $x = 5$.

(A) $\frac{1}{26}$ **(B)** $-\frac{1}{26}$

(C) 26 **(D)** -26

Ans. Option (B) is correct.

Explanation: The slope of the tangent at $x = 5$ is given by

$$\left.\frac{dy}{dx}\right|_{x=5} = 8x - 14\Big|_{x=5}$$
$$= 40 - 14$$
$$= 26$$

Therefore, slope of the normal to the curve is

$$-\frac{1}{\text{slope of the longest at } x=5} = -\frac{1}{26}$$

Q. 30. Find the slope of the tangent to the curve $y = 7x^3 - 2x^2$ at the point $x = 2$.

(A) 67 **(B)** 76

(C) 46 **(D)** 64

Ans. Option (B) is correct.

Explanation: The slope of the longest at $x = 2$ is given by

$$\left.\frac{dy}{dx}\right|_{x=2} = 21x^2 = 4x\Big|_{x-2}$$
$$= 21(2)^2 - 4(2)$$
$$= 84 - 8$$
$$= 76.$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. If $x = t^2$ and $y = t^3$ then find $\frac{d^2y}{dx^2}$. AI

Sol. Given that, $x = t^2$ and $y = t^3$

Then, $\frac{dx}{dt} = 2t$

and $\frac{dy}{dt} = 3t^2$

Thus, $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}\frac{dt}{dx}$

$\frac{3}{2}\cdot\frac{1}{2t} = \frac{3}{4t}$ **1**

Q. 2. Given $f(x) = e^x + \log x$. Find the value of $f''(x)$. R&U

Q. 3. Show that function $y = 4x - 9$ is increasing for all $x \in \mathbb{R}$. AI

Sol. Given, $y = 4x - 9$

$\frac{dy}{dx} = 4 > 0$ for all $x \in \mathbb{R}$.

Hence, function is increasing for all $x \in \mathbb{R}$. **1**

Q. 4. Show that the function given by $f(x) = 3x + 17$ is increasing on $\mathbb{R}$. R&U

Sol. Let $y = f(x) = 3x + 17$

$\frac{dy}{dx} = 3 > 0$ for all $x \in \mathbb{R}$

Hence, given function is increasing on $\mathbb{R}$. **1**

Q. 5. A firm is selling 100 units at a price of ₹ 250/unit. However, to sell 110 units, they need to cut the price down to ₹ 240/unit. What is the level of marginal revenue at this higher level of sales ? R&U

Q. 6. The total expenditure (in ₹) required for providing the cheap edition of a book for poor and deserving students is given by $C(x) = 3x^2 + 36x$, where x is the number of set of books. If the marginal expenditure is defined as $\frac{dC}{dx}$, write the marginal expenditure required for 1200 such sets. AI

Sol. Here, $C(x) = 3x^2 + 36x$, where x is the number of sets of book

$\therefore \quad \frac{dC}{dx} = 6x + 36$

$$\left(\frac{dC}{dx}\right)_{x=1200} = 6(1200) + 36 = 7,236$$

So, the marginal cost of books for 1200 such sets

= ₹ 7,236 **1**

Q. 7. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in ₹) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$. R&U

Sol. Total revenue is given by $R(x) = 3x^2 + 36x + 5$

Marginal revenue $= \frac{dR}{dx} = 6x + 36$

At $x = 5$,

$$\left(\frac{dR}{dx}\right)_{x=5} = 6 \times 5 + 36 = 66$$ **1**

Q. 8. If at $x = 1$, function $x^4 - 62x^2 + ax + 9$ attains its maximum value, in the interval $[0, 2]$, then find the value of a.

Q. 9. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 26x + 15$. Find the marginal revenue, when $x = 15$.

Sol. Here,

$R(x) = 3x^2 + 26x + 15$

These questions are for practice and their solutions are available at the end of the chapter

$$\frac{dR}{dx} = 6x + 26$$

$$\Rightarrow \left(\frac{dR}{dx}\right)_{\text{at } x=15} = 6(15) + 26$$

$$= ₹\ 116 \quad \mathbf{1}$$

Q. 10. Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 72x - 18x^2$

Sol. Let profit function

$$p(x) = 41 - 72x - 18x^2$$

$$\therefore \quad p'(x) = -72 - 36x$$

$$\Rightarrow \quad x = -\frac{72}{36} = -2$$

$$p''(x) = -36$$

Also, $p''(-2) = -36 < 0$

By second derivative test, $x = -2$ is the point of local maxima of p.

$\therefore$ Maximum profit $= p(-2)$

$= 41 - 72(-2) - 18(-2)2$

$= 41 + 144 - 72 = 113$

Hence, the maximum profit that the company can make is 113 units. **1**

Q. 11. Find the least value of a function such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2).

Sol. $f(x) = x^2 + ax + 1$

$\therefore f'(x) = 2x + a$

Now, function f is strictly increasing in (1, 2).

i.e., $f'(x) > 0$

$\therefore \quad 2x + a > 0$

$a > -2x$

$\therefore$ We have to find least value of a, such that

$x \in (1, 2)$

$\therefore \quad a > -2$

$\Rightarrow \quad a \in (-2, \infty)$

So, least value of a is -2. **1**

Q. 12. Check whether the function $g(x) = \log x$ is maxima or minima.

Q. 13. If $y = x^3 \log x$, then find $\frac{d^4y}{dx^4}$.

Sol. If $y = x^3 \log x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(x^3 \log x)$$

$$\frac{dy}{dx} = x^2(1 + 3\log x)$$

$$\frac{d^2y}{dx^2} = x^2 \frac{d}{dx}(1 + 3\log x) + (1 + 3\log x)\frac{d}{dx}x^2$$

$$\frac{d^2y}{dx^2} = x(5 + 6\log x)$$

$$\frac{d^3y}{dx^3} = x\frac{d}{dx}(5 + 6\log x) + (5 + 6\log x)\frac{d}{dx}x$$

$$\frac{d^3y}{dx^3} = 11 + 6\log x$$

$$\frac{d^4y}{dx^4} = \frac{6}{x} \quad \mathbf{1}$$

Q. 14. Given $y = \frac{x}{x^2 + 1}$, then find the value of second derivative at $x = 1$.

Sol. Let $y = \frac{x}{x^2 + 1}$, then

$$y = \frac{(x^2 + 1)\frac{d}{dx}x - x\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$\Rightarrow \quad y' = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$y' = \frac{(x^2 + 1)\frac{d}{dx}n - x - \frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$y'' = \frac{(x^2 + 1)^2(-2x) - (1 - x^2)[2(x^2 + 1)2x]}{(x^2 + 1)^4}$$

At $x = 1$,

$$y'' = \frac{(1^2 + 1)^2(-2) - (1 - 1^2)[2(1^2 + 1)2]}{(1^2 + 1)^4}$$

$$\Rightarrow \quad y'' = \frac{(2)^2(-2) - (1 - 1)[2(2)2]}{(2)^4}$$

$$\Rightarrow \quad y'' = \frac{-8 + 0}{(2)^4}$$

$$\Rightarrow \quad y'' = \frac{-1}{2} \quad \mathbf{1}$$

Short Answer Type Questions-I (2 marks each)

Q. 1. Given $y = e^x(x + 1)$. Find the value of second derivative at $x = 1$. R&U

Q. 2. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R. R&U

Sol. $f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$ ½

$= 3[x^2 - 2x + 2] = 3[(x - 1)^2 + 1]$ **1**

since $f'(x) > 0$; $x \in R$

$\therefore f(x)$ is increasing on R. ½

Q. 3. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R. R&U

Sol. $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27$ ½

$= 3(2x - 3)^2 \geq 0$; $x \in R$ **1**

$\therefore f(x)$ is increasing on R. ½

Q. 4. Find the interval in which the function f given by $f(x) = 2x^2 - 3x$ is (a) Increasing (b) Decreasing. A I

Q. 5. The total cost function (in thousands) for manufacturing x manipulators per year is given $C(x) = 375 + 25x - 0.25x^2 \quad 0 \leq x \leq 50$

(a) Use the marginal cost function to approximate the cost of producing the 31st manipulator. R&U

(b) Use the total cost function to find the exact cost of producing the 31st manipulator.

Sol. (a) The marginal cost is $C'(x) = 25 - 0.5x$.

Thus, $C'(30) = 25 - 0.5(30) = 10$.

The cost of producing of the 31st manipulator is approximately ₹ 10,000. **1**

These questions are for practice and their solutions are available at the end of the chapter

(b) The exact cost of producing the 31st manipulator is $C(31) - C(30) = 909.75 - 900 = 9.75$, i.e. ₹ 9,750.
The marginal cost of ₹ 10,000 per manipulator is a close approximation to his exact cost. **1**

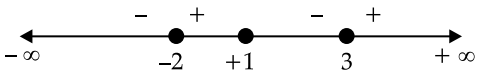
Short Answer Type Questions-II (3 marks each)

Q. 1. Find the intervals in which

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$ **is (a) strictly increasing (b) strictly decreasing.** R&U

Sol. $f'(x) = \frac{12}{10}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$ ½

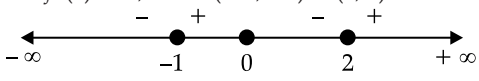
$$= \frac{6}{5}(x-1)(x+2)(x-3)$$ **1**



$f'(x) = 0$ at $x = 1, -2, 3$
$\therefore$ Intervals are $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$ ½
$\because f'(x) > 0$ for $(-2, 1) \cup (3, \infty)$
$f(x)$ is strictly increasing in $(-2, 1)$ and $(3, \infty)$ ½
$\because f'(x) < 0$ for $(-\infty, -2)$ and $(1, 3)$
$f(x)$ is strictly decreasing in $(-\infty, -2) \cup (1, 3)$ ½

Q. 2. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is (a) strictly increasing (b) strictly decreasing. R&U

Sol. $f'(x) = 12x^3 - 12x^2 - 24x$
$= 12x(x+1)(x-2)$ **1**
$f'(x) > 0, \forall\, x \in (-1, 0) \cup (2, \infty)$ ½
$f'(x) < 0, \forall\, x \in (-\infty, -1) \cup (0, 2)$ ½



$\therefore f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$
and strictly decreasing in $(-\infty, -1) \cup (0, 2)$ **1**

Q. 3. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (a) strictly increasing, (b) strictly decreasing. R&U

Q. 4. Determine for what values of x, the function $f(x) = x^3 + \frac{1}{x^3}$ $(x \neq 0)$ is strictly increasing or strictly decreasing. A

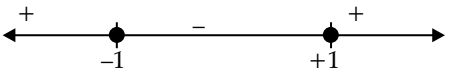
Sol. Here, $f'(x) = 3x^2 - 3x^{-4}$

$$= \frac{3(x^6 - 1)}{x^4}$$

$$= \frac{3(x^4 + x^2 + 1)}{x^4}(x+1)(x-1)$$

Critical points are -1 and 1. **1**
or $f'(x) > 0$ if $x > 1$ or $x < -1$, and $f'(x) < 0$ if $-1 < x < 1$

$$\left(\because \frac{3(x^4 + x^2 + 1)}{x^4} \text{always + ve}\right)$$



Hence, $f(x)$ is strictly increasing for $x > 1$ **1**
or $x < -1$; and strictly decreasing in $(-1, 0) \cup (0, 1)$ **1**

Q. 5. Find the intervals in which the function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is (i) Strictly increasing (ii) Strictly decreasing. R&U

Sol. $f(x) = -2x^3 - 9x^2 - 12x + 1$
Now, $f'(x) = -6x^2 - 18x - 12$
$= -6[x^2 + 3x + 2]$
$= -6[x^2 + 2x + x + 2]$
$f'(x) = -6(x+1)(x+2)$ **1**
Put, $f'(x) = 0 \Rightarrow x = -2, x = -1$
$\Rightarrow$ Intervals are $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$
Getting $f'(x) > 0$ in $(-2, -1)$ and $f'(x) < 0$ in $(-\infty, -2) \cup (-1, \infty)$ **1**
$\Rightarrow f(x)$ is strictly increasing in $(-2, -1)$
and strictly decreasing in $(-\infty, -2) \cup (-1, \infty)$ **1**

Q. 6. If x is real find the minimum value of $x^2 - 8x + 17$. A I

Q. 7. Find the maximum value of $\left(\frac{1}{x}\right)^x$. A

Sol. Let $y = \left(\frac{1}{x}\right)^x$

$\Rightarrow \log y = x.\log\frac{1}{x}$

$\therefore \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\frac{1}{x}}.\left(-\frac{1}{x^2}\right) + \log\frac{1}{x}.1$

$= -1 + \log\frac{1}{x}$

$\therefore \frac{dy}{dx} = \left(\log\frac{1}{x} - 1\right)\left(\frac{1}{x}\right)^x$ **2**

Now, $\frac{dy}{dx} = 0$

$\Rightarrow \log\frac{1}{x} = 1 = \log e$

$\Rightarrow \frac{1}{x} = e$

$\Rightarrow x = \frac{1}{e}$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$. **1**

Q. 8. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. A I

Sol. Let side of base $= x$ and depth of tank $= y$

$$V = x^2y \Rightarrow y = \frac{V}{x^2},$$

(V = Quantity of water = constant)

Cost of material is least when area of sheet used is minimum.

A (Surface area of tank) $= x^2 + 4xy = x^2 + \frac{4V}{x}$

$$\frac{dA}{dx} = 2x - \frac{4V}{x^2}$$

$$\frac{dA}{dx} = 0 \qquad \text{½+½}$$

$\Rightarrow \quad x^3 = 2V,$

$\Rightarrow \quad x^3 = 2x^2y,$

$$y = \frac{x^3}{2x^2} = \frac{x}{2} \qquad \text{½+½}$$

[as $V = x^2y$]

$$\frac{d^2A}{dx^2} = 2 + \frac{8V}{x^3} > 0, \qquad \text{½+½}$$

$\therefore$ Area is minimum, thus cost is minimum when

$$y = \frac{x}{2}$$

Commonly Made Error

- In most of the cases, candidates do not read the question attentively which results incorrect variable in area.

Answering Tip

- Give adequate practice on mensuration related concept and problems.

Q. 9. A jet of enemy is flying along the curve $y = x^2 + 2$ and a soldier is placed at the point (3, 2). Find the minimum distance between the soldier and the jet.

A

Sol. Let $P(x, y)$ be the position of the jet and the soldier is placed at $A(3, 2)$.

$$\Rightarrow \quad AP = \left|\sqrt{(x-3)^2 + (y-2)^2}\right| \qquad \ldots(i)$$

As given, $y = x^2 + 2$

$$\Rightarrow \quad y - 2 = x^2 \qquad \ldots(ii)$$

$$\therefore \quad AP^2 = (x-3)^2 + x^4 = z \text{ (say)} \qquad ½$$

$$\therefore \quad \frac{dz}{dx} = 2(x-3) + 4x^3 \qquad ½$$

For maximum and minimum value,

$$\frac{dz}{dx} = 0$$

$\therefore 2x - 6 + 4x^3 = 0$

Put $x = 1$

$$2 - 6 + 4 = 0$$

$\therefore$ $x - 1$ is a factor **1**

and $\frac{d^2z}{dx^2} = 12x^2 + 2$

Now, $\frac{d^2z}{dx^2}$ (at $x = 1$) > 0

$\therefore$ z is minimum, when $x = 1$, $y = 1 + 2 = 3$ [from (ii)]

$\therefore$ Minimum distance $= \left|\sqrt{(1-3)^2 + (3-2)^2}\right|$

$= \sqrt{5}$ units **[from (i)] 1**

Q. 10. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 m^3. If building of tank costs ₹ 100 per square metre for the base and ₹ 50 per square metre for the sides, find the cost of least expensive tank. A

Q. 11. Suppose the total cost $C(x)$ (in millions) for manufacturing x air-planes per year is given by the function

$$C(x) = 6 + \sqrt{4x + 4} \qquad 0 \le x \le 30$$

(a) Find the marginal cost at a production level of x air-planes per year.

(b) Find the marginal cost at a production level of 15 and 24 air-planes per year, and interpret the results. R&U

Sol. (a) The marginal cost at a production level of x air-planes is

$$C'(x) = \frac{d}{dx}(6 + \sqrt{4x+4}),$$

$$= \frac{2}{\sqrt{4x+4}} \qquad \textbf{1}$$

(b) The marginal cost at a production level of 15 air-planes is

$$C'(15) = \frac{2}{\sqrt{4(15)+4}} = 0.25$$

At a production level of 15 air-planes per year, the total cost is increasing at the rate of ₹ 250,000 per one air-plane.

The marginal cost at a production level of 24 air-planes is

$$C'(24) = \frac{2}{\sqrt{4(24)+4}} = 0.2$$

At a production level of 24 air-planes per year, the total cost is increasing at the rate of ₹ 200,000 per one air-plane. **2**

Long Answer Type Questions (5 marks each)

Q. 1. If $x \cos(a + y) = \cos y$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. Hence, show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a + y)\frac{dy}{dx} = 0$. A I

Sol. Given, $x \cos(a + y) = \cos y$

or $x = \dfrac{\cos y}{\cos(a+y)}$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}\cos y - \cos y \frac{d}{dy}\cos(a+y)}{\cos^2(a+y)} \quad \textbf{1}$$

[by using quotient rule of derivative]

$$= \frac{\cos(a+y)\times(-\sin y) + \cos y \times \sin(a+y)}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\cos^2(a+y)} \quad ½$$

or $$\frac{dy}{dx} = \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$= \frac{\sin a}{\cos^2(a+y)}$$

$[\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$

or $$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a} \quad \text{...(i)} \ \textbf{1}$$

Again, on differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\frac{d}{dx}\cos^2(a+y)$$

$$= \frac{1}{\sin a}\times\frac{d}{dy}\cos^2(a+y)\times\frac{dy}{dx} \quad \textbf{1}$$

$$= \frac{1}{\sin a} \times 2\cos(a+y)[-\sin(a+y)] \times \frac{dy}{dx}$$

$$= -\frac{2\sin(a+y)\cos(a+y)}{\sin a}\times\frac{dy}{dx} \quad ½$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-\sin 2(a+y)}{\sin a}\frac{dy}{dx} \ [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$\therefore \sin a \dfrac{d^2y}{dx^2} + \sin 2(a+y)\dfrac{dy}{dx} = 0$ **Hence proved. 1**

Q. 2. If $x = \sin t$ and $y = \sin pt$, then prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$. R&U

Q. 3. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence find the intervals in which $f(x)$ is :

(i) strictly increasing

(ii) strictly decreasing. R&U

Sol. $f'(x) = 2x - 1$ **1**

$$f'(x) > 0, \forall\, x \in \left(\frac{1}{2}, 1\right) \quad \textbf{1}$$

$$f'(x) < 0, \forall\, x \in \left(-1, \frac{1}{2}\right) \quad \textbf{1}$$

$\therefore f(x)$ is neither increasing nor decreasing in $(-1, 1)$. **1**

$f(x)$ is strictly increasing on $\left(\dfrac{1}{2}, 1\right)$

and $f(x)$ is strictly decreasing on $\left(-1, \dfrac{1}{2}\right)$. **1**

Q. 4. Show that $y = \log(1+x) - \dfrac{2x}{2+x}, x > -1$, is an increasing function of x throughout its domain. A I

Sol. Given, $y = \log(1+x) - \dfrac{2x}{2+x}, x > -1$

Differentiating, w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{[(2+x)(2) - 2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{[4+2x-2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(2+x)^2(1+x)}$$

$$= \frac{4+x^2+4x-4-4x}{(2+x)^2(1+x)}$$

$$= \frac{x^2}{(2+x)^2(1+x)} \quad \textbf{2}$$

For increasing function,

$$\frac{dy}{dx} \geq 0$$

$$\frac{x^2}{(2+x)^2(x+1)} \geq 0$$

$$\Rightarrow \left(\frac{x}{2+x}\right)^2 \cdot \left(\frac{1}{x+1}\right) > 0$$

$$\Rightarrow \left(\frac{1}{x+1}\right) > 0$$

This is possible only when $1 + x > 0$ *i.e.*, $x > -1$

So, $\dfrac{dy}{dx} > 0$ for $x > -1$

When $x > -1$ **2**

$\dfrac{dy}{dx}$ is always greater than zero.

$\therefore y = \log(1+x) - \dfrac{2x}{2+x}$ is always increasing through out its domain. **1**

Q. 5. A cone is inscribed in a sphere of radius 12 cm. If the volume of the cone is maximum, find its height. A

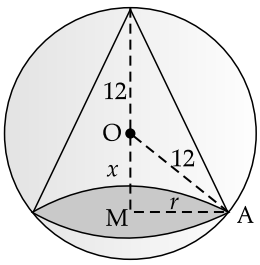
These questions are for practice and their solutions are available at the end of the chapter

Sol. Let r be the radius of cone and h be the height of cone, let x be the distance between centre of sphere and centre of base of cone,

Given,

radius of sphere $R = 12$ cm

$\therefore h = 12 + x$ ...(i)



1

Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(12^2 - x^2)(12 + x)$

$[\because$ In ΔOMA, $12^2 = x^2 + r^2]$

$= \frac{1}{3}\pi(144 - x^2)(12 + x)$ **1**

$\therefore \frac{dV}{dx} = \frac{1}{3}\pi\left[(144 - x^2).1 + (12 + x)(-2x)\right]$

$= \frac{1}{3}\pi\left[144 - x^2 - 24x - 2x^2\right]$

$= \frac{1}{3}\pi\left[144 - 24x - 3x^2\right]$

For maximum or minimum volume, put $\frac{dV}{dx} = 0$ **1**

i.e., $\frac{1}{3}\pi[144 - 24x - 3x^2] = 0$

$\Rightarrow 144 - 24x - 3x^2 = 0$

$\Rightarrow x^2 + 8x - 48 = 0$

$\Rightarrow (x + 12)(x - 4) = 0$

$\Rightarrow x = -12$ and $x = 4$ **1**

Now,

$$\frac{d^2V}{dx^2} = \frac{1}{3}\pi[-24 - 6x]$$

$$\therefore \left(\frac{d^2V}{dx^2}\right)_{x=4} = \frac{1}{3}\pi[-24 - 24] < 0$$

Hence, at $x = 4$ volume of cone is maximum and height of cone $(h) = 12 + x = 12 + 4 = 16$ cm. **1**

Commonly Made Error

- Most of the candidates attempted this question incorrectly. Some candidates could not express volume of the cone as function in mathematical form. A number of calculation mistakes were also made while differentiating.

Answering Tip

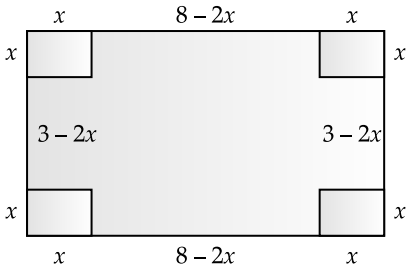
- The concept of maxima and minima need to be understood carefully. Do adequate practice of problems based on maxima and minima. Also, practice problem based on mensuration related concept.

Q. 6. Show that the altitude of a right circular cone of maximum that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. A

Q. 7. An open topped box is to be made by removing equal squares from each corner of a 3 m by 8 m rectangle sheet of aluminium and by folding up the side. Find the volume of the largest such box. A

Sol. Given rectangle is of breadth = 3 m

and length = 8 m



1

Let x be the length of each square removed from the corners

Volume of box $= V(x) = (8 - 2x)(3 - 2x)x$

$= x(24 - 16x - 6x + 4x^2)$

$= 24x - 22x^2 + 4x^3$

$V'(x) = 24 - 44x + 12x^2$ **1**

For max. and min. volume

$V'(x) = 0$

$\Rightarrow 12x^2 - 44x + 24 = 0$

$\Rightarrow 3x^2 - 11x + 6 = 0$

$\Rightarrow 3x^2 - 9x - 2x + 6 = 0$

$\Rightarrow (x - 3)(3x - 2) = 0$

$\Rightarrow x = 3$ (not possible) or $3x = 2$

$\Rightarrow x = \frac{2}{3}$ **1**

Now, $V''(x) = -44 + 24x$

at $x = \frac{2}{3}$ $V''\left(\frac{2}{3}\right) = -44 + 24\left(\frac{2}{3}\right) < 0$

$\therefore$ Volume is maximum when $x = \frac{2}{3}$ **1**

$$\text{Volume } (V) = \left(8 - 2\left(\frac{2}{3}\right)\right)\left(3 - 2\left(\frac{2}{3}\right)\right)\left(\frac{2}{3}\right)$$

$$= \left(8 - \frac{4}{3}\right)\left(3 - \frac{4}{3}\right)\left(\frac{2}{3}\right)$$

$$= \frac{20}{3} \times \frac{5}{3} \times \frac{2}{3}$$

$$= \frac{200}{27} \text{ m}^3$$ **1**

Q. 8. Show that the surface area of closed cuboid with square base and given volume is minimum when it is a cube. AI

Sol. Let the length and breadth be x units (square base), height of cuboid be h unit.

Volume of cuboid $(v) = x^2h$

$\therefore h = \frac{v}{x^2}$.

Surface area of cuboid

$(s) = 2(x^2 + xh + xh)$

$\therefore s = 2x^2 + 4xh$ **2**

$\Rightarrow s = 2x^2 + 4x \times \frac{v}{x^2}$

$\Rightarrow s = 2x^2 + 4v\frac{1}{x}$

$\therefore \frac{ds}{dx} = 4x - 4v\frac{1}{x^2}$ **1**

For maximum and minimum surface, $\frac{ds}{dx} = 0$

$\therefore 4x - 4v.\frac{1}{x^2} = 0$

$\Rightarrow 4x = \frac{4v}{x^2}$

$\Rightarrow x^3 = v$

$\Rightarrow x^3 = x^2h$

$\Rightarrow x = h$ **1**

Now, $\frac{d^2s}{dx^2} = 4 + \frac{8v}{x^3}$

At $x = h$, $\frac{d^2s}{dx^2} = 4 + \frac{8v}{h^3} > 0$

$\therefore$ Surface area is minimum, when $x = h$.

i.e., when $x = h$, it is a cube, surface area will be minimum. $x = h$ **Hence Proved. 1**

Commonly Made Error

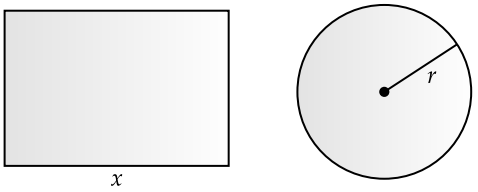
- Many candidates did not express volume and surface area of cube with square base by correct formula. Most of the candidates did not show the working of second order derivatives and failed to prove $\frac{d^2s}{dx^2} > 0$.

Answering Tips

- Learn the formulas for area, perimeter, surface and volume of 2-D and 3-D geometrical figures.
- Express the function to be optimized in terms of a single variable by using given data.

Q. 9. A wire of length 50 m is cut into two pieces. One piece of the wire is bent in the shape of a square and the other in the shape of a circle. What should be the length of each piece so that the combined area of the two is minimum ? A

Sol.



Length of wire = 50 m (Given)

Let length of one piece for shape of square = x m

$\therefore$ Length of other piece for shape of circle

$= (50 - x)$ m

Now perimeter of square = $4a = x$

$\Rightarrow a = \frac{x}{4}$

and circumference of circle = $2\pi r = 50 - x$

$\Rightarrow r = \frac{50 - x}{2\pi}$ **1**

Combined Area = $a^2 + \pi r^2$

$$= \frac{x^2}{16} + \pi \cdot \left(\frac{50 - x}{2\pi}\right)^2$$

$$= \frac{x^2}{16} + \pi \cdot \frac{(50 - x)^2}{4\pi^2}$$

$$A = \frac{x^2}{16} + \frac{(50 - x)^2}{4\pi}$$ **1**

Differentiating w.r. t. x, we get

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(50 - x)(-1)}{4\pi}$$

$$= \frac{x}{8} + \frac{(x - 50)}{2\pi}$$

$$= \frac{\pi x + 4x - 200}{8\pi}$$

$$= \frac{x(4 + \pi) - 200}{8\pi}$$ **1**

For maximum and minimum value of A, $\frac{dA}{dx} = 0$

$\therefore \quad x(4 + \pi) - 200 = 0$

$$x = \frac{200}{4 + \pi}$$

At $x = \frac{200}{4+\pi}$, $\frac{d^2A}{dx^2} > 0$

A is minimum at $\frac{200}{4+\pi}$ **1**

$\therefore$ Length of square wire, $x = \frac{200}{4+\pi}$ m

and length of circle wire $= 50 - x$

$= 50 - \frac{200}{4+\pi}$

$= \frac{50\pi}{4+\pi}$ m **1**

Commonly Made Error

- The required sum of areas of the two parts had to be expressed correctly and in terms of a single variable. Some candidates failed in this. Differentiation and simplification errors were also made by several candidates.

Answering Tips

- Differentiation rules for different functions and forms need continuous revision and practice. Obtaining the result in the required form needs simplification, cross multiplication, substitution, etc. Hence enough revision and practices is required.
- If $4x + 2\pi r = 50$ then $r = \frac{50-4x}{2\pi}$. Hence $A = x^2 + \pi\left(\frac{50-4x}{2\pi}\right)^2$. In this manner 'A' can be expressed in terms of a single variable 'x'. Differentiation and solving follows. Also, second order derivative needs to be shown as negative for maximum, and positive for minimum. So, practice these problems regularly.

Q. 10. The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}$, x and $2x$ is constant. Show that the sum of their volumes is minimum if x is equal to the three times the radius of sphere. A I A

Sol. Given $S = 4\pi r^2 + 2\left[\frac{x^2}{3} + 2x^2 + \frac{2x^2}{3}\right]$

$S = 4\pi r^2 + 6x^2$ or $x^2 = \frac{S-4\pi r^2}{6}$ **1**

and $V = \frac{4}{3}\pi r^3 + \frac{2x^3}{3}$

$\therefore V = \frac{4}{3}\pi r^3 + \frac{2}{3}\left(\frac{S-4\pi r^2}{6}\right)^{3/2}$ **1**

$\frac{dV}{dr} = 4\pi r^2 + \left(\frac{S-4\pi r^2}{6}\right)^{1/2}\left(\frac{-8\pi r}{6}\right)$ **1**

$\frac{dV}{dr} = 0\left(\frac{S-4\pi r^2}{6}\right)^{1/2} = 3r$

or $r = \sqrt{\frac{S}{54+4\pi}}$ **1**

Now $\frac{d^2V}{dr^2} = 8\pi r + \left(\frac{-8\pi}{6}\right)\left(\frac{S-4\pi r^2}{6}\right)^{1/2} + \frac{1}{2}\left(\frac{S-4\pi r^2}{6}\right)^{-1/2}\cdot\left(\frac{-8\pi r}{6}\right)\left(\frac{-8\pi r}{6}\right)$

at $r = \sqrt{\frac{S}{54+4\pi}}$; $\frac{d^2V}{dr^2} > 0$

$\therefore$ for $r = \sqrt{\frac{S}{54+4\pi}}$ volume is minimum

i.e., $r^2(54 + 4\pi) = S$

or $r^2(54 + 4\pi) = 4\pi r^2 + 6x^2$

or $6x^2 = 54r^2$

or $x^2 = 9r^2$

or $x = 3r$ **1**

Commonly Made Error

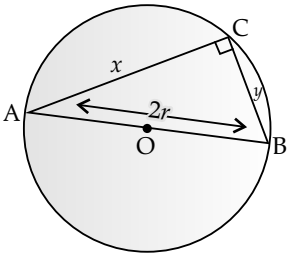
- The required sum of areas of the two parts has to be expressed correctly and in terms of single variable. Sometimes Candidates fail in this.

Answering Tip

- Obtaining the result in the required form needs simplification, cross-multiplication, substitution., etc.

Q. 11. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles. A

Sol.



1

Let the sides of rt. ΔABC be x and y.

$\therefore \quad x^2 + y^2 = 4r^2$

and A = Area of $\Delta = \frac{1}{2}xy$ **1**

Let, $S = A^2 = \frac{1}{4}x^2y^2$

$$= \frac{1}{4}x^2(4r^2 - x^2)$$

$$= \frac{1}{4}(4r^2x^2 - x^4) \quad \textbf{1}$$

$$\therefore \frac{dS}{dx} = \frac{1}{4}[8r^2x - 4x^3]$$

or $\frac{dS}{dx} = 0$ or $x^2 = 2r^2$ or $x = \sqrt{2}r$ **1**

and $y^2 = 4r^2 - 2r^2 = 2r^2$ or $y = \sqrt{2}r$

i.e. $x = y$ and $\frac{d^2S}{dx^2} = (2r^2 - 3x^2) = 2r^2 - 6r^2 < 0$ **1**

or Area is maximum, when Δ is isosceles.

Hence Proved.

Q. 12. *ABC* is a right angled triangle of given area *S*. Find the sides of the triangle for which the area of circumscribed circle is least. A I

Q. 13. A closed right circular cylinder has volume $\frac{539}{2}$ cubic units. Find the radius and the height of the cylinder so that the total surface area is minimum. A

Sol. Let r be the radius and h be the height of the cylinder.

$$V = \pi r^2 h = \frac{539}{2}$$

$$\Rightarrow h = \frac{539}{2\pi r^2} \quad \textbf{1}$$

Let S be the total surface area, then

$$S = 2\pi rh + 2\pi r^2$$

$$= 2\pi r\left(\frac{539}{2\pi r^2}\right) + 2\pi r^2$$

$$= \frac{539}{r} + 2\pi r^2$$

$$\frac{dS}{dr} = -\frac{539}{r^2} + 4\pi r \quad \textbf{1}$$

For maxima or minima, $\frac{dS}{dr} = 0$

$$\Rightarrow -\frac{539}{r^2} + 4\pi r = 0$$

$$\Rightarrow -539 + 4\pi r^3 = 0$$

$$\Rightarrow r^3 = \frac{539}{4\pi} = \frac{539 \times 7}{4 \times 22}$$

$$\Rightarrow r^3 = \left(\frac{7}{2}\right)^3$$

$$\Rightarrow r = \frac{7}{2} \text{ units} \quad \textbf{1}$$

$$\therefore \frac{d^2S}{dr^2} = \frac{1078}{r^3} + 4\pi,$$

$$\left(\frac{d^2S}{dr^2}\right)_{r=\frac{7}{2}} = \frac{1078}{\left(\frac{7}{2}\right)^3} + 4\pi > 0 \quad \textbf{1}$$

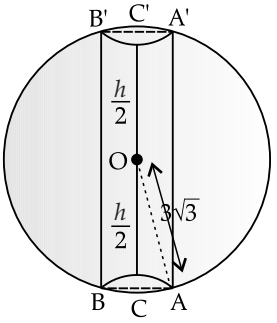
$\therefore S$ is minimum, at $r = \frac{7}{2}$ units

and $h = \frac{539 \times 7 \times 2 \times 2}{2 \times 22 \times 7 \times 7} = 7$

$\therefore$ Radius of the cylinder be $\frac{7}{2}$ units and height be 7 units. **1**

Q. 14. Find the maximum volume of the cylinder which can be inscribed in a sphere of radius $3\sqrt{3}$ cm. (Leave the answer in terms of π). A

Sol. Let h units be the height of the cylinder and R unit be the radius of the cylinder. Given, $3\sqrt{3}$ cm is the radius of the sphere.



If V is the volume of the cylinder, then

$$V = \pi R^2 h \quad \text{...(i)} \ \textbf{1}$$

Let O be the centre of the sphere and $C'OC \perp B'A'$ as well as BA.

From right angled ΔOCA

$$(3\sqrt{3})^2 = \left(\frac{h}{2}\right)^2 + R^2$$

$$\Rightarrow R^2 = 27 - \frac{h^2}{4} \quad \text{...(ii)}$$

$$\therefore V = \pi\left(27 - \frac{h^2}{4}\right).h$$

$$V = 27\pi h - \pi\frac{h^3}{4} \quad \textbf{1}$$

$$\therefore \frac{dV}{dh} = 27\pi - \frac{3\pi h^2}{4}$$

and $\frac{d^2V}{dh^2} = -\frac{6\pi h}{4}$ **1**

For Maxima/Minima, $\frac{dV}{dh} = 0$

$$\therefore 27\pi = \frac{3\pi h^2}{4}$$

$$h^2 = \frac{4 \times 27}{3}$$

$$h^2 = 36$$
$$h = 6, -6$$

and $\left(\frac{d^2V}{dh^2}\right)_{h=6} = \frac{-6\pi \times 6}{4} < 0$ **1**

$\therefore$ V is maximum when $h = 6$, putting $h = 6$ in equation (ii),

$$R^2 = 27 - \frac{36}{4}$$
$$R^2 = 18$$

$\therefore$ From (i), $V = \pi R^2 h$
$$= \pi \times 18 \times 6$$
$$= 108\pi \text{ cm}^3$$ **1**

Q. 15. The cost function for the manufacture of x number of goods by a company is $C(x) = x^3 - 9x^2 + 24x$. Find the level of output at which the marginal cost is minimum. Further, if the selling price of a unit is $2x^3 + 9x^2$, find the average profit. AI

Q. 16. Gymnast Clothing manufactures expensive hockey jerseys for sale to college bookstores in runs of up to 150. Its cost (in rupees) for a run of x hockey jerseys is

$$C(x) = 1500 + 10x + 0.2x^2, (0 \le x \le 150)$$

(a) Gymnast Clothing sells the jerseys at ₹ 90 each. Find the revenue function.

(b) Find the profit function.

(c) How many should Gymnast Clothing manufacture to make a profit? (Round your answer up to the nearest whole number.) AI

Sol. (a) Since the manufacturer sells the jerseys for ₹ 90 each, the revenue function is

$$R(x) = 90x$$ **1**

(b) Profit is defined to be revenue minus cost, so the profit function is

$$P(x) = R(x) - C(x)$$
$$= 90x - (1500 + 10x + 0.2x^2)$$
$$= -1500 + 80x - 0.2x^2$$ **1**

(c) In order to make a profit, $P(x)$ must be greater than zero. So, to find how many jerseys we need to make in order to make a profit, we should find the break-even point. So, we put $P(x) = 0$, *i.e.*,

$-0.2x^2 + 80x - 1500 = 0$

$$x = \frac{-80 \pm \sqrt{80^2 - 4(-0.2)(-1500)}}{2(-0.2)}$$

This simplifies to $x = 19.7$ or $x = 380.2$, but since the problem specifies that the domain is between $x = 0$ and $x = 150$, we can reject the larger answer. Rounding the other answer up, we get $x = 20$. So, if we make 20 jerseys, we will make a profit. **3**

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

A farmer has a piece of land. He observed that he got 600 units of fruits per tree by planting upto 25 trees and when 26 trees were grown, he received 15210 units of fruits, for 27 trees he ended up with 15390 fruits, for 28 trees he got 15540 fruits and this sequence of production of fruits continues in the same pattern as more trees, in excess of 25, were grown. **[CBSE Text Book 2022]**

Q. 1. If 'x' more trees, in excess of 25 are grown, then the number of fruits produced per tree is

(A) $600 - 15x$ **(B)** $600 + 15x$

(C) $600x - 15$ **(D)** $600x + 15$

Ans. Option (A) is correct.

Explanation: When no. of trees = 25, then no. of units of fruits per tree = 600

When no. of trees = 26, then no. of units of fruits per tree $= \frac{15210}{26} = 585 = (600 - 15)$

When no. of trees = 27, then no. of units of fruits per tree $= \frac{15390}{27} = 570 = (600 - 15 \times 2)$

When no. of trees = 28, then no. of units of fruits per tree $= \frac{15540}{28} = 555 = (600 - 15 \times 3)$

Thus, when x more trees, in excess of 25 are grown then the no. of fruits produced per tree is $600 - 15x$

Q. 2. The product of entire garden if 'x' more trees, in excess of 25, are planted

(A) $(25 + x)(600 + 15x)$

(B) $(25 - x)(600 - 15x)$

(C) $(25 + x)(600 - 15x)$

(D) $(25 + x)(15x - 600)$

Ans. Option (C) is correct.

Explanation: We know that, if 'x' more tree in excess of 25 are grown, then the number of fruits produced per tree is $600 - 15x$.

So, the production of entire garden, if 'x' more trees, in excess of 25, are planted is given by

$$p = (25 + x)(600 - 15x)$$

These questions are for practice and their solutions are available at the end of the chapter

Q. 3. The marginal production of the garden when 'x' more trees, in excess of 25, are

(A) $225 + 30x$ **(B)** $225 - 30x$

(C) $225x + 30$ **(D)** $225x - 30$

Ans. Option (B) is correct.

Explanation: Since, $P = (25 + x)(600 - 15x)$

$\therefore$ Marginal production,

$$\frac{dP}{dx} = (600 - 15x) + (25 + x)(-15)$$

$$= (600 - 375) - 30x$$

$$= 225 - 30x$$

Q. 4. The critical point of producing 'x' more units of tree is

(A) 7 **(B)** 8

(C) 7.5 **(D)** 8.5

Ans. Option (C) is correct.

Explanation: For critical point, Put $\frac{dP}{dx} = 0$

i.e., $225 - 30x = 0$

$\Rightarrow$ $x = \frac{225}{30}$

$= 7.5$

Q. 5. The number of tree to be grown to get maximum production is

(A) 30 or 31 trees **(B)** 32 or 33 trees

(C) 33 or 34 trees **(D)** 34 or 35 trees

Ans. Option (B) is correct.

Explanation:

$\therefore$ $\frac{dP}{dx} = 225 - 30x$

$\frac{d^2P}{dx^2} = -30 < 0$

(Production is maximum)

Thus, no. of trees to be grown to get maximum production is $25 + x = 25 + 7.5 = 32.5$

Hence, required no. of trees = 32 or 33

II. Read the following text and answer the following questions on the basis of the same:

A manufacturing company manufactures toys, the company observed the following costs at different production levels, **[CBSE Text Book, 2022]**

Number of toys manufactured	Cost of raw material (₹)	Cost of Production (₹)	Cost of freight (₹)	Property tax (₹)	Salaries (₹)
100	800	2000	1000	5000	20000
150	1200	3000	1500	5000	20000
200	1600	4000	2000	5000	20000
250	2000	5000	2500	5000	20000
300	2400	6000	3000	5000	20000

Q. 1. Which of the following is the fixed cost

(A) Number of toys manufactured

(B) Cost of raw material

(C) Cost of production supply

(D) Salaries

Ans. Option (D) is correct.

Explanation: Salary always comes under fixed cost for a company.

Q. 2. Total cost C(x) of toys for 'x' units of production is

(A) $C(x) = 8x^2 + 30x + 25000$

(B) $C(x) = 8x^2 + 30x + 20000$

(C) $C(x) = 38x + 25000$

(D) $C(x) = 28x + 25000$

Ans. Option (C) is correct.

Explanation: From the given table, we get

Cost of raw material per toy = ₹ 8

Cost of production supply per toy = ₹ 20

Cost of freight per toy = ₹ 10

Fixed cost = property fax + Salaries

= ₹ (5000 + 20000)

= ₹ 25000

Hence, Cost function,

$C(x) = (8 + 20 + 10)x + 25000$

or $C(x) = 38x + 25000$

Q. 3. If the company observes the price 'p' per unit of item sold $p = 5000 - 10x$, where 'x' is the number of units sold. Then the revenue function $R(x)$ is given by.

(A) $R(x) = 5000x - 10x^2$ **(B)** $R(p) = 5000p - 10x^2$

(C) $R(x) = 5000 - 10x^2$ **(D)** $R(p) = 5000 - 10p^2$

Ans. Option (A) is correct.

Explanation: Revenue,

$R(x) = p.x$

$= (5000 - 10x)x$

$= 5000x - 10x^2$

Q. 4. The marginal revenue (MR) of the company is given by

(A) $5000 - 20x$ **(B)** $5000 - 20p$

(C) $-20x$ **(D)** $-20p$

Ans. Option (A) is correct.

Explanation: Marginal revenue,

$$\text{MR} = \frac{dR(x)}{dx}$$

$$= \frac{d}{dx}(5000x - 10x^2)$$

$$= 5000 - 20x$$

Q. 5. If the profit function $P(x) = R(x) - C(x)$, then it is given by

(A) $-18x^2 + 4970x - 25000$

(B) $-10x^2 + 4962x - 20000$

(C) $10x^2 + 4962x - 25000$

(D) $-10x^2 + 4962x - 25000$

Ans. Option (D) is correct.

Explanation:

$P(x) = (5000x - 10x^2) - (38x + 25000)$

$= -10x^2 + 4962x - 25000$

III. Read the following text and answer the following questions on the basis of the same:

Four friends Rohan, Riya, Kartik and Mayank are playing number game. One by one they are giving

the riddles to others based on the numbers. Now, its Riya's chance to give a number riddle. Riya said: "The sum of three positive numbers is 26. The second number is thrice as large as the first and sum of the squares of the number is least".

Q. 1. If x, y, z are first, second and third number respectively, then which of the following is true.

(A) $x + y + z = 26$ and $y = 3x$
(B) $x + y + z = 26$ and $x = 3y$
(C) $x + y + z = 26$ and $z = 3x$
(D) $x + y + z = 26$ and $y = 3z$

Ans. Option (A) is correct.

Explanation: If x, y, z are first, second and third number respectively, then according to question: $x + y + z = 26$ and $y = 3x$

Q. 2. If S is the sum of the squares of the three numbers, then which of the following relation is true.

(A) $S = 26x^2 + 208x + 676$
(B) $S = 26x^2 - 208x - 676$
(C) $S = 26x^2 - 208x + 676$
(D) $S = -26x^2 - 208x + 676$

Ans. Option (C) is correct.

Explanation: Since, $x + y + z = 26$ and $y = 3x$

Then, $4x + z = 26$

$\Rightarrow$ $z = 26 - 4x$

Sum of the squares of three numbers,

$$S = x^2 + y^2 + z^2$$
$$= x^2 + (3x^2) + (26 - 4x)^2$$
$$= x^2 + 9x^2 + 676 + 16x^2 - 208x$$

$\Rightarrow$ $S = 26x^2 - 208x + 676$

Q. 3. The second derivative of S with respect to x is:

(A) 52 **(B)** 25
(C) $52x - 208$ **(D)** $52x + 208$

Ans. Option (A) is correct.

Explanation: Since, $S = 26x^2 - 208x + 676$

On differentiating both sides w.r.t. x, we get

$$\frac{dS}{dx} = 52x - 208$$

Now, $$\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{dS}{dx}\right)$$

$$= \frac{d}{dx}(52x - 208) = 52$$

Q. 4. At what values of x, the sum of the square of the numbers is least ?

(A) 5 **(B)** 2
(C) 8 **(D)** 4

Ans. Option (D) is correct.

Explanation: For maxima or minima,

put $$\frac{dS}{dx} = 0$$

$\Rightarrow$ $52x - 208 = 0$

$\Rightarrow$ $52x = 208$

$\therefore$ $x = 4$

Now, at $x = 4$ $$\frac{d^2S}{dx^2} = 52 > 0$$

Hence, at $x = 4$, S becomes minimum.

Q. 5. The remaining two numbers y and z are respectively:

(A) 10, 12 **(B)** 12, 10
(C) 10, 10 **(D)** 26, 12

Ans. Option (B) is correct.

Explanation: For $x = 4$

$y = 3 \times 4 = 12$

and $z = 26 - 4 \times 4$

$= 26 - 16 = 10$

IV. Read the following text and answer the following questions on the basis of the same:

The market research department of a company recommends manufacture and market a new toy-car. The financial department provides the following cost function (in rupees) $C(x) = 7{,}000 + 2x$; $0 \le x \le 10{,}000$ where ₹ 7,000 is the estimate of fixed costs and ₹ 2 is the estimate of variable cost per toy-car. The estimate of revenue function (in rupees) is $R(x) = x(10 - 0.001x)$; $0 \le x \le 10{,}000$.

Q. 1. The marginal cost function is:

(A) $2x$ **(B)** 2
(C) $7000 + 2x$ **(D)** 7000

Ans. Option (B) is correct.

Explanation: The marginal cost is

$C(x) = 7000 + 2x$

$C'(x) = 2$

Since the is a constant, it cost an additional ₹ 2 to produce one more toy-car at any production level.

Q. 2. The marginal revenue function is:

(A) $10 - 0.002x$ **(B)** $x(10 - 0.001x)$
(C) $-0.002x$ **(D)** 10

Ans. Option (A) is correct.

Explanation: The marginal revenue is

$R'(x) = 10 - 0.002x$.

Q. 3. The marginal revenue at $x = 2000, 5000, 7000$

(A) 6, – 4, 0 **(B)** 0, – 4, 6

(C) 6, 0, – 4 **(D)** none of these

Ans. Option (C) is correct.

Explanation: For production level of $x = 2{,}000, 5000,$ and 7000, we have $R(x) = 10 - 0.002x$

$$R'(2{,}000) = 6$$
$$R'(5{,}000) = 0$$
$$R'(7{,}000) = -4$$

Q. 4. The profit function $P(x)$ is given by

(A) $0.001x^2 + 8x + 7000$

(B) $0.001x^2 - 8x - 7000$

(C) $-0.001x^2 + 8x - 7000$

(D) $-0.001x^2 - 8x - 7000$

Ans. Option (C) is correct.

Explanation: The profit function is

$$P(x) = R(x) - C(x)$$
$$= -0.001x^2 + 8x - 7000$$

Q. 5. The marginal profit function at $x = 1000$

(A) 6 **(B)** 0

(C) – 4 **(D)** none of these

Ans. Option (A) is correct.

Explanation: The marginal profit function is

$$P'(x) = -0.002x + 8.$$

Therefore, $P'(1{,}000) = 6$

AI V. Read the following text and answer the following questions on the basis of the same:

A group of people decided to start a new project related to print media. They all have decided to publish a newspaper which has only college related news. They named their newspaper 'The Collegiate Investigator'. This newspaper has fixed production cost of ₹ 70 per edition and marginal printing and distribution cost of ₹ 0.40/per copy. The collegiate Investigator sells for ₹ 0.50/per copy.

Q. 1. The associated cost function is:

(A) $C(x) = 0.40x + 70$

(B) $C(x) = 0.40x - 70$

(C) $C(x) = -0.40 + 70x$

(D) $C(x) = -0.40 - 70x$

Ans. Option (A) is correct.

Explanation: Since, the marginal cost is ₹ 0.40 per copy and the fixed costs are ₹ 70. So, the cost function is $C(x) = 0.40x + 70$

Q. 2. The associated revenue function is:

(A) $R(x) = 0.40x$

(B) $R(x) = 0.50x$

(C) $R(x) = 0.70x$

(D) $R(x) = 0.50x + 70$

Ans. Option (B) is correct.

Explanation: Since, the selling price of newspaper is ₹ 0.50 per copy. So, the revenue function is

$$R(x) = 0.50x$$

Q. 3. The associated profit function is:

(A) $P(x) = -0.10x - 70$

(B) $P(x) = -0.10x + 50$

(C) $P(x) = 0.10x - 70$

(D) $P(x) = -0.10x + 70$

Ans. Option (C) is correct.

Explanation: Since, profit is defined to be revenue minus cost, the profit function is

$$P(x) = 0.50x - (0.40x + 70) = 0.10x - 70$$

Q. 4. What profit (or loss) results from the sale of 500 copies of The Collegiate Investigator ?

(A) 20 loss **(B)** 20 profit

(C) 70 loss **(D)** 70 profit

Ans. Option (A) is correct.

Explanation: The profit from selling 500 copies is $P(500)$

i.e., $P(500) = 0.10(500) - 70 = -20$

This means that if they sell 500 newspaper, it will result in a loss of ₹ 20.

Q. 5. How many copies should be sold in order to break even ?

(A) 500 **(B)** 700

(C) 400 **(D)** 200

Ans. Option (B) is correct.

Explanation: To find break even quantity, we can either set revenue equal to cost and solve for x, or we can set profit equal to zero and solve for x. Here, we set profit equal to zero, we get

$$P(x) = 0$$

or $0.10x - 70 = 0$

$\Rightarrow$ $x = 700$

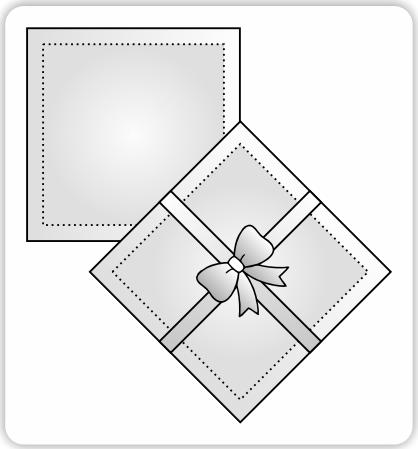
So, they must sell 700 newspapers in order to break even.

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:

(Each Sub-part carries 2 marks)

Revathi is going to her best friend's birthday party. She has made a beautiful painting for her friend as a birthday gift. Now, she wants to prepare a hand made gift box for that painting. For making lower part of the box, she takes a square piece of cardboard of side 30 cm.



Q. 1. If x can be the length of each side of the square cardboard which is to be cut off from corners of the square piece of side 30 cm, then write the expression for volume of the open box formed by folding up the cutting corners.

Also, find $\frac{dV}{dx}$.

Sol. Given, side of square is of length 30 cm.

Also, height of the open box $= x$ cm

Length of open box $= 30 - 2x$

and width of open box $= 30 - 2x$

$\therefore$ Volume (V) of open box $= x \times (30-2x) \times (30-2x)$

$= x(30 - 2x)^2$

Now, $\frac{dV}{dx} = x[2(30 - 2x)(-2)] + 1(30 - 2x)^2$ **1**

or, $\frac{dV}{dx} = -4x(30 - 2x) + (30 - 2x)^2$

or, $\frac{dV}{dx} = (30 - 2x)(-4x + 30 - 2x)$

or, $\frac{dV}{dx} = (30 - 2x)(30 - 6x)$ **1**

Q. 2. Revathi is inserted in maximising the volume of the box. So, what should be the side of square to be cut off so that volume of the box is maximum?

Sol. Since, $\frac{dV}{dx} = (30 - 2x)(30 - 6x)$

[from above]

$\therefore$ $\frac{d^2V}{dx^2} = -2(30 - 6x) + (30 - 2x)(-6)$

or, $\frac{d^2V}{dx^2} = -2(30 - 6x + 90 - 6x)$

or, $\frac{d^2V}{dx^2} = -2(120 - 12x)$

or, $\frac{d^2V}{dx^2} = -24(10 - x)$

Now, put $\frac{dV}{dx} = 0$

$\Rightarrow (30 - 2x)(30 - 6x) = 0$

$\Rightarrow x = 15$ or $x = 5$

At $x = 15$, $\frac{d^2V}{dx} = -24(10 - 15) = -24 \times (-5)$ **1**

$= 120 > 0$

At $x = 5$, $\frac{d^2V}{dx} = -24(10 - 15) = -24 \times 5$

$= -120 < 0$ **1**

So, volume will be maximum when $x = 5$ cm.

II. Read the following text and answer the following questions on the basis of the same:

The Government declare that farmers can get ₹ 500 per quintal for their onions on 1st April and after that, the price will be dropped by ₹ 4 per quintal per extra day.

Roshan's father has 120 quintal of onions in the field on 1st April and estimates that crop is increasing at the rate of ₹ 1 per day.

Q. 1. If x is the number of extra days after 1st April, then find the express for price and quantity of onions. Also, find the excess for reverse, R in terms of x_1,

Sol. Given, x is the number of extra days after 1st April

$\therefore$ Price, $P(x) = ₹ (500 - 4x)$ ½

and Quantity, $Q(x) = 120$ quintals

$\times x$(1 quintal per day)

$= (120 + x)$ quintals ½

Now, revenue, $R(x) =$ Quantity $\times$ Price

$= Q(x) \times P(x)$

$= (120 + x)(500 - 4x)$

$= 60000 - 480x + 500x - 4x^2$

$= 60{,}000 + 20x - 4x^2$

$\therefore$ $R(x) = 60{,}000 + 20x - 4x^2$ **1**

Q. 2. Find the number of days after 1st April, when Roshan's father attain maximum revenue.

Sol. We have, $R(x) = 60{,}000 + 20x - 4x^2$

$\therefore$ $R'(x) = 20 - 8x$

and $R''(x) = -8$ **1**

For $R(x)$ to be maximum, $R'(x) = 0$

$\Rightarrow 20 - 8x = 0$

$\Rightarrow x = \frac{20}{8} = 2.5$

Thus, Roshan's father will attain maximum revenue after 2.5 days. **1**

III. Read the following text and answer the following questions on the basis of the same:

In a residential society comprising of 100 houses, there were 60 children between the ages of 10-15 years. They were inspired by their teachers to start

composting to ensure that biodegradable waste is recycled, For this purpose, instead of each child doing it for only his/her house, children convinced the Residents welfare association to do it as a society initiative. For this they identified a square area in the local park. Local authorities charged amount of ₹50 per square metre for space so that there is no misuse of the space and Resident welfare association takes it seriously. Association hired a labourer for digging out 250 m^3 and he charged ₹400 X (depth)^2. Association will like to have minimum cost.

Q. 1. Let side of square plot is x m and its depth is h metres, then find cost equation C for the pit. Also, find second derivative of cost C w.r.t. h.

Sol. $$C = \frac{250 \times 50}{h} + 400 \times h^2$$ **1**

$$C = \frac{12500}{h} + 400h^2$$

$$\therefore \quad \frac{dC}{dh} = \frac{-12500}{h^2} + 800h$$

And $$\frac{d^2C}{dh^2} = \frac{-(-2) \times 12500}{h^3} + 800$$

$$\Rightarrow \quad \frac{d^2C}{dh^2} = \frac{25000}{h^3} + 800$$ **1**

Q. 2. Find the value of x (in m) for which cost is minimum.

Sol. For minimum cost, put $\frac{dC}{dh} = 0$, we get

$$\therefore \quad \frac{-12500}{h^2} + 800h = 0$$

$$\Rightarrow \quad 800h^3 = 12500$$

$$\Rightarrow \quad h^3 = \frac{125}{8}$$

$$\Rightarrow \quad h = \frac{5}{2} = 2.5 \text{ m}$$

$$h = 2.5 \text{ m}$$

At $h = 2.5, \frac{d^2C}{dh^2} > 0$ **1**

(Hence, minimum)

Value of x at minimum cost

$$x = \frac{400 \times (2.5)^2}{250}$$

$$= \frac{2500}{250} = 10 \text{ m}$$ **1**

Solutions for Practice Questions

Very Short Answer Type Questions

Sol. 2: Given, $f(x) = e^x + \log x$

The first derivative will be,

$$f'(x) = e^x + \frac{1}{x}$$

Differentiating it again,

$$f''(x) = e^x - \frac{1}{x^2}$$ **1**

Sol. 5: The total revenue function for 100 units as:

$$R(100) = 100 \times 250 = ₹\ 25000$$

Similarly, for 110 units:

$$R(110) = 110 \times 240 = ₹\ 26400$$

Thus, Marginal Revenue $= \frac{26400 - 25000}{10}$ **1**

Sol. 8: $$f(x) = x^4 - 62x^2 + ax + 9$$

Therefore, $$f'(x) = 4x^3 - 124x + a$$

f attains its maximum value on the interval [0, 2] at $x = 1$.

Therefore, $$f'(1) = 0$$

$$4 - 124 + a = 0$$

$$a = 120$$ **1**

Sol. 12: $$g(x) = \log x$$

$$\therefore \quad g'(x) = \frac{1}{x}$$

Since log x is defined for a positive number x, $g'(x) > 0$ for any x.

Therefore, there does not exist $c \in R$ such that $g'(c)$.

Hence, function g does not have maxima or minima. **1**

Short Answer Type Questions-I

Sol. 1: Given, $y = e^x(x + 1)$

Since this function is product of two functions, we will use multiplication rule for derivative.

$$\frac{d(f(x)g(x))}{dx} = f(x)\frac{d(g(x))}{dx} + g(x)\frac{d(f(x))}{dx}$$

$$y' = e^x(x + 1) + e^x$$ **1**

Now we can differentiate it again to get the second derivative.

$$y'' = \frac{d(e^x(x+1))}{dx} + \frac{d(e^x)}{dx}$$

$$y'' = e^x(x + 3)$$

At $x = 1$

$$y'' = e^1(1 + 3) = 4e$$ **1**

Sol. 4: Let, $y = f(x) = 2x^2 - 3x$

$$\frac{dy}{dx} = 4x - 3 > 0$$

For increasing function, $\frac{dy}{dx} > 0$ or $4x - 3 > 0$ or $x > \frac{3}{4}$

Hence, given function is increasing in $\left(\frac{3}{4}, \infty\right)$. **1**

For decreasing function, $\frac{dy}{dx} < 0$ or $4x - 3 < 0$ or $x < \frac{3}{4}$

Hence, given function is increasing in $\left(-\infty, \frac{3}{4}\right)$. **1**

Short Answer Type Questions-II

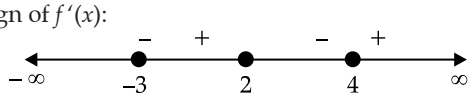
Sol. 3: $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

$f'(x) = x^3 - 3x^2 - 10x + 2$

$= (x - 2)(x - 4)(x + 3)$ **1**

$f'(x) = 0 \Rightarrow x = -3, 2, 4.$

sign of $f'(x)$:



1

$\therefore$ $f(x)$ is strictly increasing on $(-3, 2) \cup (4, \infty)$ ½

and $f(x)$ is strictly decreasing on $(-\infty, -3) \cup (2, 4)$ ½

Sol. 6: Let $f(x) = x^2 - 8x + 17$

On differentiating with respect to x, we get

$f'(x) = 2x - 8$

So, $f'(x) = 0$

$\Rightarrow$ $2x - 8 = 0$

$\Rightarrow$ $2x = 8$

$\therefore$ $x = 4$ **2**

Now, Again on differentiating with respect to x, we get

$f''(x) = 2 > 0, \forall x$

So, $x = 4$ is the point of local minima.

Minimum value of $f(x)$ at $x = 4$

$f(4) = 4 \times 4 - 8 \times 4 + 17 = 1$ **1**

Alter:

$f(x) = x^2 - 8x + 17$

$= (x - 4^2) + 1$

$\therefore$ Minimum value is 1

Sol. 10: Let l, b, h be the length, breadth and depth of the tank, respectively.

$\therefore$ $l \times b \times 3 = 75$

$\Rightarrow l \times b = 25$ ½

Let C be the cost, then

$C = 100(l \times b) + 50 \times 2\,[h(b + l)]$

$= 100\left(l \times \frac{25}{l}\right) + 300\left(\frac{25}{l} + l\right)$

$= 2500 + 300\left(\frac{25}{l} + l\right)$ ½

Differentiating w.r.t. l,

$$\frac{dC}{dl} = 0 + 300\left(\frac{-25}{l^2} + 1\right)$$ ½

For maximum and minimum cost,

$$\frac{dC}{dl} = 0$$

$\Rightarrow$ $300\left(-\frac{25}{l^2} + 1\right) = 0$

$\Rightarrow$ $l^2 = 25$ or $l = \pm 5$ ½

Getting $\left(\frac{d^2C}{dl^2}\right) = 300\left(\frac{50}{l^3}\right)$

$\Rightarrow \left(\frac{d^2C}{dl^2}\right)_{\text{at } l=5} = \frac{15000}{125} > 0$

i.e., C is minimum, when $l = 5$

$\Rightarrow$ $b = 5$ ½

$\therefore$ $C = 100(25) + 300(10)$

$= 2{,}500 + 3{,}000$

$= 5{,}500$

Hence, the minimum cost is ₹ 5,500. ½

Long Answer Type Questions

Sol. 2: Given, $x = \sin t$ and $y = \sin pt$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt} = \cos t$$

and $$\frac{dy}{dt} = \cos pt.p$$

or $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos pt.p}{\cos t}$$ **1**

Now on differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} =$$

$$p\frac{[\cos t(-\sin pt.p) - \cos pt(-\sin t)]}{\cos^2 t} \cdot \frac{dt}{dx}$$

or $$\frac{d^2y}{dx^2} = \frac{p[\cos pt.\sin t - \cos t \sin pt.t]}{\cos^2 t} \cdot \frac{1}{\cos t}$$ **1**

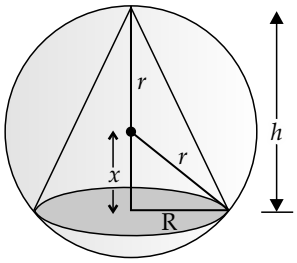
or $$\cos^2 t \frac{d^2y}{dx^2} = \frac{p[\cos pt.\sin t - p^2 \cos t \sin pt]}{\cos t}$$

or $(1-\sin^2 t)\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}.x - p^2y$

$\left[\because \dfrac{dy}{dx} = \dfrac{p\cos pt}{\cos t},\ x = \sin t \text{ and } y = \sin pt\right]$ **1**

$\therefore (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2 = 0$ **1**

Sol. 6: Let R be the radius of base and h be the height of the cone, let x be the distance between the centre of sphere and centre of base of cone.



$\therefore \quad h = r + x$

and $R^2 = r^2 - x^2$ **1**

Volume of cone $V(x) = \dfrac{1}{3}\pi R^2 h$

$= \dfrac{1}{3}\pi (r^2 - x^2)(r + x)$ ½

$\therefore \quad V'(x) = \dfrac{1}{3}\pi \{-2x(r + x) + (r^2 - x^2)\}$

$= \dfrac{1}{3}\pi(r + x)(-2x + r - x)$

$= \dfrac{1}{3}\pi(r + x)(r - 3x)$ **1**

For max. and min. volume

$V'(x) = 0$

$\Rightarrow \quad (r + x)(r - 3x) = 0$

$\Rightarrow \quad r = -x$ (not possible) or $r = 3x$

$\Rightarrow \quad x = \dfrac{r}{3}$

Now, $V''(x) = \dfrac{1}{3}\pi\{(r - 3x) - 3(r + x)\}$

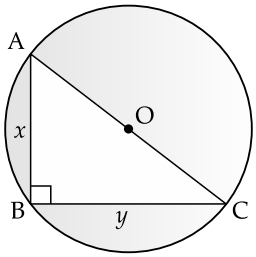
at $x = \dfrac{r}{3}$, $V''\left(\dfrac{r}{3}\right) < 0$ **1½**

$\therefore$ Volume is maximum when $x = \dfrac{r}{3}$

$\therefore \quad h = r + x = r + \dfrac{r}{3} = \dfrac{4r}{3}$

i.e., height of cone is $\dfrac{4r}{3}$. **Hence Proved. 1**

Sol. 12: Suppose ABC is a right angled triangle whose area is S.



$S = \dfrac{1}{2} \times \text{Base} \times \text{Height}$

$= \dfrac{1}{2} \times y \times x$ or, $y = \dfrac{2S}{x}$ ...(i) **1**

Circumscribed circle of ΔABC is one which passes through points A, B and C.

Let O be the centre of circumscribing circle.

$\therefore OA = OB = OC \Rightarrow OA = OC$

Since, O is the mid-point of AC.

$\therefore \ OA = OC = \dfrac{1}{2}AC$

Now, radius of circumscribing circle

$= \dfrac{1}{2} \times$ (Diameter)

$= \dfrac{1}{2} \times AC = \dfrac{1}{2} \times \sqrt{x^2 + y^2}$

[using Pythagoras theorem] ½

Area of circumscribing circle (A)

$= \pi \times$ (Radius of circle)^2

$= \pi \times \left(\dfrac{1}{2}\sqrt{x^2 + y^2}\right)^2$

$= \dfrac{\pi}{4}(x^2 + y^2)$ ...(ii) ½

On putting the value of y in eq. (ii), from eq. (i) we get

$A = \dfrac{\pi}{4}\left(x^2 + \dfrac{4S^2}{x^2}\right)$

On differentiating both sides w.r.t. x, we get

$\dfrac{dA}{dx} = \dfrac{\pi}{4}\left[2x + \left(\dfrac{-8S^2}{x^3}\right)\right]$

$$= \frac{\pi}{4}\left(2x - \frac{8S^2}{x^3}\right) \quad \textbf{1}$$

For maximum and minimum area, $\frac{dA}{dx}$ must be zero.

i.e., $\frac{\pi}{4}\left(2x - \frac{8S^2}{x^3}\right) = 0$

$\Rightarrow \; x - \frac{4S^2}{x^3} = 0$

$x^4 = 4S^2$

$\therefore \quad S = \frac{1}{2}x^2$ **1**

Now, $\frac{d^2A}{dx^2} = \frac{\pi}{4}\left(2 + \frac{24S^2}{x^4}\right)$

At $S = \frac{1}{2}x^2$, $\frac{d^2A}{dx^2} > 0$

Hence, area is least. ½

On putting $S = \frac{1}{2}x^2$ in Eq. (i) we get,

$$\frac{1}{2}x^2 = \frac{1}{2} \times xy$$

$\Rightarrow x = y$

Hence, area of circumscribed circle is least, when $x = y$. ½

Sol. 15: We calculate the marginal cost as

$$\frac{dc(x)}{dx} = \frac{d}{dx}(x^3 - 9x^2 + 24x)$$

$$= 3x^2 - 18x + 24 \quad \textbf{1}$$

In order to be a minimum at $x = x_0$ (say), its derivative must vanish at x_0.

Thus, $\left[\frac{d}{dx}\right]_{x=x_0} = 0$

or $(3x^2 - 18x + 24) = 0$

Thus, $x_0 = 2, 4$ **1**

Now, $\frac{d^2}{dx^2}(x^3 - 9x^2 + 24x) = 6x - 18$

$\left[\frac{d^2}{dx^2}\right]_{x_0=2} = -6$ and $\left[\frac{d^2}{dx^2}\right]_{x_0=4} = 6$ **1**

By the second derivative test, we can conclude that at $x = 4$, the function assumes a minima. Thus, for an output = 4 finished goods, the marginal cost would be minimum. Now, the average profit as

$$\frac{p(x)}{x} = \frac{SP(x) - CP(x)}{x}$$

$$\Rightarrow \frac{(2x^3 + 9x^2) - (x^3 - 9x^2 + 24x)}{x}$$

$\Rightarrow \frac{x^3 - 24x}{x} \Rightarrow x^2 - 24$ **2**

REFLECTIONS

(a) Will you be able to calculate the higher order derivatives easily?

(b) Do you think, derivatives can be used in medical field?

(c) Can you apply derivatives to calculate profit and loss in business?

(d) Can you easily apply second derivative test to find the maximum and minimum area or volume of geometrical figures?

CHAPTER

6

INTEGRALS AND IT'S APPLICATIONS

Syllabus

Integration, Indefinite integral as family of curves, definite integral as area under the curves, Application of integration.

In this chapter you will study

- *Understand and determine indefinite integrals of simple functions as anti-derivative*
- *Evaluate indefinite integrals of simple algebraic functions by method of: i) substitution ii) partial fraction iii) by parts*
- *Define definite integral as area under the curve*
- *Understand fundamental theorem of Integral calculus and apply it to evaluate the definite integral*
- *Apply properties of definite integrals to solve the problems*
- *Identify the region representing C.S. and P.S. graphically*
- *Apply the definite integral to find consumer surplus-producer surplus*

List of Topics

Topic-1: Indefinite Integrals **Page No. 106**

Topic-2 : Definite Integrals **Page No. 118**

Topic-3 : Application of Integrals **Page No. 123**

Topic-1 Indefinite Integrals

Concepts Covered • *Formulae for Indefinite Integrals, Integration by substitution, Integration by parts, Integration by partial fraction.*

Revision Notes

Meaning of Integral of Function

If differentiation of a function $F(x)$ is $f(x)$ i.e., if $\frac{d}{dx}[F(x)] = f(x)$, then we say that one integral or primitive or **anti-derivative** of $f(x)$ is $F(x)$ and in symbols, we write, $\int f(x)dx = F(x) + C$.

Scan to know more about this topic

Indefinite Integral

Therefore, we can say that integration is the inverse process of differentiation.

Indefinite Integral: Let $f(x)$ be a function. Then family of all its primitives or anti derivatives is called the indefinite integral of $f(x)$ and denoted by $\int f(x)dx$.

Formulae for Indefinite Integrals

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + k,\ n \neq -1$

(b) $\int \frac{1}{x} dx = \log|x| + k$

(c) $\int a^x dx = \frac{1}{\log a} a^x + k$

(d) $\int e^{ax} dx = \frac{1}{a} e^{ax} + k$

Key Word

Anti-derirative: In calculus, an anti-derivative, inverse derivative, primitive function, primitive integral or indefinite integral of a function f is a differentiable function F whose derivative is equal to the original function f. This can be stated symbolically as $F' = f$.

Integrals and It's Applications-1

Integration by Parts

For functions $f(x)$ and $g(x)$, we have
$\int [f(x).\ g(x)]\ dx = f(x).\int g(x).\ dx - \int \{f'(x).\int g(x).\ dx\}.\ dx$
Here, we can choose first function according to its position in ILATE, where
I = Inverse trigonometric function
L = Logarithmic function
A = Algebraic function
T = Trigonometric function
E = Exponential function

Integration by Substitution

$\int f(x).\ dx = \int f\{g(t)\}.\ g'(t)\ dt$, if we substitute $x = g(t)$, such that $dx = g'(t)\ dt$.

(i) $\int \{f(x)\}^n.\ f'(x).\ dx = \dfrac{\{f(x)\}^{n+1}}{n+1} + C,\ n \neq -1$

(ii) $\int \dfrac{f'(x)}{f(x)}.\ dx = \log|f(x)| + C,\ f(x) \neq 0$

Some Examples

Integration by parts

Evaluate: $\int e^{-2x} \sin x\ dx$
Method:-
(i) Use integration by part method i.e. consider sin x as 1st function and e^{-2x} as 2nd function constant

(ii) Use the formula: $\int \underset{\text{I}}{u}.\underset{\text{II}}{v}\,dx = u\int v.dx - \int\left[\dfrac{d}{dx}(u).\int v\,dx\right]dx$

(iii) Again further use the integration by parts and simplify it

Integration by substitution

Evalute : $\int \dfrac{1}{x+\sqrt{x}}\,dx$

Method:- To solve this problem, we use the concept of method of substitution. Here first of all put $x = t^2$, then apply the formula i.e.

$\int \dfrac{1}{x}dx = \log|x| + C$

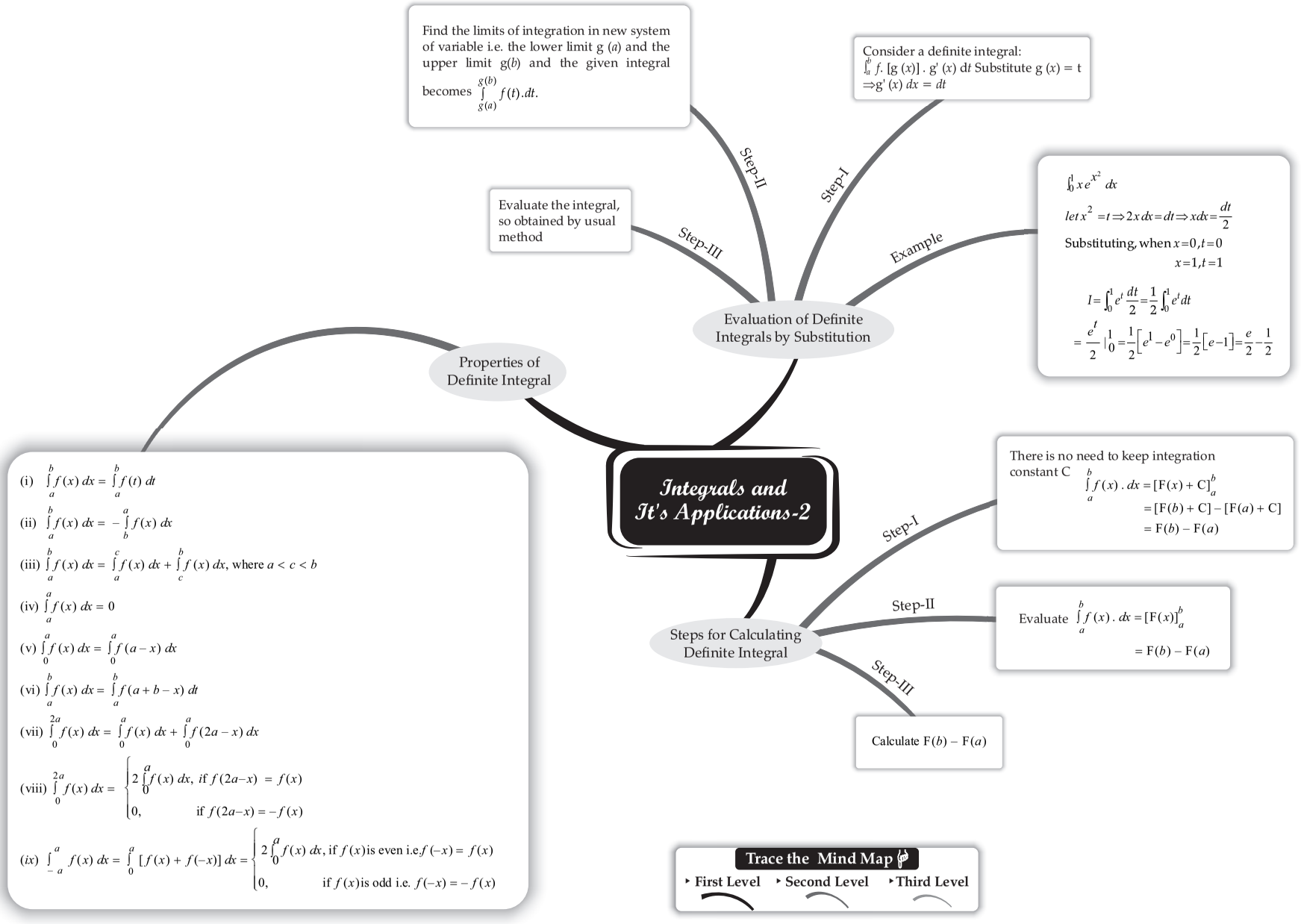
Integration as process of differentiation

$\dfrac{d}{dx}[F(x)] = f(x)$, then $\int f(x).dx = F(x) + C$

Basic Formulae of Integrals

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$
2. $\int e^x\,dx = e^x + C$
3. $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + C$
4. $\int a^x\,dx = \dfrac{a^x}{\log a} + C$
5. $\int \dfrac{1}{x}dx = \log|x| + C$
6. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
7. $\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$
8. $\int \dfrac{1}{a^2 - x^2}dx = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$
9. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C$
10. $\int \sqrt{x^2 - a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 - a^2} - \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$
11. $\int \sqrt{x^2 + a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$
12. $\int (ax+b)^n\,dx = \dfrac{1}{a}\dfrac{(ax+b)^{n+1}}{n+1} + C,\ n \neq -1$

Trace the Mind Map
First Level Second Level Third Level


Integrals and It's Applications-2
Evaluation of Definite Integrals by Substitution
Step-I
Consider a definite integral: $\int_a^b f.\,[g(x)]\,.\,g'(x)\,dt$ Substitute $g(x) = t \Rightarrow g'(x)\,dx = dt$
Step-II
Find the limits of integration in new system of variable i.e. the lower limit g (a) and the upper limit g(b) and the given integral becomes $\int_{g(a)}^{g(b)} f(t).dt.$
Step-III
Evaluate the integral, so obtained by usual method
Example
$\int_0^1 x e^{x^2}\,dx$
$let\ x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$
Substituting, when $x=0, t=0$
$x=1, t=1$
$I = \int_0^1 e^t \frac{dt}{2} = \frac{1}{2}\int_0^1 e^t dt$
$= \frac{e^t}{2}\Big|_0^1 = \frac{1}{2}\left[e^1 - e^0\right] = \frac{1}{2}[e-1] = \frac{e}{2} - \frac{1}{2}$
Properties of Definite Integral
(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$
(iv) $\int_a^a f(x)\,dx = 0$
(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
(vi) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dt$
(vii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$
(viii) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$
(ix) $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even i.e. } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is odd i.e. } f(-x) = -f(x) \end{cases}$
Steps for Calculating Definite Integral
Step-I
There is no need to keep integration constant C
$\int_a^b f(x)\,.\,dx = [F(x) + C]_a^b$
$= [F(b) + C] - [F(a) + C]$
$= F(b) - F(a)$
Step-II
Evaluate $\int_a^b f(x)\,.\,dx = [F(x)]_a^b$
$= F(b) - F(a)$
Step-III
Calculate $F(b) - F(a)$
Trace the Mind Map
First Level
Second Level
Third Level

Integrals and It's Applications-3

Consumer's Surplus

The consumers surplus is the difference between what the consumers would be willing to pay for a commodity and what they actually pay for them.
The consumers' surplus is given by

$$CS = \int_0^{Qe} D(x)\,dx - Q_e P_e$$

where D is the demand function, Q_e is the equilibrium quantity and P_e is equilibrium price.

Example

Example : Suppose that the demand function for producing a can of tennis balls is $P(x)=20-0.05x$ and that the current price level is P_e = Rs.8. Find the consumers surplus.
Sol.: First, we need to find the value of Q_e that corresponds to P_e =Rs.8. Setting P = 8 and solving for x gives
$8 = 20 - 0.05x \Rightarrow 0.05x = 12 \Rightarrow x = 240$ or $Q_e = 240$
Using the integral 'formula' for consumer surplus, we find that

$$CS = \int_0^{240} (20 - 0.05x)\,dx - 240.8$$

$$= 20x\Big|_0^{240} - 0.025x^2\Big|_0^{240} - 1920 = \text{Rs.}1,440.$$

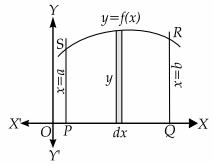
Area under simple curves

The area of the region bounded by the curve $y = f(x)$, X-axis and the lines $x=a$ and $x = b\,(b > a)$ is given by

$$A = \int_a^b y\,dx \quad \text{or} \quad \int_a^b f(x)dx$$

eg : the area bounded by $y=x^2$, X-axis in I quadrant and the lines $x=2$ and $x=3$ is -

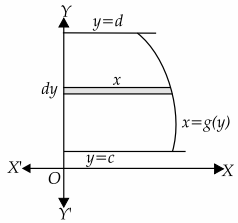
$$A = \int_2^3 y\,dx = \int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{1}{3}(27-8) = \frac{19}{3} \text{ Sq.units.}$$



The area of the region bounded by the curve $x = f(y)$, $Y-axis$ and the lines $y=c$ and $y = d\,(d > c)$ is given by $A = \int_c^d x\,dy$ or $\int_c^d f(x)dy$

eg : the area bounded by $x = y^3$, $Y-axis$ in the I quadrant and the lines $y=1$ and $y=2$ is

$$\int_a^b f(x)dx = \int_a^b f^3\,dy = \left[\frac{1}{4}y^4\right]_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4} \text{ Sq. units}$$



Producer's Surplus

Producer's surplus is the difference between what producers are willing and able to supply a good for and the price they actually receive.
The producers surplus is given by

$$PS = Q_e P_e - \int_0^{Q_e} S(x)\,dx$$

Where S is the demand function, Q_e is the equilibrium quantity and P_e is equilibrium price.

Trace the Mind Map

- First Level
- Second Level
- Third Level

(e) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b|+k$

(f) $\int \lambda dx = \lambda x + k$, where 'λ' is a none zero constant.

(g) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+k$

(h) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+k$

(i) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right|+k$

(j) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left|x+\sqrt{x^2+a^2}\right|+k$

(k) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+k$

(l) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+k$

Example 1: Find: $\int \frac{dx}{4x^2-9}$

Sol. Let, $I = \int \frac{dx}{4x^2-9}$

$= \frac{1}{4}\int \frac{dx}{x^2-\frac{9}{4}}$

$= \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{3}{2}\right)^2}$

$= \frac{1}{4}\frac{1}{2\times\frac{3}{2}}\log\left|\frac{x-\frac{3}{2}}{x+\frac{3}{2}}\right|+C$

$= \frac{1}{12}\log\left|\frac{2x-3}{2x+3}\right|+C$

Key Facts

- In real life, integrations are used in various fields such as engineering, where engineers use integrals to find the shape of building.
- In Physics, integrals used in the centre of gravity etc.
- Integrals are used in the field of graphical representation, where three-dimensional models are demonstrated.

- **Methods of Integration**

(a) Integration by Substitution Method: In this method, we change the integral $\int f(x)dx$, where independent variable is x, to another integral in which independent variable is t (say different from x) such that x and t are related by $x = g(t)$.

Scan to know more about this topic

Integration by Substitution

Let $u = \int f(x)dx$ then, $\frac{du}{dx} = f(x)$

Again as $x = g(t)$ so we have $\frac{dx}{dt} = g'(t)$

Now $\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} = f(x)\cdot g'(t)$

On integrating both sides w.r.t. t, we get

$$\int\left(\frac{du}{dt}\right)dt = \int f(x)g'(t)dt$$

$\Rightarrow u = \int f[g(t)]g'(t)dt$

i.e., $\int f(x)dx = \int f[g(t)]g'(t)dt$, where $x = g(t)$.

So, it is clear that substituting $x = g(t)$ in $\int f(x)\,dx$ will give us the same result as obtained by putting $g(t)$ in place of x and $g'(t)dt$ in place of dx.

Example 2: Find: $\int \frac{x}{e^{x^2}}dx$

Sol. Let, $I = \int \frac{x}{e^{x^2}}dx$

$= \int e^{-x^2}x\,dx$

Put $-x^2 = t$

$\Rightarrow -2x\,dx = dt$

$\Rightarrow x\,dx = -\frac{1}{2}dt$

$\therefore I = \int e^t\left(-\frac{1}{2}dt\right)$

$= -\frac{1}{2}\int e^t dt$

$= -\frac{1}{2}e^t + C$

$= -\frac{1}{2}e^{-x^2} + C$

Scan to know more about this topic

Integration by Parts

(b) Integration by Parts:

If u and v be two functions of x, then

$$\int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\,dx = u\left(\int vdx\right) - \int\left\{\frac{du}{dx}\int vdx\right\}dx$$

Example 3: Find: $\int \log x\,dx$

Sol. Let us assume here $\log x$ is the first function and constant 1 is the second function. Then the integral of the second function is x.

Therefore,

$\int(\log x.1)dx = \log x\int 1dx - \int\left[\frac{d}{dx}(\log x)\int 1dx\right]dx$

$= (\log x).x - \int\frac{1}{x}x\,dx$

$= x\log x - x + C$

Note

We can choose first function as the one whose initials comes first in the word. 'ILATE', where
I - Inverse Trigonometric function
L - Logarithm function
A - Algebraic function
T - Trigonometric function
E - Exponential function

(C) Integration by partial Fractions: Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

If the degree of numerator i.e., $f(x)$ is greater than or equal to the degree of denominator i.e., $g(x)$ then, this type of rational function is called an improper rational function. And if degree of $f(x)$ is smaller than the degree of denominator i.e., $g(x)$ then this type of rational function is called a proper rational function.

In rational polynomial function if the degree (i.e., highest power of the variable) of numerator (Nr) is greater than or equal to the degree of denominator (Dr), then (without) any doubt always perform the division i.e,. divide the Nr by Dr before doing anything and thereafter use the following:

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Table Demonstrating Partial Fractions of Various Forms

From of the Rational Functions	Form of the Rational Functions
$\frac{px+q}{(x-a)(x-b)}, x \neq a,b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2} x \neq a$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} x \neq a,b,c$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)} x \neq a,b$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} x \neq a$ where $x^2 + bx + c$ can't be factorized further.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Example 4: Evaluate: $\int \frac{3x+1}{(x+1)^2(x+3)}dx$

Sol. Let $I = \int \frac{3x+1}{(x+1)^2(x+3)}dx$

Let $\frac{3x+1}{(x+1)^2(x+3)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$ ...(i)

$3x + 1 = A(x + 1)(x + 3) + B(x + 3) + C(x + 1)^2$

$\Rightarrow$ $3x + 1 = x^2(A + C) + x(4A + B + 2C) + (3A + 3B + C)$

$\Rightarrow$ $A + C = 0, 4A + B + 2C = 3, 3A + 3B + C = 1$

On solving, we get $A = 2, B = -1, C = -2$

From (ii),

$$\frac{3x+1}{(x+1)^2(x+3)} = \frac{2}{x+1}-\frac{1}{(x+1)^2}-\frac{2}{x+3}$$

$\therefore$ $I = \int \frac{2}{x+1}dx - \int \frac{1}{(x+1)^2}dx - \int \frac{2}{x+3}dx$

$= 2\log|x + 1| + \frac{1}{x+1} - 2\log|x + 3| + C$

$= 2\log\frac{(x+1)}{(x+3)} + \frac{1}{x+1} + C$

Special form of Integrals:

(a) $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$

(b) $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. The value of: $\int (x+3)(x+2)dx$ **is:**

(A) $\frac{x^3}{3}+\frac{5x^2}{2}+2x+C$ **(B)** $x^3+\frac{5}{2}x^2+x+C$

(C) $\frac{x^3}{3}+\frac{x^2}{2}+2x+C$ **(D)** $\frac{x^3}{3}+\frac{x^2}{2}+x+C$

Ans. Option (A) is correct.

Explanation: Let

$$I = \int (x+3)(x+2)dx$$
$$= \int (x^2+5x+2)dx$$
$$= \int x^2dx + 5\int xdx + 2\int dx$$
$$= \frac{x^3}{3}+\frac{5x^2}{2}+2x+C$$

Q. 2. The value of: $\int\left[\sqrt{x}+\frac{1}{\sqrt{x}}\right]^2 dx$ **is:**

(A) $x^2-2x+\log|x|+C$

(B) $\frac{x^2}{2}+2x+\log|x|+C$

(C) $x^2+2x-\log|x|+C$

(D) $x^2-x+\log|x|+C$

Ans. Option (B) is correct.

Explanation: Let

$$I = \int\left[\sqrt{x}+\frac{1}{\sqrt{x}}\right]^2 dx$$
$$= \int\left(x+\frac{1}{x}+2\right)dx$$
$$= \int xdx + \int\frac{1}{x}dx + 2\int dx$$
$$= \frac{x^2}{2}+\log|x|+2x+C$$

Q. 3. The value of: $\int\left[a^x+a^{-x}\right]^2 dx$ **is:**

(A) $\frac{a^x}{2}+\frac{a^{-x}}{2}+2x+C$

(B) $\frac{a^{2x}}{2}+\frac{a^{-2x}}{2}+2x+C$

(C) $\frac{a^{2x}}{2\log a}-\frac{a^{-2x}}{2\log a}+2x+C$

(D) $2\frac{a^x}{\log a}-\frac{a^{-x}}{2\log a}+2x+C$

Ans. Option (C) is correct.

Explanation: Let

$$I = \int\left[a^x+a^{-x}\right]^2 dx$$
$$= \int\left(a^{2x}+a^{-2x}+2a^x.a^{-x}\right)dx$$
$$= \int\left(a^{2x}+a^{-2x}+2\right)dx$$
$$= \frac{a^{2x}}{2\log a}+\frac{a^{-2x}}{-2\log a}+2x+C$$
$$= \frac{a^{2x}}{2\log a}-\frac{a^{-2x}}{2\log a}+2x+C$$

Q. 4. The value of: $\int 3^{3^x}\,3^x dx$ **is:**

(A) $\frac{3^{3^x}}{(\log 3)^2}+C$ **(B)** $\frac{3^x}{(\log 3)^2}+C$

(C) $\frac{3^{3x}}{\log 3}+C$ **(D)** $\frac{3^{3x}}{\log 3^x}+C$

Ans. Option (A) is correct.

Explanation: Let $I = \int 3^{3^x}\,3^x dx$

Put $3^x = t$

$\Rightarrow$ $3^x \log 3\, dx = dt$

$\therefore$ $I = \int 3^t \frac{dt}{\log 3}$

$$= \frac{1}{\log 3}\int 3^t dt$$
$$= \frac{1}{\log 3}\left(\frac{3^t}{\log 3}\right)+C$$
$$= \frac{3^{3^x}}{(\log 3)^2}+C$$

Q. 5. $\int\sqrt{1+x^2}dx$ **is equal to:**

(A) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right|+C$

(B) $\frac{2}{3}(1+x^2)^{3/2}+C$

(C) $\frac{2}{3}x(1+x^2)^{3/2}+C$

(D) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left(x+\sqrt{1+x^2}\right)+C$

Ans. Option (A) is correct.

Explanation: It is known that,

$$\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\therefore\int\sqrt{1+x^2}dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$$

Q. 6. $\int x^2e^{x^3}dx$ **is equal to:**

(A) $\frac{1}{3}e^{x^3}+C$ **(B)** $\frac{1}{3}e^{x^2}+C$

(C) $\frac{1}{2}e^{x^3}+C$ **(D)** $\frac{1}{2}e^{x^2}+C$

Ans. Option (A) is correct.

Explanation:

Let $I=\int x^2e^{x^3}dx$

Also, let $x^3=t$

$\Rightarrow 3x^2\,dx=dt$

$\Rightarrow I=\frac{1}{3}\int e^t dt$

$=\frac{1}{3}\left(e^t\right)+C$

$=\frac{1}{3}e^{x^3}+C$

Q. 7. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$ **is equal to:**

(A) $\frac{e^x}{1+x^2}+C$ **(B)** $\frac{-e^x}{1+x^2}+C$

(C) $\frac{e^x}{\left(1+x^2\right)^2}+C$ **(D)** $\frac{-e^x}{\left(1+x^2\right)^2}+C$

Ans. Option (A) is correct.

Explanation:

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx=\int e^x\frac{1+x^2-2x}{(1+x^2)^2}dx$$

$$=\int e^x\left[\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right]dx$$

$$=\int e^x[f(x)+f'(x)]dx,$$

$$\left[\text{where } f(x)=\frac{1}{1+x^2}\right]$$

$$=e^xf(x)+C$$

$$=\frac{e^x}{1+x^2}+C$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Evaluate: $\int\frac{1}{x(1+\log x)}dx$ R&U [O.E.B.]

Sol. Let $1+\log x=t$

$$\left(\frac{1}{x}\right)dx=dt$$

$$dx=x\,dt$$

$$I=\int\frac{1}{x(t)}x\,dt$$

$$=\int\frac{1}{t}dt$$

$$=\log|t|+C$$

$$=\log|1+\log x|+C \quad \mathbf{1}$$

Q. 2. Evaluate: $\int xe^{(1+x^2)}dx$ R [O.E.B.]

Sol. Let $(1+x^2)=t$

so, $2x\,dx=dt$

$\Rightarrow I=\frac{1}{2}\int e^t\,dt$

$=\frac{1}{2}e^t+c$

$=\frac{1}{2}e^{(1+x^2)}+c$ **1**

Q. 3. Evaluate: $\int\frac{3x}{3x-1}dx$ R&U [O.E.B.]

Q. 4. Evaluate: $\int_e^{e^2}\frac{dx}{x\log x}$ R&U [O.E.B.]

Sol. Put $\log x=t$

or $dt=\frac{1}{x}dx$

Put $x=e^2$ or $t=2$ and $x=e$ or $t=1$

$\therefore\int_e^{e^2}\frac{dx}{x\log x}=\int_1^2\frac{dt}{t}$

$=\log 2-\log 1$

$=\log 2$ **1**

These questions are for practice and their solutions are available at the end of the chapter

Q. 5. Find $\int \frac{2^{x+1}-5^{x-1}}{10^x}dx$. A I

Sol.
$$\int \frac{2^{x+1}-5^{x-1}}{10^x}dx = \int \frac{2^{x+1}-5^{x-1}}{2^x.5^x}dx$$
$$= \int\left(\frac{2^{x+1}}{2^x.5^x}-\frac{5^{x-1}}{2^x.5^x}\right)dx$$
$$= \int \frac{2}{5^x}dx - \int \frac{1}{5\cdot 2^x}dx$$
$$= 2\int 5^{-x}dx - \frac{1}{5}\int 2^{-x}dx$$
$$= 2\times\frac{-5^{-x}}{\log 5} - \frac{1}{5}\times\frac{-2^{-x}}{\log 2}$$
$$= \frac{1}{5.2^x\log 2} - \frac{2}{5^x\log 5} + C \quad \mathbf{1}$$

Q. 6. Find: $\int \frac{dx}{\sqrt{x}+x}$ U

Sol.
$$\int \frac{dx}{\sqrt{x}+x}dx = \int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$$

Let, $1+\sqrt{x} = t$

$$0+\frac{1}{2\sqrt{x}}dx = dt$$
$$\frac{1}{\sqrt{x}}dx = 2dt$$
$$= 2\int\frac{1}{t}dt$$
$$= 2\log t + C$$
$$= 2\log(1+\sqrt{x}) + C \quad \mathbf{1}$$

Commonly Made Error

- Mostly students find a difficulty in finding the correct substitution.

Answering Tip

- Practice more problems based on substitution.

Q. 7. Find $\int xe^{(1+x^2)}dx$. A I

Sol. Let $(1+x^2) = t$

so $2x\,dx = dt$

$\Rightarrow$ $I = \frac{1}{2}\int e^t\,dt$

$$= \frac{1}{2}e^t + c$$
$$= \frac{1}{2}e^{(1+x^2)} + c \quad \mathbf{1}$$

Q. 8. Find: $\int \frac{2x}{\sqrt[3]{x^2+1}}dx$ U

Sol. Let, $x^2+1 = t^3$ $\quad\therefore 2xdx = 3t^2dt$

$$\int \frac{2x}{\sqrt[3]{x^2+1}}dx = \int\frac{1}{t^3}dt$$
$$\int\frac{3t^2dt}{\sqrt[3]{t^3}} = 3\int tdt = \frac{3t^2}{2}+C$$
$$= \frac{3}{2}(x^2+1)\frac{2}{3}+C$$

Short Answer Type Questions-I (2 marks each)

Q. 1. Evaluate: $\int \frac{e^x(x-3)}{(x-1)^3}dx$ R&U [O.E.B.]

Sol.
$$\int \frac{e^x(x-3)}{(x-1)^3}dx = \int e^x\left[\frac{(x-1)-2}{(x-1)^3}\right]dx$$
$$= \int e^x\left[\frac{1}{(x-1)^2}+\left(\frac{-2}{(x-1)^3}\right)\right]dx \quad \mathbf{1}$$
$$\left(\because \int e^x[f(x)+f'(x)]dx = e^xf(x)+c\right)$$
$$1 = \frac{e^x}{(x-1)^2}+C \quad \mathbf{1}$$

Commonly Made Error

- Some candidates use Product Rule in $\int e^x\{f(x)+f'(x)\}$ in both parts which is wrong. Students should apply the rule that $\int e^x\{f(x)+f'(x)\} = e^xf(x)+C$ only.

Answering Tip

- $\int e^x\{f(x)+f'(x)\}$ is specific situation in integration by parts. This needs to be sufficiently practiced by students.

Q. 2. Find: $\int \frac{(x^4-x)^{\frac{1}{4}}}{x^5}dx$ R&U [O.E.B.]

Sol. Let $I = \int \frac{(x^4-x)^{\frac{1}{4}}}{x^5}dx$

$$= \int\left[x^4\left(1-\frac{x}{x^4}\right)\right]^{\frac{1}{4}}\frac{1}{x^5}dx$$

$$= \int x\left(1-\frac{1}{x^3}\right)^{\frac{1}{4}}\frac{1}{x^5}dx$$

$$= \int\left(1-\frac{1}{x^3}\right)^{\frac{1}{4}}\frac{1}{x^4}dx \quad \mathbf{1}$$

Put $1-\frac{1}{x^3} = t$

So that, $\frac{3}{x^4}dx = dt$

$\Rightarrow$ $dx = \frac{x^4}{3}dt$

$$= \frac{1}{3}\int t^{\frac{1}{4}}dt = \frac{1}{3}.\frac{4}{5}t^{\frac{5}{4}}$$

$$= \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}}+C \quad \mathbf{1}$$

Q. 3. Evaluate: $\int\left(\frac{1-x}{1+x^2}\right)^2 e^x dx$ R&U **[O.E.B.]**

Q. 4. Evaluate: $\int\frac{dx}{5-8x-x^2}$ R&U **[O.E.B.]**

Sol. $\int\frac{dx}{5-8x-x^2} = \int\frac{dx}{(\sqrt{21})^2-(x+4)^2}$ **1**

$$= \frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+(x+4)}{\sqrt{21}-(x+4)}\right|+c \quad \mathbf{1}$$

Commonly Made Error

- Some candidates make errors in simplifying the expression before integration.

Answering Tip

- When denominator is given in quadratic form, then make it perfect square then apply the property of integral.

Q. 5. Evaluate: $\int\sqrt{x^2-2x}\,dx$ R&U **[O.E.B.]**

Sol. $I = \int\sqrt{(x-1)^2-1^2}\,dx$ **1**

$$= \frac{(x-1)}{2}\sqrt{x^2-2x}-\frac{1}{2}\log\left|x-1+\sqrt{x^2-2x}\right|+C \quad \mathbf{1}$$

Q. 6. Evaluate: $\int\frac{x^3+5x^2+4x+1}{x^2}dx$ R&U **[O.E.B.]**

Sol. Let $I = \int\frac{x^3+5x^2+4x+1}{x^2}dx$

$$= \int\left(x+5+\frac{4}{x}+\frac{1}{x^2}\right)dx$$

$$= \int x\,dx+5\int 1dx+4\int\frac{1}{x}dx+\int\frac{1}{x^2}dx \quad \mathbf{1}$$

$$= \frac{x^2}{2}+5x+4\log|x|+\left(\frac{-1}{x}\right)+C$$

$$= \frac{x^2}{2}+5x+4\log|x|-\frac{1}{x}+C \quad \mathbf{1}$$

Commonly Made Error

- This question was attempted correctly by most of the candidates; some of the candidates made errors in simplifying the expression before integration.

Answering Tip

- Do practice to simplify different types of algebraic expressions in the form of required simplest form for integration.

Q. 7. Evaluate: $\int\frac{e^{2x}}{2+e^x}dx$ **[O.E.B.]**

Sol. Given, $\int\frac{e^{2x}}{2+e^x}dx$

Let $e^x+2 = t$

$e^x = t-2$

$\Rightarrow$ $e^x dx = dt$ **1**

$\therefore$ $\int\frac{t-2}{t}dt = t-2\log|t|+C_1$

$= e^x+2-2\log|t|+C_1$

$= e^x-2\log|e^x+2|+C$

[Since $e^{2x}dx = e^x.e^x dx = (t-2)\,dt$] $(C = C_1+2)$ **1**

Q. 8. Evaluate: $\int\frac{x^2}{x^2-4}dx$. A **[O.E.B.]**

These questions are for practice and their solutions are available at the end of the chapter

Short Answer Type Questions-II (3 marks each)

Q. 1. Evaluate: $\int \frac{2x+1}{(x+1)(x-2)}dx$ [O.E.B.]

Sol. Let $\frac{2x+1}{(x+1)(x-2)} = \frac{A}{x+1}+\frac{B}{(x-2)}$

$\Rightarrow \quad 2x + 1 = A(x-2) + B(x+1)$

Equating the coefficients of x and constant term on both sides, we get

$A + B = 2$ and $-2A + B = 1$

Solving these equations, we get $A = \frac{1}{3}$ and $B = \frac{5}{3}$

$\therefore \quad \frac{2x+1}{(x+1)(x-2)} = \frac{1/3}{(x+1)}+\frac{5/3}{(x-2)}$ **2**

$\Rightarrow \quad \int \frac{2x+1}{(x+1)(x-2)}dx = \frac{1}{3}\int\frac{dx}{x+1}+\frac{5}{3}\int\frac{dx}{x-2}$

$= \frac{1}{2}\log|x+1| + \frac{5}{3}\log|x-2| + C$

1

Q. 2. Evaluate: $\int \frac{x^3}{x^4+3x^2+2}dx$ [O.E.B.]

Q. 3. Evaluate: $\int \frac{dx}{x(x^5+3)}$ [O.E.B.]

Sol. Let $I = \int \frac{dx}{x(x^5+3)} = \int \frac{x^4}{x^5(x^5+3)}dx$

Put $x^5 = t \Rightarrow 5x^4\,dx = dt \Rightarrow x^4\,dx = \frac{1}{5}dt$

$\therefore \quad I = \frac{1}{5}\int\frac{dt}{t(t+3)}$

Now, $\frac{1}{t(t+3)} = \frac{A}{t}+\frac{B}{t+3}$

$\Rightarrow \quad 1 = A(1+3) + Bt$

Equating the coefficients of t and constant terms, we get

$A + B = 0$ and $3A = 1$

$\Rightarrow \quad A = \frac{1}{3}$ and $B = -\frac{1}{3}$

$\therefore \quad \frac{1}{t(t+3)} = \frac{1}{3t}-\frac{1}{3(t+3)}$

$\therefore \quad I = \frac{1}{5}\cdot\frac{1}{3}\int\left(\frac{1}{t}-\frac{1}{t+3}\right)dt$ **2**

$= \frac{1}{15}(\log|t| - \log|t+3|)$

$= \frac{1}{15}\log\left|\frac{t}{t+3}\right|+C$

$= \frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right|+C$ **1**

Q. 11. Find: $\int \frac{(2x-5)e^{2x}}{(2x-3)^3}dx.$ A I

Sol. Let $2x = t$

$I = \frac{1}{2}\int\frac{(t-5)}{(t-3)^3}e^t dt$ **1**

$= \frac{1}{2}\int\left[\frac{1}{(t-3)^2}-\frac{2}{(t-3)^3}\right]e^t dt$ **1**

$= \frac{1}{2}\frac{1}{(t-3)^2}e^t + C$

$= \frac{1}{2}\frac{1}{(2x-3)^2}e^{2x} + C$ **1**

Commonly Made Error

- Several candidates commit the mistake of ignoring the constant of integration and use of partial fraction.

Answering Tip

- Reduce the given expression in proper fraction then apply partial fraction.

Long Answer Type Questions (5 marks each)

Q. 1. Find: $\int \frac{\sqrt{x^2+1}\{\log(x^2+1)-2\log x\}}{x^4}dx.$

A I [O.E.B.]

Sol. Let $I = \int \frac{\sqrt{x^2+1}\{\log(x^2+1)-2\log x\}}{x^4}dx$

$= \int\sqrt{1+\frac{1}{x^2}}\left(\log\left(1+\frac{1}{x^2}\right)\right)\frac{1}{x^3}dx$

Let, $1 + \frac{1}{x^2} = t^2$

or $\frac{-2}{x^3}dx = 2\,tdt$

or $\frac{1}{x^3}dx = -\,tdt$ **2**

$= -\int t(2\log t)t\,dt$

$= -2\int \log t\cdot t^2 dt$

$= -2\log t\cdot\frac{t^3}{3} + \int 2\frac{1}{t}\cdot\frac{t^3}{3}dt$ **1**

$$= -\frac{2}{3}\log t \cdot t^3 + \frac{2}{9}t^3 + C$$

$$= -\frac{2}{3}t^3\left(\log t - \frac{1}{3}\right) + C \quad \textbf{1}$$

$$= -\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left(\log\left(1+\frac{1}{x^2}\right) - \frac{1}{3}\right) + C \quad \textbf{1}$$

Q. 2. Evaluate: $\int \frac{3x^2-5}{9x^4-20x^2+25}dx$ **[O.E.B.]**

Q. 3. Evaluate: $\int \frac{3x+1}{(x^3-x^2-x+1)}dx$ **[O.E.B.]**

Sol. Let $I = \int \frac{3x+1}{(x^3-x^2-x+1)}dx$

$\Rightarrow I = \int \frac{3x+1}{(x-1)^2(x+1)}dx$

Resolving the integrand into partial fractions.

$$\frac{3x+1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad \text{...(i) } \textbf{1}$$

$3x + 1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$\Rightarrow 3x + 1 = x^2(A + C) + x(B - 2C) + (-A + B + C)$

$\Rightarrow A + C = 0, B - 2C = 3, -A + B + C = 1$ **1**

On solving we get $A = \frac{1}{2}$, $B = 2$, $C = -\frac{1}{2}$

From (i),

$$\int \frac{3x+1}{(x^3-x^2-x+1)}dx$$

$$I = \int\left(\frac{1}{2(x-1)} + \frac{2}{(x-1)^2} + \frac{1}{2}\frac{-1}{x+1}\right)dx$$

$$= \frac{1}{2}\log|x-1| - \frac{2}{x-1} - \frac{1}{2}\log|x+1| + C \quad \textbf{1}$$

$$= \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) - \frac{2}{x-1} + C \quad \textbf{1}$$

Q. 4. Evaluate: $\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

[O.E.B.]

Sol. Let $\frac{2x-1}{(x-1)(x+2)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}$

$\Rightarrow 2x - 1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

$\Rightarrow 2x - 1 = A(x^2 - x - 6) + B(x^2 - 4x + 3) + C(x^2 + x - 2)$

Equating the coefficients of x^2, x and constant terms, we get

$A + B + C = 0,$

$-A - 4B + C = 2$

and $-6A + 3B - 2C = -1$ **1**

Solving these equations, we get

$A = -\frac{1}{6}$, $B = -\frac{1}{3}$ and $C = \frac{1}{2}$

$$\therefore \frac{2x-3}{(x-1)(x+2)(x+3)} = \frac{-\frac{1}{6}}{x-1} + \frac{-\frac{1}{3}}{x+2} + \frac{\frac{1}{2}}{x-3} \quad \textbf{2}$$

$$\Rightarrow \int \frac{2x-1}{(x-1)(x+2)(x+3)}dx = -\frac{1}{6}\int\frac{1}{x-1}dx - \frac{1}{3}\int\frac{1}{x+2}dx + \frac{1}{2}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}|\log x - 3| + C \quad \textbf{2}$$

Q. 5. Evaluate: $\int \frac{3x-2}{(x+1)^2(x+3)}dx$ **[O.E.B.]**

Sol. Let, $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$\Rightarrow 3x - 2 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$

$\Rightarrow 3x - 2 = A(x^2 + 4x + 3) + B(x+3) + (x^2 + 2x + 1)$

Equating coefficients of x^2, x and constant terms, we get

$A + C = 0,$

$4A + B + 2C = 3$

and $3A + 3B + C = -2$ **2**

Solving these equations, we get

$A = \frac{11}{4}$, $B = \frac{-5}{2}$

and $C = \frac{-11}{4}$

$$\therefore \frac{3x-2}{(x+1)^2(x+3)} = \frac{\frac{11}{4}}{x+1} - \frac{\frac{5}{2}}{(x+1)^2} - \frac{\frac{11}{4}}{x+3} \quad \textbf{2}$$

$$\Rightarrow \int \frac{3x-2}{(x+1)^2(x+3)}dx = \frac{11}{4}\int\frac{dx}{x+1} - \frac{5}{2}\int\frac{1}{(x+1)^2}dx - \frac{11}{4}\int\frac{dx}{x+3}$$

$$= \frac{11}{4}\log|x+1| - \frac{5}{2}\cdot\frac{(x+1)^{-1}}{-1} - \frac{11}{4}\log|x+3| + C$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C \quad \textbf{1}$$

Q. 6. Evaluate: $\int \frac{2x+3}{(2-x)(x^2+3)}dx$ **[O.E.B.]**

Sol. Let $\frac{2x+3}{(2-x)(x^2+3)} = \frac{A}{2-x} + \frac{Bx+C}{x^2+3}$

$\Rightarrow 2x + 3 = A(x^2 + 3) + (Bx + C)(2 - x)$
$\Rightarrow 2x + 3 = A(x^2 + 3) + B(2x - x^2) + C(2 - x)$
Equating coefficients of x^2, x and constant terms, we get

$$A - B = 0,$$
$$2B - C = 2,$$
and $$3A + 2C = 3$$ **2**

Solving these equations, we get

$$A = 1, B = 1 \text{ and } C = 0$$

$$\therefore \quad \frac{2x+3}{(2-x)(x^2+3)} = \frac{1}{2-x} + \frac{x}{x^2+3}$$

$$\Rightarrow \int \frac{2x+3}{(2-x)(x^2+3)}dx = \int \frac{1}{2-x}dx + \int \frac{x}{x^2+3}dx$$

$$= \int \frac{1}{2-x}dx + \frac{1}{2}\int \frac{2x}{x^2+3}dx \quad \textbf{2}$$

$$= \frac{\log|2-x|}{-1} + \frac{1}{2}\log(x^2+3) + C \quad \textbf{1}$$

$$= -\log|2-x| + \frac{1}{2}\log(x^2+3) + C$$

Q. 7. Evaluate: $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx$ **[O.E.B.]**

Sol. Put $x^2 = t$
$\Rightarrow 2x\,dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+2)^2}dx = \int \frac{1}{(t+1)(t+2)^2}dt$$

Let, $$\frac{1}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{(t+2)^2}$$

$\Rightarrow 1 = A(t + 2)^2 + B(t + 1)(t + 2) + C(t + 1)$
$\Rightarrow 1 = A(t^2 + 4t + 4) + B(t^2 + 3t + 2) + C(t + 2)$
Equating the coefficients of t^2, t and constant terms, we get

$$A + B = 0,$$
$$4A + 3B + C = 0,$$
and $$4A + 2B + C = 1$$

Solving these equations for A, B and C, we get $A = 1$, $B = 1$, $C = 1$

$$\therefore \quad \frac{1}{(t+1)(t+2)^2} = \frac{1}{t+1} - \frac{1}{t+2} - \frac{1}{(t+2)^2}$$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+2)^2}dx$$

$$= \int \frac{1}{t+1}dx - \int \frac{1}{t+2}dt - \int \frac{1}{(t+2)^2}dt \quad \textbf{2}$$

$$= \log|t+1| - \log|t+2| - \frac{(t+2)^{-1}}{-1} + C$$

$$= \log(x^2+1) - \log(x^2+2) + \frac{1}{x^2+2} + C$$

$$= \log\frac{x^2+1}{x^2+2} + \frac{1}{x^2+2} + C \quad \textbf{1}$$

Topic-2 Definite Integrals

Concepts Covered • *Properties of definite integrals*

Revision Notes

Meaning of Definite Integral of Function

If $\int f(x)dx = F(x)$, be an integral of $f(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$. Moreover, the definite integral gives a unique and definite value (numeric value) of anti-derivative of the function between the given intervals. It acts as a substitute for evaluating the area analytically.

Scan to know more about this topic

Meaning of Definite Integral of function

Properties of definite Integral

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ **(b)** $\int_a^b f(x)dx = \int_a^b f(t)dt$ **(c)** $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

(d) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **(e)** $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(f) $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(x) \text{ is an even function i.e., } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is an odd function i.e., } f(-x) = -f(x) \end{cases}$ **(g)** $\int_{-a}^a f(x)dx = \int_0^a \{f(x) + f(-x)\}dx$

(h) $\int_0^{2a} f(x)dx = \int_0^{2a} \{f(x) + f(2a-x)\}dx$ **(i)** $\int_o^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a - x) = f(x) \\ 0 \quad ; & \text{if } f(2a-x) = -f(x) \end{cases}$

Mnemonics

Concept:

SeCond FundAmental Theorem of Definite Integration

f c c F a d F b F a

Continuous Closed anti derivative

Mnemonics:

fcc is small **f**ashionable **C**lothes

Counter **F**or **a**dorable **d**resses

at **Fbb** **Fa**shion

Closed anti Continuous derivative

F(b) – F(a)

Interpretations: Let *f* be a continuous function defined on a closed interval [*a,b*] and F be an anti derivative of *f*. Then $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$, where *a* and *b* are called limit of Integration.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. The value of: $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$ **is:**

(A) $\log\sqrt{2}$ **(B)** $\log\left(1+\sqrt{2}\right)$

(C) $\log\left(1-\sqrt{2}\right)$ **(D)** $\frac{1}{\log(1+\sqrt{2})}$

Ans. Option (B) is correct.

Explanation:

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}dx = \log\left|x+\sqrt{1+x^2}\right|_0^1$$

$$= \log\left|1+\sqrt{1+1^2}\right| - \log\left|0+\sqrt{1+0}\right|$$

$$= \log\left(1+\sqrt{2}\right) - \log 1$$

$$= \log(1+\sqrt{2}) \qquad [\because \log 1 = 0]$$

Q. 2. The value of: $\int_{-1}^1 x^2\sqrt{x^3+1}\,dx$ **is:**

(A) $\frac{4}{9}$ **(B)** $\frac{4\sqrt{2}}{3}$

(C) $\frac{4\sqrt{2}}{9}$ **(D)** $-\frac{4}{9}$

Ans. Option (C) is correct.

Explanation:

Let $I = \int_{-1}^1 x^2\sqrt{x^3+1}\,dx$

Let $x^3 + 1 = t \Rightarrow 3x^2dx = dt \Rightarrow x^2dx = \frac{dt}{3}$

Also, when $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$$= \frac{1}{3}\left[\frac{t^{3/2}}{3/2}\right]_0^2$$

$$= \frac{2}{9}\left[2^{3/2} - 0\right]$$

$$= \frac{4\sqrt{2}}{9}$$

Q. 3. If $f(a + b - x) = f(x)$**, then** $\int_a^b xf(x)\,dx$ **is equal to**

(A) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$ **(B)** $\frac{a+b}{2}\int_a^b f(b+x)\,dx$

(C) $\frac{b-a}{2}\int_a^b f(x)\,dx$ **(D)** $\frac{a+b}{2}\int_a^b f(x)\,dx$

Ans. Option (D) is correct.

Explanation:

Let $I = \int_a^b xf(x)dx$ ...(i)

$$I = \int_a^b (a+b-x)f(a+b-x)dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$\Rightarrow$ $I = \int_a^b (a+b-x)f(x)dx$

$\Rightarrow$ $I = (a+b)\int_a^b f(x)dx - I$ [Using Eq. (i)]

$\Rightarrow$ $I + I = (a+b)\int_a^b f(x)dx$

$\Rightarrow$ $2I = (a+b)\int_a^b f(x)dx$

$\Rightarrow$ $I = \left(\frac{a+b}{2}\right)\int_a^b f(x)dx$

Q. 4. Find the value of $\int_1^4 |x-5|\,dx$**.**

(A) 2 **(B)** $\frac{15}{2}$

(C) $\frac{2}{15}$ **(D)** 15

Ans. Option (B) is correct.

Explanation:

$$\int_1^4 |x-5|\,dx = \int_1^4 -(x-5)dx$$

$$= -\frac{1}{2}[(x-5)^2]_1^4$$

$$\int_1^4 |x-5|\,dx = -\frac{1}{2}[1-16] = \frac{15}{2}$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Evaluate: $\int_1^2 \frac{x^3-1}{x^2}dx$ R&U [O.E.B.]

Sol. Let $I = \int_1^2\left(\frac{x^3}{x^2}-\frac{1}{x^2}\right)dx$

$$= \int_1^2\left(x-\frac{1}{x^2}\right)dx$$

$$= \left[\frac{x^2}{2}+\frac{1}{x}\right]_1^2$$

$$I = \left(2+\frac{1}{2}\right)-\left(\frac{1}{2}+1\right)$$

$$= \frac{5}{2}-\frac{3}{2}$$

or $I = 1$ **1**

Q. 2. Evaluate: $\int_2^4 \frac{x}{x^2+1}dx$ A I [O.E.B.]

Q. 3. Evaluate: $\int_{-2}^2 (x^3+1)dx$ R [O.E.B.]

Sol. $\int_{-2}^2 (x^3+1)dx = \int_{-2}^2 (x^3)dx + \int_{-2}^2 1\,dx = I_1 + I_2$

$= 0 + [x]_{-2}^2$ (As I_1 is odd function)

$= 2 + 2$

$= 4$ **1**

Q. 4. Evaluate: $\int_0^1 e^{x^2}x\,dx$ A [O.E.B.]

Sol. Let, $I = \int_0^1 e^{x^2}x\,dx$

Put, $x^2 = t$

or $x\,dx = \frac{1}{2}dt$

Also when $x = 0$ or $t = 0$ and when $x = 1 \Rightarrow t = 1$

$\therefore \quad I = \frac{1}{2}\int_0^1 e^t dt$

or $I = \frac{1}{2}\left[e^t\right]_0^1$

$= \frac{1}{2}(e-1)$ **1**

Q. 5. Evaluate: $\int_2^3 3^x dx$ R&U [O.E.B.]

Short Answer Type Questions-I (2 marks each)

Q. 1. Evaluate: $\int_{-1}^2 \frac{|x|}{x}dx$ R [O.E.B.]

Sol. Let $I = \int_{-1}^2 \frac{|x|}{x}dx$

Since, $\frac{|x|}{x} = \begin{cases}\frac{-x}{x}, & x<0\\ \frac{x}{x}, & x>0\end{cases}$

$= \begin{cases}-1, & x<0\\ 1, & x>0\end{cases}$

$\therefore \quad I = \int_{-1}^0 (-1)dx + \int_0^2 (1)dx$ **1**

$= [-x]_{-1}^0 + [x]_0^2$

$= -[0-(-1)] + (2-0)$

$= -1 + 2$

$= 1$ **1**

Q. 2. If [x] denotes the greatest integer function, then find $\int_0^{3/2}[x^2]dx$. A I

Sol. $\int_0^{3/2}[x^2]dx$

For $0 \le x < 1, 0 \le x^2 < 1$, hence $[x^2] = 0$

For $1 \le x < \sqrt{2}, 1 \le x^2 < 2$, hence $[x^2] = 1$

For $\sqrt{2} \le x < \frac{3}{2}, 2 \le x^2 < \frac{9}{4}$, hence $[x^2] = 2$

$$\int_0^{3/2}[x^2]dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{3/2} 2\,dx$$ **1**

$= 0[x]_0^1 + 1[x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{3/2}$

$= 0 + \sqrt{2} - 1 + 3 - 2\sqrt{2}$

$= 2 - \sqrt{2}$ **1**

These questions are for practice and their solutions are available at the end of the chapter

Q. 3. Evaluate $\int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right] e^{2x}.dx$ U

Sol. Put $2x = t$,

$\therefore \quad dx = \frac{1}{2}dt$ ½

$\therefore \quad I = \int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right] e^{2x} dx$

$= \int_2^4 \left[\frac{1}{t} - \frac{1}{t^2}\right] e^t dt$ ½

$= \left[\frac{1}{t} e^t\right]_2^4 = \frac{e^4}{4} - \frac{e^2}{2}$ ½ + ½

Commonly Made Error

- Some students make the substitution, but not change the limit.

Answering Tip

- In definite integrals, change the limit on making substitution.

Q. 4. Prove that: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Short Answer Type Questions-II (3 marks each)

Q. 1. Evaluate: $\int_0^1 \frac{xe^x}{(1+x)^2} dx$ A I R&U **[O.E.B.]**

Sol. $I = \int_0^1 \frac{xe^x}{(1+x)^2} dx$

$= \int_0^1 \frac{(x+1)-1}{(1+x)^2} . e^x dx$

$= \int_0^1 \left[\frac{(x+1)e^x}{(x+1)^2} - \frac{e^x}{(x+1)^2}\right] dx$

$= \int_0^1 e^x \left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] dx$ **1**

$= \left[e^x \frac{1}{1+x}\right]_0^1$

$[\because \int e^x [f(x) + f'(x)]dx = e^x f(x) + c]$ **1**

$= \frac{e^1}{1+1} - \frac{e^0}{1+0} = \frac{e}{2} - 1$

$\therefore \quad I = \frac{e}{2} - 1$ **1**

Q. 2. Evaluate: $\int_1^3 |x^2 - 2x| dx.$ A I U **[O.E.B.]**

Q. 3. Evaluate: $\int_{-1}^2 |x^3 - x| dx$. A I U **[O.E.B.]**

Sol. $\because \int_{-1}^2 |x^3 - x| dx$

$= \int_{-1}^0 (x^3 - x)dx - \int_0^1 (x^3 - x)dx + \int_1^2 (x^3 - x)dx$ **1½**

$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^0 - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^2$

$= \left[(0-0) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] - \left[\left(\frac{1}{4} - \frac{1}{2}\right) - (0-0)\right]$

$+ \left[(4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$

$= \frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4} = \frac{11}{4}$ **1½**

Q. 4. Evaluate: $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$ R&U **[O.E.B.]**

Sol. Let $I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

$= \int_0^1 \log\left(\frac{1-x}{x}\right) dx$

Using property,

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we have

$I = \int_0^1 \log\left[\frac{1-(1-x)}{1-x}\right] dx$ **1**

$= \int_0^1 \log\left(\frac{x}{1-x}\right) dx$

$= \int_0^1 \log\left(\frac{1-x}{x}\right)^{-1} dx$

$= \int_0^1 (-1)\log\left(\frac{1-x}{x}\right) dx$

$I = -\int_0^1 \log\left(\frac{1-x}{x}\right) dx$

$I = -I$

$2I = 0$

$\therefore \quad I = 0$

$\therefore \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = 0$ **2**

These questions are for practice and their solutions are available at the end of the chapter

Commonly Made Error

- Use of the definite integral property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ was tried but unsuccessfully by some candidates. Some tried direct integration or integration by substitution and failed after a point.

Answering Tip

- Use properties of integrals to simplify the given integral into easier form. Effective use of properties of definite integrals need to understood.

Q. 5. Evaluate: $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx$ A [O.E.B.]

Sol. Let $I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx$ ...(i)

Using properties,

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}dx$$

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx \quad \text{...(ii) } \mathbf{1}$$

Adding eqs. (i) and (ii),

$$2I = \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx = \int_1^2 dx$$

$$= [x]_1^2 = 2-1 = 1$$

$\therefore \quad I = \frac{1}{2}$ **2**

Q. 6. Evaluate: $\int_{-3}^{3} |x+2|dx$ R&U [O.E.B.]

Sol. We have $|x+2| = \begin{cases} (x+2) & \text{if } x+2 \geq 0 \\ -(x+2) & \text{if } x+2 \leq 0 \end{cases}$

$$= \begin{cases} (x+2) & \text{if } x \geq -2 \\ -(x+2) & \text{if } x \leq -2 \end{cases}$$

$\therefore \int_{-3}^{3}|x+2|dx = \int_{-3}^{-2}-(x+2)dx + \int_{-2}^{3}(x+2)dx$ **1½**

$$= -\left[\frac{x^2}{2}+2x\right]_{-3}^{-2} + \left[\frac{x^2}{2}+2x\right]_{-2}^{3}$$

$$= -\left[2-4-\frac{9}{2}-2(-3)\right] + \left[\frac{9}{2}+6-2+4\right]$$

$$= -\left[4-\frac{9}{2}\right] + \left[\frac{9}{2}+8\right] = 4+2\times\frac{9}{2}$$

$$= 4+9$$

$$= 13 \quad \mathbf{1½}$$

Q. 7. Evaluate: $\int_a^b \frac{\log x}{x}dx$ A I R&U [O.E.B.]

Long Answer Type Questions (5 marks each)

Q. 1. Evaluate: $\int_{-1}^{1} \frac{x+|x|+1}{x^2+2|x|+1}.dx$ R&U [O.E.B.]

Q. 2. $\int_1^4 \{|x-1|+|x-2|+|x-4|\}dx$ R&U [O.E.B.]

Sol. $I = \int_1^4 \{|x-1|+|x-2|+|x-4|\}dx$

$$= \int_1^4 (x-1)dx - \int_1^2 (x-2)dx + \int_2^4 (x-2)dx$$

$$-\int_1^4 (x-4)dx \quad \mathbf{1½}$$

$$= \frac{(x-1)^2}{2}\Big]_1^4 - \frac{(x-2)^2}{2}\Big]_1^2 + \frac{(x-2)^2}{2}\Big]_2^4 - \frac{(x-4)^2}{2}\Big]_1^4 \quad \mathbf{2}$$

$$= \frac{9}{2}+\frac{1}{2}+2+\frac{9}{2} = 11\frac{1}{2} \text{ or } \frac{23}{2} \quad \mathbf{1½}$$

Q. 3. Evaluate: $\int_{-1}^{2} |x^3-3x^2+2x|dx$

Sol. The given definite integral

$$= \int_{-1}^{2} |x(x-1)(x-2)|dx$$

$$= \int_{-1}^{0} |x(x-1)(x-2)|dx + \int_0^1 |x(x-1)(x-2)|dx$$

$$+\int_1^2 |x(x-1)(x-2)|dx \quad \mathbf{1½}$$

$$= -\int_{-1}^{0} (x^3-3x^2+2x)dx + \int_0^1 (x^3-3x^2+2x)dx$$

$$-\int_1^2 (x^3-3x^2+2x)dx \quad ½$$

$$= -\left[\frac{x^4}{4}-x^3+x^2\right]_{-1}^{0} + \left[\frac{x^4}{4}-x^3+x^2\right]_0^1$$

$$-\left[\frac{x^4}{4}-x^3+x^2\right]_1^2$$

1

$$= \frac{9}{4}+\frac{1}{4}+\frac{1}{4} = \frac{11}{4} \quad \mathbf{2}$$

Topic-3 Application of Integrals

Concepts Covered • *Area under simple curves,* • *Consumer surplus,* • *Producer surplus*

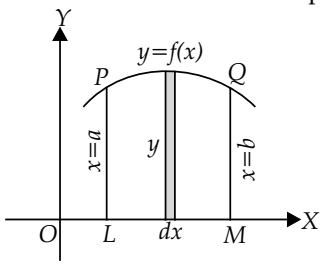
Revision Notes

► **Area Under Simple Curves:**

(i) Let us find the area bounded by the **curve** $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$. Consider the area under the curve as composed by large number of thin vertical strips.

Scan to know more about this topic

Area under simple curve



Key Words

Curve: A **curve** is a continuous and smooth flowing line without any sharp turns. One way to recognize a curve is that it bends and changes its direction at least once.

- A **open curve** does not enclose any area within itself and it has two endpoints. Some of the open curves are given in the figure below.
- A **closed curve** has no end points and encloses an area (or a region). It is formed by joining the end points of an open curve together. e.g.: Circles, ellipses are formed from closed curves.
- A **simple curve** changes direction but does not cross itself while changing direction. A simple curve can be open and closed both.
- A **non-simple curve** crosses its own path

Arbitrary: In mathematics, "arbitrary" just means "for all".

For example: "For all $a, b, a + b = b + a$".

Another way to say this would be "$a + b = b + a$ for arbitrary a, b."

Ordinate: The Cartesian coordinate obtained by measuring parallel to the y-axis.

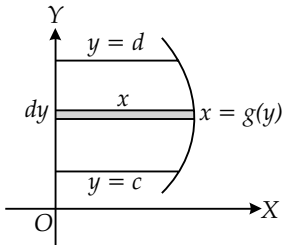
Let there be an **arbitrary** strip of height y and width dx.

Area of elementary strip $dA = y\,dx$, where $y = f(x)$. Total area A of the region between X-axis ordinates $x = a$, $x = b$ and the curve $y = f(x)$ = sum of areas of elementary thin strips across the region $PQML$.

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

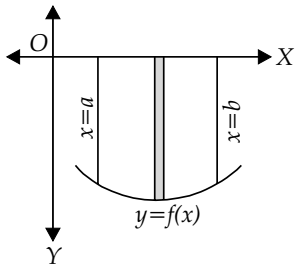
(ii) The area A of the region bounded by the curve $x = g(y)$, Y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d x\,dy$$

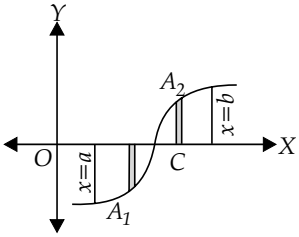


(iii) If the curve under consideration lies below X-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the **ordinates** $x = a$, $x = b$ and X-axis is negative. But, if the numerical value of the area is to be taken into consideration, then

$$\text{Area} = \left|\int_a^b f(x)dx\right|$$



(iv) It may also happen that some portion of the curve is above X-axis and some portion is below X-axis as shown in the figure. Let A_1 be the area below X-axis and A_2 be the area above the X-axis. Therefore, area bounded by the curve $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$ is given by $A = |A_1| + |A_2|$



Example 5 : Calculate the area under the curve of a function, $f(x) = 7 - x^2$, the limit is given as $x = -1$ to 2.

Sol.: Given function is, $f(x) = 7 - x^2$ and limit is $x = -1$ to 2

$$\text{Area} = \int_{-1}^{2}(7 - x^2)dx$$

$$= \left(7x - \frac{1}{3}x^3\right)\Big|_{-1}^{2}$$

$$= \left[7.2 - \frac{1}{3}(8)\right] - \left[7(-1) - \frac{1}{3}(-1)\right]$$

$$= \left[\frac{(42-8)}{3}\right] - \left[\frac{(1-21)}{3}\right]$$

$$= \frac{(34+20)}{3}$$

$$= \frac{54}{3}$$

$$= 18 \text{ sq. units}$$

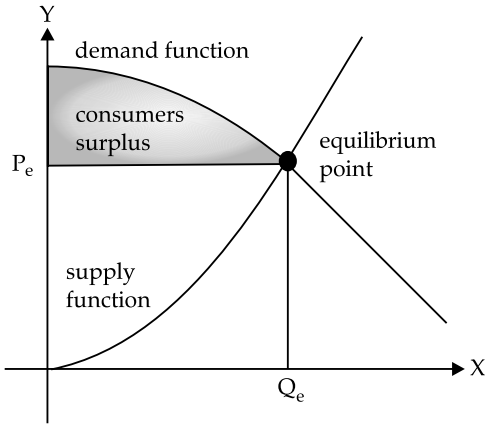
Consumer Surplus

Scan to know more about this topic

Consumer surplus

- It can be defined as the surplus that is retained with the consumer after he purchases a product for which he paid lesser than what he was able to.
- This is the difference between what the consumer pays and what he would have been willing to pay.

For example: If we would be willing to pay Rs. 50 for a ticket to see a drama, but we can buy a ticket for ₹ 40. In this case, the consumer surplus is ₹ 10.



In the graph, we can see that consumer's surplus is the area of the region bounded above by the demand function and below by the line that represents the unit market price.

The consumers' surplus is given by

$$CS = \int_0^{Q_e} D(x)dx - Q_e.P_e$$

Where, $D(x)$ = Demand Function
Q_e = Quantity Sold
P_e = Unit Market Price

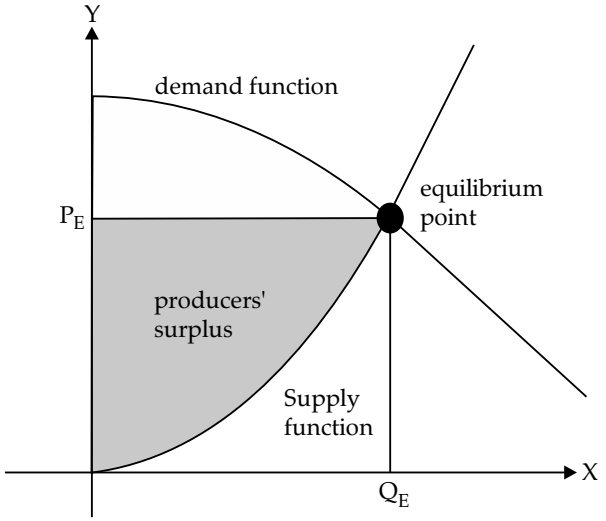
Producer Surplus

Scan to know more about this topic

Producer surplus

- It can be defined as the surplus that is retained with the producer after he sells a product for which he accepted more than what he was expected to receive.
- This is the difference between the price a firm receives and the price it would be willing to sell it at.

For example: If a firm would sell a good at ₹ 4, but the market price is ₹ 7, the producer surplus is ₹ 3.



In the graph, we can see that producer's surplus is the area of the region bounded above by the line that represents the price and below by the supply curve.

The producer's surplus is given by

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)dx$$

Where, $S(x)$ = Supply Function
Q_e = Quantity Supplied
P_e = Unit Market Price

Example 6 : Find the producer's surplus defined by the supply curve $S(x) = 4x + 8$ for the supply of 5 units.

Sol.: Here,

$$S(x) = 4x + 8 \text{ and } Q_e = 5$$
$$P_e = S(5) = 4(5) + 8 = 28$$

Since,

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)dx$$

Therefore, $$PS = (5 \times 28) - \int_0^5 (4x+8)dx$$

$$= 140 - \left[4\left(\frac{x^2}{2}\right) + 8x\right]_0^5$$

$$= 140 - (50 + 40)$$

$$= 50 \text{ units}$$

Hence the producer's surplus = 50 units.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. The area of the region bounded by the curve $x^2 = 4y$ and the straight-line $x = 4y - 2$ is:

(A) $\frac{3}{8}$ sq. units **(B)** $\frac{5}{8}$ sq. units

(C) $\frac{7}{8}$ sq. units **(D)** $\frac{9}{8}$ sq. units

Ans. Option (D) is correct.

Explanation: $x^2 = x + 2$

$$x^2 - x - 2 = 0$$

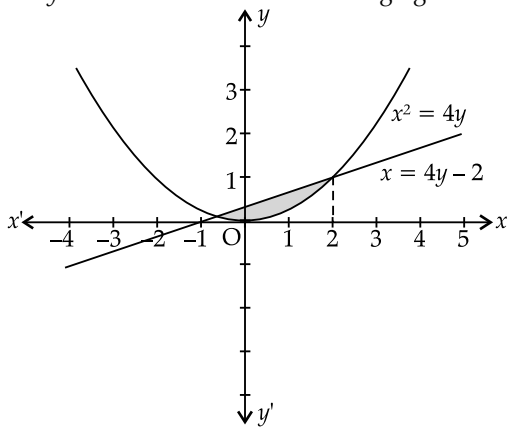
$$(x-2)(x+1) = 0$$

$$x = -1,\ 2$$

For $x = -1,\ y = \frac{1}{4}$ and for $x = 2,\ y = 1$

Points of intersection are $(-1, \frac{1}{4})$ and (2, 1).

Graphs of parabola $x^2 = 4y$ and the straight line $x = 4y - 2$ are shown in the following figure:



$$A = \int_{-1}^{2}\left[\frac{x+2}{4} - \frac{x^2}{4}\right]dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\frac{2^2}{2} + 2\times 2 - \frac{2^3}{3} - \frac{(-1)^2}{2} - 2\times(-1) + \frac{(-1)^3}{3}\right]$$

$$= \frac{1}{4}\left[6 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[8 - 3 - \frac{1}{2}\right]$$

$$= \frac{9}{8} \text{ sq. units}$$

Q. 2. The area of the region bounded by parabola $y^2 = x$ and the straight line $2y = x$ is:

(A) $\frac{4}{3}$ sq. units **(B)** 1 sq. units

(C) $\frac{2}{3}$ sq. units **(D)** $\frac{1}{3}$ sq. units

Ans. Option (A) is correct.

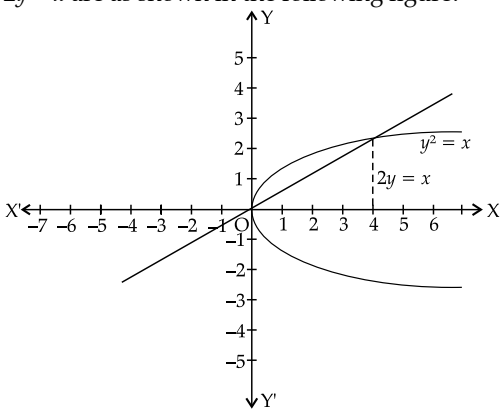
Explanation: When $y^2 = x$ and $2y = x$

Solving we get $y^2 = 2y$

$\Rightarrow y = 0,\ 2$ and when $y = 2,\ x = 4$

So, points of intersection are (0, 0) and (4, 2).

Graphs of parabola $y^2 = x$ and the straight line $2y = x$ are as shown in the following figure:



From the figure, area of the shaded region,

$$A = \int_0^4\left[\sqrt{x} - \frac{x}{2}\right]dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}.\frac{x^2}{2}\right]_0^4$$

$$= \frac{2}{3}.(4)^{3/2} - \frac{16}{4} - 0$$

$$= \frac{16}{3} - 4$$

$$= \frac{4}{3} \text{ sq. units}$$

Q. 3. If the demand function for a commodity is $p = 35 - 2x - x^2$. Then the consumer's surplus at equilibrium price $p_0 = 20$ is given by:

(A) 20 **(B)** 27

(C) 25 **(D)** 35

Ans. Option (B) is correct.

Explanation: Given, the demand function is

$$p = 35 - 2x - x^2 \quad \text{...(i)}$$

and the equilibrium price, $p_0 = 20$.

Note: At equilibrium demand function is

$$p_0 = 35 - 2x_0 - x_0^2$$

Substituting this value of p_0 in (*i*), we get

$$20 = 35 - 2x_0 - x_0^2$$

$$\Rightarrow \quad x_0^2 + 2x_0 - 15 = 0$$

$\Rightarrow \quad (x_0 + 5)(x_0 - 3) = 0$
$\Rightarrow \quad x_0 = -5$, or $x_0 = 3$
$\because \quad x_0 \neq -5$, so $x_0 = 3$
So, consumer's Surplus

$$= \int_0^3 (35 - 2x - x^2)dx - 3 \times 20$$

$$= \left[35x - x^2 - \frac{x^3}{3}\right]_0^3 - 60$$

$$= (105 - 9 - 9) - 60$$
$$= 27$$

Q. 14. If the supply function for a commodity is $p = 4 + x$, and 12 units of goods are sold, then the producer's surplus is given by:
(A) 70 **(B)** 72
(C) 12 **(D)** 48
Ans. Option (B) is correct.

Explanation: Given, the supply function is

$$p = 4 + x \quad \text{...(i)}$$

and the market demand

$$x_0 = 12.$$

At equilibrium, $p_0 = 4 + x_0$
Substituting this value of x_0 in (i), we get

$$p_0 = 4 + 12$$
$\Rightarrow \quad p_0 = 16$

So, Producer's Surplus

$$= 12 \times 16 - \int_0^{12} (4 + x)dx$$

$$= 192 - \left[4x + \frac{x^2}{2}\right]_0^{12}$$

$$= 192 - (48 + 72) + 0$$
$$= 192 - 120$$
$$= 72$$

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. What do you understand by Consumer Surplus ? R [O.E.B.]

Sol. Consumer surplus is the extra amount of money that consumers are willing to pay for a goods above the equilibrium price, it is the satisfaction gained from a product after accounting for its price.

For example, Soniya decided to buy a laptop with a 2.4 GHz CPU and a 15.4" screen and is willing to spend up to ₹ 35,000. As she browses through various electronics stores, she finds one for ₹ 30,000 that meets all her exact criteria (2.4 GHz CPU and a 15.4" screen), saving her ₹ 5,000 compared to what she was willing to spend. The ₹ 5,000 is her consumer surplus, which she can now save or spend on other goods or services. **1**

Q. 2. Sarah is selling her used scooter. The minimum amount she needs to be paid for the scooter is ₹ 5,000. She advertises the scooter on olx.com for ₹ 8,000 and eventually sells the scooter for ₹ 6,000. Find her producer surplus. R&U [O.E.B.]

Q. 3. Mahesh is selling his car. The minimum amount he needs to be paid for the car is ₹ 15,500. He find a buyer for who is willing to pay ₹ 22,400, but this buyer insists that Mahesh pays for delivery of the car. The cost of delivery is ₹ 700. Find his producer surplus from selling his car. R&U [O.E.B.]

Sol. Mahesh's(Producer's) Surplus

$$= ₹\,22,400 - ₹\,15,500 - ₹\,700$$
$$= ₹\,6,200 \quad \textbf{1}$$

Q. 4. Define Producer Surplus. R

Sol. Producer surplus is the extra amount of money producers are paid to supply a product above what they are willing to supply the product for.

A simple example of producer surplus would be when we sell an item for which we intend to charge ₹ 200, but the consumer has paid ₹ 250. In this case, we have a producer surplus of ₹ 50. **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$. A [O.E.B.]

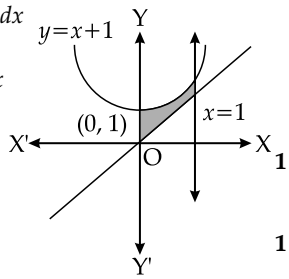
Sol. The upper boundary curve is $y = x^2 + 1$ and the lower boundary curve is $y = x$.
Using the formula for area bounded by curves,

$$A = \int_0^1 \left[(x^2 + 1) - x\right] dx$$

$$= \int_0^1 (x^2 - x + 1)\, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x \Big]_0^1 \quad \textbf{1}$$

$$= \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6} \quad \textbf{1}$$



Q. 2. What is the area bounded by the curve $y = \log x$, $X-$axis and the ordinates $x = 1$, $x = 2$? A I R&U [O.E.B.]

Sol. Given curve $y = \log x$ and $x = 1$, $x = 2$

Hence, required area $= \int_1^2 \log x\, dx$

$$= (x \log x - x)|_1^2 \quad \textbf{1}$$
$$= (2 \log 2 - 1)$$
$$= \log 4 - 1 \text{ sq. unit} \quad \textbf{1}$$

These questions are for practice and their solutions are available at the end of the chapter

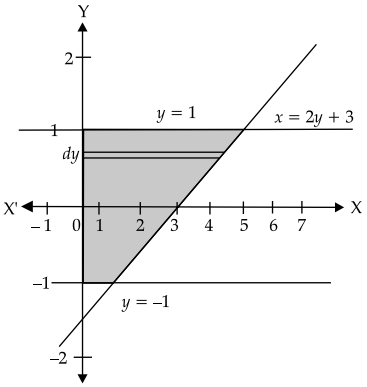
Q. 3. Find the area of the region bounded by the graph of $f(x) = x^5$ and the X-axis between $x = -1$ and $x = 1$.

A [O.E.B.]

Q. 4. Find the area of region bounded by the curve $x = 2y + 3$, Y-axis and the lines $y = 1$ and $y = -1$.

A I [O.E.B.]

Sol.



From the figure, area of the shaded region,

$$A = \int_{-1}^{1} (2y+3)\,dy$$

$$= \left[y^2 + 3y\right]_{-1}^{1}$$

$$= [1+3-1+3]$$

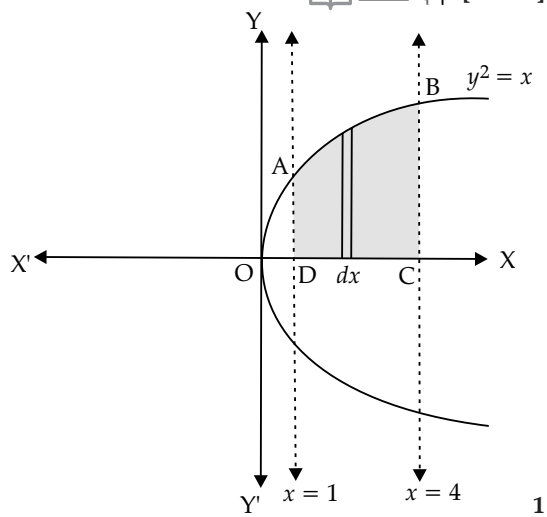
$$= 6 \text{ sq. units} \qquad \textbf{2}$$

Short Answer Type Questions-II (3 marks each)

Q. 1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the X-axis.

A I R&U [O.E.B.]

Sol.



1

The area of the region bounded by the curve, $y^2 = x$, the lines $x = 1$ and $x = 4$, and the x-axis is the area ABCD.

$$\text{Area of } ABCD = \int_1^4 y\,dx = \int_1^4 \sqrt{x}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$

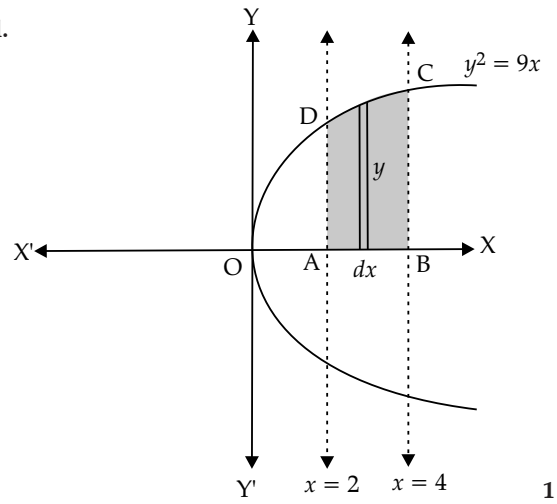
$$= \frac{2}{3}[8-1]$$

$$= \frac{14}{3} \text{ units}^2 \qquad \textbf{2}$$

Q. 2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$, and the X-axis in the first quadrant. A

[O.E.B.]

Sol.



1

The area of the region bounded by the curve, $y^2 = 9x$, $x = 2$ and $x = 4$, and the X-axis is the area ABCD.

$$\text{Area of } ABCD = \int_2^4 y\,dx = \int_2^4 3\sqrt{x}\,dx$$

$$= 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$= 2\left[x^{\frac{3}{2}}\right]_2^4$$

$$= 2\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$

$$= 2\left[8 - 2\sqrt{2}\right]$$

$$= \left(16 - 4\sqrt{2}\right) \text{ units}^2 \qquad \textbf{2}$$

Q. 3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $x = 4$, and the Y-axis in the first quadrant. A

[O.E.B.]

These questions are for practice and their solutions are available at the end of the chapter

Q. 4. If the demand curve is given by $D(x) = 50 - 0.06x^2$. Find the surplus or profit of the consumers if the level of sale amounts to twenty units.

[O.E.B.]

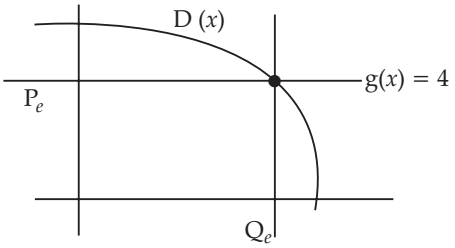
Long Answer Type Questions (5 marks each)

Q. 1. Suppose the demand for the certain product is given by $p = -0.01x^2 - 0.1x + 6$, where p is the unit price given in rupees and x is the quantity demanded per month given in the units of 1000. The unit market price for the product is ₹ 4 per unit.

(a) Find the quantity demanded at the given price.

(b) Find the consumer's surplus if the market price for the product is ₹ 4 per unit. [O.E.B.]

Sol. (a) First we find the point of intersection of demand curve and market price for the product *i.e.*, (D, g).



Since, p is the unit price so we put $p = 4$ in demand curve, $p = -0.01x^2 - 0.1x + 6$,

i.e., $4 = -0.01x^2 - 0.1x + 6$

$\Rightarrow 0.01x^2 + 0.1x - 2 = 0$

$\Rightarrow x^2 + 10x - 200 = 0$

$\Rightarrow (x + 20)(x - 10) = 0$

$\Rightarrow x = 10$ (Neglecting $x = -20$)

So, $(D, g) = (10, 4)$ **2½**

Hence, the quantity demanded at the given price $(Q_e, P_e) = (1000, 4)$

(b) Consumer's surplus is given by

$$CS = \int_0^{Q_e} D(x)\,dx - Q_e.P_e$$

Here, $D(x) = -0.01x^2 - 0.1x + 6$

$Q_e = 10$

$P_e = p = 4$

Thus,

$$CS = \int_0^{10} (-0.01x^2 - 0.1x + 6)\,dx - 10 \times 4$$

$$= -0.01\left[\frac{x^3}{3}\right]_0^{10} - 0.1\left[\frac{x^2}{2}\right]_0^{10} + 6[x]_0^{10} - 40$$

$$= \frac{-0.01}{3}(10^3 - 0) - \frac{0.1}{2}(10^2 - 0) + 6(10 - 0) - 40$$

$$= \frac{-10}{3} - \frac{10}{2} + 60 - 40$$

$$= -3.33333 - 5 + 20 = 11.6667$$

For 1000 units, $11.6667 \times 1000 = 11666.7 = 11667$ **2½**

Q. 2. The demand and supply function of a commodity are $P_d = 18 - 2x - x^2$ and $P_s = 2x - 3$. Find the consumer's surplus and producer's surplus at equilibrium price. [O.E.B.]

Q. 3. The demand and supply function of an article are $D(q) = 1000 - 0.4q^2$ and $S(q) = 42q$. Find the consumer's surplus and producer's surplus at equilibrium price. R&U [O.E.B.]

Sol. Given, $D(q) = 1000 - 0.4q^2$ and $S(q) = 42q$

At equilibrium $D(q) = S(q)$

$1000 - 0.4q^2 = 42q$

or, $0.4q^2 + 42q - 1000 = 0$

or, $4q^2 + 420q - 10000 = 0$

or, $q^2 + 105q - 2500 = 0$

or, $(q + 125)(q - 20) = 0$

$q = -125$ or 20 **1**

The value of q cannot be negative, $q = 20$

When $q = Q_e = 20$

$D(20) = 1000 - 0.4(20)^2$

$= 1000 - 0.4 \times 400$

$= 1000 - 160 = 840 = P_e$ **1**

Consumer's surplus is given by

$$CS = \int_0^{Q_e} D(q)dq - Q_e.P_e$$

$$= \int_0^{20} (1000 - 0.4q^2)dq - 20 \times 840$$

$$= 1000[x]_0^{20} - 0.4\left[\frac{q^3}{3}\right]_0^{20} - 16800$$

$$= 1000(20 - 0) - 0.4\left(\frac{20^3}{3} - 0\right) - 16800$$

$$= 1000 \times 20 - 0.4 \times \frac{8000}{3} - 16800$$

$= 20000 - 1066.666 - 16800$

$= 20000 - 17866.666$

$= 2133.334$

$= 2133$ units **2**

Producer's surplus is given by

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)\,dx$$

$$= 20 \times 840 - \int_0^{20} 42q\,dq$$

$$= 16800 - 42\left[\frac{q^2}{2}\right]_0^{20}$$

$$= 16800 - 42\left(\frac{20^2}{2} - 0\right)$$

$= 16800 - 21 \times 400$

$= 16800 - 8400$

$= 8400$ units **1**

Hence, at equilibrium price, (i) the consumer's surplus is 2133 units and (ii) the producer's surplus is 8400 units.

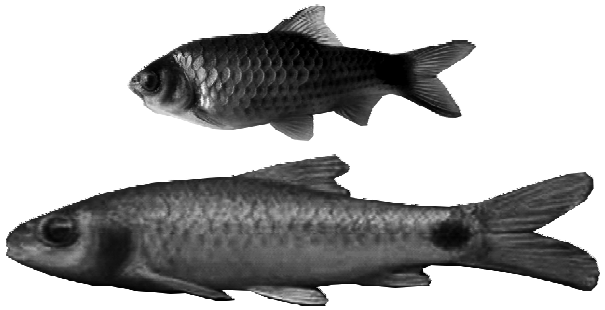
COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

The second new species named Puntius euspilurus is an edible freshwater fish found in the Mananthavady river in Wayanad. The epithet euspilurus is a Greek word referring to the distinct black spot on the caudal fin. The slender bodied fish prefers fast flowing, shallow and clear waters and occurs only in unpolluted areas. It appears in great numbers in paddy fields during the onset of the Southwest monsoon.



Suppose that the supply schedule of this Fish is given in the table below which follows a linear relationship between price and quantity supplied.

Price P per kg (in ₹)	Quantity (X) of Fish supplied (in kg)
25	800
20	700
15	600
10	500
5	400

Suppose that this Fish can be sold only in the Kerala. The Kerala demand schedule for this Fish is as follows and there is a linear relationship between price and quantity demanded.

Price P per kg (in ₹)	Quantity (X) of Fish Demanded (in kg)
25	200
20	400
15	600
10	800
5	1000

Q. 1. Which of the following represents the Price (p) – supply (x) relationship?

(A) $p = 65 - \frac{x}{20}$ **(B)** $p = 65 + \frac{x}{20}$

(C) $p = -15 + \frac{x}{20}$ **(D)** $p = 15 - \frac{x}{20}$

Ans. Option (C) is correct.

Explanation: From the given table of price and supply we have the following price and supply coordinates i.e., (25, 800), (20, 700), (15, 600), (10, 500), (5, 400)

All these coordinates, satisfy the equation

$$p = -15 + \frac{x}{20}$$

Q. 2. The equation of demand curve can be given by:

(A) $p = 30 - \frac{x}{40}$ **(B)** $p = 30 + \frac{x}{40}$

(C) $p = 20 - \frac{x}{40}$ **(D)** $p = 20 + \frac{x}{40}$

Ans. Option (A) is correct.

Explanation: From the given table of price and demand, we have the following price and demand coordinates, i.e., (25, 200), (20, 400), (15, 600), (10, 800), (5, 1000)

All of these coordinates, lie on the equation

$$p = 30 - \frac{x}{40}$$

Q. 3. The value of x at equilibrium is:

(A) 1400/3 **(B)** 600

(C) 15 **(D)** 200/3

Ans. Option (B) is correct.

Explanation: At equilibrium Demand = Supply

Therefore, $30 - \frac{x}{40} = -15 + \frac{x}{20}$

$\Rightarrow \quad 30 + 15 = \frac{x}{20} + \frac{x}{40}$

$\Rightarrow \quad 45 = \frac{2x + x}{40}$

$\Rightarrow \quad 3x = 45 \times 40$

$\Rightarrow \quad x = 600$

Q. 4. The equilibrium price is:

(A) 400 **(B)** 20

(C) 600 **(D)** 15

Ans. Option (D) is correct.

Explanation: At $x = 600$

$$p = 30 - \frac{600}{40}$$

$$= 30 - 15$$

$$= 15$$

Q. 5. The consumers' surplus at equilibrium price is:

(A) 18009 **(B)** 13500

(C) 9000 **(D)** 4500

Ans. Option (D) is correct.

Explanation: Consumer's Surplus

$$CS = \int_0^{q_e} D(x)dx - q_e p_e$$

Here, $q_e = 600, p_e = 15,$

$$D(x) = 30 - \frac{x}{40}$$

$$\therefore \quad CS = \int_0^{600}\left(30 - \frac{x}{40}\right)dx - 600\times 15$$

$$= \left[30x - \frac{x^2}{2\times 40}\right]_0^{600} - 9000$$

$$= \left[30\times 600 - \frac{(600)^2}{80} - 0\right] - 9000$$

$$= 18000 - \frac{360000}{80} - 9000$$

$$= 18000 - 4500 - 9000$$

$$= 4500$$

II. Read the following text and answer the following questions on the basis of the same:

Rishi and Nitish were discussing integration, in there discussion following points were discussed.

In question like $\int e^x[f(x)+f'(x)]dx$ we should result $e^x f(x) + C$

If we have irrational terms in integration question, then we should try doing rationalising these terms.

Q. 1. $\int \frac{1}{\sqrt{3x+4}-\sqrt{3x+1}}dx$ **is equal to:**

(A) $\frac{2}{27}\{(x+4)^{3/2} + (3x+1)^{3/2}\} + C$

(B) $\frac{2}{27}\{(3x-4)^{1/2} + (3x+1)^{3/2}\} + C$

(C) $\frac{2}{7}\{(3x+4)^{3/2} + (3x+1)^{3/2}\} + C$

(D) $\frac{2}{27}\{(3x+4)^{3/2} + (3x+1)^{3/2}\} + C$

Ans. Option (D) is correct.

Explanation: Rationalising denominator we get:

$$= \int \frac{\sqrt{3x+4}+\sqrt{3x+1}}{3}dx$$

$$= \frac{1}{3}\int[\sqrt{3x+4}+\sqrt{3x+1}]dx$$

$$= \frac{1}{3}\left[\frac{(3x+4)^{3/2}}{3\times 3/2} + \frac{(3x+1)^{3/2}}{3\times 3/2} + C\right]$$

$$= \frac{2}{27}\{(3x+4)^{3/2} + (3x+1)^{3/2}\} + C$$

Q. 2. $\int e^x\left[\log x + \frac{1}{x}\right]dx$ **equal to:**

(A) $e^x \log x + C$ **(B)** $\frac{e^x}{x}\log x + C$

(C) $\frac{\log e}{e^x} + C$ **(D)** $\frac{e^x}{\log x} + C$

Ans. Option (A) is correct.

Explanation: Let $I = \int e^x\left[\log x + \frac{1}{x}\right]dx$

Here $f(x) = \log x$ and $f'(x) = \frac{1}{x}$

$\therefore$ By using the rule

$$\int e^x[f(x)+f'(x)]dx = e^x(fx) + C$$

We get, $I = e^x \log x + C$

Q. 3. $\int \frac{1}{\sqrt{1-2x}+\sqrt{3-2x}}dx$ **equal to:**

(A) $\frac{1}{6}(1-2x)^{3/2} + \frac{1}{5}(3-2x)^{3/2} + C$

(B) $\frac{1}{5}(1-2x)^{3/2} + \frac{1}{6}(3-2x)^{3/2} + C$

(C) $\frac{1}{6}(1-2x)^{3/2} - \frac{1}{6}(3-2x)^{3/2} + C$

(D) $\frac{1}{6}(1-x)^{3/2} + \frac{1}{6}(3-2x)^{3/2} + C$

Ans. Option (C) is correct.

Explanation: Rationalising the denominator, we get:

$$\int \frac{\sqrt{1-2x}-\sqrt{3-2x}}{-2}dx$$

$$= -\frac{1}{2}\int\sqrt{1-2x}dx + \frac{1}{2}\int\sqrt{3-2x}dx$$

$$= \frac{1}{6}(1-2x)^{3/2} - \frac{1}{6}(3-2x)^{3/2} + C$$

Q. 4. $\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$ **is equal to:**

(A) $\frac{\log x}{x}+C$ **(B)** $x\log x+C$

(C) $\frac{x}{\log x}+C$ **(D)** $e^x\log x+C$

Ans. Option (C) is correct.

Explanation: Let I $= \int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$

Let $\log x = t$

$\Rightarrow x = e^t$

$\Rightarrow dx = e^t\,dt$

Thus, $I = \int e^t\left[\frac{1}{t}-\frac{1}{t^2}\right]dt$

Here $f(t) = \frac{1}{t}$ and $f'(t) = -\frac{1}{t^2}$

$I = e^t f(x)$

$= e^{\log x}\times\frac{1}{\log x}+C$

$= \frac{x}{\log x}+C$

Q. 5. $\int e^x\left(\frac{1}{x^2}-\frac{2}{x^3}\right)dx$ **is equal to:**

(A) $\frac{e^x}{2x^2}+C$ **(B)** $\frac{e^x}{x^2}+C$

(C) $\frac{3e^x}{5x^2}+C$ **(D)** $\frac{e^x}{7x^2}+C$

Ans. Option (B) is correct.

Explanation:

Derivative of $\frac{1}{x^2} = \frac{-2}{x^3}$

Therefore, $\int e^x\left(\frac{1}{x^2}-\frac{2}{x^3}\right)dx = e^x\left(\frac{1}{x^2}\right)+C$

III. Read the following text and answer the following questions on the basis of the same: AI

The demand function for a popular make of 12- speed bicycle is given by p = D(x) = $-0.001x^2 + 250$, where p is the unit price in Rupees and x is the quantity demanded in units of a thousands. The supply function for the same product is given by p = S(x) = $0.0006x^2 + 0.02x + 100$, where p is the unit price in Rupees and x is the quantity supplied in units of a thousands.

Q. 1. If at equilibrium, curves of demand function and supply function intersect at (x, y), then what does the point of intersection represents ?

(A) x = equilibrium quantity, y = equilibrium price

(B) x = equilibrium price, y = equilibrium quantity

(C) x = equilibrium demand, y = equilibrium price

(D) x = equilibrium supply, y = equilibrium demand

Ans. Option (A) is correct.

Explanation: At equilibrium, the intersection point (x, y) of demand curve and supply curve represents equilibrium quantity (Q_e) and equilibrium price (P_e), respectively.

Q. 2. Find point of intersection.

(A) (300, 160) **(B)** (160, 300)

(C) (300, 300) **(D)** (160, 160)

Ans. Option (A) is correct.

Explanation: At equilibrium,

$D(x) = S(x)$

$\Rightarrow -0.001x^2 + 250 = 0.0006x^2 + 0.02x + 100$

$\Rightarrow (0.0006 + 0.001)x^2 + 0.02x + (100 - 250) = 0$

$\Rightarrow 2x^2 + 25x - 187500 = 0$

$\Rightarrow 2x^2 + 625x - 600x - 187500 = 0$

$\Rightarrow (2x + 625)(x - 300) = 0$

$\Rightarrow x = \frac{-625}{2}$ or $x = 300$

The value of x cannot be negative, So

$x = 300$

Now, substituting $x = 300$ in $D(x)$ or $S(x)$, we get $p = 160$

Thus, point of intersection is (300, 160).

Q. 3. Formula for Consumer's Surplus is:

(A) $CS = \int_0^{Q_e} D(x)dx + Q_e.P_e$

(B) $CS = \int_0^{Q_e} D(x)dx - Q_e.P_e$

(C) $CS = -\int_0^{Q_e} D(x)dx - Q_e.P_e$

(D) $CS = -\int_0^{Q_e} D(x)dx + Q_e.P_e$

Ans. Option (B) is correct.

Explanation: Formula for Consumer's Surplus is

$$CS = \int_0^{Q_e} D(x)dx - Q_e.P_e$$

Where, $D(x)$ = Demand Function
P_e = Equilibrium Price
Q_e = Equilibrium Quantity

Q. 4 Formula for Producer's Surplus is:

(A) $PS = Q_e.P_e - \int_0^{Q_e} D(x)dx$

(B) $PS = Q_e.P_e - \int_0^{Q_e} S(x)dx$

(C) $PS = Q_e.P_e + \int_0^{Q_e} S(x)dx$

(D) $PS = -Q_e.P_e + \int_0^{Q_e} D(x)dx$

Ans. Option (B) is correct.

Explanation: Formula for Producer's Surplus is

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)dx$$

Where, $S(x)$ = Supply Function
P_e = Equilibrium Price
Q_e = Equilibrium Quantity

Q. 5. Producer's Profit (in Rs.) is:

(A) 21,700,000 **(B)** 11,700

(C) 1,17,000 **(D)** 11,700,000

Ans. Option (D) is correct.

Explanation: Producer's Profit is

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)dx$$

Here, $S(x) = 0.0006\,x^2 + 0.02\,x + 100$
$P_e = 160$
and $Q_e = 300$
So, $PS = 300 \times 160$

$$-\int_0^{300}(0.0006x^2 + 0.02x + 100)dx$$

$$= 48000 - 0.0006\left[\frac{x^3}{3}\right]_0^{300} - 0.02\left[\frac{x^2}{2}\right]_0^{300} - 100[x]_0^{300}$$

$$= 48000 - 0.0006\left(\frac{300^3}{3} - 0\right) - 0.02\left(\frac{300^2}{2} - 0\right) + 100(300 - 0)$$

$$= 48000 - 5400 - 900 - 30000$$
$$= 11700$$

Since, the quantity supplied in units of a thousands, so

$$PS = 11{,}700 \times 1000 = 11{,}700{,}000$$

IV. Read the following text and answer the following questions on the basis of the same:

Rakesh and Kartik are final year engineering students. Both are selected in their campus interviews and got placement in a reputed company. But both of them decide to start their own business. For this they are surveying the market. During their survey they find the demand curve for a certain item is

$$p = D(q) = \frac{20}{q+1}$$

and the supply curve is

$$p = S(q) = q + 2.$$

Q. 1. At equilibrium which of the following is true ?

(A) $D(q) = S(q)$ **(B)** $D(q) > S(q)$

(C) $D(q) < S(q)$ **(D)** none of these

Ans. Option (A) is correct.

Explanation: At the equilibrium point, supply and demand curves meet.

So, $D(q) = S(q)$.

Q. 2. Find equilibrium quantity ?

(A) 5 **(B)** 3

(C) – 6 **(D)** – 3

Ans. Option (B) is correct.

Explanation: To find the equilibrium quantity, we let D(q) = S(q) to obtain

$$\frac{20}{q+1} = q + 2.$$

Clearing the denominator gives $20 = (q + 1)(q + 2)$, which simplifies to $q^2 + 3q - 18 = 0$.

The positive solution gives the equilibrium quantity $q_e = 3$.

Q. 3. Find equilibrium price ?

(A) 5 **(B)** 3

(C) – 6 **(D)** – 3

Ans. Option (A) is correct.

Explanation: Put $q = 3$ in $D(q)$ or $S(q)$ to get equilibrium price (p_e).

$$S(3) = 3 + 2 = 5$$

Q. 4. Consumer's Profit (in ₹) is:

(A) 12 **(B)** 12.52

(C) 12.73 **(D)** 12.11

Ans. Option (C) is correct.

Explanation:

$$CS = \int_0^{q_e} D(q)dq - p_e q_e$$

$$= \int_0^3 \frac{20}{q+1}dq - (5)(3)$$

$$= 20 \ln (q+1)\Big|_0^3 - 15$$

$$= 20\, ln\, 4 - 15$$

$= 20\ 2\log 2 - 15$
$= 20\ 2\ (0.3010) - 15$
$\approx 12.73.$

Q. 5. Producer's Profit (in ₹) is:
(A) 2.50 **(B)** 3
(C) 4 **(D)** 4.50

Ans. Option (D) is correct.
Explanation:

$$PS = p_e q_e - \int_0^{q_e} S(q)dp$$
$$= (5)(3) - \int_0^3 (q+2)dq$$
$$= 15 - \left(\frac{q^2}{2} + 2q\right)\Big|_0^3$$
$$= 15 - \left(\frac{9}{2} + 6\right) = 4.50.$$

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:
Each sub-part 2 marks.

Let's say that we want to evaluate $\int [P(x)/Q(x)]\ dx$, where $P(x)/Q(x)$ is a proper rational fraction. In such cases, it is possible to write the integrand as a sum of simpler rational functions by using partial fraction decomposition. Post this, integration can be carried out easily. The following image indicates some simple partial fractions which can be associated with various rational functions:

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, x \neq a,b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2} x \neq a$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$ $x \neq a,b,c$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)} x \neq a,b$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} x \neq a$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$,
	where, x^2+bx+c cannot be factorised further	

In the above table, A, B and C are real numbers to be determined suitably.

Q. 1. Evaluate: $\int \frac{dx}{(x+1)(x+2)}$

Sol. We write,

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \quad ...(i)$$

where, real number A and B are to be determined suitably. This gives

$$1 = A(x+2) + B(x+1)$$

Equating the coefficients of x and the constant term, we get

$$A + B = 0$$
and $$2A + B = 1$$

Solving these equations, we get $A = 1$ and $B = -1$. Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2} \quad \textbf{1}$$

Therefore,

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$
$$= \log|x+1| - \log|x+2| + C$$
$$= \log\left|\frac{x+1}{x+2}\right| + C \quad \textbf{1}$$

Q. 2. Evaluate: $\int \frac{1}{x^2-9} dx$

Sol. Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$$1 = A(x-3) + B(x+3)$$

Equating the coefficients of x and constant term, we obtain

$$A + B = 0$$
$$-3A + 3B = 1$$

On solving, we obtain

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\therefore \quad \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \quad \textbf{1}$$

$$\Rightarrow \quad \int \frac{1}{(x^2-9)} dx = \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)}\right) dx$$
$$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C$$
$$= \frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right| + C \quad \textbf{1}$$

II. Read the following text and answer the following questions on the basis of the same:

Mohan is the student of class XII of Sunshine School. In the mathematics class, teacher defines the method for evaluation of definite integral, she said, To evaluate the definite integral $\int_a^b f(x)dx$ of a continuous function $f(x)$ defined on $[a, b]$, we may use the following algorithm.

Step 1: Find the indefinite integral $\int_a^b f(x)dx$

Let this be P(x). There is no need to keep the constant of integration.

Step 2: Evaluate $\phi(b)$ and $\phi(a)$.

Step 3: Calculate $\phi(b) - \phi(a)$.

The number obtained in step 3 is the value of definite integral $\int_a^b f(x)dx$.

Q. 1. Evaluate: $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}dx$

Sol. $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}dx$

$= \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})}$

$= \int_0^1 (\sqrt{1+x}-\sqrt{x})dx$

$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right]_0^1$ **1**

$= \left[\frac{2}{3}(1+1)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1+0)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}\right]$

$= \frac{2}{3}(2^{\frac{3}{2}}-1) - \frac{2}{3}(1-0)$

$= \frac{2}{3}(2\sqrt{2}-2) = \frac{4}{3}(\sqrt{2}-1)$ **1**

Q. 2. If $\int_1^a (3x^2+2x+1)dx = 1$**, find real values of** a**.**

Sol. Given $\int_1^a (3x^2+2x+1)dx = 11$

$\Rightarrow [x^3 + x^2 + x]_1^a = 11$

$\Rightarrow (a^3 + a^2 + a) - (1 + 1 + 1) = 11$

$\Rightarrow a^3 + a^2 + a - 3 = 11$ **1**

$\Rightarrow a^3 + a^2 + a - 14 = 0$

$\Rightarrow (a-2)(a^2 + 3a + 7) = 0$

$\Rightarrow a = 2$

$[\because a^2 + 3a + 7 \neq 0$, for any $a \in R]$ **1**

Solutions for Practice Questions (Topic-1)

Very Short Answer Type Questions

Sol. 3: $\int \frac{3x}{3x-1}dx = \int \frac{3x-1+1}{3x-1}dx$

$= \int \left(1 + \frac{1}{3x-1}\right)dx$

$= x + \frac{1}{3}\log|3x-1| + C$

Short Answer Type Questions-I

Sol. 3: Let $I = \int \left(\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right)e^x dx$ **1**

$= \frac{1}{1+x^2}e^x + c$

$\left(\because \int e^x[f(x)+f'(x)]dx = e^x f(x) + c\right)$

$\left[\text{as } \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2}\right]$ **1**

Sol. 8: Given, $\int \frac{x^2}{x^2-4}dx$

$\therefore \int \frac{x^2-4+4}{x^2-4}dx = \int \left(\frac{x^2-4}{x^2-4} + \frac{4}{x^2-4}\right)dx$ **1**

$\Rightarrow \int \left(1 + \frac{4}{x^2-4}\right)dx = x + 4 \times \frac{1}{2\times 2}\log\left|\frac{x-2}{x+2}\right| + C$

$= x + \log\left|\frac{x-2}{x+2}\right| + C$ **1**

Short Answer Type Questions-II

Sol. 2: Put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{1}{2}dt$

$\int \frac{x^3}{x^4+3x^2+2}dx = \frac{1}{2}\int \frac{t\,dt}{t^2+3t+2}$

Now, $\frac{t}{t^2+3t+2} = \frac{t}{(t+2)(t+1)}$

$= \frac{A}{(t+2)} + \frac{B}{(t+1)}$

$\Rightarrow t = A(t+1) + B(t+2)$

Equating the coefficients of t and constant terms, we get

$A + B = 1$

and $A + 2B = 0$

$\Rightarrow A = 2$ and $B = -1$

$\therefore \int \frac{x^3}{x^4+3x^2+2}dx = \frac{1}{2}\int \left(\frac{2}{t+2} - \frac{1}{t+1}\right)dt$ **2**

$= \frac{1}{2}2\log|t+2| - \frac{1}{2}\log|t+1| + C$

$= \log|x^2+2| - \frac{1}{2}\log|x^2+1| + C$ **1**

Long Answer Type Questions

Sol. 2: Let $I = \int \frac{3x^2-5}{9x^4-20x^2+25}dx$

Divide each term by x^2

$I = \int \frac{3 - \frac{5}{x^2}}{9x^2 - 20 + \frac{25}{x^2}}dx$ **1**

$$= \int \frac{\left(3-\frac{5}{x^2}\right)dx}{\left(3x+\frac{5}{x}\right)^2-50} \quad \textbf{1}$$

Let $3x+\frac{5}{x} = t$

$\left(3-\frac{5}{x^2}\right)dx = dt$ **1**

$I = \int \frac{dt}{t^2-(\sqrt{50})^2}$

$= \frac{1}{2\sqrt{50}} \log\left|\frac{t-\sqrt{50}}{t+\sqrt{50}}\right| + C$ **1**

$= \frac{1}{10\sqrt{2}} \log\left|\frac{3x+\frac{5}{x}-5\sqrt{2}}{3x+\frac{5}{x}+5\sqrt{2}}\right| + C$

$= \frac{1}{10\sqrt{2}} \log\left|\frac{3x^2-5\sqrt{2}x+5}{3x^2+5\sqrt{2}x+5}\right| + C$ **1**

Solutions for Practice Questions (Topic-2)

Very Short Answer Type Questions

Sol. 2: Put, $1 + x^2 = t$

or $2xdx = dt$

or $xdx = \frac{dt}{2}$

$\therefore \int_2^4 \frac{x}{x^2+1}dx = \int_2^4 \frac{dt}{2t} = \frac{1}{2}[\log t]_2^4$

$\frac{1}{2}\left[\log(1+x^2)\right]_2^4 = \frac{1}{2}\log\frac{17}{5}.$ **1**

Sol. 5: $\int_2^3 3^x dx = \left[\frac{3^x}{\log 3}\right]_2^3$

$= \frac{18}{\log 3}$ **1**

Short Answer Type Questions-I

Sol. 4: Let $I = \int_0^a f(a-x)dx$

Put $a - x = t$

$\Rightarrow -dx = dt$

$I = \int_a^0 f(t)dt$

$= \int_0^a f(t)dt$

$= \int_0^a f(x)dx$

Short Answer Type Questions-II

Sol. 2: Consider $I = \int_1^3 |x^2-2x|\,dx$

$$|x^2-2x| = \begin{cases} -(x^2-2x) & \text{where } 1 \le x < 2 \\ (x^2-2x) & \text{where } 2 \le x \le 3 \end{cases}$$

$I = \int_1^2 |x^2-2x|\,dx + \int_2^3 |x^2-2x|\,dx$ **1**

$I = \int_1^2 -(x^2-2x)\,dx + \int_2^3 (x^2-2x)\,dx$

$I = -\left[\frac{x^3}{3}-x^2\right]_1^2 + \left[\frac{x^3}{3}-x^2\right]_2^3$ **1**

$I = -\left[\frac{8}{3}-4-\frac{1}{3}+1\right] + \left[9-9-\frac{8}{3}+4\right]$

$= -\frac{7}{3}+3-\frac{8}{3}+4$ **1**

$I = 7 - 5 = 2$

Commonly Made Error

- Most of the students carelessly neglect the modulus sign or are unable to change the modulus.

Answering Tip

- Learn comprehensively all standard methods of integrals, problems based on them, definite integrals, their basic properties and the problems based on them.

Sol. 7: Let $I = \int_a^b \frac{\log x}{x}dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

When $x = a$, then $t = \log a$

When $x = b$, then $t = \log b$

$\therefore I = \int_{\log a}^{\log b} t\,dt$ **1**

$= \left[\frac{t^2}{2}\right]_{\log a}^{\log b}$

$= \frac{1}{2}[(\log b)^2 - (\log a)^2]$

$I = \frac{1}{2}[(\log b + \log a)(\log b - \log a)]$

$I = \frac{1}{2}\left[\log ab \cdot \log\frac{b}{a}\right]$ $[\because \log m + \log n = \log mn$ and $\log m - \log n = \log\frac{m}{n}]$

$I = \frac{1}{2}\log ab \cdot \log\frac{b}{a}.$ **2**

Long Answer Type Questions

Sol. 1: $I = \int_{-1}^{1} \frac{x+|x|+1}{x^2+2|x|+1}dx$

$= \int_{-1}^{1} \frac{x}{x^2+2|x|+1}dx + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}dx$

$= I_1 + I_2$ (Say) ...(i) **1**

Now, $I_1 = \int_{-1}^{1} \frac{x}{x^2+2|x|+1}dx$

Let $f(x) = \frac{x}{x^2+2|x|+1}dx$

$f(-x) = \frac{-x}{(-x)^2+2|-x|+1}$

$= \frac{-x}{x^2+2|x|+1} = -f(x)$

$\because$ $f(x)$ is odd function. **1½**

Hence, $I_1 = 0$ ..(ii)

Also, $I_2 = \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}dx$

Let $g(x) = \frac{|x|+1}{x^2+2|x|+1}$

$\Rightarrow$ $g(-x) = \frac{|-x|+1}{(-x)^2+2|-x|+1}$

$g(-x) = \frac{|x|+1}{x^2+2|x|+1} = g(x)$

$\because$ $g(x)$ is even function **1½**

$\therefore$ $I_2 = 2\int_{0}^{1} \frac{x+1}{x^2+2x+1}dx$

$= 2\int_{0}^{1} \frac{1}{x+1}dx$

$= 2\left[ln|x+1|\right]_0^1 = 2\left[ln 2 - ln 1\right]$

$I_2 = 2ln 2$ ...(iii)

From (i), (ii) and (iii), we get

$I = 2ln 2$ **1**

Solutions for Practice Questions (Topic-3)

Very Short Answer Type Questions

Sol. 2: Sarah's (Producer's) Surplus = ₹ 6000 – ₹ 5000

= ₹ 1000 **1**

Short Answer Type Questions-I

Sol. 3: Required area $= \int_{-1}^{1} x^5\,dx$

$= 2\int_0^1 x^5 dx$

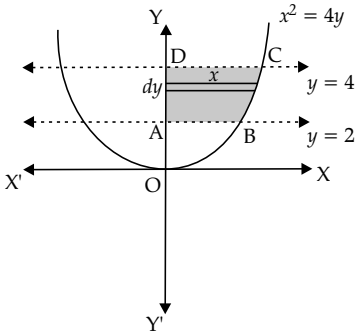
$= 2.\left(\frac{x^6}{6}\right)\Big|_0^1$ **1**

$= 2.\frac{1}{6}(1^6 - 0)$

$= \frac{1}{3}$ **1**

Short Answer Type Questions-II

Sol. 3:



1

The area of the region bounded by the curve, $x^2 = 4y$, $y = 2$ and $y = 4$, and the y-axis is the area ABCD.

Area of $ABCD = \int_2^4 x\,dy = \int_2^4 2\sqrt{y}\,dy$

$= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$

$= \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$

$= \frac{4}{3}\left[8 - 2\sqrt{2}\right]$

$= \left(\frac{32 - 8\sqrt{2}}{3}\right)$units^2 **2**

Sol. 4: As the number of units is 20,

its price rises up $= D(20)$

$= 50 - 0.06 \times 20^2 = 26.$

The profit of consumer's: **1**

$CS = \int_0^{Q_e} D(x)\,dx - Q_e.P_e$

$= \int_0^{20} (50 - 0.06x^2)dx - 20.26$

$= 50[x]_0^{20} - 0.06\left[\frac{x^3}{3}\right]_0^{20} - 520$

$= 50(20 - 0) - 0.06\left[\frac{20^3}{3} - 0\right] - 520$

$$= 50(20) - 0.06\left[\frac{20^3}{3}\right] - 520$$

$$= 1000 - 160 - 520$$

$$= 1000 - 680$$

$$= 320$$

The consumer gain is ₹ 320, if the level of sales is twenty units. **2**

Long Answer Type Questions

Sol. 2: Given $P_d = 18 - 2x - x^2$;

$P_s = 2x - 3$

We know that at equilibrium prices $P_d = P_s$

$18 - 2x - x^2 = 2x - 3$

$x^2 + 4x - 21 = 0$

$(x - 3)(x + 7) = 0$

$x = -7$ or 3

The value of x cannot be negative, $x = 3$

When $x = Q_e = 3$

$\therefore \quad P_e = 18 - 2(3) - (3)^2 = 3$ **1**

consumer's surplus is given by

$$CS = \int_0^{Q_e} D(x)\,dx - Q_e.P_e$$

Here, $D(x) = P_d$

So, $$CS = \int_0^3 (18 - 2x - x^2)dx - 3 \times 3$$

$$= \left[18x - x^2 - \frac{x^3}{3}\right]_0^3 - 9$$

$$= 18(3) - (3)^2 - \left(\frac{3^3}{3}\right) - 9$$

$$= 27 \text{ units}$$ **2**

Producer's surplus is given by

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)\,dx$$

Here, $S(x) = P_S$

$$= (3 \times 3) - \int_0^3 (2x - 3)dx$$

$$= 9 - \left(x^2 - 3x\right)_0^3$$

$$= 9 - (9 - 9)$$

$$= 9 \text{ units}$$ **2**

Hence, at equilibrium price, (i) the consumer's surplus is 27 units and (ii) the producer's surplus is 9 units.

REFLECTIONS

(a) Will you be able to apply the partial fraction method easily to solve the given integral?

(b) Can you handle the problems based on consumer's surplus and producer's surplus?

(c) Will you be able to find the area under the curves using definite integral?

CHAPTER

7

DIFFERENTIAL EQUATIONS AND MODELING

Syllabus

Differential equations, Formulation and solving linear differential equation, Application of differential equations.

In this chapter you will study

- *The concept of differential equations using the idea of differential calculus.*
- *Identify order and degree of a differential equations when solution is not given.*
- *Method to solve differential equations.*
- *Growth and Decay Model in Biological sciences, Economics and business.*

List of Topics

Topic-1: Basic Differential Equations
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Topic-1 Basic Differential Equations

Concepts Covered • *Order of differential Equation,* • *Degree of differential Equation,* • *Solution of differential equation,* • *Variable separation method*

Revision Notes

► **Differential Equation:**
An equation consisting of an independent variable, dependent variable and differential coefficients of dependent variable with respect to the independent variable is known as differential equation.

Scan to know more about this topic
Differential equation, order and degree

e.g. : **(i)** $\frac{d^2y}{dx^2} = -a^2y$,

(ii) $\frac{dy}{dx} = \frac{x+y}{x^2}$,

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = p\frac{dy}{dx}$

► **Order of Differential Equation:** The order of a differential equation is the order of the highest derivative appearing in the differential equation.

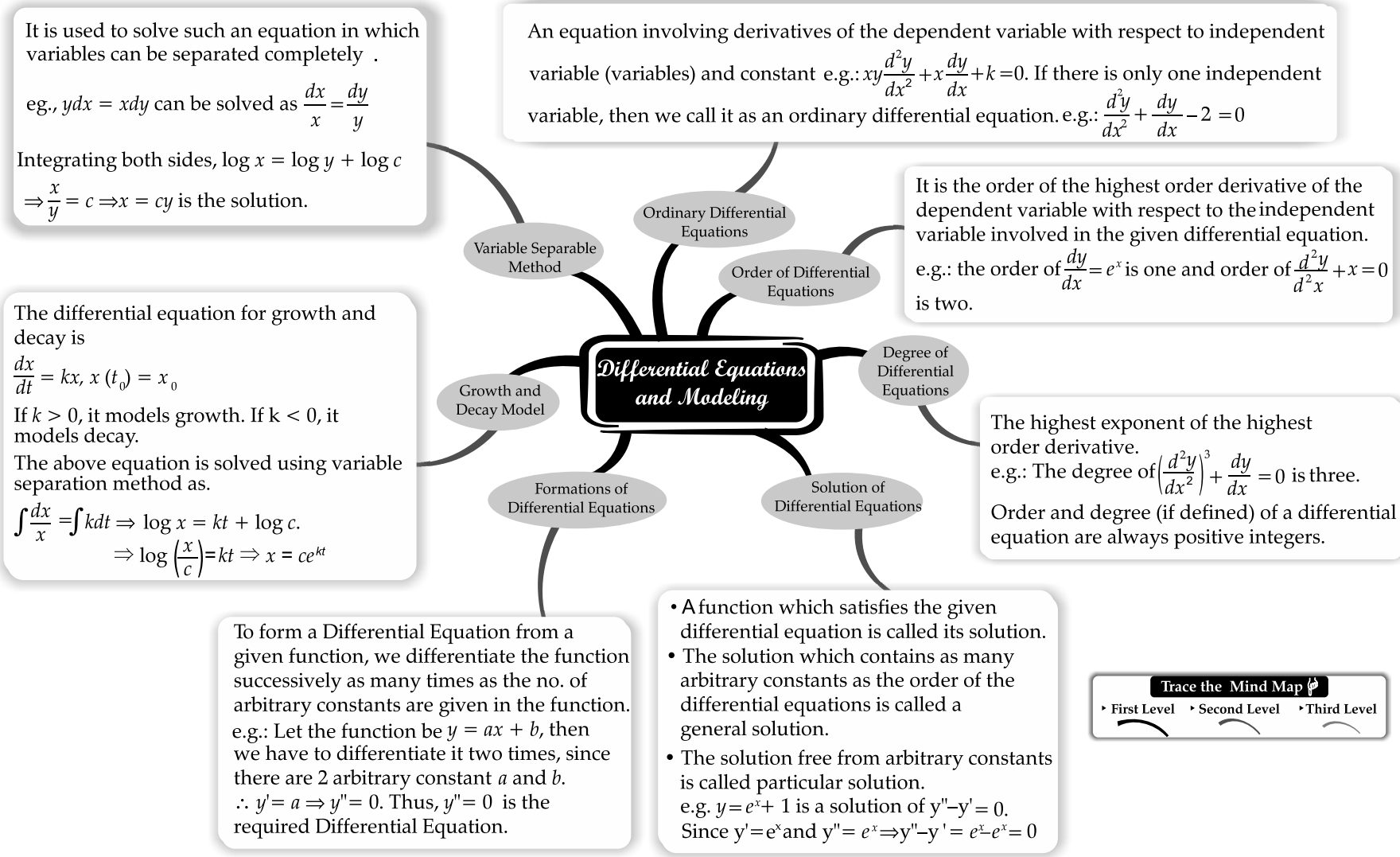
e.g.: $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^3 + 2 = 0$ is the differential equation of order 3 because highest order derivative of y w.r.t. x is $\frac{d^3y}{dx^3}$.

Scan to know more about this topic
General and particular solution of differential equation

► **Degree of Differential Equation:** The degree of the differential equation is the degree (power) of the highest order derivative, when the **differential coefficient** has been made free from the radicals and fractions.

Key Word

Differential coefficient: Differential coefficient is the result of mathematical differentiation.


Differential Equations and Modeling
Ordinary Differential Equations
An equation involving derivatives of the dependent variable with respect to independent variable (variables) and constant e.g.: $xy\frac{d^2y}{dx^2}+x\frac{dy}{dx}+k=0$. If there is only one independent variable, then we call it as an ordinary differential equation. e.g.: $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2=0$
Variable Separable Method
It is used to solve such an equation in which variables can be separated completely .
eg., $ydx = xdy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$
Integrating both sides, $\log x = \log y + \log c$
$\Rightarrow \frac{x}{y} = c \Rightarrow x = cy$ is the solution.
Order of Differential Equations
It is the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.
e.g.: the order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{d^2x}+x=0$ is two.
Growth and Decay Model
The differential equation for growth and decay is
$\frac{dx}{dt} = kx,\ x(t_0) = x_0$
If $k > 0$, it models growth. If $k < 0$, it models decay.
The above equation is solved using variable separation method as.
$\int\frac{dx}{x} = \int kdt \Rightarrow \log x = kt + \log c.$
$\Rightarrow \log\left(\frac{x}{c}\right) = kt \Rightarrow x = ce^{kt}$
Degree of Differential Equations
The highest exponent of the highest order derivative.
e.g.: The degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three.
Order and degree (if defined) of a differential equation are always positive integers.
Formations of Differential Equations
To form a Differential Equation from a given function, we differentiate the function successively as many times as the no. of arbitrary constants are given in the function.
e.g.: Let the function be $y = ax + b$, then we have to differentiate it two times, since there are 2 arbitrary constant a and b.
$\therefore y' = a \Rightarrow y'' = 0$. Thus, $y'' = 0$ is the required Differential Equation.
Solution of Differential Equations
• A function which satisfies the given differential equation is called its solution.
• The solution which contains as many arbitrary constants as the order of the differential equations is called a general solution.
• The solution free from arbitrary constants is called particular solution.
e.g. $y = e^x + 1$ is a solution of $y'' - y' = 0$.
Since $y' = e^x$ and $y'' = e^x \Rightarrow y'' - y' = e^x - e^x = 0$
Trace the Mind Map
First Level
Second Level
Third Level

e.g.: $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}} = 3\frac{d^2y}{dx^2}$ is the differential equation of degree 3, because the power of highest order derivative $\frac{d^2y}{dx^2}$ is 3 (after cubing).

- **Formation of the Differential Equation:**

If the equation of the family of curves is given then its differential equation is obtained by **eliminating arbitrary constants** occurring in equation with the help of equation of the curve and the equations obtained by differentiating the equation of the curve.

Scan to know more about this topic

Formation of differential equation

- **Algorithm for the Formation of the Differential Equation:**

Step 1: Write down the given equation of the curve.

Step 2: Differentiate the given equation with respect to the independent variable as many times as the number of arbitrary constants.

Step 3: Eliminate the arbitrary constants by using given equation and the equations obtained by the differentiation in step 2.

Example 1 : Find the differential equation formed from the equation $y = mx + c$, where m and c are arbitrary constants.

Sol. Given, $y = mx + c$

or $\frac{dy}{dx} = m$

or $\frac{d^2y}{dx^2} = 0$

So, the required differential equation is $\frac{d^2y}{dx^2} = 0$.

- **Solution of Differential Equations:**

(a) General solution: The solution which contains as many as arbitrary constants as the order of the differential equations, e.g., $y=\alpha \cos x+\beta \sin x$ is the general solution of $\frac{d^2y}{dx^2}+y=0$.

Scan to know more about this topic

Separation of variables method

Here, the differential equation is of second order and there are two arbitrary constants *i.e.*, α and β in the general solution.

(b) Particular solution: Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

e.g., $y = 3\cos x + 2\sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2}+y=0$.

(c) Solution of Differential Equation by Variable Separable Method: A variable separable form of the differential equation is the one which can be expressed in the form of $f(x)\,dx = g(y)\,dy$. The solution is given by $\int f(x)dx = \int g(y)dy + k$, or $\int g(y)\,dy = \int f(n)\,dn + k$, where k is the constant of integration.

Example 2 : Find the solution of the differential equation $x\frac{dy}{dx} = \frac{y}{1+\log x}$.

Sol. $x\frac{dy}{dx} = \frac{y}{1+\log x}$

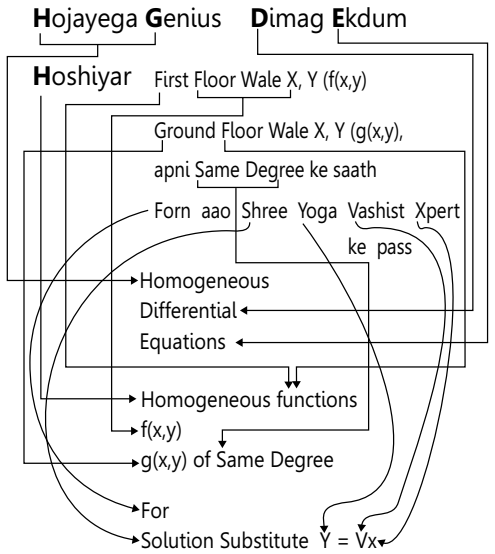
$$\int \frac{dy}{y} = \int \frac{1}{x(1+\log x)}dx$$

$$\log|y| = \log|1+\log x| + \log C$$

$$y = C(1+\log x)$$

Mnemonic

Homogeneous Differential Equation



Interpretation :

Differential equation can be expressed in the form $\frac{dy}{dx}=f(x,y)$ or $\frac{dx}{dy}=g(x,y)$ where $f(x,y)$ and $g(x,y)$ are homogeneous functions of sum is called a homogeneous Differential equation. These equations can be solved by substituting $y = vx$ so that dependent variable y is changed to another variable v, where v is some unknown function.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. The order of the differential equation

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0 \text{ is:}$$

(A) 2 **(B)** 1
(C) 0 **(D)** Not defined

Ans. Option (A) is correct.

Explanation: $2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

Q. 2. The order and degree (if defined) of differential equation $\frac{dy}{dx}=ky$, where k is a scalar are:

(A) 1, 1 **(B)** 0, 1
(C) 1, 0 **(D)** Not defined

Ans. Option (A) is correct.

Explanation: The highest order derivative present is $\frac{dy}{dx}$ and it is raised to power 1. So, its order is 1 and degree is also 1.

Q. 3. The order and degree (if defined) of differential equation $\left(\frac{dS}{dx}\right)^4+2S\frac{d^2S}{dt^2}=0$ are:

(A) 1, 2 **(B)** 2, 1
(C) 1, 1 **(D)** 4, 2

Ans. Option (B) is correct.

Explanation: The highest order derivative present is $\frac{d^2S}{dt^2}$ and it is raised to power 1. So, its order is 2 and degree is 1.

Q. 4. The order and degree (if defined) of differential equation $\left(1+\frac{dy}{dx}\right)^3=\left(\frac{d^2y}{dx^2}\right)^2$ are:

(A) 3, 2 **(B)** 2, 2
(C) 3, 3 **(D)** Not defined

Ans. Option (B) is correct.

Explanation: The highest order derivative present is $\frac{d^2y}{dx^2}$ and it is raised to power 2. So, its order is 2 and degree is also 2.

Q. 5. The order and degree (if defined) of differential equation $y\,dx+\left[x\log\left(\frac{y}{x}\right)-2x\right]dy=0$ are:

(A) 1, 1 **(B)** 1, 0
(C) 0, 1 **(D)** Not defined

Ans. Option (A) is correct.

Explanation: Given differential equation,

$$y\,dx+x\log\left(\frac{y}{x}\right)-2x\,dy=0$$

$$\Rightarrow \left[2x-x\log\left(\frac{y}{x}\right)\right]dy=y\,dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2x-x\log\left(\frac{y}{x}\right)}$$

The highest order derivative present is $\frac{dy}{dx}$ and it is raised to power 1. So, its order is 1 and degree is also 1.

Q. 6. The order and degree (if defined) of differential equation $x\frac{dy}{dx}+2y=x^2$ $x\neq 0$ are:

(A) 1, 1 **(B)** 1, 0
(C) 0, 1 **(D)** Not defined

Ans. Option (A) is correct.

Explanation: The highest order derivative present is $\frac{dy}{dx}$ and its raised to power 1. So, its degree is 1 and order is also 1.

Q. 7. The order and degree (if defined) of differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ are:

(A) 1, 1 **(B)** 2, 1
(C) 1, 2 **(D)** 2, 2

Ans. Option (B) is correct.

Explanation: The highest order derivative present is $\frac{d^2y}{dx^2}$ and its raised to power 1. So, its degree is 1 and order is 2.

Q. 8. The order and degree (if defined) of differential equation $\left(\frac{dy}{dx}\right)^4+3y\left(\frac{d^2y}{dx^2}\right)=0$ are:

(A) 4, 1 **(B)** 1, 4
(C) 4, 4 **(D)** 2, 1

Ans. Option (D) is correct.

Explanation: The highest order derivative present is $\frac{d^2y}{dx^2}$ and its raised to power 1. So, its degree is 1 and order is 2.

Q. 9. The order and degree (if defined) of differential equation $(y''')^2+(y'')^3+(y')^4+y^5=0$ are:

(A) 3, 1 **(B)** 1, 3
(C) 3, 2 **(D)** 2, 3

Ans. Option (C) is correct.

Explanation: The highest order derivative present is y''' and its raised to power 2. So, its order is 3 and degree is 2.

Q. 10. The order and degree (if defined) of differential equation $y''' + y^2 + e^{y'} = 0$ are:

(A) 3, 1 **(B)** 3, not defined
(C) not defined, 3 **(D)** 1, 3

Ans. Option (B) is correct.

Explanation: The highest order derivative present is y''', so its order is 3. Here the term $e^{y'}$ is not a polynomial in y', so the degree of the given differential equation is not defined.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Write the order of the differential equation:

$$\log\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^3 + x$$ R&U

Sol. The highest order derivative is $\frac{d^2y}{dx^2}$. Thus, the order of given differential equation is 2. **1**

Q. 2. Write the sum of the order and degree of the differential equation $1+\left(\frac{dy}{dx}\right)^4 = 7\left(\frac{d^2y}{dx^2}\right)^3$. A I

Sol. Degree of the given differential equation = 3

Order of the given differential equation = 2 ½

Hence, the sum of order and degree = 2 + 3

= 5 ½

Q. 3. Write the order and the degree of the following differential equation:

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

Q. 4. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$. A I

Sol. Given, differential equation is

$$x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$$

The highest order derivative occurs in the given differential equation is $\frac{d^2y}{dx^2}$

Hence, order of differential equation = 2

Also, the power of highest order derivative is 1.

Hence, degree of differential equation is 1. **1**

Q. 5. Find the order and degree (if defined) of the differential equation.

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$$ A I

Sol. Given differential equation is

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. So it is of order 2.

Clearly, the differential equation is not expressible as polynomial in $\frac{d^2y}{dx^2}$. So, its degree is not defined.

Hence, Order = 2

Degree is not defined. **1**

Commonly Made Error

- Mostly students write the degree as 1.

Answering Tip

- Note the questions in which degree is not defined.

Q. 6. Write the differential equation obtained by eliminating the arbitrary constant C in the equation representing the family of curves $xy = C\cos x$. R&U

Sol. On differentiating the given equation, we get

$$x\frac{dy}{dx} + y = -C\sin x$$ ½

Eliminating C using above equation, we get

$$x\frac{dy}{dx} + y + xy\tan x = 0$$ ½

Q. 7. Find the differential equation of the family of lines passing through the origin. R&U

Sol. General equation of family of lines passing through origin is

$$y = mx \quad \Rightarrow \quad m = \frac{y}{x}$$

$$\frac{dy}{dx} = m$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$x\frac{dy}{dx} - y = 0$$ **1**

Q. 8. Find the differential equation representing the family of curves $V = \frac{A}{r} + B$, where A and B are arbitrary constants. R&U

Q. 9. If m and n are the order and degree, respectively of the differential equation $y\left(\frac{dy}{dx}\right)^3 + x^3\left(\frac{d^2y}{dx^2}\right)^2 - xy = \sin x$, then write the value of $m + n$. A I

Sol. $m + n = 4$

$\because \quad m = 2$ (second order derivative) ½

$\therefore \quad n = 2$

(degree of the highest order derivative) ½

Q. 10. Find the sum of the order and the degree of the following differential equations :

$$\frac{d^2y}{dx^2} + \sqrt[3]{\frac{dy}{dx}} + (1 + x) = 0$$ A I R&U

Sol. Here, $\left\{\frac{d^2y}{dx^2} + (1 + x)\right\}^3 = -\frac{dy}{dx}$ ½

Thus, order is 2 and degree is 3. So, the sum is 5 ½

Q. 11. Find the differential equation of the family of concentric circles $x^2 + y^2 = a^2$. R&U

Q. 12. Find the solution of the differential equation $\frac{dy}{dx} = x^3 e^{-2y}$. R&U

Sol. $$\int \frac{dy}{e^{-2y}} = \int x^3 dx$$

$$\int e^{2y} dy = \int x^3 dx$$ ½

or $$\frac{e^{2y}}{2} = \frac{x^4}{4} + C$$

or $$\frac{1}{2}e^{2y} = \frac{x^4}{4} + C$$ ½

$$2e^{2y} = x^4 + C_1 \quad \text{where } (C_1 = 4C)$$

Q. 13. Write the solution of the differential equation $\frac{dy}{dx} = 2^{-y}$. R&U

Sol. Given differential equation is

$$\frac{dy}{dx} = 2^{-y}$$

On separating the variables, we get

$$2^y\, dy = dx$$

On integrating both sides, we get

$$\int 2^y dy = \int dx$$

or $$\frac{2^y}{\log 2} = x + C_1$$

or $2^y = x \log 2 + C_1 \log 2$

$\therefore 2^y = x \log 2 + C$, where $C = C_1 \log 2$ **1**

Q. 14. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$. A I

Sol. Given differential equation is

$$\frac{dy}{dx} = e^{x+y}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x e^y$$

$$\Rightarrow \quad \frac{dy}{e^y} = (e^x)dx$$

$$\Rightarrow \quad (e^{-y})dy = (e^x)dx$$

Integrating both sides, we get

$$\int (e^{-y})dy = \int (e^x)dx$$

$$\Rightarrow \quad -e^{-y} = e^x + c'$$

$$\Rightarrow \quad e^{-y} = -e^x + c \text{ [where } c = -c'] \quad \mathbf{1}$$

Short Answer Type Questions-I (2 marks each)

Q. 1. Find the differential equation of the family of curves $y^2 = 4ax$. R&U

Sol. Differential both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \quad \mathbf{1}$$

Eliminating $4a$, we get

$$y^2 = 2y\frac{dy}{dx}.x$$

or $$2xy\frac{dy}{dx} - y^2 = 0 \quad \mathbf{1}$$

Q. 2. Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where 'a' and 'b' are arbitrary constants. A I

Sol. $$y = e^{2x}(a + bx) \quad ...(i)$$

Differentiating both sides w.r.t. x we get

$$\frac{dy}{dx} = e^{2x}(b) + (a + bx)2e^{2x}$$

$$= be^{2x} + 2y$$

$$\frac{dy}{dx} - 2y = be^{2x} \quad ...(ii) \ \mathbf{1}$$

Differentiating again,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 2be^{2x}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 2\left(\frac{dy}{dx} - 2y\right)$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 \quad \mathbf{1}$$

Q. 3. From the differential equation of equation $y = a\cos 2x + b\sin 2x$, where a and b are constants. R&U

Q. 4. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants. U

These questions are for practice and their solutions are available at the end of the chapter

Sol. Differentiating $y = Ae^{2x} + Be^{-2x}$, we get

$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$,

Differentiate again to get, **1**

$\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x} = 4y$ *or* $\frac{d^2y}{dx^2} - 4y = 0$. **1**

Q. 5. Form the differential equation representing the family of curves $y = ae^{bx+5}$, where 'a' and 'b' are arbitrary constants. U

Sol. $\frac{dy}{dx} = bae^{bx+5}$

$\frac{dy}{dx} = by$ **1**

$\Rightarrow \frac{d^2y}{dx^2} = b\frac{dy}{dx}$ ½

$\therefore$ The differential equation is: $y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. ½

Q. 6. Verify that $ax^2 + by^2 = 1$ is a solution of differential equation $x(yy_2 + y_1^2) = yy_1$. A I

Sol. $ax^2 + by^2 = 1$ or $2ax + 2byy_1 = 0$

or $ax + byy_1 = 0$ ...(i) **1½**

or $a + b[yy_2 + y_1^2] = 0$ or $a = -b[yy_2 + y_1^2]$

Substituting this value, for a in the equation (i), we get,

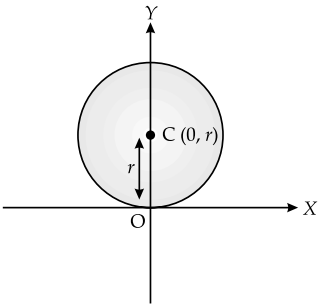
$-b[yy_2 + y_1^2]x + byy_1 = 0$

or $x[yy_2 + y_1^2] = yy_1$. **Hence verified** ½

Q. 7. Form the differential equation of all circles which is touching the X-axis at the origin. R&U

Sol. $(x-0)^2 + (y-r)^2 = r^2$ ½

$\Rightarrow x^2 + y^2 = 2ry$ ...(i)



Differentiating both sides w.r.t. x, we get

$2x + 2yy' = 2ry'$ **1**

$\Rightarrow r = \frac{x + yy'}{y'}$ ...(ii)

Substituting r from (ii) in (i), we get ½

$(x^2 + y^2)y' = 2y(x + yy')$

Commonly Made Error

- Many candidates do not have an ideal of formation of differential equation and make errors while solving it.

Answering Tip

- Learn the fundamentals on formation of differential equations and concepts of identifying the order and degree of the equations.

Q. 8. Find the general solution of the differential equation.

$xy\frac{dy}{dx} = (x+2)(y+2)$. R&U

Sol. $\frac{y}{y+2}dy = \frac{(x+2)}{x}dx$ or

$\int\left(1 - \frac{2}{y+2}\right)dy = \int\left(1 + \frac{2}{x}\right)dx$ **1**

$y - 2\log|y+2| = x + 2\log|x| + C$ **1**

Q. 9. Find the solution of $\frac{dy}{dx} = 2^{y-x}$. A I

Q. 10. Solve the differential equation

$\frac{dy}{dx} = 1 - xy + y - x$ A I

Sol. Given, $\frac{dy}{dx} = 1 - xy + y - x$

$= 1 + y - x(1+y)$

$\Rightarrow \frac{dy}{dx} = (1+y)(1-x)$

$\Rightarrow \frac{dy}{1+y} = (1-x)dx$ **1**

Integrating both sides we get

$\int\frac{dy}{1+y} = \int(1-x)dx$

$\log(1+y) = x - \frac{x^2}{2} + c$ **1**

Commonly Made Error

- Many candidates could not separate the variables to solve the differential equation. Some candidates attempted it by linear equation method but made errors in calculating integrating factor and further. In most of the cases, solution had been written without a constant 'c'.

Answering Tip

- Do adequate practice of various types of differential equations.

Q. 11. Solve the differential equation:

$\log\left(\frac{dy}{dx}\right) = 2x - 3y$ R&U

Short Answer Type Questions-II (3 marks each)

Q. 1. Form the differential equation of the family of circles in the second quadrant and touching the co-ordinate axes. R&U

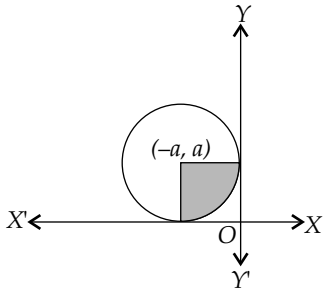
Sol. Equation of family of circle is

$$(x + a)^2 + (y - a)^2 = a^2$$

or $x^2 + y^2 + 2ax - 2ay + a^2 = 0$ ...(i) ½

Differentiating, we get

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0 \qquad ½$$



½

or $$x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right) \qquad ½$$

or $$a = \frac{x + yy'}{y' - 1},$$

where $$y' = \frac{dy}{dx}$$

Substituting the value of a in eqn. (i) and simplifying

$$(xy' - x + x + yy')^2 + (yy' - y - x - yy')^2 = (x + yy')^2$$

or $(x + y)^2[(y')^2 + 1] = (x + yy')^2$ **1**

Q. 2. Prove that $x^2 - y^2 = C(x^2 + y^2)^2$ is the general solution of the differential equation $(x^2 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, where C is a parameter. R&U

Sol. $x^2 - y^2 = C(x^2 + y^2)^2$

or $2x - 2yy' = 2C(x^2 + y^2)(2x + 2yy')$ **1**

or $(x - yy') = \frac{x^2 - y^2}{y^2 + x^2}(2x + 2yy')$

or $(y^2 + x^2)(x - yy') = (x^2 - y^2)(2x + 2yy')$

or $[-2y(x^2 - y^2) - y(y^2 + x^2)]\frac{dy}{dx}$

$= 2x(x^2 - y^2) - x(y^2 + x^2)$ **1**

or $(y^3 - 3x^2y)\frac{dy}{dx} = (x^3 - 3xy^2)$

or $(y^3 - 3x^2y)\,dy = (x^3 - 3xy^2)\,dx$ **1**

Hence $x^2 - y^2 = C(x^2 + y^2)^2$ is the solution of given differential equation.

Q. 3. Find the particular solution of the differential equation $e^x\sqrt{1 - y^2}\,dx + \left(\frac{y}{x}\right)dy = 0$ given that $y = 1$ when $x = 0$. R&U

Sol. $e^x\sqrt{1 - y^2}\,dx = \frac{-y}{x}dy$

or $xe^x dx = \frac{-y}{\sqrt{1 - y^2}}dy$ ½

Integrating both sides, we get

$$\int xe^x dx = \frac{1}{2}\int \frac{-2y}{\sqrt{1 - y^2}}dy \qquad ½$$

or $$xe^x - e^x = \sqrt{1 - y^2} + C \qquad \mathbf{1}$$

For $x = 0, y = 1, C = -1$

∴ Solution is: $e^x(x - 1) = \sqrt{1 - y^2} - 1$ ½+½

Q. 4. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$ given that $y = 0$ when $x = 1$. A I

Sol. $$\frac{dy}{dx} = 1 + x + y + xy$$

$$= (1 + x)(1 + y)$$

or $$\int \frac{dy}{1 + y} = \int (1 + x)dx \qquad \mathbf{1}$$

∴ $$\log|1 + y| = x + \frac{x^2}{2} + C \qquad \mathbf{1}$$

When, $x = 1, y = 0$ or $C = -\frac{3}{2}$ ½

∴ Solution is:

$$\log|1 + y| = x + \frac{x^2}{2} - \frac{3}{2} \qquad ½$$

Q. 5. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$, when $x = 0$. R&U

Sol. Given differential equation can be written as

$$\frac{dy}{dx} = e^{(3x + 4y)} = e^{3x} \cdot e^{4y}$$

∴ $$\int e^{-4y}dy = \int e^{3x}dx \qquad \mathbf{1}$$

or $$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C \qquad \mathbf{1}$$

∴ $4e^{3x} + 3e^{-4y} + 12C = 0$

Taking $x = 0, y = 0$, we get

$$C = -\frac{7}{12} \qquad ½$$

∴ The solution is $4e^{3x} + 3e^{-4y} - 7 = 0$. ½

Q. 6. Find the particular solution of the following differential equation:

$(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$; $y = 0$ when $x = 0$. A I

Sol. Given equation can be written as

$$\int \frac{dy}{2e^{-y}-1} = \int \frac{dx}{x+1}$$

$$\Rightarrow \int \frac{e^y}{2-e^y}dy = \int \frac{dx}{x+1} \quad \text{½}$$

$$\Rightarrow -\log|2-e^y| + \log c = \log|x+1|$$

$$\Rightarrow (2-e^y)(x+1) = c \quad \text{1½}$$

When $x = 0, y = 0 \Rightarrow c = 1$

$\therefore$ Solution is $(2-e^y)(x+1) = 1$ **1**

Q. 7. Find the particular solution of the following differential equation :

$$xy\frac{dy}{dx} = (x+2)(y+2);\ y = -1, \text{when } x = 1.$$ AI

Q. 8. Solve: $(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0$ R&U

Sol. Given, $(x^2 - yx^2)\,dy + (y^2 + xy^2)dx = 0$

$$\Rightarrow x^2(1-y)\,dy + y^2(1+x)dx = 0$$

$$\Rightarrow \frac{1-y}{y^2}dy + \frac{1+x}{x^2}dx = 0$$

On integration, we get

$$\int\frac{1-y}{y^2}dy + \int\frac{1+x}{x^2}dx = 0$$

$$\Rightarrow \int\frac{1}{y^2}dy - \int\frac{1}{y}dy + \int\frac{1}{x^2}dx + \int\frac{1}{x}dx = 0 \quad \textbf{1}$$

$$\Rightarrow -\frac{1}{y} - \log y - \frac{1}{x} + \log x = C \quad \textbf{1}$$

$$\Rightarrow \log\left(\frac{x}{y}\right) - \frac{1}{x} - \frac{1}{y} = C$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = C + \frac{1}{x} + \frac{1}{y} \quad \textbf{1}$$

Q. 9. Solve the differential equation: AI

$$(x+1)dy - 2xy\,dx = 0$$

Sol. $(x+1)\,dy = 2xy\,dx$

$$\Rightarrow \int\frac{dy}{y} = \int\frac{2x}{x+1}dx$$

$$\Rightarrow \log y = 2\int\frac{x+1-1}{x+1}dx$$

$$= 2\int\left(1 - \frac{1}{x+1}\right)dx \quad \textbf{1}$$

$\Rightarrow \log y = 2[x - \log(x+1)] + \log c$ **1**

$\log y + 2\log(x+1) - \log c = 2x$

$$\Rightarrow \log\frac{y(x+1)^2}{c} = 2x$$

$$\Rightarrow \frac{y(x+1)^2}{c} = e^{2x}$$

$$\Rightarrow y = \frac{ce^{2x}}{(x+1)^2} \quad \textbf{1}$$

Q. 10. Solve the differential equation:

$$\frac{dy}{dx} = e^{x+y} + x^2e^y$$ R&U

Long Answer Type Questions (5 marks each)

Q. 1. Obtain the differential equation of all the circles of radius *r*. R&U

Q. 2. Solve the following differential equation.

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$ AI R&U

Sol. Given differential equation is

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

or $$\sqrt{(1+x^2)+y^2(1+x^2)} = -xy\frac{dy}{dx}$$

or $$\sqrt{(1+x^2)(1+y^2)} = -xy\frac{dy}{dx}$$

or $$\sqrt{1+x^2}.\sqrt{1+y^2} = -xy\frac{dy}{dx}$$

or $$\frac{y}{\sqrt{1+y^2}}dy = -\frac{\sqrt{1+x^2}}{x}dx \quad \textbf{1}$$

On integrating both sides, we get

$$\int\frac{y}{\sqrt{1+y^2}}dy = -\int\frac{\sqrt{1+x^2}}{x^2}.xdx$$

On putting $1 + y^2 = t$ and $1 + x^2 = u^2$

or $2y\,dy = dt$ and $2x\,dx = 2u\,du$

$$\Rightarrow y\,dy = \frac{dt}{2}$$

and $x\,dx = u\,du$ **1**

$$\therefore \frac{1}{2}\int t^{-1/2}dt = -\int\frac{u}{u^2-1}.u\,du$$

or $$\frac{1}{2}\int t^{-1/2}dt = -\int\frac{u^2}{u^2-1}du$$

or $$\frac{1}{2}\frac{t^{1/2}}{\frac{1}{2}} = -\int\frac{(u^2-1+1)}{u^2-1}du \quad \textbf{1}$$

or $$t^{1/2} = -\int\frac{u^2-1}{u^2-1}du - \int\frac{1}{u^2-1}du$$

or $$\sqrt{1+y^2} = -\int du - \int\frac{1}{u^2-(1)^2}du$$

or $$\sqrt{1+y^2} = -u - \frac{1}{2}\log\left|\frac{u-1}{u+1}\right| + C \quad \textbf{1}$$

$$\left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

These questions are for practice and their solutions are available at the end of the chapter

$$\therefore \quad \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$$

which is the required solution. **1**

Q. 3. Solve the differential equation:

$$(x - y - 2)dx - (2x - 2y - 3)dy = 0$$

Sol. Given, $(x - y - 2)dx - (2x - 2y - 3)dy = 0$

$\Rightarrow \quad (x - y - 2)dx - [2(x - y) - 3]dy = 0$

$$\Rightarrow \quad \frac{x-y-2}{2(x-y)-3} = \frac{dy}{dx} \qquad \text{½}$$

Let $x - y - 2 = t$

$$\Rightarrow \quad 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 - \frac{dt}{dx} \qquad \text{½}$$

$$\frac{t}{2(t+2)-3} = 1 - \frac{dt}{dx}$$

$$\left(\because\ x - y - 2 = t \right.$$
$$\left. \quad x - y = t + 2 \right)$$

$$\Rightarrow \quad \frac{t}{2t+4-3} = 1 - \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{t}{2t+1} = 1 - \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{dt}{dx} = 1 - \frac{t}{2t+1} \Rightarrow \frac{dt}{dx} = \frac{2t+1-t}{2t+1}$$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{t+1}{2t+1}$$

$$\Rightarrow \quad \frac{2t+1}{t+1}dt = dx \qquad \text{1½}$$

Integrating both sides

$$\int \frac{2t+1}{t+1}dt = \int dx$$

$$\Rightarrow \quad \int \frac{2t+2-2+1}{t+1}dt = \int dx$$

$$\Rightarrow \quad \int \frac{2(t+1)-1}{t+1}dt = x + c$$

$$\Rightarrow \quad \int 2\,dt - \int \frac{1}{t+1}dt = x + c$$

$$2t - \log|t+1| = x + c \qquad \text{1½}$$

$\Rightarrow 2(x - y - 2) - \log|x - y - 2 + 1| = x + c$

$\Rightarrow \quad 2x - 2y - 4 - \log|x - y - 1| = x + c$

$\Rightarrow \quad x - 2y - 4 - \log|x - y - 1| = c$

$\Rightarrow \quad x - 2y - \log|x - y - 1| = C_1\ (C_1 = c + 4)$

is the required solution. **1**

Topic-2 Application of Differential Equations

Concepts Covered • *Growth and Decay model*

Revision Notes

- **Law of Natural Growth:** Let $x(t)$ be the population at any time t. Assume that population grows at a rate directly proportional to the amount of present population. Then the differential equation governing this phenomenon is the linear equation.

Scan to know more about this topic

Growth and decay model

$$\frac{dx}{dt} = kx,$$ where k is the proportionality constant

Here, $k > 0$, since this is phenomenon.

The solution of the D.E. is

$x(t) = Ce^{kt}$,

Where C is constant of integration. Here, C, k are determined from the two given (initial).

Law of Natural Decay:

The differential equation

$$\frac{dm}{dt} = -km, k > 0$$

describes the decay phenomenon, where it is assumed that the material $m(t)$ at any time t decays at a rate which is proportional to the amount present. The solution is

$$m(t) = ce^{-kt}$$

It initially at, $t = 0$, m_0 is the amount present then

$$m(t) = m_0e^{-kt}$$

Example 3: A population grows at the rate of 8% per year. How long does it take for the population to double ? Use differential equation for it.

Sol. Let P_0 be the initial population and let the population after t years be P. Then,

$$\frac{dP}{dt} = \frac{8P}{100}$$

$$\frac{dP}{dt} = \frac{2P}{25}$$

$$\frac{dP}{P} = \frac{2dt}{25}$$

Amazing Facts

1. Differential equations began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's `fluxional equations' in the 1670s. Applications were made largely to geometry and mechanics; is operimetrical problems were exercises in optimization.
2. A differential equation will typically have an infinite number of solutions, for essentially the same reason that a function will typically have an infinite number of anti-derivatives.
3. For many important differential equations, it is not possible to find a formula for a solution, for essentially the same reason that, for some functions, we are not able to find a formula for an anti-derivative.

   e.g.: $\frac{dy}{dt} = e^{-t^2}$.

$$\int \frac{dP}{P} = \frac{2}{25}\int dt$$

$$\log P = \frac{2}{25}t + C \quad \text{...(i)}$$

At $t = 0$, $P = P_0$

Therefore, $\log P_0 = \frac{2}{25}(0) + C$

$\Rightarrow \quad C = \log P_0$

Substituting the value of C in eq. (i), we get

$$\log P = \frac{2}{25}t + \log P_0$$

$$\log \frac{P}{P_0} = \frac{2}{25}t$$

$$t = \frac{25}{2}\log \frac{P}{P_0}$$

When $P = 2P_0$, $t = \frac{25}{2}\log \frac{2P_0}{P_0} = \frac{25}{2}\log 2$

Thus, the population is doubled in $\frac{25}{2}\log\ 2$ years.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. Systems that exhibit exponential growth increase according to the mathematical model $y = y_0e^{kt}$ where y_0 represents the initial state of the system and $k > 0$ is a constant, is called the……constant.

(A) Growth **(B)** Decay
(C) Simple **(D)** Normal

Ans. Option (A) is correct.

Q. 2. Systems that exhibit exponential decay follow a model of the form:

(A) $y = y_0\, e^{kt}$ **(B)** $y = y_0\, e^{-kt}$
(C) $y = y_0\, e^{k/t}$ **(D)** None of these

Ans. Option (B) is correct.

Q. 3. Systems that exhibit exponential decay have a constant half-life, which is given by:

(A) $\frac{\log 2}{k}$ **(B)** $\frac{k}{\log 2}$

(C) $k.\log 2$ **(D)** None of these

Ans. Option (A) is correct.

Explanation: If a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $\frac{\log 2}{k}$.

Q. 4. Systems that exhibit exponential growth have a constant doubling time, which is given by:

(A) $\frac{\log 2}{k}$ **(B)** $\frac{k}{\log 2}$

(C) $k.\log 2$ **(D)** None of these

Ans. Option (A) is correct.

Explanation: If a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double. It is given by $\frac{\log 2}{k}$.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. If \$ 150 is deposited in a bank that pays 5½% annual interect compound continuously, Find the value of account after 10 year.

Sol. Given $k = 5½\% = 0.55$

$$C = 150$$

Since, $Q(t) = Ce^{k\,t}$

then, $Q(t) = 150e^{.055t}$

After 10 year the value of the account is

$$Q(10) = 150\, e^{.55} \approx \$\ 259.99$$

Q. 2. A radioactive Substance has a half-life of 1620 years. 4g, how much will be left 810 year from now ?

Sol. $Q = 4e^{-kt}$

$$k = \frac{\log_2}{1620}$$

Since, $Q = 4e^{-\left(t\frac{\log_2}{1620}\right)}$

therefore the mass left 810 year will be

$$Q(810) = 4e^{(-810\log 2/1620)}$$
$$= 4e^{(-\log 2/2)}$$
$$= 2\sqrt{5}g$$

Short Answer Type Questions-I (2 marks each)

Q. 1. There are 1000 birds on an island. They breed with a constant growth rate of 10% per year. How many birds will be on the island after seven year.

Sol. Let x be the number of the birds on the island after 1 year with continuous growth rate $k = 10\% = 0.1$

$C = 1000$

We have $\frac{dx}{dt} = 1.x$

Solving the diff. eq.

$$x(t) = Ce^{0.1t}$$
$$x(t) = 1000\, e^{0.1t}$$

$t = 7$ $\quad x(t) = 1000\, e^{0.7}$

$$= 2010$$

Q. 2. A cell culture in a biology lab currently holds 1 million cells. The cells have a constant continuous birth rate of 1.5 % and death rate of 0.5% per hour. Cells are extracted from the culture for an experiment at the rate of 5000 per hour. How many cells will be in the culture 10 hours from now ?

Sol. Let p be the population, and let t be the number of hours from now. Births, deaths and extraction respectively contribute $0.015p$, $-0.005p$ and -5000 to $\frac{dp}{dt}$. So we have $\frac{dp}{dt} = 0.015p - 0.005p - 5000 = 0.01p - 5000$.

The general solution to this differential equation is

$$p(t) = Ce^{0.01t} + \frac{5000}{0.01}$$
$$= Ce^{0.01t} + 500000.$$

Since $p = 1000000$ when t = 0, we have C = 500000, so

$p(t) = 500000\, e^{0.01t} + 500000.$

Substituting t = 10, the population in 10 hours will be $500000\, e^{0.1} \approx 1052585$.

Short Answer Type Questions-II (3 marks each)

Q. 1. The population of a town grows at the rate of 10% per year. Using differential equation, find how long will it take for the population to grow 4 times. A

Sol. Let initial population be: P_0 and population after t years be P.

$$\frac{dP}{dt} = 10\% \text{ of } P$$

$$\Rightarrow \frac{dP}{dt} = \frac{1}{10}P$$

$$\Rightarrow \frac{dP}{P} = \frac{1}{10}dt \quad \mathbf{1}$$

Integrating both sides, we get

$$\int\frac{dP}{P} = \int\frac{1}{10}dt$$

$$\log P = \frac{1}{10}t + C$$

When $t = 0$, $P = P_0 \Rightarrow \log P_0 = C$

$$\log P = \frac{1}{10}.t + \log P_0$$

$$\log\left(\frac{P}{P_0}\right) = \frac{1}{10}.t \quad \mathbf{1\frac{1}{2}}$$

Since, population grows 4 times, $P = 4P_0$

$$\therefore \log 4 = \frac{1}{10}.t$$

$$\Rightarrow t = 10\log 4$$

$\Rightarrow 20\log 2$(years) ½

Q. 2. The surface area of a balloon being inflated changes at a constant rate. If initially, its radius is 3 units and after 2 seconds, it is 5 units, find the radius after t seconds. A I

Sol. Let r be the radius and S be the surface area of the balloon at any time t. Then,

$$S = 4\pi r^2$$

$$\Rightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \quad \text{...(i) } \mathbf{1}$$

It is given that, $\frac{dS}{dt}$ = constant = k(say)

Putting $\frac{dS}{dt} = k$ in eq. (i), we get

$$k = 8\pi r\frac{dr}{dt}$$

$$\Rightarrow k = 8\pi r\frac{dr}{dt}$$

$$\Rightarrow 8\pi r dr = k dt$$

(By separating the variables)

Integrating both sides, we get

$$4\pi r^2 = kt + C \quad \text{...(ii) } ½$$

Given that at $t = 0$, $r = 3$. Putting $t = 0$ and $r = 3$ in eq. (ii), we get

$$36\pi = k(0) + C$$

$$\Rightarrow C = 36\pi$$

Putting $t = 2$ and $r = 5$ in eq. (ii), we get

$$100\pi = 2k + 36\pi$$

$$\Rightarrow k = 32\pi \quad \mathbf{1}$$

Substituting the values of C and k in eq. (ii), we get

$$4\pi r^2 = 32\pi t + 36\pi$$

$$\Rightarrow r^2 = 8t + 9$$

$$\Rightarrow r = \sqrt{8t+9} \quad ½$$

Q. 3. Doctors have shown certain drugs leave a person's bloodstream at a rate that is proportional to the amount present. In an experiment a patient is

injected with 450 mg of a substance. Seven hours later it is found that 50 mg of the substance remains. Assuming the proportional model is correct for the particular substance.

(a) Express the differential equation that models this scenario.

(b) Find the time constant, k.(log 9 = 2.197225). A

Long Answer Type Questions (5 marks each)

Q. 1. Assume that a spherical raindrop evaporates at a rate proportional to its surface area. If its radius originally in 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time. A

Sol. Let $r(t)$ denotes the radius of the raindrop after t minutes.

Since, given the radius is decreasing as t increases, the rate of change of r must be negative.

Let V be the volume of the raindrop and S be its surface area

i.e., $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ **1**

Also given, $\frac{dv}{dt} \propto S$

$$\Rightarrow \quad \frac{dv}{dt} = -kS$$

[$\therefore$ as the rate of change of radius with time is negative]

$$\therefore \quad \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = -k\,4\pi r^2 \quad \left[\because \frac{dv}{dt} = \frac{4}{3}\pi(3r^2)\right]$$

or $$\frac{dr}{dt} = -k \qquad \text{...(i) } \mathbf{1}$$

Integrating (i), we get

$$\int \frac{dr}{dt} = -\int k$$

$$\Rightarrow \quad \int dr = -\int k\,dt$$

$$\Rightarrow \quad r = -kt + C \qquad \text{...(ii) } \mathbf{2}$$

Let at time $t = 0$, radius is r_0

$\therefore$ From (ii), $r_0 = 0 + C = C$

Hence, $r = -kt + r_0$ is the required expression. **1**

Q. 2. A radio active substance has a half life of h days. Find a formula for its mass m in terms of t the time, if the initial mass is m_0. What is the initial decay rate ? A I

Q. 3. Suppose the growth of a population is proportional to the number present. If the population of a colony doubles in 25 days, in how many days will the population become triple ? A

Sol. Let P_0 be the initial population and P be the population at any time t.

Given, $$\frac{dP}{dt} \propto P$$

$$\Rightarrow \quad \frac{dP}{dt} = kP, \text{ } k \text{ is constant}$$

$$\Rightarrow \quad \frac{dP}{P} = k\,dt$$

$$\Rightarrow \quad \int \frac{dP}{P} = k\int dt$$

$$\Rightarrow \quad \log|P| = kt + C \qquad \text{...(i) } \mathbf{1\frac{1}{2}}$$

At $t = 0$, we have $P = P_0$

$\therefore$ From eq (i) $\log|P_{0|} = 0 + C$

$$\Rightarrow \quad C = \log|P_{0|}$$

Therefore, eq (i) becomes $\log P = kt + \log P_0$

$$\Rightarrow \quad \log\left|\frac{P}{P_0}\right| = kt \qquad \mathbf{1}$$

Also, given that $P = 2P_0$ when $t = 25$ days.

$\therefore$ From eq (ii), we have

$$\log\left|\frac{2P_0}{P_0}\right| = 25\,k$$

$$\Rightarrow \quad k = \frac{1}{25}\log 2$$

Put value of k in eq (ii), we get

$$\log\left|\frac{P}{P_0}\right| = \left(\frac{1}{25}\log 2\right)t \qquad \text{...(iii) } \mathbf{1}$$

Suppose the population is tripled in t_1 days i.e., $P = 3P_0$ when $t = t_1$

Put $P = 3P_0$ and $t = t_1$ in eq (ii), we get

$$\log\left|\frac{3P_0}{P_0}\right| = \left(\frac{1}{25}\log 2\right)t_1$$

$$\Rightarrow \quad t_1 = 25\left(\frac{\log 3}{\log 2}\right) \text{days}$$

Hence, population is tripled in $25\left(\frac{\log 3}{\log 2}\right)$ days . **1½**

Q. 4. It is given that radium decomposes at a rate proportional to the amount present. If P% of the original amount of radium disappears in l years, what percentage of it will remain after $2l$ years ? A

Sol. Let A_0 be the original amount of radium and A be the amount of radium at any time t. Then, the rate of decomposition of radium is $\frac{dA}{dt}$.

Given, $\frac{dA}{dt} \propto A$

$\Rightarrow \quad \frac{dA}{dt} = -kA$, where k is positive constant

$\Rightarrow \quad \frac{dA}{A} = -kdt$

$\Rightarrow \quad \int\frac{dA}{A} = -k\int dt$

$\Rightarrow \log|A| = -kt + C$ ...(i) **1½**

At $t = 0$, we have $A = A_0$, therefore from eq (i), we get

$\log|A_0| = 0 + C \Rightarrow C = \log A_0$

$\therefore$ Eq (i) becomes,

$\log A = -kt + \log A_0$

$\Rightarrow \quad \log\left|\frac{A}{A_0}\right| = -kt$ ...(ii) **1**

Given that $p\%$ of the original amount of radium disintegrates in l years.

$\therefore$ Amount of radium present at $t = l$ is

$$\left(A_0 - \frac{p}{100}\times A_0\right) = \left(1-\frac{p}{100}\right)A_0$$

Put $A = A_0\left(1-\frac{p}{100}\right)$ and $t = l$ in eq. (ii), we get

$$\log\left(1-\frac{p}{100}\right) = -kl$$

$$\Rightarrow \quad k = \frac{-1}{l}\log\left(1-\frac{p}{100}\right)$$

Substitute value of k in eq (ii), we get

$$\log\left(\frac{A}{A_0}\right) = \frac{t}{l}\log\left(1-\frac{p}{100}\right) \quad \text{...(iii) } \mathbf{1}$$

Let A be the amount of radium available after $2l$ years

Put $t = 2l$ in eq (iii), we get

$$\log\left|\frac{A}{A_0}\right| = 2\log\left|1-\frac{p}{100}\right|$$

$$\Rightarrow \quad \left(\frac{A}{A_0}\right) = \left(1-\frac{p}{100}\right)^2 \quad \mathbf{1}$$

Multiplying both sides by 100

$$\Rightarrow \quad \frac{A}{A_0}\times 100 = \left(1-\frac{p}{100}\right)^2 \times 100$$

$$\Rightarrow \quad \frac{A}{A_0}\times 100 = \left(10-\frac{p}{10}\right)^2$$

Hence, required percentage is $\left(10-\frac{p}{10}\right)^2$. ½

Q. 5. The population of a village increases at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what was its population in 2009 ?

Sol. Let the population at time t be x.

Given, rate of population increase is proportional to number of inhabitants.

i.e., $\frac{dx}{dt} \propto x$

$\Rightarrow \quad \frac{dx}{dt} = kx,$

where k is constant of proportionality

$$\frac{dx}{x} = kdt$$

$\Rightarrow \quad \int\frac{dx}{x} = k\int dt$

$\Rightarrow \quad \log|x| = kt + C$ ...(i) **1**

In 1999, $t = 0$ and population = 20,000

$\therefore \quad \log 20{,}000 = C$

Put value of C in eq (i), we get

$\log|x| = kt + \log 20{,}000$

$\Rightarrow \log|x| - \log 20{,}000 = kt$

$\Rightarrow \quad \log\left(\frac{x}{20{,}000}\right) = kt$ ...(ii) **1**

In 2004, $t = 5$, $x = 25{,}000$

$\therefore \quad \log\left(\frac{25000}{20000}\right) = k\times 5$

$\Rightarrow \quad k = \frac{1}{5}\log\left(\frac{5}{4}\right)$ **1**

$\therefore$ From eq (ii) becomes

$$\log\left|\frac{x}{20{,}000}\right| = \left(\frac{1}{5}\log\frac{5}{4}\right)t$$

In 2009, $t = 10$

$\therefore \quad \log\left|\frac{x}{20{,}000}\right| = \left(\frac{1}{5}\log\frac{5}{4}\right)\times 10$

$\Rightarrow \quad \log\left|\frac{x}{20{,}000}\right| = 2\log\left(\frac{5}{4}\right)$

$= \log\left(\frac{5}{4}\right)^2$

$= \log\left(\frac{25}{16}\right)$

$\frac{x}{20{,}000} = \frac{25}{16}$

$\Rightarrow \quad x = \frac{25\times 20{,}000}{16}$

$= 31{,}250$

Hence, in 2009 the population was 31,250. **2**

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

Read the following text and answer the following questions on the basis of the same:

I. Riya a biologist runs an experiment in the laboratory involving a culture of bacteria. She is doing this experiment first time. She notices that the mass of the bacteria in culture increases exponentially. The bacteria count in the culture is 1,00,000. The rate of growth of bacteria is proportional to the number of bacteria present in the culture. Also, the number of bacteria is increased by 10% in 2 hours.

Based on the above given information answer the following questions:

Q. 1. If y be the number of bacteria at time t, then the differential equation that models this scenario is:

(A) $\frac{dy}{dx} = ky$ **(B)** $\frac{dy}{dt} = ky$

(C) $\frac{dy}{dt} = kt$ **(D)** $\frac{dy}{dx} = kt$

Ans. Option (B) is correct.

Explanation:

If y be the number of bacteria at time t, then

$$\frac{dy}{dt} \propto y$$

$$\Rightarrow \quad \frac{dy}{dt} = ky$$

Q. 2. The general solution of the associated differential equation is:

(A) $\log y = kt + c$ **(B)** $\log x = kt + c$

(C) $\log y = kx + c$ **(D)** $\log y = kt$

Ans. Option (A) is correct.

Explanation:

$$\frac{dy}{dt} = ky$$

$$\Rightarrow \quad \frac{dy}{y} = kdt$$

Integrating both sides, we get

$$\log y = kt + c$$

Q. 3. The value of integration constant 'c' is:

(A) 1,00,000 **(B)** log 1000

(C) log 1,00,000 **(D)** 1,10,000

Ans. Option (C) is correct.

Explanation:

Put $t = 0$ and $y = 1,00,000$ in equation $\log y = kt + c$, we get

$$\log 1,00,000 = k \times 0 + c$$

$$\Rightarrow \quad c = \log 1,00,000$$

Q. 4. Count of bacteria after 2 hours:

(A) 1,00,000 **(B)** log 1000

(C) log 1,00,000 **(D)** 1,10,000

Ans. Option (D) is correct.

Explanation:

When, $t = 2$ hours

$$y = 1,00,000 + 10\% \times 1,00,000$$

$$= 1,10,000$$

Q. 5. In how many hours will the bacteria count reach 2,00,000 ?

(A) $\frac{\log 2}{\log\left(\frac{10}{11}\right)}$ **(B)** $\frac{2\log 2}{\log\left(\frac{10}{11}\right)}$

(C) $\frac{\log 2}{\log\left(\frac{11}{10}\right)}$ **(D)** $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$

Ans. Option (D) is correct.

Explanation:

Put $c = \log 1,00,000$ and $y = 1,10,000$ in $\log y = kt + c$ we get

$$\log y = kt + \log 1,00,000 \qquad ...(i)$$

Now when $t = 2$, $y = 1,10,000$

So, from equation (i), we get

$$\log 1,10,000 = 2k + \log 1,00,000$$

$$\log(1,00,000 - 1,10,000) = 2k$$

$$\Rightarrow \quad k = \frac{1}{2}\log\left(\frac{11}{10}\right)$$

$$\left[\log m - \log n = \log \frac{m}{n}\right]$$

Now finding t when bacteria = 2,00,000

$$\log 2,00,000 = \frac{1}{2}\log\left(\frac{11}{10}\right) t + \log(1,00,000)$$

$$\log\left(\frac{2,00,000}{1,00,000}\right) = \frac{1}{2}\log\left(\frac{11}{10}\right)t$$

$$\log 2 = \frac{1}{2}\log\left(\frac{11}{10}\right)t$$

$$t = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

II. Mr. Lal was 75 year old businessman. He had two sons and three daughters. On the silver jubilee of his company, he gifted an equal amount of money among his children. One of his daughters buys jewellery from the gifted money whereas the other daughter invested his money in bank.

It is known that if the interest is compounded continuously, the principal changes at the rate equal to the product of the rate of bank interest per annum and principal.

Based on the above given information answer the following questions:

Q. 1. If P denotes the principal at time t and the rate of interest be r% per annum compounded continuously, then the differential equation that models this scenario is:

(A) $\frac{dP}{dt} = \frac{Pr}{100}$ **(B)** $\frac{dP}{dt} = \frac{100}{Pr}$

(C) $\frac{dP}{dt} = Pr$ **(D)** $\frac{dt}{dP} = \frac{Pr}{100}$

Ans. Option (A) is correct.

Explanation:

If P denotes the principal at time t and the rate of interest be r% per annum compounded continuously, then according to the law given in the problem, we get

$$\frac{dP}{dt} = \frac{Pr}{100}$$

Q. 2. The particular solution of the associated differential equation is:

(A) $\log\left(\frac{P}{P_0}\right) = \frac{100}{rt}$ **(B)** $\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$

(C) $\log\left(\frac{P_0}{P}\right) = \frac{100}{rt}$ **(D)** $\log\left(\frac{P_0}{P}\right) = \frac{rt}{100}$

Ans. Option (B) is correct.

Explanation:

Since, $\frac{dP}{dt} = \frac{Pr}{100}$

$$\Rightarrow \quad \frac{dP}{P} = \frac{r}{100}dt$$

$$\Rightarrow \quad \int\frac{dP}{P} = \frac{r}{100}\int dt$$

$$\Rightarrow \quad \log P = \frac{rt}{100} + C \qquad \text{...(i)}$$

Let P_0 be the initial principal i.e., at $t = 0$, $P = P_0$.

Putting $P = P_0$ in eq. (i), we get

$$\log P_0 = C$$

Putting $C = \log P_0$ in eq. (i), we get

$$\log P = \frac{rt}{100} + \log P_0$$

$$\Rightarrow \quad \log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$$

Q. 3. If the interest is compounded continuously at 5% per annum, in how many years will ₹ 100 double itself ? ($\log_e 2 = 0.6931$)

(A) 13.86 **(B)** 13

(C) 14 **(D)** 14.82

Ans. Option (A) is correct.

Explanation:

We have, $\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$

Put $r = 5$, $P_0 = ₹\ 100$ and $P = ₹\ 200 = 2P_0$ in the above equation, we get

$$\log 2 = \frac{5t}{100}$$

$$\Rightarrow \quad t = 20\log_e 2$$
$$= 20 \times 0.6931$$
$$= 13.86 \text{ years}$$

Q. 4. At what rate interest rate will ₹ 100 double itself in 10 years ? ($\log_e 2 = 0.6931$)

(A) 6% **(B)** 7%

(C) 6.93% **(D)** 7.36%

Ans. Option (C) is correct.

Explanation:

We have,

$$\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$$

Put $t = 10$, $P_0 = ₹\ 100$ and $P = ₹\ 200 = 2P_0$ in the above equation, we get

$$\log 2 = \frac{10r}{100}$$

$$\Rightarrow \quad r = 10\log_e 2$$
$$= 10 \times 0.6931$$
$$= 6.931\% \text{ or } 6.93\ \% \text{ per annum}$$

Q. 5. How much will ₹ 1000 be worth at 5% interest after 10 years ? ($e^{0.5} = 1.648$)

(A) 1640 **(B)** 1700

(C) 1.648 **(D)** 1648

Ans. Option (D) is correct.

Explanation:

We have, $\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$

Put $r = 5\%$, $P_0 = ₹\ 1000$ and $t = 10$ in the above equation, we get

$$\log\left(\frac{P}{1000}\right) = \frac{5\times10}{100}$$
$$= \frac{1}{2} = 0.5$$

$$\Rightarrow \quad \frac{P}{1000} = e^{0.5}$$
$$\Rightarrow \quad P = 1000 \times e^{0.5}$$
$$\Rightarrow \quad = 1000 \times 1.648 = 1648$$

Case based Subjective Questions

Read the following text and answer the following questions on the basis of the same:

Each sub part carry 2 Marks.

I. Polio drops are delivered to 50K children in a district. The rate at which polio drops are given is directly proportional to the number of children who have not been administered the drops. By the end of 2nd week half the children have been given the polio drops. How many will have been given the drops by the end of 3rd week can be estimated using the solution to the differential equation $\frac{dy}{dx} = k(50 - y)$ where x denotes the number of weeks and y the number of children who have been given the drops.

Q. 1. Find the order and general solution of given differential equation $\frac{dy}{dx} = k(50 - y)$

Sol. The order of the above given differential equation is 1. ½

$$\frac{dy}{dx} = k(50 - y)$$
$$\int \frac{dy}{50 - y} = \int k dx$$
$$-\log|50 - y| = kx + C \qquad \textbf{1½}$$

Q. 2. Find in the particular solution differential equation $\frac{dy}{dx} = k(50 - y)$ if $y(0) = 0$ and $k = 0.049$.

Sol. Given, $y(0) = 0$ and $k = 0.049$

We have, $-\log|50 - y| = kx + C$

$$\log|50 - y| = -kx - C$$
$$\log|50 - 0| = 0 - C$$
$$[\because x = 0, K = 0.049, y(0) = 0]$$
$$\log 50 = -C$$
$$C = \log\frac{1}{50} \qquad \textbf{1}$$
$$-\log|50 - y| = kx + C$$
$$-\log|50 - y| = kx + \log\frac{1}{50}$$
$$\log\frac{50 - y}{50} = -kx$$
$$\frac{50 - y}{50} = e^{-kx}$$
$$50 - y = 50e^{-kx}$$
$$y = 50 - 50e^{-kx}$$
$$y = 50(1 - e^{-kx}) \qquad \textbf{1}$$

II. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours.

Q. 1. If 'N' is the number of bacteria, then find corresponding differential equation. Also, find the general and particular solution. Given N_0 is the initial count of bacteria.

Sol. Given that N is the number of bacteria.

$$\frac{dN}{dt} \propto N$$
$$\Rightarrow \quad \frac{dN}{dt} = KN$$
$$\int \frac{dN}{dt} = K\int dt$$
$$\log|N| = Kt + C \qquad \text{...(i) } \textbf{1}$$

Given when $t = 0$, $N = N_0$.

From (i), $\log|N_0| = C$

(i), $\Rightarrow$ $\log|N| = Kt + \log|N_0|$

$$\Rightarrow \quad \log\left|\frac{N}{N_0}\right| = Kt \qquad \text{...(ii)}$$

Given when $t = 5$, $N = 3N_0$.

From (ii), $\log|3| = 5K$

$$\Rightarrow \quad K = \frac{1}{5}\log 3$$

$\therefore$ The particular solution is

$$\log\left|\frac{N}{N_0}\right| = \frac{t}{5}\log 3 \qquad \text{...(iii) } \textbf{1}$$

Q. 2. Find the bacterial count after 10 hours. Also, find in how time the bacteria count will becomes 10 times.

Sol. When $t = 10$,

$$\log\left|\frac{N}{N_0}\right| = 2\log 3 = \log 9$$
$$\frac{N}{N_0} = 9$$
$$\Rightarrow \quad N = 9N_0 \qquad \textbf{1}$$

Given, $N = 10N_0$,

$$\log\left|\frac{N}{N_0}\right| = \frac{t}{5}\log 3$$

(iii) $\Rightarrow$ $\log 10 = \frac{t}{5}\log 3$

$$t = \frac{5\log 10}{\log 3} \qquad \textbf{1}$$

Solutions for Practice Questions (Topic-1)

Very Short Answer Type Questions

Sol. 3: Given, $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$

The highest order derivative is $\left(\frac{d^2y}{dx^2}\right)$, hence the order of given differential equation is 2.

Also, the power of the highest order derivative is 2, hence the degree of differential equation is 2.

Therefore, Order = 2 and Degree = 2 **1**

Commonly Made Error

- Some students take the degree as the highest power of the derivative.

Answering Tip

- Degree is the highest power of the higher order derivative.

Sol. 8: On differentiating the given equation, we get

$$\frac{dV}{dr} = -\frac{A}{r^2} \qquad \text{½}$$

$$r^2\frac{dV}{dr} = -A$$

Differentiating again, we get

$$r^2\frac{d^2V}{dr^2} + 2r\frac{dV}{dr} = 0 \qquad \text{½}$$

Sol. 11: Given equation : $x^2 + y^2 = a^2$

Differentiate *w.r.t.* x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0 \qquad \textbf{1}$$

Short Answer Type Questions-I

Sol. 3: $y = a\cos 2x + b\sin 2x$

or $\frac{dy}{dx} = -2a\sin 2x + 2b\cos 2x$ **1**

or $\frac{dy}{dx} = 2[-a\sin 2x + b\cos 2x]$

or $\frac{d^2y}{dx^2} = -4[a\cos 2x + b\sin 2x] = -4y$

$$\frac{d^2y}{dx^2} + 4y = 0 \qquad \textbf{1}$$

Sol. 9: Given that, $\frac{dy}{dx} = 2^{y-x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2^y}{2^x} \qquad \left[\because a^{m-n} = \frac{a^m}{a^n}\right]$$

$$\Rightarrow \quad \frac{dy}{2^y} = \frac{dy}{2^x}$$

On integrating both sides,we get

$$\int 2^{-y}dy = \int 2^{-x}dx$$

$$\Rightarrow \quad \frac{-2^{-y}}{\log 2} = \frac{-2^{-x}}{\log 2} + C$$

$$\Rightarrow -2^{-y} + 2^{-x} = +C\log 2$$

$$\Rightarrow \quad 2^{-x} - 2^{-y} = -C\log 2$$

$$\Rightarrow \quad 2^{-x} - 2^{-y} = k \qquad [\text{where, } k = +C\log 2]$$

Sol. 11. Given, $\log\left(\frac{dy}{dx}\right) = 2x - 3y$

$$\Rightarrow \quad \frac{dy}{dx} = e^{2x-3y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{2x}}{e^{3y}}$$

$$\Rightarrow \quad e^{3y}dy = e^{2x}dx \qquad \textbf{1}$$

Integrating both sides, we get

$$\int e^{3y}dy = \int e^{2x}dx$$

$$\Rightarrow \frac{e^{3y}}{3} = \frac{e^{2x}}{2} + C \qquad \textbf{1}$$

is the required solution.

Short Answer Type Questions-II

Sol. 7: $xy\frac{dy}{dx} = (x+2)(y+2)$

or $\frac{y}{y+2}dy = \frac{x+2}{x}dx$

or $\left(1-\frac{2}{y+2}\right)dy = \left(1+\frac{2}{x}\right)dx$ ½

On integrating it, we get

$y - 2\log(y+2) = x + 2\log x + C$ ...(i) **1**

Given $y = -1$, when $x = 1$, then from (i)

$-1 - 2\log(-1+2) = 1 + 2\log 1 + C$

or $C = -2$, as $\log 1 = 0$ **1**

Then (i) becomes:

$y - 2\log(y+2) = x + 2\log x - 2$

or $y = x - 2 + 2\{\log(y+2) + \log x\}$

or $y = x - 2 + 2\log\{x(y+2)\}$

This is the required particular solution. ½

Sol. 10: Given, $\frac{dy}{dx} = e^{x+y} + x^2e^y$

$= e^x.e^y + x^2.e^y$

$\Rightarrow \frac{dy}{dx} = e^y(e^x + x^2)$

$\int\frac{dy}{e^y} = \int(e^x + x^2)dx$

(Integrating both sides)

$\Rightarrow \int e^{-y}dy = \int e^x dx + \int x^2 dx$ **1**

$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + c$ **1**

$\Rightarrow -\frac{1}{e^y} = e^x + \frac{x^3}{3} + c$

or $e^{x+y} + \frac{e^y.x^3}{3} + ce^y + 1 = 0$

is the required solution. **1**

Long Answer Type Questions

Sol. 1: $(x-a)^2 + (y-b)^2 = r^2$ ...(i)

or $2(x-a) + 2(y-b)\frac{dy}{dx} = 0$ ...(ii) ½

or $1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ ...(iii) ½

$\therefore (y-b) = -\frac{(1+y_1^2)}{y_2}$ **1**

From (ii), $(x-a) = \frac{y_1(1+y_1^2)}{y_2}$ **1**

Putting these values in eqn. (i),

$\frac{y_1^2(1+y_1^2)^2}{y_2^2} + \frac{(1+y_1^2)^2}{y_2^2} = r^2$ **1**

or $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$ **1**

Solutions for Practice Questions (Topic-2)

Short Answer Type Questions-II

Sol. 3: **(a)** With the amount of the substance represented as A(t), we can write the differential equation as

$\frac{dA}{dt} = kA$ **1**

(b) To find the time constant, k, we first write the expression for the general solution as

$A(t) = A(0)e^{kt} = 450e^{kt}$

Where $A(0) = 450$

Next, we use the fact that 50 mg was found in the bloodstream after 7 hours to solve for k.

$A(7) = 450e^{k.7}$

$50 = 450e^{k.7}$

$\frac{1}{9} = e^{k.7}$

$\log\frac{1}{9} = k.7$

$\log\frac{1}{9} = k = \frac{-\log 9}{7} \cong -0.31$

As expected, k is negative since the substance decays over time. **2**

Long Answer Type Questions

Sol. 2: According to question,

The equation of decay is $\frac{dm}{dt} = -km$ ...(i)

$\therefore \frac{dm}{m} = -k\,dt$ **1**

Integrating both sides, we get

$\int\frac{dm}{m} = -k\int dt$

$\therefore \log m = -kt + C$ ...(ii) ½

Initially, $t = 0$, $m = m_0$ $\therefore \log m_0 = C$

and $\log m = -kt + \log m_0$ ½

Again when $t = h$ (half life), $m = \frac{m_0}{2}$

[$\because$ half life means – half mass decays in h days]

$$\Rightarrow \quad \log\left|\frac{m_0}{2}\right| = -kh + \log|m_0|$$

$$\Rightarrow \quad kh = \log|m_0| - \log\left|\frac{m_0}{2}\right|$$

$$= \log\left|\frac{m_0}{m_0/2}\right| = \log 2$$

$$\Rightarrow \quad k = \frac{\log 2}{h} \qquad \mathbf{1}$$

From eq. (ii),

$$\log m = \frac{-\log 2}{h}.t + \log m_0$$

[$\therefore C = \log m_0$]

$$\therefore \quad \log|m| - \log|m_0| = \frac{-t}{h}\log 2$$

$$\Rightarrow \quad \log\left|\frac{m}{m_0}\right| = \log 2^{-\left(\frac{t}{h}\right)}$$

$$\Rightarrow \quad \frac{m}{m_0} = 2^{-\left(\frac{t}{h}\right)}$$

$$\Rightarrow \quad m = m_0.2^{-\left(\frac{t}{h}\right)} \qquad \text{...(ii) } \mathbf{1}$$

Which is the formula for its mass m in terms of t.

The initial decay rate is given by $\frac{dm}{dt}$ when $m = m_0$ and $k = \frac{\log 2}{h}$

Hence, $\frac{dm}{dt} = -km = \frac{-\log 2}{h} \times m_0$ **1**

Long Answer Type Questions

Sol. 1: $(x-a)^2 + (y-b)^2 = r^2$ ...(i)

or $2(x-a) + 2(y-b)\frac{dy}{dx} = 0$ ...(ii) ½

or $1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ ...(iii) ½

$$\therefore (y-b) = -\frac{(1+y_1^2)}{y_2} \qquad \mathbf{1}$$

From (ii), $(x-a) = \frac{y_1(1+y_1^2)}{y_2}$ **1**

Putting these values in eqn. (i),

$$\frac{y_1^2(1+y_1^2)^2}{y_2^2} + \frac{(1+y_1^2)^2}{y_2^2} = r^2 \qquad \mathbf{1}$$

or $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$ **1**

REFLECTIONS

(a) Will you be able to solve the questions based on growth and decay model effortlessly?

(b) Can you find the order and degree of differential equation, when differential equation contains radial sign?

(c) Find any other application of differential equation apart from growth and decay model.

❑❑❑

SELF ASSESSMENT PAPER – 03

Maximum Mark: 30 **Max. Time: 1 Hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. The solution of $\int a^x da$ is

(A) $\frac{a^x}{\log_e a} + c$ **(B)** $a^x \log_e a + c$

(C) $\frac{a^{x+1}}{x+1} + c$ **(D)** $xa^{x-1} + c$

Q. 2. The value of $\int \frac{(1+\log x)^2}{x} dx$ is

(A) $(1 + \log x)^3 + c$ **(B)** $3(1 + \log x)^3 + c$

(C) $\frac{1}{3}(1+\log x)^3 + c$ **(D)** None of these

Q. 3. Area bounded by parabola $y^2 = x$ and straight line $2y = x$ is _____ sq. units.

(A) $\frac{4}{3}$ **(B)** 1 **(C)** $\frac{2}{3}$ **(D)** $\frac{1}{3}$

Q. 4. Differentiate in ln10.

(A) 1 **(B)** – 1 **(C)** 0 **(D)** None of these

Q. 5. The anti-derivative of the following function $3x^2 + 4x^3$ is:

(A) $x^3 + x^4$ **(B)** $x^3 - x^4$ **(C)** $x^3 + x^{-4}$ **(D)** $x^{-3} + x^4$

Q. 6. The order and degree (if defined) of differential equation $(y'') + (y'')^3 + (y')^4 + y^5 = 0$ are:

(A) 2, 2 **(B)** 3, 2 (C) 2, 3 **(D)** 3, 3

Case-Based MCQs

Q. 7. Attempt any 4 sub-parts from each questions. Each question carries 1 mark.

Read the following text and answer the following questions on the basis of the same.

Whatsapp became the world's most popular messaging application by 2015. As of 2021, WhatsApp is most popular global mobile messenger app worldwide with approximately two billion monthly active users. WhatsApp has also been criticized for spreading of fake news.

In a population of 5000 people, a rumour on whatsapp spreads at a rate proportional to the product of the number of people who have heard it and the number of people who have not. Also, it is given that 100 people initiate the rumour and a total of 500 people know the rumour after 2 days.

(i) Find the maximum value of $y(t)$, if $y(t)$ denote the number of people who know the rumour at an instant t.

(A) 500 **(B)** 100 **(C)** 5000 **(D)** none of these

(ii) $\frac{dy}{dt}$ = ...

(A) $(y - 5000)$ (B) $y(y - 5000)$ **(C)** $y(500 - y)$ **(D)** $y(5000 - y)$

(iii) $y(0)$ = ...

(A) 100 (B) 500 **(C)** 600 **(D)** 200

(iv) $y(2)$ = ...

(A) 100 (B) 500 **(C)** 600 **(D)** 200

(v) At any time t, the value of y is:

(A) $y = \frac{5000}{(e^{-5000kt}+1)}$ (B) $y = \frac{5000}{(1+e^{-5000kt})}$ **(C)** $y = \frac{5000}{(49e^{-5000kt}+1)}$ **(D)** $y = \frac{5000}{49(1+e^{-5000kt})}$

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. Find: $\int \frac{3x}{3x-1}\,dx$.

Q. 9. Write the general solution of differential equation $\frac{dy}{dx} = e^{x+y}$.

Q. 10. Using integration, Find the area bounded by the function $f(x) = x^3$, the X-axis and the lines $x = -1$.

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. Find the value of $\int_0^1 x(1-x)^n\,dx$.

Q. 12. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Q. 13. Find the integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$.

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. Prove that: $\int_2^3 \frac{x}{x^2+1} = \log\sqrt{2}$

Q. 15. Today there are 1000 birds on an island. They breed with a constant continuous growth rate of 10% per year. Find how many birds will be on the island after seven years ? (Given $e^{0.7} = 2.013752$)

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 16. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

CHAPTER

8

PROBABILITY

Syllabus

Probability distribution, Mathematical Expectation, Variance, Binomial Distribution, Poisson Distribution, Normal Distribution.

In this chapter you will study

- *Random Variables*
- *Probability Mass Function*
- *Expected Value*
- *Variance*
- *Binomial Distribution*
- *Poisson Distribution*
- *Normal Distribution*

List of Topics

Topic-1: Expectation and Variance of Probability Distribution **Page No. 160**

Topic-2 : Binomial and Poisson Distributions **Page No. 171**

Topic-3 : Normal Distribution **Page No. 181**

Topic-1 Expectation and Variance of Probability Distribution

Concepts Covered • *Discrete and continuous random variables,* • *Probability mass function for discrete and continuous random variables,* • *Expected Value,* • *Variance*

Revision Notes

► **Random Variable:** A random variable (r.v.) is a real valued function defined on a **sample space** S and taking value in $(-\infty, \infty)$ or whose possible values are numerical outcomes of a **random experiment**.

Key Word

Sample space: A sample space is a collection or a set of possible outcomes of a random experiment. It is a range set of values of random variable.

Random experiment: A random experiment is any well-defined procedure that produces an observable outcome that could not be perfectly predicted in advance.

► **Types of Random Variable:** Random variables are classified into two types namely discrete and continuous random variables. These are important for practical application in the field of Mathematics and statistics. The above types of random variable are defined with examples as follows.

- **Discrete random variable:** A variable which can assume finite number of possible values or an infinite sequence of countable real numbers is called a discrete random variable.

Scan to know more about this topic

Probability distribution, expectation and variance

Probability

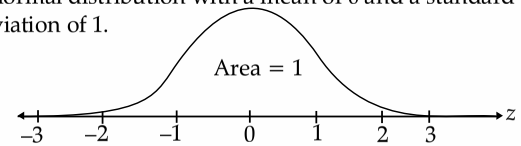
Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

μ = mean of x
σ = Standard deviation of x
$\pi \approx 3.14159....$
$e \approx 2.71828....$

Standard Normal Curve

- A normal distribution with a mean of 0 and a standard deviation of 1.



- Any x-value can be transformed into a z-score by using the formula

$$z = \frac{\text{Value} - \text{Mean}}{\text{Standard deviation}} = \frac{x - \mu}{\sigma}$$

Probability Distribution

The probability distribution of a random variable x is the system of numbers x: $x_1\, x_2$ ....., x_n
P (x): $p_1\, p_2$ .....p_n where, $p_i > 0$,
$\sum_{i=1}^{n} p_i = 1$, $i = 1, 2.....n$.

Expectation

Let x be a random variable whose possible values $x_1 x_2$,....., x_n occurs with probabilities $p_1\, p_2$ .....p_n resp. Then, mean of x, $\mu = \sum_{i=1}^{n} x_i p_i$, It is also called the expectation of x, denoted by E (x)

Variance

Let x be a random variable whose possible values $x_1 x_2$,....., x_n occurs with probabilities $p(x_1), p(x_2)$,..... $p(x_n)$, respectively, Let, $\mu = E(x)$ be the mean of x. The variance of x, var (x) or

$$\sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu)^2\ p\ (x_i) \text{ or } E\ [(x - \mu)^2]$$

Poisson Distribution

The mean number of power outages in the city or Brunswick is 4 per year. Find the probability that in a given year, there are exactly 3 outages,

$\mu = 4, x = 3$

$$P(3) = \frac{4^3 (2.71828)^{-4}}{3!}$$

≈ 0.195

$$P\ (X = x) = \frac{\mu^x e^{-\mu}}{x!}$$

where
$x = 0, 1, 2, 3,....$
μ = mean number of occurrences in the interval
e = Euler's constant ≈ 2.71828

Binomial Theorem

Let
x = the number of successes = 15
n = the number of trials = 18
p = the probability of success = 0.85

$$P\ (15) = {}^{18}C_{15} (0.85)^{15}\ (1 - 0.85)^{\ 18-15}$$

$$P\ (15) = \left(\frac{18!}{15!\ (18 - 15)!}\right) (0.85)^{15}\ [(1 - 0.85)^3]$$

$$P\ (15) = (816)\ (0.87)\ (.0036)$$

$$P\ (15) = .239$$

$$P\ (x) = \binom{n}{x}\ p^x\, q^{n-x} = \frac{n!}{(n-x)!x!}\ p^x\, q^{n-x}$$

where
n = the number of trials (or the number being sampled)
x = the number of successes desired
p = probability of getting a success in one trial
$q = 1 - p$ = the probability of getting a failure in one trial

Trace the Mind Map
First Level Second Level Third Level

Examples of discrete random variable:

- Marks obtained in a test.
- Number of red marbles in a jar.
- Number of telephone calls at a particular time.
- Number of cars sold by a car dealer in one month, etc.,

Probability Mass Function of Discrete Random Variable:

If X is a discrete random variable with distinct values $x_1, x_2, ..., x_n, ...$, then the function, denoted by $P_X(x)$ and defined by $P_X(x) = p(x)$

$$= \begin{cases} P(X = x_i) = p_i = p(x_i) & \text{if } x = x_i, i = 1, 2, ..., n, ... \\ 0 & \text{if } x \neq x_i \end{cases}$$

This is defined to be the probability mass function or discrete probability function of X. The probability mass function $p(x)$ must satisfy the following conditions

(i) $p(x_i) \geq 0 \ \forall \ i,$ **(ii)** $\sum_{i=1}^{\infty} p(x_i)$

Example 1: The number of cars in a household is given below.

No. of cars	0	1	2	3	4
No. of Household	30	320	380	190	80

Estimate the probability mass function. Verify $p(x_i)$ is a probability mass function.

Sol. Let X be the number of cars

$X = x_i$	**Number of Household**	$p(x_i)$
0	30	0.03
1	320	0.32
2	380	0.38
3	190	0.19
4	80	0.08
Total	1000	1.00

Key Facts

- A probability distribution depicts the expected outcomes of possible values for a given data generating process.
- Probability distributions help to model our world, enabling us to obtain estimates of the probability that a certain event may occur, or estimate the variability of occurrence.
- Probability distributions help to forecast power failures and network outages.

(i) $p(x_i) \geq 0 \ \forall \ i$ and

(ii) $\sum_{i=1}^{\infty} p(x_i) = p(0) + p(1) + p(2) + p(3) + p(4)$

$= 0.03 + 0.32 + 0.38 + 0.19 + 0.08 = 1$

Hence $p(x_i)$ is a probability mass function.

Note:

For $X = 0$, the probability 0.03, comes from $\frac{30}{1000}$, the other probabilities are estimated similarly.

Discrete Distribution Function

The discrete cumulative distribution function or distribution function of a real valued discrete random variable X takes the countable number of points $x_1, x_2, ...$ with corresponding probabilities $p(x_1), p(x_2), ...$ and then the cumulative distribution function is defined by

$$F_X(x) = P(X \leq x), \text{ for all } x \in R$$

i.e., $$F_X(x) = \sum_{x_i \leq x} p(x_i)$$

For instance, suppose we have a family of two children. The sample space

$S = \{bb, bg, gb, gg\}$, where b = boy and g = girl

Let X be the random variable which counts the number of boys. Then, the values (X) corresponding to the sample space are 2, 1, 1, and 0.

Hence, the probability mass function of X is

$X = x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Then, we can from a cumulative distribution function of X is

$X = x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$F_X(x) = P(X \leq x)$	$\frac{1}{4}$	$\frac{1}{4}+\frac{1}{2} = \frac{3}{4}$	$\frac{3}{4}+\frac{1}{4} = 1$

- **Continuous random variable**

  A random variable X which can take on any value (integral as well as fraction) in the interval is called continuous random variable.

  Examples of continuous random variable

- The amount of water in a 10 ounce bottle.
- The speed of a car.
- Electricity consumption in kilowatt hours.
- Height of people in a population.
- Weight of students in a class.
- The length of time taken by a truck driver to go from Chennai to Madurai, etc.

Probability Mass Function of Continuous Random Variable

The probability that a random variable X takes a value in the interval $[t_1, t_2]$ (open or closed) is given by the integral of a function called the probability density function $f_X(x)$:

$$P(t_1 \leq X \leq t_2) = \int_{t_1}^{t_2} f_X(x)dx.$$

Other names that are used instead of probability density function include density function,

continuous probability function, integrating density function.

The probability density function $f_X(x)$ or simply by $f(x)$ must satisfy the following conditions.

(i) $f(x) \geq \forall\ x$ and (ii) $\int_{-\infty}^{\infty} f(x)dx = 1.$

Example 2: A continuous random variable X has the following p.d.f $f(x) = ax, 0 \leq x \leq 1.$

Determine the constant a and also find $P\left[X \leq \frac{1}{2}\right]$

Sol. Since, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^1 ax\ dx \Rightarrow a\int_0^1 xdx = 1$$

$$\Rightarrow \quad a\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$\Rightarrow \quad \frac{a}{2}(1-0) = 1$$

$$\Rightarrow \quad a = 2$$

$$P\left[x \leq \frac{1}{2}\right] = \int_{-\infty}^{\frac{1}{2}} f(x)dx$$

$$= \int_{-\infty}^{\frac{1}{2}} 2xdx = \frac{1}{4}$$

Continuous Distribution Function

If X is a continuous random variable with the probability density function $f_X(x)$, then the function $F_X(x)$ is defined by $F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)dt, -\infty < x < \infty$ is called the distribution function (d.f.) or sometimes the cumulative distribution function (c.d.f) of the continuous random variable X.

Properties of cumulative distribution function

The function $F_X(x)$ or simply $F(x)$ has the following properties

(i) $0 \leq F(x) \leq 1, -\infty < x < \infty$

(ii) $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$ and

$F(+\infty) = \lim_{x\to-\infty} F(x) = 1.$

(iii) $F(x)$ is a monotone, non-decreasing function; that is, $F(a) \leq F(b)$ for $a < b$.

(iv) $F(x)$ is continuous from the right; that is, $\lim_{h\to 0} F(x+h) = F(x).$

(v) $F'(x) = \frac{d}{dx}F(x) = f(x) \geq 0$

(vi) $F'(x) = \frac{d}{dx}F(x) = f(x) \Rightarrow dF(x) = f(x)dx$, $dF(x)$ is know as probability differential of X.

(vii) $P(a \leq x \leq b) = \int_a^b f(x)dx$

$$= \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$$

$$= P(X \leq b) - P(X \leq a)$$

$$= F(b) - F(a)$$

Example 3: Suppose, the life in hours of a radio tube has the following p.d.f

$$f(x) = \begin{cases} \frac{100}{x^2}, & \text{when} \quad x \geq 100 \\ 0, & \text{when} \quad x < 100 \end{cases}$$

Find the distribution function.

Sol. $F(x) = \int_{-\infty}^{x} f(t)dt$

$$= \int_{100}^{x} \frac{100}{t^2}dt,\ x \geq 100$$

$$= \left[\frac{100}{-t}\right]_{100}^{x},\ x \geq 100$$

$$F(x) = \left[1 - \frac{100}{x}\right],\ x \geq 100$$

- **Expected value:** The expected value is a weighted average of the values of a random variable may assume. The weights are the probabilities.

Let X be a discrete random variable with probability mass function (p.m.f.) $p(x)$. Then, its expected value is defined by

$$E(X) = \sum_x xp(x) \qquad ...(1)$$

If X is a continuous random variable and $f(x)$ is the value of its probability density function at x, the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx \qquad ...(2)$$

- **Variance:** The variance is a weighted average of the squared deviations of a random variable from its mean. The weights are the probabilities. The mean of a random variable X, defined in (1) and (2), was a measure of central location of the density of X. The variance of a random variable X will be a measure of the spread or dispersion of the density of X or simply the variability in the values of a random variable.

The variance of X is defined by

$$Var(X) = \sum [x - E(X)]^2 p(x) \qquad ...(3)$$

If X is discrete random variable with probability mass function $p(x)$.

$$Var(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 f_X(dx)dx \qquad ...(4)$$

if X is continuos random variable with probability density function $f_X(x)$.

Expected value of $[X - E(X)]^2$ is called the variance of the random variable.

i.e., $Var(X) = E[X - E(X)]^2 = E(X^2) - [E(X)]^2$ ...(5)

where $E(X^2) = \begin{cases} \sum_x x^2 p(x), & \text{if } X \text{ is Discrete Random Variable} \\ \int_{-\infty}^{\infty} x^2 f(x)dx, & \text{if } X \text{ is Continuous Random Variable} \end{cases}$

Example 4: Determine the mean and variance of the random variable X having the following probability distribution.

$X = x$	1	2	3	4	5	6	7	8	9	10
$P(x)$	0.15	0.10	0.10	0.01	0.08	0.01	0.05	0.02	0.28	0.20

Sol. Mean of the random variable

$$X = E(X) = \sum_x xP_X(x)$$

$$= (1 \times 0.15) + (2 \times 0.10) + (3 \times 0.10) + (4 \times 0.01) + (5 \times 0.08) + (6 \times 0.01) + (7 \times 0.05) + (8 \times 0.02) + (9 \times 0.28) + (10 \times 0.20)$$

$$E(X) = 6.18$$

$$E(X^2) = \sum_x x^2 P_X(x)$$

$$= (1^2 \times 0.15) + (2^2 \times 0.10) + (3^2 \times 0.10) + (4^2 \times 0.01) + (5^2 \times 0.08) + (6^2 \times 0.01) + (7^2 \times 0.05) + (8^2 \times 0.02) + (9^2 \times 0.28) + (10^2 \times 0.20)$$

$$= 50.38$$

Variance of the Random Variable $X = V(X) = E(X^2) - [E(X)]^2$

$$= 50.38 - (6.18)^2$$

$$= 12.19$$

Therefore, the mean and variance of the given discrete distribution are 6.18 and 12.19 respectively.

Example 5: Consider a random variable X with probability density function

$$f(x) = \begin{cases} 4x^3, \text{ if } 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$$

Find E(X) and V(X).

Sol. We know that, $E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$$= \int_0^1 x\, 4x^3\, dx$$

$$= 4\left[\frac{x^5}{5}\right]_0^1$$

$$E(X) = \frac{4}{5}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$$

$$= \int_0^1 x^2\, 4x^3 dx$$

$$= 4\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{4}{6} = \frac{2}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2}{3} - \left[\frac{4}{5}\right]^2$$

$$= \frac{2}{75}$$

Mnemonics

Concept: Independent and Mutually exclusive events.

I Is not **M**E

ME Is not **I**

Here, **I**: Independent Events

ME: Mutually Exclusive events

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. Let X be a discrete random variable whose probability distribution is given below:

$X = x_i$	0	1	2	3	4	5	6	7
$P(X = x_i)$	0	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2$	$2K$

The value of K is:

(A) $\frac{1}{10}$ **(B)** -1

(C) $-\frac{1}{10}$ **(D)** $\frac{1}{5}$

Ans. Option (A) is correct.

Explanation: Here, $\Sigma P_i = 1$

$\therefore 0 + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + 2K = 1$

$\Rightarrow 9K + 10K^2 = 1$

$\Rightarrow 10K^2 + 9K - 1 = 0$

$\Rightarrow 10K^2 + 10K - K - 1 = 0$

$\Rightarrow 10K(K + 1) - 1(K + 1) = 0$

$\Rightarrow (K + 1)(10K - 1) = 0$

$\Rightarrow K = -1$ and $K = \frac{1}{10}$

Since, $K \geq 0$ therefore $K = \frac{1}{10}$

Q. 2. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. Then the possible values of X are:

(A) 0, 1, 3, 5 **(B)** 0, 2, 4, 6

(C) 0, 2, 5, 6 **(D)** 1, 3, 4, 5

Ans. Option (B) is correct.

Explanation: The coin is tossed six times and X represents the difference between the number of heads and the number of tails.

$\therefore$ $X(6H, 0T) = |6 - 0| = 6$

$X(5H, 1T) = |5 - 1| = 4$

$X(4H, 2T) = |4 - 2| = 2$

$X(3H, 3T) = |3 - 3| = 0$

$X(2H, 4T) = |2 - 4| = 2$

$X(1H, 5T) = |1 - 5| = 4$

$X(0H, 6T) = |0 - 6| = 6$

Thus, possible values are 0, 2, 4, 6.

Q. 3. The mean of the probability distribution of the number of doublets in 4 throws of a pair of dice, is:

(A) 1 **(B)** $\frac{2}{3}$

(C) $1\frac{3}{5}$ **(D)** $2\frac{2}{3}$

Ans. Option (B) is correct.

Explanation: Let X : Number of doublets

$X = 0, 1, 2, 3, 4$

Total number of possible outcomes = 36

Doublets = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

$P(\text{Doublet}) = \frac{6}{36} = \frac{1}{6}$

$P(\text{Not a Doublet}) = \frac{5}{6}$

X	P(X)	X.P(X)
0	$\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6} = \frac{625}{1296}$	0
1	$4\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6} = \frac{500}{1296}$	$\frac{500}{1296}$
2	$6\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6} = \frac{150}{1296}$	$\frac{300}{1296}$
3	$4\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6} = \frac{20}{1296}$	$\frac{60}{1296}$
4	$\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6} = \frac{1}{1296}$	$\frac{4}{1296}$

$\therefore$ Mean $= \Sigma XP(X)$

$= 0 + \frac{500}{1296} + \frac{300}{1296} + \frac{60}{1296} + \frac{4}{1296}$

$= \frac{864}{1296} = \frac{2}{3}$

Q. 4. A candidate claims 70% of the people in her constituency would vote for her. If 120000 valid votes are polled, then the number of votes she expects from her constituency is:

(A) 100000 **(B)** 84000

(C) 56000 **(D)** 36000

Ans. Option (B) is correct.

Explanation: Expected number of votes

$= np$

$= \frac{70}{100}\times 120000$

$= 84000$

Q. 5. Let X denotes the number of hours a student devotes to self-study during a randomly selected school day. The probability that X takes the value x, where k is some unknown constant is:

$$P(X = x) = \begin{cases} k & \text{if } x = 0 \\ kx & \text{if } x = 1 \text{ or } 2 \\ k(5 - x) & \text{if } x = 3 \text{ or } 4 \\ 0 & \text{otherwise} \end{cases}$$

The probability that a student studies at least 3 hours on a particular day is:

(A) $\frac{1}{7}$ **(B)** $\frac{2}{7}$

(C) $\frac{3}{7}$ **(D)** $\frac{1}{2}$

Ans. Option (C) is correct.

Explanation: $\Sigma p_i = 1$

$\Rightarrow 7k = 1$

$\Rightarrow \quad k = \frac{1}{7}$

Now, $P(x \geq 3) = 3k = 3 \times \frac{1}{7}$

$= \frac{3}{7}$

Q. 6. A sales promotion company sells tickets for ₹ 100 each to win a prize of ₹ 5 lakhs. If a person buys one of the 10,000 tickets sold, then his expected gain in rupees is

(A) –50 **(B)** 0

(C) 50 **(D)** 100

Ans. Option (A) is correct.

Explanation:

Prize (x_i)	p_i	$x_i p_i$
500000	$\frac{1}{10000}$	50
0	$\frac{9999}{10000}$	0

So, $\sum x_i p_i = 50$

Net expected gain = 50 – 100

= – 50

∴ No gain

∴ Loss is ₹ 50

Q. 7. The expectation of a random variable X (continuous or discrete) is given by.....

(A) $\sum X f(x), \int X f(X)$ **(B)** $\sum X^2 f(X), \int X^2 f(X)$

(C) $\sum f(X), \int f(X)$ **(D)** $\sum X f(X^2), \int X f(X^2)$

Ans. Option (A) is correct.

Explanation: The expectation of a random variable X is given by the summation (integral) of X times the function in its interval. If it is a continuous random variable, then summation is used and if it is discrete random variable, then integral is used.

Q. 8. Variance of a random variable X is given by .....

(A) $E(X)$ **(B)** $E(X^2)$

(C) $E(X^2) - (E(X))^2$ **(D)** $(E(X))^2$

Ans. Option (C) is correct.

Explanation: Variance of a random variable is nothing but the expectation of the square of the random variable subtracted by the expectation of X (mean of X) to the power 2. Therefore, the variance is given by $E(X^2) - (E(X))^2$.

Q. 9. Mean of a constant '*a*' is .....

(A) 0 **(B)** a

(C) $a/2$ **(D)** 1

Ans. Option (B) is correct.

Explanation: Let $f(x)$ be the pdf of the random variable X.

Now, $E(a) = \int af(x) = a \int f(x) = a.1 = a$

Q. 10. Find the mean and variance of X ?

$X = x$	0	1	2	3	4
$f(X)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

(A) 2, $\frac{4}{3}$ **(B)** 3, $\frac{4}{3}$

(C) 2, $\frac{2}{3}$ **(D)** 3, $\frac{2}{3}$

Ans. Option (A) is correct.

Explanation:

Mean = $E(X) = \sum X f(X)$

$= 0\left(\frac{1}{9}\right) + 1\left(\frac{2}{9}\right) + 2\left(\frac{3}{9}\right) + 3\left(\frac{2}{9}\right) + 4\left(\frac{1}{9}\right)$

$= 2$

Variance = $E(X^2) - (E(X))^2$

$= \left(0 + \frac{2}{9} + \frac{12}{9} + \frac{18}{9} + \frac{16}{9}\right) - 4$

$= \frac{4}{3}$

Q. 11. Find the expectation of a random variable X.

X	0	1	2	3
$f(X)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

(A) 0.5 **(B)** 1.5

(C) 2.5 **(D)** 3.5

Ans. Option (B) is correct.

Explanation:

$E(X) = 0\left(\frac{1}{6}\right) + 1\left(\frac{2}{6}\right) + 2\left(\frac{2}{6}\right) + 3\left(\frac{1}{6}\right)$

$= 0 + \frac{2}{6} + \frac{4}{6} + \frac{3}{6}$

$= \frac{9}{6} = \frac{3}{2} = 1.5$

Q. 12. A table which contains possible value of a random variable and its corresponding probabilities is called ..... .

(A) Probability Mass Function

(B) Probability Density Function

(C) Cumulative Distribution Function

(D) Probability Distribution

Ans. Option (D) is correct.

Explanation: The given statement is the definition of a probability distribution.

Q. 13. A variable that can assume any value between two given points is called ..... .

(A) Continuous Random Variable

(B) Discrete Random Variable

(C) Irregular Random Variable

(D) Uncertain Random Variable

Ans. Option (A) is correct.

Explanation: This is the definition of a continuous random variable.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. For the following probability distribution:

X	1	2	3	4
$P(X)$	1/10	1/5	3/10	2/5

Find $E(X^2)$. R&U

Sol. $E(X_2) = \sum X^2 P(X)$

$$= 1.\frac{1}{10} + 4.\frac{1}{5} + 9.\frac{3}{10} + 16.\frac{2}{5}$$

$$= \frac{1}{10} + \frac{4}{5} + \frac{27}{10} + \frac{32}{5}$$

$$= \frac{1+8+27+64}{10} = 10 \quad \textbf{1}$$

Q. 2. For the following probability distribution:

X	−4	−3	−2	−1	0
$P(X)$	0.1	0.2	0.3	0.2	0.2

Find E(X). A I

Q. 3. A discrete random variable X has the probability distribution given below :

X	0.5	1	1.5	2
$P(X)$	k	k^2	$2k^2$	k

Find the value of k. R

Sol. We know that, $\sum_{i=1}^{n} P_i = 1$, where $P_i \geq 0$

$\Rightarrow \quad P_1 + P_2 + P_3 + P_1 = 1$

$\Rightarrow \quad k + k^2 + 2k^2 + k = 1$

$\Rightarrow 3k^2 + 2k - 1 = 0$

$\Rightarrow (3k - 1)(k + 1) = 0$

$\Rightarrow k = \frac{1}{3}$. (as k is ≥ 0) **1**

Q. 4. Three balls are drawn from a bag containing 2 red and 5 black balls, if the random variable X represents the number of red balls drawn, then how many values X can take ? U

Q. 5. The probability distribution of the discrete variable X is given as :

X	2	3	4	5
$P(X)$	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

Find the value of k. R

Sol. $\Sigma P(X) = 1$

$$\Rightarrow \frac{5}{k} + \frac{7}{k} + \frac{9}{k} + \frac{11}{k} = 1$$

$\Rightarrow k = 32$. **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. The random variable X has a probability distribution $P(X)$ of the following form, where k is some number :

$$P(X) = \begin{cases} k, \text{ if } x = 0 \\ 2k, \text{ if } x = 1 \\ 3k, \text{ if } x = 2 \\ 0, \text{ otherwise} \end{cases}$$

(i) Find the value of k

(ii) Find $P(X < 2)$ R&U

Sol. Here,

(i) Since $P(0) + P(1) + P(2) = 1$, we have

$$k + 2k + 3k = 1$$

i.e., $6k = 1$, or $k = \frac{1}{6}$. **1**

(ii) $P(X < 2) = P(0) + P(1)$

$$= k + 2k = 3k = \frac{1}{2}. \quad \textbf{1}$$

Q. 2. Find the probability distribution of X, the number of heads in a simultaneous toss of two coins. A I

Sol. Let X be the number of heads.

Possible values of X are 0, 1, 2

$P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$. The probability distribution of X is :

X	0	1	2
$P(X)$	1/4	1/2	1/4

2

Q. 3. If $p(x) = \begin{cases} \frac{x}{20}, & x = 0, 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$

Find (i) $P(X < 3)$ and (ii) $P(2 < X \leq 4)$ R&U

Sol. (i) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= 0 + \frac{1}{20} + \frac{2}{20} = \frac{3}{20} \quad \textbf{1}$$

(ii) $P(2 < X \leq 4) = P(X = 3) + P(X = 4)$

$$= \frac{3}{20} + \frac{4}{20} = \frac{7}{20} \quad \textbf{1}$$

Q. 4. If you toss a fair coin three times, the outcome of an experiment consider as random variable which counts the number of heads on the upturned faces. Find out the probability mass function and check the properties of the probability mass function. R&U

These questions are for practice and their solutions are available at the end of the chapter

Q. 5. A continuous random variable X has p.d.f $f(x) = 5x^4$, $0 \le x \le 1$. Find a_1 and a_2 such that

(i) $P[X \le a_1] = P[X > a_1]$

(ii) $P[X > a_2] = 0.05$. A I

Sol. (i) Since $P[X \le a_1] = P[X > a_1]$

$$P[X \le a_1] = \frac{1}{2}$$

$$i.e., \int_0^{a_1} f(x)dx = \frac{1}{2}$$

$$i.e., \int_0^{a_1} 5x^4 dx = \frac{1}{2}$$

$$5\left[\frac{x^5}{5}\right]_0^{a_1} = \frac{1}{2}$$

$$a_1 = (0.5)^{\frac{1}{5}}$$ **1**

(ii) $P(X > a_2] = 0.05$

$$\int_{a_2}^{1} f(x)dx = 0.05$$

$$\int_{a_2}^{1} 5x^4 dx = 0.05$$

$$5\left[\frac{x^5}{5}\right]_{a_2}^{1} = 0.05$$

$$a_1 = [0.95]^{\frac{1}{5}}$$ **1**

Q. 6. The time to failure in thousands of hours of an important piece of electronic equipment used in a manufactured DVD payer has the density function.

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected life of the piece of equipment. A I

Sol. We know that

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_0^{\infty} x3e^{-3x}dx$$

$$= 3\int_0^{\infty} xe^{-3x}dx$$

$$= 3\left\{\left[x\frac{e^{-3x}}{-3}\right] - \int_0^{\infty}\left(\frac{e^{-3x}}{-3}\right)dx\right\}$$

$$\left(\because \int u\,dv = uv - \int v\,du\right)$$

$$= \int_0^{\infty} e^{-3x}dx$$

$$= \frac{1}{3}$$

Therefore, the expected life of the piece of equipment is $\frac{1}{3}$ hrs (in thousands). **2**

Q. 7. A commuter train arrives punctually at a station every 25 minutes. Each morning, a commuter leaves his house and casually walks to the train station. Let X denote the amount of time, in minutes, that commuter waits for the train from the time he reaches the train station. It is known that the probability density function of X is

$$f(x) = \begin{cases} \frac{1}{25}, & \text{for } 0 < x > 25 \\ 0, & \text{otherwise} \end{cases}$$

Obtain and interpret the expected value of the random variable X. A

Short Answer Type Questions-II (3 marks each)

Q. 1. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability of the random variable X ? Find the mean. R&U

Sol.

$X=x$	14	15	16	17	18	19	20	21
$P(x)$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

2

$$\text{Mean of } x = \Sigma xP(X)$$

$$= \frac{1}{15}[28 + 15 + 32 + 51 + 18 + 38 + 60 + 21]$$

$$= \frac{1}{15}[263] = 17{\cdot}53$$ **1**

Q. 2. The random variable X can take only values 0, 1, 2, 3. Given that $P(X = 0) = P(X = 1) = p$ and $P(X = 2) = P(X = 3)$ such that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p. R&U

Sol.

x	$P(x)$
0	p
1	p
2	k
3	k

$$\Sigma p(x) = 1 \text{ or } 2p + 2k = 1$$

or $k = \frac{1}{2} - p$

x_i	p_i	p_ix_i	$p_ix_i^2$
0	p	0	0
1	p	p	p
2	$\frac{1}{2}-p$	$1-2p$	$2-4p$
3	$\frac{1}{2}-p$	$\frac{3}{2}-3p$	$\frac{9}{2}-9p$
		$\frac{5}{2}-4p$	$\frac{13}{2}-12p$

2

Given, $\Sigma p_ix_i^2 = 2\Sigma p_ix_i$

$$\frac{13}{2} - 12p = 5 - 8p$$

$$-4p = \frac{-13}{2} + 5 = \frac{-3}{2}$$

$\therefore p = \frac{3}{8}$ **1**

Q. 3. A biased die is such that $P(4) = \frac{1}{10}$ and other scores are equally likely. The die is tossed twice. If X is the 'number of fours obtained', find the variance of X. R&U

Sol.

X	0	1	2
$P(X)$	$\left(\frac{9}{10}\right)^2 = \frac{81}{100}$	$2\times\frac{9}{10}\times\frac{1}{10} = \frac{18}{100}$	$\left(\frac{1}{10}\right)^2 = \frac{1}{100}$
$XP(X)$	0	$\frac{18}{100}$	$\frac{2}{100}$
$X^2P(X)$	0	$\frac{18}{100}$	$\frac{4}{100}$

2

$$\text{Variance} = \Sigma X^2P(X) - [\Sigma XP(X)]^2$$

$$= \frac{22}{100} - \left(\frac{20}{100}\right)^2 = \frac{18}{100} = 0.18$$ **1**

Q. 4. Let X denote the number of colleges where you will apply after your results and $P(X = x)$ denotes your probability of getting admission is x number of colleges. It is given that :

$$P(X=x) = \begin{cases} kx & \text{, if } x = 0 \text{ or } 1 \\ 2kx & \text{, if } x = 2 \\ k(5-x) & \text{, if } x = 3 \text{ or } 4 \\ 0 & \text{, if } x > 4 \end{cases}$$

where k is a positive constant. Find the value of k. Also, find the probability that you will get admission in (i) exactly one college (ii) at most 2 colleges (iii) at least 2 colleges. R&U

Long Answer Type Questions (5 marks each)

Q. 1. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X). R&U

Ans. Let X denotes the number of aces obtained. Therefore, X can take any of the values of 0, 1, or 2. In a deck of 52 cards, 4 cards are aces. Therefore, there are 48 non-ace cards. ½

$\therefore$ $P(X = 0) = P(0 \text{ ace and } 2 \text{ non-ace cards})$

$$= \frac{{}^4C_0 \times {}^{48}C_2}{{}^{52}C_2} = \frac{1128}{1326}$$ **1**

$P(X = 1) = P(1 \text{ ace and } 1 \text{ non-ace cards})$

$$= \frac{{}^4C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{192}{1326}$$ **1**

$P(X = 2) = P(2 \text{ ace and } 0 \text{ non-ace cards})$

$$= \frac{{}^4C_2 \times {}^{48}C_0}{{}^{52}C_2} = \frac{6}{1326}$$ **1**

Thus, the probability distribution is as follows :

X	0	1	2
$P(X)$	1128/1326	192/1326	6/1326

½

Then, $E(X) = \sum p_ix_i$

$$= 0\times\frac{1128}{1326} + 1\times\frac{192}{1326} + 2\times\frac{6}{1326}$$

$$= \frac{204}{1326} = \frac{2}{13}$$ **1**

Q. 2. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find the mean and variance of the number of red cards. R&U

Sol. Number of red cards = 26

Let X be a random variable which can take values 0, 1, 2, where X is the no. of red cards selected

$\therefore$ $X = 0$ means 0 red cards

$$P(X=0) = \frac{{}^{26}C_2}{{}^{52}C_2} = \frac{26\times25}{52\times51} = \frac{25}{102}$$ **1**

$$P(X=1) = \frac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2} = \frac{26\times26\times2}{52\times51} = \frac{52}{102}$$ **1**

$$P(X=2) = \frac{{}^{26}C_2}{{}^{52}C_2} = \frac{26\times25}{52\times51} = \frac{25}{102}$$ **1**

Probability distribution of random variable X is

X	0	1	2
$P(X)$	$\frac{25}{102}$	$\frac{52}{102}$	$\frac{25}{102}$

These questions are for practice and their solutions are available at the end of the chapter

$$\text{Mean} = \Sigma xP(X) = \frac{52+50}{102} = 1 \quad \textbf{1}$$

$$\text{Variance} = \Sigma x^2P(X) - \{xP(X)\}^2$$

$$= \frac{152}{102} - 1$$

$$= \frac{50}{102} \text{ or } \frac{25}{51} \quad \textbf{1}$$

Q. 3. Two numbers are selected at random (without replacement) from first 7 natural numbers. If X denotes the smaller of the two numbers obtained, find the probability distribution of X. Also, find mean of the distribution. A I R&U

Sol. Let X denotes the smaller of the two numbers obtained

So X can take values 1, 2, 3, 4, 5, 6 ½

$P(X = 1$ is smaller number)

$$P(X=1) = \frac{6}{{}^7C_2} = \frac{6}{21} = \frac{2}{7} \quad ½$$

(Total cases when two numbers can be selected from first 7 numbers are ${}^7C_2$)

$$P(X=2) = \frac{5}{{}^7C_2} = \frac{5}{21} \quad ½$$

$$P(X=3) = \frac{4}{{}^7C_2} = \frac{4}{21} \quad ½$$

$$P(X=4) = \frac{3}{{}^7C_2} = \frac{3}{21} = \frac{1}{7} \quad ½$$

$$P(X=5) = \frac{2}{{}^7C_2} = \frac{2}{21} \quad ½$$

$$P(X=6) = \frac{1}{{}^7C_2} = \frac{1}{21} \quad ½$$

x_i	p_i	p_ix_i
1	$\frac{6}{21}$	$\frac{6}{21}$
2	$\frac{5}{21}$	$\frac{10}{21}$
3	$\frac{4}{21}$	$\frac{12}{21}$
4	$\frac{3}{21}$	$\frac{12}{21}$
5	$\frac{2}{21}$	$\frac{10}{21}$
6	$\frac{1}{21}$	$\frac{6}{21}$

½

$$\text{Mean} = \Sigma p_ix_i = \frac{6}{21}+\frac{10}{21}+\frac{12}{21}+\frac{12}{21}+\frac{10}{21}+\frac{6}{21}$$

$$= \frac{56}{21}$$

$$= \frac{8}{3} \quad \textbf{1}$$

Q. 4. Three numbers are selected at random (without replacement) from first six positive integers. Let X denote the largest of the three numbers obtained. Find the probability distribution of X. Also, find the mean and variance of the distribution. R&U

Q. 5. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of kings. A I

Sol. X = no. of kings = 0, 1, 2

$$P(X=0) = P(\text{one king}) = \frac{48}{52}\times\frac{47}{51} = \frac{188}{221} \quad \textbf{1}$$

$P(X = 1) = P$(one king and one non-king)

$$= \frac{4}{52}\times\frac{48}{51}\times 2 = \frac{32}{221} \quad \textbf{1}$$

$$P(X=2) = P(\text{two kings}) = \frac{4}{52}\times\frac{3}{51} = \frac{1}{221} \quad \textbf{1}$$

Probability distribution is given by

X	0	1	2
$P(X)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$$\text{Now, Mean} = \Sigma X \cdot P(X)$$

$$= \frac{34}{221} \text{ or } \frac{2}{13} \quad \textbf{1}$$

$$\text{and Var (X)} = \Sigma X^2 \cdot P(X) - \left[\Sigma X.P(X)\right]^2$$

$$= \frac{36}{221} - \left(\frac{34}{221}\right)^2$$

$$= \frac{6800}{48841} \text{ or } \frac{400}{2873} \quad \textbf{1}$$

Q. 6. Two numbers are selected at random (without replacement) from the first six positive integers. Let x denote the larger of the two numbers obtained. Find the probability distribution of random variable x and hence find the mean of the distribution. A I

Sol. First 6 positive integers = (1, 2, 3, 4, 5, 6)

Random Variable = X = larger of 2 numbers

Sample space S = {(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (4, 2), (5, 2), (6, 2), (3, 4), (3, 5), (3, 6), (4, 3), (5, 3), (6, 3), (4, 5), (4, 6), (5, 4), (6, 4), (5, 6), (6, 5)} **1**

These questions are for practice and their solutions are available at the end of the chapter

X	$P(X)$	X_iP_i	$X_i^2P_i$
2	$\frac{2}{30}=\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$
3	$\frac{4}{30}=\frac{2}{15}$	$\frac{6}{15}$	$\frac{18}{15}$
4	$\frac{6}{30}=\frac{3}{15}$	$\frac{12}{15}$	$\frac{48}{15}$
5	$\frac{8}{30}=\frac{4}{15}$	$\frac{20}{15}$	$\frac{100}{15}$
6	$\frac{10}{30}=\frac{5}{15}$	$\frac{30}{15}$	$\frac{180}{15}$

2

$$\Sigma X_iP_i = \frac{70}{15} \quad \Sigma X_i^2P_i = \frac{350}{15}$$

$$\text{Mean} = \Sigma X_iP_i = \frac{14}{3} = 4.66$$ **1**

$$\text{Variance} = \Sigma X_i^2P_i - (\mu)^2$$
$$= \frac{350}{15} - \frac{70}{15}\times\frac{70}{15}$$
$$= \frac{70}{15}\left[5 - \frac{70}{15}\right]$$
$$= \frac{70}{15}\left[\frac{75-70}{15}\right]$$
$$= \frac{70}{15}\times\frac{5}{15} = \frac{14}{9}$$
$$= 1.55$$
$$\text{Variance} = 1.55$$ **1**

Q. 7. A random variable X has the following probability function

Values of X	$p(x)$
0	0
1	a
2	$2a$
3	$2a$
4	$3a$
5	a^2
6	$2a^2$
7	$7a^2 + a$

(i) Find a, Evaluate (ii) P(X < 3), (iii) P(X > 2) and (iv) P(2 < X ≤ 5). A I

Sol. (i) Since the condition of probability mass function is

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

$$\therefore \quad \sum_{i=0}^{7} p(x_i) = 1$$

$$0 + a + 2a + 2a + 3a + a^2 + 2a^2 + 7a^2 + a = 1$$ **1**

$$10a^2 + 9a - 1 = 0$$

$$(10a - 1)(a + 1) = 0$$

$$a = \frac{1}{10} \text{ and } -1$$

Since $p(x)$ cannot be negative, $a = -1$ is not applicable.

Hence, $a = \frac{1}{10}$ **1**

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$
$= 0 + a + 2a$
$= 3a$
$= \frac{3}{10} \qquad \left(\because a = \frac{1}{10}\right)$ **1**

(iii) $P(X > 2) = 1 - P(X \le 2)$
$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$
$= 1 - \frac{3}{10}$
$= \frac{7}{10}$ **1**

(iv) $P(2 < X \le 5) = P(X = 2) + P(X = 4) + P(X = 5)$
$= 2a + 3a + a^2$
$= 5a + a^2$
$= \frac{5}{10} + \frac{1}{100} = \frac{51}{100}$ **1**

Q. 8. If $f(x)$ is defined by $f(x) = ke^{-2x}$, $0 \le x < \infty$ is a density function. Determine the constant k and also find mean. R&U

Topic-2 Binomial and Poisson Distributions

Concepts Covered • *Bernoulli Trials,* • *Binomial Distribution,* • *Poisson Distribution*

Revision Notes

Bernoulli Trials

Trials of a random experiment are called Bernoulli trials, if they satisfy the following four conditions:

(a) The trials should be finite in numbers.

(b) The trials should be independent of each other.

(c) Each of the trial yields exactly two outcomes *i.e.*, success or failure.

These questions are for practice and their solutions are available at the end of the chapter

(d) The probability of success or the failure remains the same in each of the trial.

If an experiment is repeated n times under the similar conditions, we say that n trials of the experiment have been made.

Binomial Distribution

Scan to know more about this topic

Binomial & Poisson distribution

Let E be an event. Let p = probability of success in one trial (*i.e.*, occurrence of event E in one trial) and, $q = 1 - p$ = probability of failure in one trial (*i.e.*, non-occurrence of event E in one trial).

Let X = number of successes (*i.e.*, number of times event E occurs in n trials)

Then, Probability of X successes in n trials is given by the relation:

$$P(X = r) = P(r) = {}^nC_r p^r q^{n-r},$$

where $r = 0, 1, 2, 3, ..., n$; p = probability of success in one trial and $q = 1 - p$ = probability of failure in one trial.

Note :

The result $P(X = r) = P(r) = {}^nC_r p^r q^{n-r}$ can be used only when :

(i) the probability of success in each trial is the same.

(ii) each trial must surely result in either a success or a failure.

Special Cases

- $P(X = r)$ or $P(r)$ is also called the probability of occurrence of event E exactly r times in n trials.
- Here ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- Note that ${}^nC_r p^r q^{n-r}$ is the $(r + 1)^{th}$ term in the binomial expansion of $(q + p)^n$.
- Mean $= \sum_{r=0}^{n} r.P(r) = np$
- Variance $= \sum_{r=0}^{n} r^2.P(r) - (\text{Mean})^2 = npq$
- Recurrence formula, $P(x = r + 1)$

  $= \left(\dfrac{n-r}{r+1}\right)\left(\dfrac{p}{q}\right)P(r)$
- A Binomial distribution with n Bernoulli trials and probability of success in each trial as p is denoted by $B(n, p)$. Here n and p are known as the parameters of binomial distribution.
- The expression $P(x = r)$ or $P(r)$ is called the probability function of the binomial distribution.

Example 6 : In a box containing 100 bulbs, 10 are defective. Find the probability that out of a sample of 5 bulbs, none is defective **is.**

Sol. The repeated selections of defective bulbs from a box are Bernoulli trials. Let X denotes the number of defective bulbs out of a sample of 5 bulbs.

Probability of getting a defective bulb,

$$P = \frac{10}{100} = \frac{1}{10}$$

$$\therefore \quad q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Clearly, X has a binomial distribution with $n = 5$ and $p = \dfrac{1}{10}$

$$\therefore \quad P(X = x) = {}^nC_x q^{n-x} p^x = {}^5C_x \left(\frac{9}{10}\right)^{5-x} \left(\frac{1}{10}\right)^x$$

$$P(\text{none of the bulbs is defective}) = P(X=0)$$

$$= {}^5C_0 \cdot \left(\frac{9}{10}\right)^5 = 1.\left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^5$$

Poisson Distribution

Scan to know more about this topic

Poisson distribution

It is a discrete probability distribution. The Poisson distribution is a particular limiting form of Binomial distribution when p (or q) is very small and n is large enough so that np (or nq) is a finite constant say m.

A probability distribution of a random variable x is called Poisson distribution if x can assume non-negative integral values only and the distribution is given by

$$P(r) = P(X = r) = \begin{cases} \dfrac{e^{-m}m^r}{r!}, & r = 0, 1, 2... \\ 0, & r \neq 0, 1, 2... \end{cases}$$

Here, the value of $e = 2.7183$, it is the base of the natural system of logarithms.

Examples of Poisson Variates

(i) The number of cars passing through a certain street in time t.

(ii) The number of defective screws per box of 100 screws.

(iii) The number of deaths in a district in one year by a rare disease.

(iv) The number of printing mistakes at each page of the book.

- **Requirements for a Poisson Distribution**
  (1) Random variable (x) is the number of occurrences of an events **over some interval**.
  (2) Occurrences must be **random**
  (3) Occurrences must be **independent** of each other
  (4) Occurrences must be **uniformly distributed** over the interval being used.
- **Constants of the Poisson Distribution**
  (i) Mean, $E(X) = m = np$
  (ii) Variance, $V(X) = m = np$

Example 7 : If $P(X = 2) = 9P(X = 4) + 90\ P(X = 6)$ in Poisson distribution, then find $E(X)$.

Sol. Let $P(X = r) = \dfrac{e^{-m}m^r}{r!}$ $r = 0, 1, 2 ..., \infty$

Given, $P(X = 2) = 9P(X = 4) + 90P(X = 6)$

$\Rightarrow \dfrac{e^{-m}m^2}{2!} = 9\dfrac{e^{-m}m^9}{9!} + 90\dfrac{e^{-m}m^6}{6!}$

$\Rightarrow m^4 + 3m^2 - 4 = 0$

$\Rightarrow (m^4 + 4)(m^2 - 1) = 0$

$\Rightarrow m = 1$

($\because$ other values are not admissible)

Thus, $E(X) = m = 1$

- **Difference between Binomial and Poisson Distributions**
  - The binomial distribution is affected by the sample size n and the probability p, whereas the Poisson distribution is affected only by the mean μ.
  - In a binomial distribution the possible values of the random variable x are 0, 1, ... n but a Poisson distribution has possible x values of 0, 1, ..., with no upper limit.

Key Terms

- Expansion of e^x

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

- Expansion of e^{-x}

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...$$

Key Facts

- Binomial distribution describes the distribution of binary data from a finite sample. Thus, it gives the probability of getting r events out of n trials.
- Poisson distribution describes the distribution of binary data from a infinite sample. Thus, it gives the probability of getting r events in a population.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. If 'm' is the mean of Poisson distribution, then its standard deviation is given by:

(A) $\sqrt{m}$ **(B)** m^2

(C) m **(D)** $\dfrac{m}{2}$

Ans. Option (A) is correct.

Explanation: In case of Poisson distribution

$$\text{S.D} = \sqrt{\text{mean}}$$

Thus, S.D. $= \sqrt{m}$ [Given, mean $= m$]

Q. 2. In a box of 100 bulbs, 10 are defective. What is the probability that out of a sample of 5 bulbs, none is defective?

(A) $\left(\dfrac{9}{10}\right)^5$ **(B)** $\dfrac{9}{10}$

(C) 10^{-5} **(D)** $\left(\dfrac{1}{2}\right)^2$

Ans. Option (A) is correct.

Explanation: Probability of getting a defective bulb,

$$p = \frac{10}{100} = \frac{1}{10}$$

$\therefore$ $q = 1 - p$

$$= 1 - \frac{1}{10} = \frac{9}{10}$$

Here, $n = 5$

By binomial distribution probability of getting no defective bulb $= P(X = 0) = {}^nC_0\, p^0\, q^{n-0}$

$$= {}^5C_0\left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^5$$

Q. 3. The mean $E(x)$ of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is:

(A) 1 **(B)** 2

(C) 5 **(D)** $\dfrac{8}{3}$

Ans. Option (B) is correct.

Explanation: Total no. of observations = 6

$\therefore$ $P(X = 1) = \dfrac{3}{6} = \dfrac{1}{2}$

$P(X = 2) = \dfrac{2}{6} = \dfrac{1}{3}$

$P(X = 5) = \dfrac{1}{6}$

(where X be a random variable representing a number on die)

$\therefore$ Mean, $E(X) = \sum p_i X_i$

$$= \frac{1}{2}\times 1 + \frac{1}{3}\times 2 + \frac{1}{6}\times 5$$

$$= \frac{1}{2} + \frac{2}{3} + \frac{5}{6}$$

$$= \frac{3+4+5}{6}$$

$$= \frac{12}{6} = 2$$

Q. 4. If the mean of a binomial distribution is 81, then the standard deviation lies in the interval:

(A) [0, 9) **(B)** (0, 9]

(C) [0, 3] **(D)** (0, 3]

Ans. Option (A) is correct.

Explanation: Standard deviation, $\sigma = \sqrt{npq} \geq 0$

Now, mean $= np = 81$ and $q < 1$

So, $\sigma = \sqrt{npq} < \sqrt{np} = \sqrt{81} = 9$

$\therefore \quad 0 \le \sigma < 9$

Hence, σ lies in [0, 9)

Q. 5. If a random variable *X* has the Poisson distribution with mean 2. Then, *P*(*X* > 1.5) is:

(A) $2e^{-2}$ **(B)** $3e^{-2}$

(C) $1 - 2e^{-2}$ **(D)** $1 - 3e^{-2}$

Ans. Option (D) is correct.

Explanation: We know that,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Given that mean $= 2 = \lambda$

$\therefore \quad P(X > 1.5) = P(X = 2) + P(X = 3) + ...$

$\Rightarrow \quad P(X > 1.5) = 1 - [P(X = 0) + P(X = 1)]$

$= 1 - [e^{-2} + 2e^{-2}] = 1 - 3e^{-2}$

Q. 6. There are 50 telephone lines in an exchange. The probability that any one of them will be busy is 0·1. The probability that all the lines are busy is:

(A) $\frac{5^0 e^{-5}}{0!}$ **(B)** $1 - \frac{5^0 e^{-5}}{0!}$

(C) $\frac{5^{50} e^{-5}}{50!}$ **(D)** $1 - \frac{5^{50} e^{-5}}{50!}$

Ans. Option (C) is correct.

Explanation: Given, 50 telephone line in an exchange probability that any of them will be busy is 0·1.

Let X be the Poisson variable.

$\therefore \quad P(X = r) = \frac{e^{-m} m^r}{r!}, r = 0, 1, 2, ...$

Here, $n = 50, p = 0.1$

$\therefore \quad m = np = 50 \times 0.1 = 5$

$\therefore P(X = r) = \frac{e^{-5}(5)^r}{r!}, \ r = 0, 1, 2, ..., 50$

Thus, P(all lines are busy) $= P(X = 50)$

$\therefore \quad = \frac{e^{-5}(5)^r}{r!}$

For $r = 50$, $P = \frac{e^{-5} 5^{50}}{50!}$

Q. 7. An automatic machine produces 20000 pins per day. On rare occasion it produces a perfect pin whose chance is $\frac{1}{10000}$. Assuming Poisson distribution, the mean and variance of the number of perfect pins are __ and __, respectively

(A) $\sqrt{2}, \sqrt{2}$ **(B)** 2, 2

(C) 2, 4 **(D)** 4, 2

Ans. Option (B) is correct.

Explanation: For Poisson distribution

Mean = variance

$= np$

$= 20000 \times \frac{1}{10000}$

$= 2$

Q. 8. For a Poisson distribution with mean λ, $\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}$ is equal to

(A) -1 **(B)** 0

(C) $\frac{1}{2}$ **(D)** 1

Ans. Option (D) is correct.

Explanation:

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = \text{Total probability}$$

$= 1$

Q. 9. A TV manufacturer tests a random sample of 6 picture tubes to determine any defect. Past experience suggests the probability of defective picture tube is 0.05. The probability that there is at least one defective picture tube in the sample is:

(A) $\left(\frac{19}{20}\right)^6$

(B) $1 - \left(\frac{19}{20}\right)^6$

(C) $1 - \left[\left(\frac{19}{20}\right)^6 + \frac{3}{10}\left(\frac{19}{20}\right)^5\right]$

(D) $\left(\frac{1}{20}\right)^6$

Ans. Option (B) is correct.

Explanation: $p = 0.05$

$= \frac{1}{20}$

$q = \frac{19}{20}$

$P(x \ge 1) = 1 - P(0)$

$= 1 - 6_{c_0}\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^6$

$= 1 - \left(\frac{19}{20}\right)^6$

Q. 10. An insurance company has found that 50% of its claims are for damages resulting from accidents. The probability that a random sample of 10 claims will contain fewer than 2 for accidents is:

(A) $\frac{1}{1024}$ **(B)** $\frac{5}{512}$

(C) $\frac{11}{1024}$ **(D)** $\frac{15}{1024}$

Ans. Option (C) is correct.

Explanation:

$P(r < 2) = P(0 \text{ or } 1)$

$= {}^{10}C_0\left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10}$

$$= \frac{1+10}{1024}$$

$$= \frac{11}{1024}$$

Q. 11. During a pandemic, 10% of the patients who have the disease get complications. If 100 patients of a locality get infected by the disease, then the standard deviation of the number of patient getting complications is:

(A) 10 **(B)** 9

(C) 6 **(D)** 3

Ans. Option (D) is correct.

Explanation:

$$n = 100,$$

$$p = \frac{1}{10}$$

$$q = \frac{9}{10}$$

$$\sigma = \sqrt{npq}$$

$$= \sqrt{100 \times \frac{1}{10} \times \frac{9}{10}}$$

$$= 3$$

Q. 12. A river passing near a town floods it on an average twice every 10 years. Assuming Poisson distribution find the probability that the town faces flooding at least once in 10 years.

(A) 0.0198 **(B)** 0.1353

(C) 0.5657 **(D)** 0.8647

Ans. Option (D) is correct.

Explanation:

$$P(X \geq 1) = 1 - P(0)$$

$$= 1 - \frac{e^{-2}(2)^0}{0!}$$

$$= 1 - e^{-2}$$

$$= 0.8647$$

Q. 13. $E(X) = npq$ is for which distribution ?

(A) Bernoulli's **(B)** Binomial

(C) Poisson's **(D)** None of these

Ans. Option (B) is correct.

Explanation: In Binomial distribution, probability of success is given by p and that of future is given by q and the event is done n times. The mean of the distribution is given by npq.

Q. 14. $E(X) = \lambda$ is for which distribution ?

(A) Bernoulli's **(B)** Binomial

(C) Poisson's **(D)** None of these

Ans. Option (C) is correct.

Explanation: In Poisson's distribution, there is a positive constant λ which is the mean of the distribution and variance of the distribution.

Q. 15. In a Binomial distribution, if 'n' is the number of trials and 'p' is the probability of success, then the mean value is given by ..... .

(A) np **(B)** n

(C) p **(D)** $np(1-p)$

Ans. Option (A) is correct.

Explanation: For a discrete probability function, the mean value or the expected value is given by

$$\text{Mean } (\mu) = \sum_{x=0}^{n} xp(x)$$

For Binomial distribution $P(X) = nxp^x q^{n-x}$, substitute in above equation and solve to get $\mu = np$.

Q. 16. Binomial distribution is a .....

(A) Continuous distribution

(B) Discrete distribution

(C) Irregular distribution

(D) Not a Probability distribution

Ans. Option (B) is correct.

Explanation: It is applied to discrete random variable, hence it is a discrete distribution.

Q. 17. For larger values of 'n', Binomial Distribution ..... .

(A) loses its discreteness

(B) tends to Poisson's Distribution

(C) Stays as it is

(D) gives oscillatory values

Ans. Option (B) is correct.

Explanation: $P(X) = \lim_{n \to \infty} {}^{n}C_x \, p^x \, q^{n-x}$

$$= \frac{e^{-m} m_x}{x!}$$

where $m = np$ is the mean of Poisson Distribution.

Q. 18. If 'm' is the mean of a Poisson Distribution, then variance is given by ..... .

(A) m^2 **(B)** $m^{1/2}$

(C) m **(D)** $\frac{m}{2}$

Ans. Option (C) is correct.

Explanation: Mean and variance of Poisson distribution are equal.

Q. 19. If 'm' is the mean of Poisson Distribution, the standard. deviation is given by ..... .

(A) $\sqrt{m}$ **(B)** m^2

(C) m **(D)** $\frac{m}{2}$

Ans. Option (A) is correct.

Explanation: Standard deviation $= \sqrt{\text{Variance}}$

$$= \sqrt{m}$$

($\because$ For Poisson Distribution mean = Variance)

Q. 20. The *pdf* of Poisson Distribution is given by ..... .

(A) $\frac{e^{-m} m^x}{x!}$ **(B)** $\frac{e^{-m} x!}{m^x}$

(C) $\frac{x!}{m^x \, e^{-m}}$ **(D)** $\frac{e^m \, m^x}{x!}$

Ans. Option (A) is correct.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch ? R&U

Sol. Probability of defective watch from a lot 100 watches

$$= \frac{10}{100} = \frac{1}{10}$$

$$\therefore p = \frac{1}{10}\ q = \frac{9}{10}\ n = 8 \text{ and } r \geq 1$$

$$\therefore P(r \geq 1) = 1 - P(r = 0)$$

$$= 1 - {}^8C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{8-0}$$

$$= 1 - \frac{8!}{0!8!}\cdot\left(\frac{9}{10}\right)^8 = 1 - \left(\frac{9}{10}\right)^8 \quad \mathbf{1}$$

Q. 2. A die is rolled twice. Find the probability of getting an odd number exactly once. A I

Sol. $n = 2, p = \frac{1}{2}, q = \frac{1}{2}$

$$P(x = 1) = 2C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^1$$

$$P(x = 1) = \frac{1}{2} \quad \mathbf{1}$$

Q. 3. Find the probability of getting a doublet twice when a pair of dice is thrown 4 times. A I

Sol. $n = 4, p = \frac{6}{36} = \frac{1}{6}$

$$q = \frac{5}{6}$$

$$P(x = 2) = {}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$$

$$= \frac{25}{216} \quad \mathbf{1}$$

Q. 4. The probability that a student is not a swimmer is 1/5. Then the probability that out of five students, four are swimmers. R&U

Q. 5. The probability of getting at least one head when a fair coin is tossed 5 times. A I

Sol. $P(x \geq 1) = 1 - P(x = 0)$

$$= 1 - 5C_0\left(\frac{1}{2}\right)^5$$

$$= 1 - \frac{1}{32}$$

$$= \frac{31}{32} \quad \mathbf{1}$$

Q. 6. Eight coins are tossed together. The probability of getting exactly 3 heads. R&U

Sol. We know that, probability distribution

$$P(X = r) = {}^nC_r\,(p)^r\,q^{n-r}$$

Here, $n = 8, r = 3, p = \frac{1}{2}$ and $q = \frac{1}{2}$

$$\therefore \text{Required probability} = {}^8C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{8-3}$$

$$= \frac{8!}{5!3!}\left(\frac{1}{2}\right)^8$$

$$= \frac{8.7.6}{3.2}\cdot\frac{1}{2^8} = \frac{7}{32}. \quad \mathbf{1}$$

Q. 7. If X follows binomial distribution with parameters $n = 5$, p and $P(X = 2) = 9\,P(X = 3)$, then find the value of p. R&U

Q. 8. Use a Poisson Distribution to find probability. If $\mu = 0.5$, find $P(2)$. R&U

Sol. $P(X) = \frac{\mu^X.e^{-\mu}}{X!}$

$$P(2) = \frac{0.5^2.e^{-0.5}}{2!}$$

$$= 0.0758 \quad \mathbf{1}$$

Short Answer Type Questions-I (2 marks each)

Q. 1. Find the probability of guessing correctly at least 8 out of 10 answers on a true-false type examination. R&U

Sol. We know that,

$$P(X = r) = {}^nC_r\,(p)^r\,(q)^{n-r}$$

Here, $n = 10, p = \frac{1}{2}, q = \frac{1}{2}$ and $r \geq 8$

i.e., $r = 8, 9, 10$

$$\Rightarrow P(X = r) = P(r = 8) + P(r = 9) + P(r = 10)$$

$$= {}^{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^{10-9}$$

$$+ {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10-10} \quad \mathbf{1}$$

$$= \left(\frac{10!}{8!2!} + \frac{10!}{9!1!} + 1\right)\left(\frac{1}{2}\right)^{10}$$

$$= [45 + 10 + 1]\left(\frac{1}{2}\right)^{10}$$

$$= 56\left(\frac{1}{2}\right)^{10} = \frac{7}{128} \quad \mathbf{1}$$

These questions are for practice and their solutions are available at the end of the chapter

Q. 2. A coin is tossed 5 times. What is the probability of getting (i) 3 heads, (ii) atmost 3 heads ? A I

Q. 3. *A* die is thrown 5 times. Find the probability that an odd number will come up exactly three times. A I

Sol. Here, $n = 5$

$$p = \frac{1}{2}$$

and $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Also, $r = 3$

$\therefore P(X = r) = {}^nC_r\,(p)^r\,(q)^{n-r}$

$$= {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3}$$

$$= \frac{5!}{3!2!}\cdot\frac{1}{8}\cdot\frac{1}{4}$$

$$= \frac{10}{32} = \frac{5}{16} \qquad \textbf{2}$$

Q. 4. In a Poisson distribution probability for $x = 10$ %. Find the mean, given that $\log_e 10 = 2.3026$. A I

Sol. Let m be the mean. Then

$$P(X = x) = e^{-m}\frac{m^x}{x!}$$

$\Rightarrow P(0) = e^{-m}$ **1**

$$e^{-m} = 10\% = \frac{10}{100} = 0.1$$

or $e^{-m} = 10$

$$m = \log_e 10 = 2.3026 \qquad \textbf{1}$$

Q. 5. A car-hire-firm has two cars, which it hires, out by the day by day. The number of demands for a car on each day is distributed as Poisson distribution with mean 1.5. Calculate the proportion of days on which neither car is used and proportion of days on which some demand is refused. [$e^{-1.5} = 0.2231$] A

Sol. The proportion of days when no car will be required is

$$e^{-m}\frac{m^0}{0!} = 0.2231 \qquad ½$$

The probability when no car, one car, two cars will be required is

$$= e^{-m}\frac{m^0}{0!} + e^{-m}\frac{m^1}{1!} + e^{-m}\frac{m^2}{2!}$$

$$= e^{-m}\left(1 + m + \frac{1}{2}m^2\right)$$

$$= 0.2231(1 + 1.5 + 1.125)$$

$$= 0.80807375 \qquad \textbf{1}$$

$\therefore$ The proportion of days on which some demand is refused $= 1 - 0.80807375 = 0.1912625$ ½

Q. 6. A telephone switch board handles 600 calls on the average during a rush hour. The board can make a maximum of 20 connections per minute. Use poisson distribution to estimate the probability the board will be over taxed during any given minute. [$e^{-1} = 0.00004539$] A

Sol. Mean (per minute) $= \frac{600}{60} = 10$ ½

Hence, the probability for using 0 to 20 calls per minute

$$= \sum_0^{20} e^{-m}\frac{m^x}{x!}$$

$$= e^{-10}\sum_0^{20}\frac{10^x}{x!}$$

$$= 0.00004539\sum_0^{20}\frac{10^x}{x!} \qquad \textbf{1}$$

Thus, the probability that the board will be over taxed during any given minute *i.e.* when the calls will be more than 20 connections per minute.

$$= 1 - 0.00004539\sum_0^{20}\frac{10^x}{x!} \qquad ½$$

Q. 7. Six coins are tossed 6400 times. Using Poisson distribution, find approximate probability of getting six heads x times and 2 times. A

Q. 8. If $P(X = 0) = P(X = 1) = a$ in a Poisson distribution, show that $a = \frac{1}{e}$. A I

Sol. Let, $P(X = x) = e^{-m}\frac{m^x}{x!}$

$x = 0, 1, 2, \ldots, \infty$

Then, $P(X = 0) = e^{-m}$

and $P(X = 1) = e^{-m}\,m$

Since, given $P(X = 0) = P(X = 1)$, therefore

$$e^{-m} = e^{-m}\,m$$

$\Rightarrow \quad m = 1$

Also, $P(X = 0) = a$

$\Rightarrow \quad e^{-1} = a$

$\Rightarrow \quad a = \frac{1}{e}$ **2**

Short Answer Type Questions-II (3 marks each)

Q. 1. A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of (i) 5 successes? (ii) atmost 5 successes? A I

Sol. Let p be the probability of success *i.e.*, getting an odd number

$\therefore n = 6$

$$p = \frac{n(1, 3, 5)}{n(1, 2, 3, 4, 5, 6)} = \frac{3}{6} = \frac{1}{2}$$

and $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Let x be the binomial variable 'number of successes'

$\therefore$ By Binomial distribution,

$$P(x = r) = {}^nC_r\,p^r\,q^{n-r},\ 0 \le r \le n$$

These questions are for practice and their solutions are available at the end of the chapter

$\therefore P(x = r) = {}^6C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{6-r}, 0 \le r \le 6$

$= {}^6C_r \left(\frac{1}{2}\right)^6, 0 \le r \le 6$

$= \left(\frac{1}{64}\right) {}^6C_r, 0 \le r \le 6$ **1**

(i) P(exactly 5 successes) $= P(x = 5)$

$= \left(\frac{1}{64}\right) {}^6C_5 = \left(\frac{1}{64}\right) \times 6 = \frac{3}{32}$ **1**

(ii) P(atmost 5 successes) $= P(x \le 5)$

$= P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) + P(x = 5)$

$= \frac{1}{64} [{}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5]$

$= \frac{1}{64} [1 + 6 + 15 + 20 + 15 + 6]$

$= \frac{63}{64}$ **1**

Q. 2. Four cards are drawn successively with replacement from a well shuffled deck of 52 cards. What is the probability that:

(i) All the four cards are spades ?

(ii) Only 2 cards are spades ? R&U

Q. 3. For 6 trails of an experiment, let X be a binomial variate which satisfies the relation $9P(X = 4) = P(X = 2)$. Find the probability of success. R&U

Sol. Let p = Probability of success, q = Probability of failure and X be a random variable that denote the number of success in 6 trials.

As, X follows binomial distribution with parameters, $n = 6$, p and q.

$\therefore P(X = r) = {}^6C_r p^r q^{6-r}$

where, $X = 0, 1, 2, 3, 4, 5, 6$ 1

and $p + q = 1$

Also given, $9P(X = 4) = P(X = 2)$

$\therefore \quad 9 \times ({}^6C_4 p^4 q^2) = {}^6C_2 p^2 q^4$

or $9p^2 = q^2$ **[$\because {}^6C_4 = {}^6C_2$] 1**

or $9p^2 = (1 - p)^2$ **[$\because p + q = 1$]**

or $9p^2 = 1 + p^2 - 2p$

or $8p^2 + 2p - 1 = 0$ **1**

or $8p^2 + 4p - 2p - 1 = 0$

or $4p(2p + 1) - 1(2p + 1) = 0$

or $(2p + 1)(4p - 1) = 0$

$\therefore \quad p = -\frac{1}{2}$ or $p = \frac{1}{4}$

Thus, $p = \frac{1}{4}$ [$\because$ probability cannot be negative] **1**

Q. 4. Five cards are drawn one by one, with replacement, from a well-shuffled deck of 52 cards. Find the probability that

(i) all the five cards are diamonds.

(ii) only 3 cards are diamonds.

(iii) none is diamond. R&U

Sol. Here, $n = 5$, $p = \frac{13}{52} = \frac{1}{4}$

$q = 1 - \frac{1}{4} = \frac{3}{4}$

$P(x = 5) = {}^5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0$

$= \frac{1}{1024}$ **1**

$P(x = 3) = {}^5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2$

$= \frac{90}{1024} = \frac{45}{512}$ **1**

$P(x = 0) = {}^5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5$

$= \frac{243}{1024}$ **1**

Q. 5. From a lot of 30 bulbs which includes 6 defectives, a sample of 4 bulbs is drawn at random one by one with replacement. Find the probability distribution of the number of defective bulbs. Hence find the mean of the distributions. R&U

Q. 6. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of diamond cards drawn. Also, find the mean and the variance of the distribution. R&U

Sol. Let X = denote the random variable. $X = 0, 1, 2$, $n = 2$, $p = 1/4$, $q = 3/4$

x_i	0	1	2	Total
p_i	${}^2C_0 \left(\frac{3}{4}\right)^2 = \frac{9}{16}$	${}^2C_1 \frac{1}{4} \left(\frac{3}{4}\right) = \frac{6}{16}$	${}^2C_2 \left(\frac{1}{4}\right)^2 = \frac{1}{16}$	
$x_i p_i$	0	6/16	2/16	1/2
$x_i^2 p_i$	0	6/16	4/16	5/8

1½

These questions are for practice and their solutions are available at the end of the chapter

$$\text{Mean} = \Sigma x_i \cdot p_i = \frac{1}{2}$$

$$\text{Variance} = \Sigma x_i^2 p_i - (\Sigma x_i p_i)^2$$

$$= \frac{5}{8} - \frac{1}{4} = \frac{3}{8} \qquad 1\frac{1}{2}$$

Long Answer Type Questions (5 marks each)

Q. 1. Five bad oranges are accidently mixed with 20 good ones. If four oranges are drawn one by one successively with replacement, then find the probability distribution of number of bad oranges drawn. Hence, find the mean and variance of the distribution. A I

Sol. Total number of oranges = 25

number of good oranges = 20

number of bad oranges = 5

Probability of getting a bad orange,

$$P(\text{bad oranges}) = \frac{5}{25} = \frac{1}{5}$$

Now $p = \frac{1}{5}$

$$q = 1 - p = \frac{4}{5}$$

Let X be the random variable of "Number of bad oranges".

or $X = 0, 1, 2, 3, 4$.

$$P(X = r) = {}^nC_r . q^{n-r} . p^r$$

$$P(X = 0) = {}^4C_0 \left(\frac{4}{5}\right)^4 = \frac{256}{625}$$

$$P(X = 1) = 4 \times \frac{64}{125} \times \frac{1}{5} = \frac{256}{625}$$

$$P(X = 2) = {}^4C_2 \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^2$$

$$= 6 \times \frac{16}{625} = \frac{96}{625}$$

$$P(X = 3) = {}^4C_3 \left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3$$

$$= 4 \times \frac{4}{625} = \frac{16}{625}$$

$$P(X = 4) = {}^4C_4 \left(\frac{4}{5}\right)^0 \left(\frac{1}{5}\right)^4$$

$$= \frac{1}{625} \qquad 2$$

$\therefore$ Probability distribution is

X	$P(X)$	$XP(X)$	$X^2P(X)$
0	256/625	0	0
1	256/625	256/625	256/625
2	96/625	192/625	384/625
3	16/625	48/625	144/625
4	1/625	4/625	16/625

1

$\therefore$ $\text{Mean} = \Sigma X P(X)$

$$= \frac{500}{625} = \frac{20}{25} = \frac{4}{5}$$

$$\text{Var.}(X) = E(X^2) - [E(X)]^2 \qquad 1$$

$$= \frac{800}{625} - \frac{16}{25}$$

$$\text{Var.}(X) = \frac{32}{25} - \frac{16}{25} = \frac{16}{25} \qquad 1$$

Commonly Made Error

- Most of the students are not clear with concept of Binomial distribution.

Answering Tip

- Revision must be done on concepts of mean and variance of binomial probability distribution.

Q. 2. An experiment succeeds thrice as often as it fails. Find the probability that in the next 5 trials, there will be atleast 3 successes. R&U

Sol. Let probability of success be p and that of failure be q

$\therefore p = 3q$, and $p + q = 1$

$\therefore p = \frac{3}{4}$ and $q = \frac{1}{4}$ **1**

P(atleast 3 successes)

$= P(r \geq 3) = P(3) + P(4) + P(5)$

$$= {}^5C_3 \left(\frac{1}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^3 + {}^5C_4 \left(\frac{1}{4}\right)^1 \cdot \left(\frac{3}{4}\right)^4 + {}^5C_5 \left(\frac{3}{4}\right)^5 \qquad 2$$

$$= \frac{270}{1024} + \frac{405}{1024} + \frac{243}{1024} = \frac{918}{1024} = \frac{459}{512} \qquad 2$$

Q. 3. A box has 20 pens of which 2 are defective. Calculate the probability that out of 5 pens drawn one by one with replacement, at most 2 are defective. R&U

These questions are for practice and their solutions are available at the end of the chapter

Q. 4. An unbiased coin is tossed 4 times. Find the mean and variance of number of heads obtained. R&U

Sol. Given $n = 4$

Getting $p = \frac{1}{2}$ and $q = \frac{1}{2}$

No. of Successes (x)	0	1	2	3	4	Total
$p(x)$	${}^4C_0\left(\frac{1}{2}\right)^4 = \frac{1}{16}$	${}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{4}{16}$	${}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{6}{16}$	${}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 = \frac{4}{16}$	${}^4C_4\left(\frac{1}{2}\right)^4 = \frac{1}{16}$	
$x\,p(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	$\frac{32}{16}$
$x^2p(x)$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$	$\frac{80}{16}$

3

$$\text{Mean} = \Sigma x\,p(x) = \frac{32}{16} = 2$$

$$\text{Variance} = \Sigma x^2\,p(x) - \{\Sigma x p(x)\}^2$$

$$= \frac{80}{16} - (2)^2 = 5 - 4 = 1$$ **2**

Q. 5. In the proportion of defective in a bulk is 4%. Find the probability of not more than 2 defective in a sample of 10. It is known that $e^{-4} = 0.6703$. A I

Sol. Proportion of defective units

$$p = \frac{4}{100} = 0.4$$

$$m = np = 0.04 \times 10 = 0.4$$ **1**

Probability of not more than 2 defective means the sum of probabilities of 0, 1 and 2 defectives:

No. of defective Units	Probability
0	$e^{-4} = 0.6703$
1	$e^{-4} \times \frac{4}{1} = 0.6703 \times 4 = 0.2681$
2	$e^{-4} \times \frac{(0.4)^2}{2!} = \frac{0.6703 \times 0.4 \times 0.4}{2 \times 1} = 0.0536$
Probability of defective upto 2 terms	$= 0.9920$

3

Hence the probability of not more than 2 defectives

$$= 0.9920 \text{ or } 99.2\%$$ **1**

Topic-3 Normal Distribution

Concepts Covered • *Normal Distribution,* • *Standard Normal Distribution*

Revision Notes

- Normal distribution is a continuous probability distribution in which the relative frequencies of a continuous variable are distributed according to normal probability. In simple words, it is a symmetrical distribution in which the frequencies are distributed evenly about the mean of distribution.

e.g.,: (i) Height and intelligence are approximately normally distributed.

(ii) Measurement of errors also often have a normal distribution.

Scan to know more about this topic

Normal distribution

The Normal (or Gaussian) Distribution is defined by the probability density function

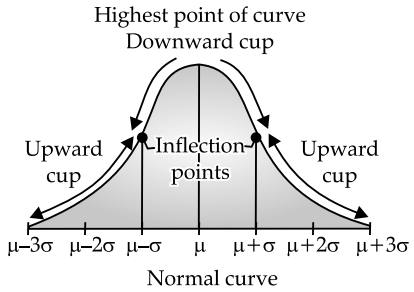
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ for } -\infty < x < \infty$$

where μ and $\sigma > 0$ are parameters of the distribution.

Clearly, $f(x)$ is non-negative and $\int_{-\infty}^{\infty} f(x)dx = 1$.

The notation $N(\mu, \sigma^2)$ means normally distributed with mean μ and variance σ^2. If we say $x \sim N(\mu, \sigma^2)$ we mean that x is distributed $N(\mu, \sigma^2)$.

- **Normal distribution is diagrammatically represented as follows:**



Normal curve

Normal distribution is limiting case of binomial distribution under the following conditions:

(i) n, the number of trials is infinitely large, *i.e.*, $n \to \infty$

(ii) neither p(nor q) is very small,

The normal distribution of a variable when represented graphically, takes the shape of a symmetrical curve, known as the Normal Curve. The curve is asymptotic to X-axis on its either side.

Chief Characteristics or Properties of Normal Probability distribution and Normal probability Curve.

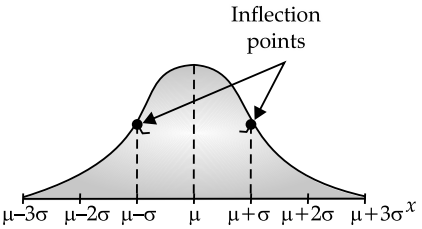
The normal probability curve with mean μ and standard deviation σ has the following properties:

(i) The curve is bell- shaped and symmetrical about the line $x = \mu$

(ii) Mean, median and mode of the distribution coincide.

(iii) x-axis is an asymptote to the curve, (tails of the curve never touches the horizontal (X) axis).

(iv) No portion of the curve lies below the X-axis as $f(x)$ being the probability function can never be negative.



(v) The points of inflection of the curve are $x = \mu \pm \sigma$

(vi) The curve of a normal distribution has a single peak *i.e.*, it is a unimodal.

(vii) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x = \mu$ and is given by $[p(x)]\max = \frac{1}{\sigma\sqrt{2\pi}}$

(viii) The total area under the normal curve is equal to unity and the percentage distribution of area under the normal curve is given below.

(a) About 68.27% of the area falls between $\mu - \sigma$ and $\mu + \sigma$

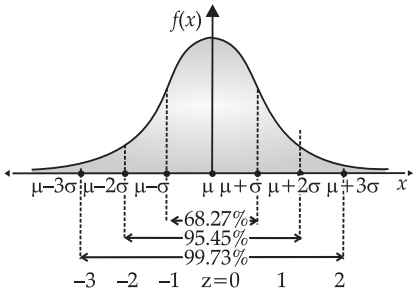
$$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$$

(b) About 95.5% of the area falls between $\mu - 2\sigma$ and $\mu + 2\sigma$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$

(c) About 99.7% of the area falls between $\mu - 3\sigma$ and $\mu + 3\sigma$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$



Standard Normal Distribution

A random variable $Z = (X - \mu)/\sigma$ follows the standard normal distribution. Z is called the standard normal variate with mean 0 and standard deviation 1 *i.e.*, $Z \sim N(0, 1)$. Its Probability density function is given by :

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$

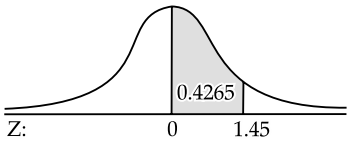
Scan to know more about this topic

Area under normal curve

1. The area under the standard normal curve in equal to 1.
2. 68.26% of the area under the standard normal curve lies between Z = –1 and Z = 1
3. 95.44% of the area lies between Z = – 2 and Z = 2
4. 99.74% of the area lies between Z = – 3 and Z = 3

Standard Normal Distribution Table

This table provides the area between the mean and some Z score.
For example, when Z score = 1.45
the area = 0.4265.



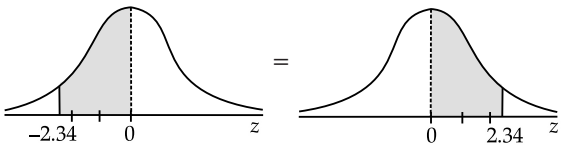
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3980	0.3997	0.4015
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.5	04938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	04985	0.4986	0.4986

3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.489	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

Example: Find the area under the standard normal curve from $z = -2.34$ to 0.

Sol. The area from $z = -2.34$ to 0 is the same as the area from $z = 0$ to 2.34. (See Figure) By Table 1, the area from 0 to 2.34 is 0.4904. Therefore, the area from $z = -2.34$ to 0 is also 0.4904.

Area from $z = -2.34$ to 0 Equals Area from $z = 0$ to 2.34



Key Facts

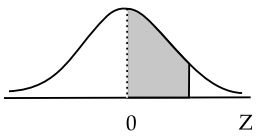
- Normal distribution is also called as Gaussian distribution, named after mathematician **Karl Friedrich Gauss**'.

To find areas other than those between a given z value and $z = 0$, we use Table A together with addition or subtraction of areas we find in Table 1.

1. For areas extending from one side of the mean $z = 0$ to the other side, we add areas found in Table 1.
2. For areas completely on one side of the mean $z = 0$ (but not bordering $z = 0$), we subtract areas found in Table 1.
3. The area extending from $z = 0$ and including the entire right half of the graph is 0.5000. Likewise, the area extending from z = 0 and including the entire left half of the graph is 0.5000.

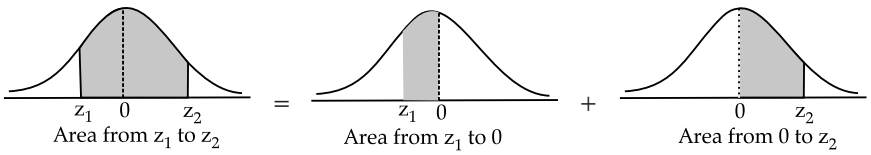
- **Patterns for Finding Areas Under the Standard Normal Curve:**

(a) Area between a given z value and 0

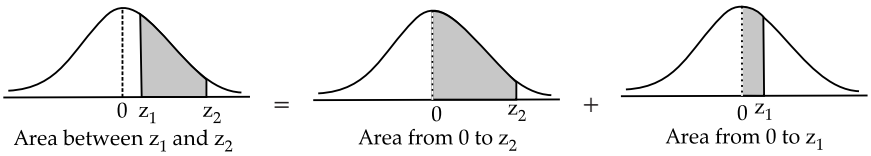


use Table 1

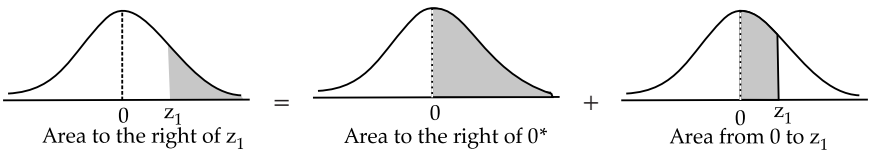
(b) Area between z values on either side of 0



(c) Area between z value on same side of 0.

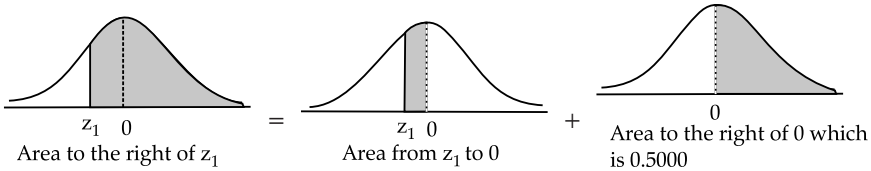


(d) Area to the right of a positive z value or to the left of a negative z value



This area = 0.5000 since the area under the entire curve is 1 and the area to the right of 0 is half the area under the entire curve

(e) Area of the right of a negative z value or to the left of a positive z value.



OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. The normal distribution curve is symmetrical about:

(A) $X = \mu$ **(B)** $X = \sigma$

(C) $X = \frac{\mu}{\sigma}$ **(D)** $X = \frac{\sigma}{\mu}$

Ans. Option (A) is correct.

Explanation: Normal distribution is symmetrical about mean.

Q. 2. If X is a normal variate with mean μ and standard deviation $\sigma > 0$, then the new random variate $Z = \frac{X - \mu}{\sigma}$ is a variate with:

(A) Mean = 1, Standard deviation = 0.
(B) Mean = 1, Standard deviation = 1.
(C) Mean = 2, Standard deviation = 1.
(D) Mean = 0, Standard deviation = 1.

Ans. Option (D) is correct.

Q. 3. It is known from the past experience that the number of telephone calls made daily in a certain community between 3 p.m. and 4 p.m. has a mean of 352 and a standard deviation of 31. What percentage of the time will there be more than 400 telephone calls made in the community between 3 pm to 4 pm? [Use : $P(0 \le Z \le 1.5) = 0.4394$]

(A) 11.4% **(B)** 9.6%
(C) 7.08% **(D)** 6.06%

Ans. Option (D) is correct.

Explanation: Let

$$Z = \frac{x - \mu}{\sigma} = \frac{x - 352}{31}$$

Now, when $x = 400$,

$$Z = \frac{400 - 352}{31}$$

$$Z = \frac{48}{31}$$

$$Z = 1.55$$

$\therefore$ Probability (more than 400 telephone call are made)

$= p(x > 400) = p(z > 1.55)$
$= p(z \ge 0 - p(0 < z < 1.55)$
$= .5 - .4394$
$= .0606$
(given $p(0 \le z \le 1.55) = .4394$)
$= .0606$

$\therefore$ Required% $= .0606 \times 100$
$= 6.06\%$

Q. 4. The total area under the normal distributed curve above the base line *i.e.*, $\int_{-\infty}^{\infty} f(x)dx$ is

(A) 0 **(B)** 0.5
(C) 0.75 **(D)** 1

Ans. Option (D) is correct.

Explanation: The total area under the normal distribution curve above the base line is 1.

Q. 5. An electrical supplier distributor has found the daily demand for fluorescent light bulbs is normally distributed with a mean of 432 and standard deviation of 86. Find the probability that the demand on a particular day exceeds 518 bulbs.

(A) 0.1587 **(B)** 0.3413
(C) 0.7587 **(D)** 0.8413

Ans. Option (A) is correct.

Explanation:

$P(x > 518) = 1 - p(x < 518)$
$= 1 - P(z < 1)$
$= 1 - 0.8413$
$= 0.1587$

Q. 6. The value of mortgage loans made by a certain bank is normally distributed with mean of ₹ 36 lacs and a standard deviation of ₹ 12 lacs. The probability that a randomly selected mortgage loan is less than 54 lacs is

(A) 85.26% **(B)** 93.32%
(C) 97.42% **(D)** 98.04%

Ans. Option (B) is correct.

Explanation:

$P(x < 54) = P(z < 1.5)$
$= 0.9332$
$= 93.32\ \%$

Q. 7. The height of certain species of plant is normally distributed with mean of 20 cm and standard deviation of 4 cm. what is the probability that the height of a plant chosen at random lies between 10 cm and 30 cm

(A) 0.0062 **(B)** 0.5341
(C) 0.9876 **(D)** 0.9938

Ans. Option (C) is correct.

Explanation:

$P(10 < X < 30) = P(-2.5 < Z < 2.5)$
$= P(z < 2.5) - P(z < -2.5)$
$= 0.9876$

Q. 8. Normal Distribution is applied for ..... .

(A) Continuous Random Distribution
(B) Discrete Random Variable
(C) Irregular Random Variable
(D) Uncertain Random Variable

Ans. Option (A) is correct.

Q. 9. The shape of the Normal curve is ..... .

(A) Bell shaped **(B)** Flat
(C) Circular **(D)** Spiked

Ans. Option (A) is correct.

Explanation: Due to the nature of the probability mass function, a bell shaped curve is obtained.

Q. 10. Skewness of Normal distribution is ..... .

(A) Negative **(B)** Positive
(C) 0 **(D)** Undefined

Ans. Option (C) is correct.

Explanation: Since the normal curve is symmetric about its mean, its skewness is zero.

Q. 11. The shape of the normal curve depends on its .....
(A) Mean Deviation **(B)** Standard Deviation
(C) Quartile Deviation **(D)** Correlation.

Ans. Option (B) is correct.
Explanation: This can be seen in the *pdf* of normal distribution where standard deviation is a variable.

Q. 12. The value of constant '*e*' appearing in Normal Distribution is ..... .
(A) 2.5185 **(B)** 2.7836
(C) 2.1783 **(D)** 2.7183

Ans. Option (D) is correct.
Explanation: This is a standard constant.

Q. 13. In Normal distribution, the highest value of ordinate occurs at ..... .
(A) Mean
(B) Variance
(C) Extremes
(D) Same value occurs at all points

Ans. Option (A) is correct.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Write the formula for Normal distribution. R

Sol. The probability density function of Normal or Gaussian distribution is given by;

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

where,
x is the variable
μ is the mean
σ is the standard deviation

Q. 2. If the value of random variable is 2, mean is 5 and the standard deviation is 4, then find the probability density function of the normal distribution. A I

Sol. Given,
Variable, $x = 2$
Mean $\mu = 5$ and
Standard deviation $\sigma = 4$
By the formula of the probability density of normal distribution, we can write;

$$f(2, 5, 4) = \frac{1}{4\sqrt{2\pi}} e^{\frac{-(2-2)^2}{2\times 4^2}}$$

$$= \frac{1}{4\sqrt{2\pi}} e^0 = 0.0997 \quad \textbf{1}$$

Q. 3. Calculate the probability density function of normal distribution using the following data. $x = 3$, $\mu = 4$ and $\sigma = 2$. R&U

Sol. Given, variable, x = 3
Mean $\mu = 4$ and
Standard deviation $\sigma = 2$
By the formula of the probability density of normal distribution, we can write;

$$f(3,4,2) = \frac{1}{2\sqrt{2\pi}} e^{\frac{-(3-4)^2}{2\times 2^2}}$$

$$= \frac{1}{2\sqrt{2\pi}} e^{\frac{-1}{8}}$$

$$= \frac{1}{2\sqrt{2\pi}} e^{-0.125}$$

$$= \frac{1}{2\sqrt{2\pi}} \times 0.88250$$

$$= 0.1761 \quad \textbf{1}$$

Q. 4. Weights of fish caught by a traveller are approximately normally distributed with a mean weight of 2.25 kg and a standard deviation of 0.25 kg. What percentage of fish weigh less than 2 kg? A I

Q. 5. The average daily procurement of milk by village society in 800 litres with a standard deviation of 100 litres. Find out proportion of societies procuring milk between 800 litres to 1000 litres per day. A I

Sol. We are given mean $\mu = 800$ and standard deviation $\sigma = 100$. Probability that the procurement of milk between 800 litres to 1000 litres per day is

$$P(800 < X < 1000) = P\left(\frac{800-800}{100} < z < \frac{1000-800}{100}\right)$$

$$= P(0 < Z < 2)$$
$$= 0.4772 \text{ (table value)}$$

Therefore 47.75 percent of societies procure milk between 800 litres to 1000 litres per day.

Short Answer Type Questions-I (2 marks each)

Q. 1. Assume the mean height of children to be 69.25 cm with a variance of 10.8 cm. How many children in a school of 1,200 would you expect to be over 74 cm tall ? A I

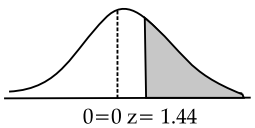
Sol. Let the distribution of heights be normally distributed with mean 68.22 and standard deviation = 3.286

$$Z = \frac{X-\mu}{\sigma} = \frac{X-69.25}{3.286}$$

When $X = 74$

$$Z = \frac{X-\mu}{\sigma} = \frac{74-69.25}{3.286}$$

$$= 1.4455 \quad \textbf{1}$$



These questions are for practice and their solutions are available at the end of the chapter

Now $P(Z > 74) = P(Z > 1.44)$
$= 0.5 - 0.4251$
$= 0.0749$

Expected number of children to be over 74 cm out of 1200 children
$= 1200 \times 0.0749$
$= 90$ children **1**

Q. 2. The speeds of cars are measure using a radar unit, on a motorway. The speeds are normally distributed with a mean of 90 km/hr and a standard deviation of 10 km/hr. What is the probability that a car selected at chance is moving at more than 100 km/hr? A

Q. 3. For some computers, the time period between charges of the battery is normally distributed with a mean of 50 hours and a standard deviation of 15 hours. Rohan has one of these computers and needs to know the probability that the time period will be between 50 and 70 hours. A I

Sol. Let x be the random variable that represents the time period.
Given Mean, $\mu = 50$ and standard deviation, $\sigma = 15$
By using the transformation equation, we know;

$$z = (X - \mu)/\sigma$$

For $x = 50$, $z = \dfrac{(50-50)}{15} = 0$

For $x = 70$, $z = \dfrac{(70-50)}{15} = 1.33$

$P(50 < x < 70) = P(0 < z < 1.33)$
$= P(z = 1.33)$
$= 0.4082$ (from table)

The probability that Rohan's computer has time period between 50 and 70 hours is equal to 0.4082. **2**

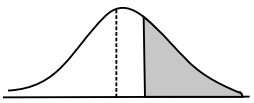
Short Answer Type Questions-II (3 marks each)

Q. 1. The marks obtained in a certain exam follow normal distribution with mean 45 and SD 10. If 1,300 students appeared at the examination, calculate the number of students scoring (i) less than 35 marks and (ii) more than 65 marks. A I

Sol. Let X be the normal variate showing the score of the candidate with mean 45 and standard deviation 10.

(i) less than 35 marks
When $X = 35$

$$Z = \frac{X-\mu}{\sigma} = \frac{35-45}{10} = -1$$



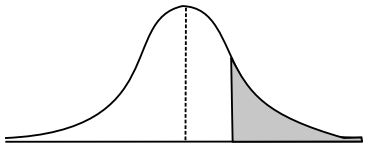
$z = 0$ $z = 1.0$

$P(X < 35) = P(Z < -1)$
$P(Z > 1) = 0.5 - P(0 < Z < 1)$
$= 0.5 - 3413$
$= 0.1587$

Expected number of students scoring less than 35 marks are $0.1587 \times 1300 = 206$ **1½**

(ii) more than 65 marks
When $X = 65$

$$Z = \frac{X-\mu}{\sigma} = \frac{65-45}{10} = 2.0$$



$z = 0$ $z = 2.0$

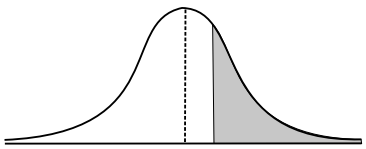
$P(X > 65) = P(Z > 2.0)$
$= 0.5 - P(0 < Z < 2.0)$
$= 0.5 - 0.4772$
$= 0.0228$

Expected number of students scoring more than 65 marks are $0.0228 \times 1300 = 30$ **1½**

Q. 2. 900 light bulbs with a mean life of 125 days are installed in a new factory. Their length of life is normally distributed with a standard deviation of 18 days. What is the expected number of bulbs expire in less than 95 days? R&U

Sol. Let X be the normal variate of light bulbs with mean 125 and standard deviation 18.
For less than 95 days
When $X = 95$

$$Z = \frac{X-\mu}{\sigma} = \frac{95-125}{18} = -1.667$$ **1**



$z = 0$ $z = 1.67$

$P(X < 95) = P(Z < -1.667)$
$= P(Z > 1.667)$
$= 0.5 - P(0 < Z < 1.67)$
$= 0.5 - 0.4525$
$= 0.0475$

No. of bulbs expected to expire in less than 95 days out of 900 bulbs is $900 \times .0475 = 43$ bulbs. **2**

Q. 3. Assume that the mean height of soldiers is 69.25 inches with a variance of 9.8 inches. How many soldiers in a regiment of 6,000 would you expect to be over 6 feet tall ? A

Q. 4. A bank manager has observed that the length of time the customers have to wait for being attended by the teller is normally distribution with mean time of 5 minutes and standard deviation of 0.7 minutes. Find the probability that a customer has to wait

(i) for less than 6 minutes

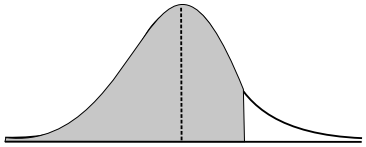
(ii) between 3.5 and 6.5 minutes A I

Sol. Let X be the waiting time of a customer in the queue and it is normally distributed with mean 5 and SD 0.7.

(i) for less than 6 minutes

$$Z = \frac{X-\mu}{\sigma} = \frac{6-5}{0.7} = 1.4285$$

These questions are for practice and their solutions are available at the end of the chapter



$z = 0 \quad z = 1.43$

$$P(X < 6) = P(Z < 1.43)$$
$$= 0.5 + 0.4236$$
$$= 0.9236 \qquad 1\frac{1}{2}$$

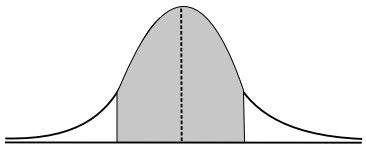
(ii) between 3.5 and 6.5 minutes

When $X = 3.5$

$$Z = \frac{X - \mu}{\sigma} = \frac{3.5 - 5}{0.7} = -2.1429$$

When $X = 6.5$

$$Z = \frac{X - \mu}{\sigma} = \frac{6.5 - 5}{0.7} = 2.1429$$



$z = -2.14 \quad z = 0 \quad z = 2.14$

$$P(3.5 < X < 6.5) = P(-2.1429 < Z < 2.1429)$$
$$= P(0 < Z < 2.1429) + P(0 < Z < 2.1429)$$
$$= 2\,P(0 < Z < 2.1429)$$
$$= 2 \times .4838$$
$$= 0.9676 \qquad 1\frac{1}{2}$$

Long Answer Type Questions (5 marks each)

Q. 1. What is the probability that a standard normal variate Z will be

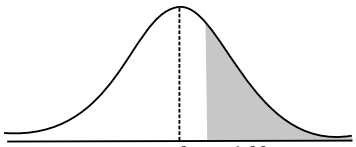
(i) greater than 1.09

(ii) less than – 1.65

(iii) lying between – 1.00 and 1.96 R&U

Sol. (i) greater than 1.09

The total area under the curve is equal to 1, so that the total area to the right $Z = 0$ is 0.5 (since the curve is symmetrical). The area between Z = 0 and 1.09 (from tables) is 0.3621.



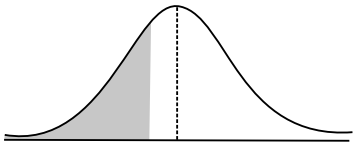
$z = 0 \quad z = 1.09$

$$P(Z > 1.09) = 0.5000 - 0.3621$$
$$= 0.1379$$

The shaded area to the right of $Z = 1.09$ is the probability that Z will be greater than 1.09. $1\frac{1}{2}$

(ii) less than –1.65

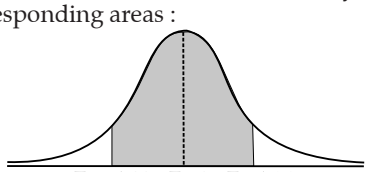
The area between –1.65 and 0 is the same as area between 0 and 1.65. In the table the area between zero and 1.65 is 0.4505 (from the table). Since the area to the left of zero is 0.5, P(Z < 1.65) = 0.5000 – 0.4505 = 0.0495.



$z = 1.65 \quad z = 0$ $1\frac{1}{2}$

(iii) lying between –1.00 and 1.96

The probability that the random variable Z in between –1.00 and 1.96 is found by adding the corresponding areas :



$Z = -1.00 \quad Z = 0 \quad Z = 1.96$

Area between –1.00 and 1.96 = area between (–1.00 and 0) + area between (0 and 1.96)

$$P(-1.00 < Z < 1.96) = P(-1.00 < Z < 0) + P(0 < Z < 1.96)$$
$$= 0.3413 + 0.4750 \quad \text{(by tables)}$$
$$= 0.8163 \qquad 2$$

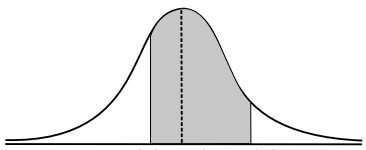
Q. 2. If X is a normal variate with mean 30 and SD 5. Find the probabilities that (i) 26 ≤ X ≤ 40 (ii) X > 45. A I

Sol. Here mean $\mu = 30$ and standard deviation $\sigma = 5$

(i) When $X = 26 \qquad Z = \frac{(X - \mu)}{\sigma}$

$$= \frac{(26 - 30)}{5} = -0.8$$

And when X = 40, $Z = \frac{40 - 30}{5} = 2$ **1**



$z = -0.8 \quad z = 0 \quad z = 2.0$

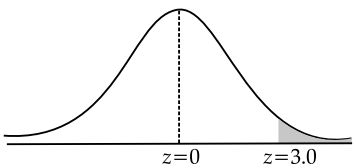
Therefore,

$$P(26 < X < 40) = P(-0.8 \le Z < 2)$$
$$= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$$
$$= P(0 \le Z \le 0.08) + P(0 \le Z \le 2)$$
$$= 0.2881 + 0.4772 \quad \text{(By tables)}$$
$$= 0.7653 \qquad 1\frac{1}{2}$$

(ii) The probability that $X \ge 45$

When $X = 45$

$$X = \frac{X - \mu}{\sigma} = \frac{45 - 30}{5} = 3 \qquad \frac{1}{2}$$



$z = 0 \quad z = 3.0$

$$P(X \geq 45) = P(Z \geq 3)$$
$$= 0.5 - 0.49865$$
$$= 0.00135$$ **1**

Q. 3. The average daily sale of 550 branch offices was ₹ 150 thousand and standard deviation is ₹ 15 thousand. Assuming the distribution to be normal, indicate how many branches have sales between

(i) ₹ 1,25,000 and ₹ 1,45,000

(ii) ₹ 1,40,000 and ₹ 1,60,000 R&U

Q. 4. A sample of 125 dry battery cells tested to find the length of life produced the following results with mean 12 and SD 3 hours. Assuming that the data to be normal distributed, what percentage of battery cells are expected to have life

(i) more than 13 hours

(ii) less than 5 hours

(iv) between 9 and 14 hours A I

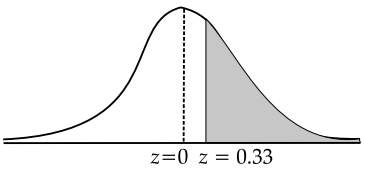
Sol. Let X denote the length of life of dry battery cells follows normal distribution with mean 12 and SD 3 hours

(i) more than 13 hours

$P(X > 13)$

When $X = 13$

$$Z = \frac{X-\mu}{\sigma} = \frac{13-12}{3} = 0.333$$



$$P(X > 13) = P(Z > 0.333)$$
$$= 0.5 - 0.1293$$
$$= 0.3707$$

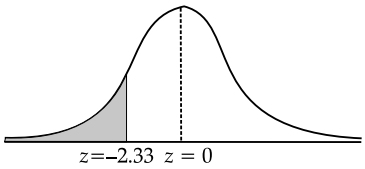
The expected battery cells life to have more than 13 hours is $125 \times 0.3707 = 46.34\%$ **2**

(ii) less than 5 hours

$P(X < 5)$

When $X = 5$

$$Z = \frac{X-\mu}{\sigma} = \frac{5-12}{3} = -2.333$$



$$P(X < 5) = P(Z < -2.333)$$
$$= P(Z > 2.333)$$
$$= 0.5 - 0.4901$$
$$= 0.0099$$

The expected battery cells life to have more than 13 hours is $125 \times 0.0099 = 1.23\%$ **2**

(iii) between 9 and 14 hours

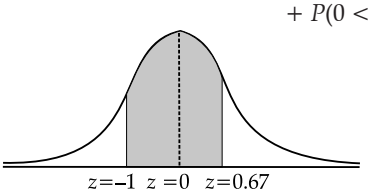
When $X = 9$

$$Z = \frac{X-\mu}{\sigma} = \frac{9-12}{3} = -1$$

When $X = 14$

$$Z = \frac{X-\mu}{\sigma} = \frac{14-12}{3} = 0.667$$

$$P(9 < X < 14) = P(-1 < Z < 0.667)$$
$$= P(0 < Z < 1) + P(0 < Z < 0.667)$$



$$= 0.3413 - 0.2486$$
$$= 0.5899$$

The expected battery cells life to have more than 13 hours is $125 \times 0.5899 = 73.73\%$ **2**

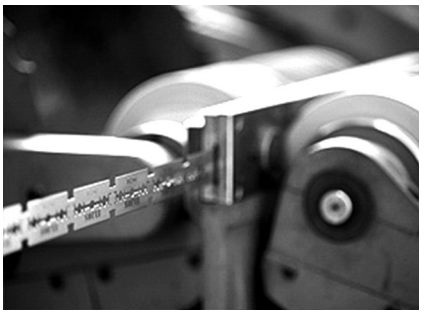
COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

Mr. Rama Roy a owner of a factory lives in Karnataka. In his factory, he manufactures razor blades. There is a small chance that $\frac{1}{500}$ for any blade to be defective. The blades are in the packets of 10.



Q. 1. The probability that packet contain no defective blade is:

(A) $e^{-0.02}$ **(B)** 0.02

(C) $0.02\, e^{-0.02}$ **(D)** $0.002\, e^{-0.02}$

These questions are for practice and their solutions are available at the end of the chapter

Ans. Option (A) is correct.

Explanation: Here given $N = 1000$, $n = 10$, $p = \frac{1}{500} = 0.002$

$\Rightarrow \quad m = np = 10 \times 0.002$

$\Rightarrow \quad m = 0.02$

P(No defective blade) $= P(X = 0)$

$= e^{-0.02}(0.002)^0 = e^{-0.02}$

Q. 2. The probability that packet contain one defective blade is:

(A) $e^{-0.02}$ **(B)** $0.02e^{-0.02}$

(C) $0.02e^{-0.002}$ **(D)** $0.002e^{-0.02}$

Ans. Option (B) is correct.

Explanation: P(one defective blade) $= P(X = 1)$

$= \frac{e^{-0.02}(0.02)^1}{1!}$

$= 0.02e^{-0.02}$

Q. 3. Find the probability that packet contain two detective blade, when it is given that $e^{-0.02} = 0.9802$.

(A) 0.0002 **(B)** 0.002

(C) 0.2 **(D)** None of these

Ans. Option (A) is correct.

Explanation: P(two defective blade) $= P(X = 2)$

$= \frac{e^{-0.02}(0.02)^2}{2!}$

$= \frac{0.9802 \times 0.0004}{2}$

$= 0.000196$

$= 0.0002$

Q. 4. Find the approximate number of packets containing no defective blade, when there are 10000 packets in a consignment. (Use $e^{-0.02} = 0.9802$)

(A) 9802 **(B)** 9800

(C) 2 **(D)** 196

Ans. Option (A) is correct.

Explanation: P (no defective blade) $= e^{-0.02}$

$= 0.9802$

So, the approximate number of packets containing no defective blade

$= 10{,}000 \times 0.9802 = 9802$

Q. 5. Find the approximate number of packets containing one defective blade, when there are 20,000 packets in the consignment. (Use $e^{-0.02} = 0.9802$)

(A) 196 **(B)** 392

(C) 932 **(D)** 619

Ans. Option (B) is correct.

Explanation: $\because$ P(one defective blade)

$= 0.02e^{-0.02}$

$= 0.02 \times 0.9802$

$= 0.0196$

So, the approximate number of packets containing 1 defective blade $= 20{,}000 \times 0.0196$

$= 392.$

II. Read the following text and answer the following questions on the basis of the same:

We take 8 identical slips of paper; write the number 0 on one of them, the number 1 on three of the slips, the number 2 on three of the slips and the number 3 on one of the slip. These slips are folded, put in a box and thoroughly mixed. One slip is drawn at random from the box. If X is the random variable denoting the number written on the drawn slip, answer the following questions:

Q. 1. How many total values X can take?

(A) 1 **(B)** 2

(C) 3 **(D)** 4

Ans. Option (D) is correct.

Explanation: Clearly X can take values 0, 1, 2, 3. So, in total X can take total 4 values.

Q. 2. Find the probability when X = 0.

(A) $\frac{1}{8}$ **(B)** $\frac{3}{8}$

(C) $\frac{1}{2}$ **(D)** $\frac{1}{4}$

Ans. Option (A) is correct.

Explanation: $P(X = 0)$ = (Probability of getting a slip written 0 on it) $= \frac{1}{8}$.

Q. 3. Find the probability when X = 1.

(A) $\frac{1}{8}$ **(B)** $\frac{3}{8}$

(C) $\frac{1}{2}$ **(D)** $\frac{1}{4}$

Ans. Option (B) is correct.

Explanation: $P(X = 1)$ = (Probability of getting a slip written 1 on it) $= \frac{3}{8}$.

Q. 4. The probability distribution is given as:

(A)

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(B)

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$

(C)

X	0	1	2	3
P(X)	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(D)

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{3}{8}$

Ans. Option (A) is correct.

Explanation: $P(X = 0)$ = (Probability of getting a slip written 0 on it) = $\frac{1}{8}$.

$P(X = 1)$ = (Probability of getting a slip written 1 on it) = $\frac{3}{8}$.

$P(X = 2)$ = (Probability of getting a slip written 2 on it) = $\frac{3}{8}$.

$P(X = 3)$ = (Probability of getting a slip written 3 on it) = $\frac{1}{8}$.

Thus, probability distribution is given as:

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Q. 5. The expectation value is

(A) $\frac{3}{8}$ **(B)** $\frac{1}{2}$

(C) $\frac{2}{3}$ **(D)** $\frac{3}{2}$

Ans. Option (D) is correct.

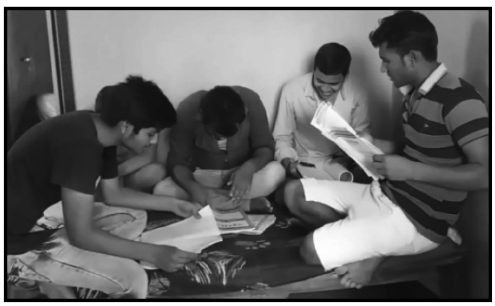
Explanation: $E(X) = \sum_{i=0}^{3} X_i P(X_i)$

$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$

III. Read the following text and answer the following questions on the basis of the same:

Today in mathematics class Mr. Lal teaches the topic of Binomial distribution. Four friends Rohan, Rohit, Roshan and Raman have few doubts in this topic, so they decided for group study at Rohan's place. They are trying to solve a question in which a pair of dice is thrown 7 times. If getting a total 7 is considered a success, then answer the following questions:



Q. 1. If X denote the number of success in 7 throws of a pair of dice, then X Binomial variate with parameters:

(A) $n = 7, p = \frac{1}{6}$ **(B)** $n = 2, p = \frac{1}{6}$

(C) $n = 7, p = \frac{5}{6}$ **(D)** $n = 2, p = \frac{5}{6}$

Ans. Option (A) is correct.

Explanation: Let p denote the probability of getting a total of 7 in a single throw of a pair of dice. Then

$$p = \frac{6}{36} = \frac{1}{6}$$

[$\because$ The sum can be 7 in any one of the ways : (1, 6), (6, 1), (2, 5), (5, 2), (3, 4) and (4, 3)]

So, the parameters are:

$$n = 7 \text{ and } p = \frac{1}{6}$$

Q. 2. What is the probability of no success ?

(A) $\left(\frac{6}{5}\right)^7$ **(B)** $\left(\frac{5}{6}\right)^7$

(C) $\left(\frac{1}{5}\right)^7$ **(D)** $\left(\frac{5}{6}\right)$

Ans. Option (B) is correct.

Explanation: The X is a binomial variate with parameters $n = 7$ and $p = \frac{1}{6}$

$$P(X = r) = {}^7C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{7-r}$$

$r = 0, 1, 2, \ldots,$ ...(i)

$\left[\because q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}\right]$

Probability of no success $= P(X = 0)$

$= {}^7C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{7-0}$

$= \left(\frac{5}{6}\right)^7$

Q. 3. What is the probability of 6 success ?

(A) $\left(\frac{1}{6}\right)^7$ **(B)** $35\left(\frac{1}{6}\right)$

(C) $35\left(\frac{1}{6}\right)^7$ **(D)** $35\left(\frac{5}{6}\right)^7$

Ans. Option (C) is correct.

Explanation: From eq. (i),

Probability of 6 success $= P(X = 6)$

$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{7-6}$

$= 35\left(\frac{1}{6}\right)^7$

Q. 4. What is the probability of at least 6 success ?

(A) $\left(\frac{1}{6}\right)^5$ **(B)** $\left(\frac{1}{6}\right)^7$

(C) $\left(\frac{1}{6}\right)^6$ **(D)** $\left(\frac{5}{6}\right)^5$

Ans. Option (A) is correct.

Explanation: Probability of at least 6 success

$= P(X \geq 6)$

$= P(X = 6) + P(X = 7)$

$= {}^7C_6\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^{7-6} + {}^7C_7\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^{7-7}$

$= {}^7C_6\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right) + {}^7C_7\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^0$

$= 7\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^7$

$= \left(\frac{1}{6}\right)^6\left(\frac{35}{6} + \frac{1}{6}\right) = \left(\frac{1}{6}\right)^5$

Q. 5. What is the probability of at most 6 success ?

(A) $1 - \left(\frac{1}{6}\right)^5$ **(B)** $1 - \left(\frac{1}{6}\right)^7$

(C) $1 - \left(\frac{1}{6}\right)^6$ **(D)** $1 - \left(\frac{5}{6}\right)^5$

Ans. Option (B) is correct.

Explanation: From eq. (i),

Probability of at most 6 success $= P(X \leq 6)$

$= 1 - P(X > 6)$

$= 1 - P(X = 7)$

$= 1 - {}^7C_7\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^{7-7}$

$= 1 - \left(\frac{1}{6}\right)^7$

Case based Subjective Questions

Each sub-part carries 2 marks.

I. In regards to product warranties, manufacturer defines the life span of a product to be the length of time that a product is in production and/or has parts in supply to support the repair or replacement of that product. The life span of certain kinds of electronic devices have mean of 300 hours and standard derivation of 25 hours. Assuming that the distribution of these life spans, which are measured to the nearest hour, can be approximated closely with a normal curve.

Q. 1. Find the probability that any one of these electronic devices will have a life span of more than 350 hours. [Given $P(0 \leq Z \leq 2) = 0.4772$]

Sol. Let X denotes the life spans of the given electronic devices. Then X is normally distributed with mean, $\mu = 300$ and standard deviation, $\sigma = 25$

Let Z be the standard normal variate. Then

$$Z = \frac{X - \mu}{\sigma}$$

$$\Rightarrow \quad Z = \frac{X - 300}{25}$$

when $X = 350$, we get

$$Z = \frac{350 - 300}{25}$$

$$= \frac{50}{25} = 2 \qquad \textbf{1}$$

$\therefore \quad P(X > 350) = P(Z > 2)$

$= 0.5 - P(0 \leq Z \leq 2)$

$= 0.5 - 0.4772$

$= 0.0228$ **1**

Q. 2. What percentage will have life spans from 220 to 260 hours.
[Given, $P(0 \leq Z \leq 3.2) = 0.4993$ and $P(0 \leq Z \leq 1.6) = 0.4452$]

Sol. When $X = 220$, we get

$$Z = \frac{220 - 300}{25} = -3.2 \qquad ½$$

and when $X = 260$, we get

$$Z = \frac{260 - 300}{25} = -1.6 \qquad ½$$

$\therefore P(220 \leq X \leq 260) = P(-3.2 \leq Z \leq -1.6)$

$= P(1.6 \leq Z \leq 3.2)$

$= P(0 \leq Z \leq 3.2) - P(0 \leq Z \leq 1.6)$

$= 0.4993 - 0.4452$

$= 0.0541$

Thus, 5.41% of the electronic devices will have life spans from 220 to 260.

II. Injection site reaction are the most common adverse events following immunisation. These include pain, etching, swelling or redness around the site of injection. These reactions are usually mild and last for 1-2 days. If the probability that an individual suffers a bad reaction from an injection of a given serum is 0.001, then

Q. 1. Determine the probability that out of 2000 individuals exactly 3 individuals suffer from bad reaction. [Given, $e^{-2} = 0.13534$]

Sol. Let p be the probability that an individual suffers a bad reaction from an injection of a given serum and n be the total number of individuals who took the injection.

Given, $p = 0.001$ and $n = 2000$

Since, p is very small and n is large, therefore, we apply Poisson's distribution.

We have, $m = np = 2000 \times 0.001 = 2$ **1**

Let X denote the number of individuals who suffer from bad reaction. Then, X is a Poisson variate such that

$$P(X = r) = \frac{m^r e^{-m}}{r!}, r = 0, 1, 2, ... \infty$$

$$\therefore \quad P(X = r) = \frac{2^r e^{-2}}{r!}, r = 0, 1, 2, ... \infty$$

$\therefore$ Required probability $= P(X = 3)$

$$= 2^3 \frac{e^{-2}}{3!} = \frac{4}{3}e^{-2}$$

$$= \frac{4}{3} \times 0.13534$$

$$= 0.18044 \quad \mathbf{1}$$

Q. 2. Find the probability that out of 2000 individuals more than 2 individuals suffer from bad reaction.

Sol. Required probability

$$= P(X > 2)$$

$$= 1 - P(X \le 2)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \quad \mathbf{1}$$

$$= 1 - \left[e^{-2} + 2\frac{e^{-2}}{1!} + 2^2\frac{e^{-2}}{2!}\right]$$

$$= 1 - \frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}$$

$$= 1 - \frac{5}{e^2}$$

$$= 1 - 5 \times 0.13534$$

$$= 1 - 0.67670$$

$$= 0.323 \quad \mathbf{1}$$

Solutions for Practice Questions (Topic-1)

Very Short Answer Type Questions

Sol. 2: $E(X) = \sum X\, P(X)$

$$= -4 \times (0.1) + (-3 \times 0.2) + (-2 \times 0.3) + (-1 \times 0.2) + (0 \times 0.2)$$

$$= -0.4 - 0.6 - 0.6 - 0.2 + 0 = -1.8 \quad \mathbf{1}$$

Sol. 4: As there are two red balls, the maximum number of red balls can be 2.

X can take values 0, 1, 2. **1**

Short Answer Type Questions-I

Sol. 4: Let X is the random variable which counts the number of heads on the upturned faces. The outcomes are stated below

Outcomes	Values of X	Outcomes	Values of X
(HHH)	3	(HTT)	1
(HHT)	2	(THT)	1
(HTH)	2	(TTH)	1
(THH)	2	(TTT)	0

These values are summarized in the following Probability table. **1**

Value of X	0	1	2	3	Total
$p(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\sum_{i=0}^{3} p(x_i) = 1$

(i) $p(x_i) \ge 0 \; \forall \; i$ and

(ii) $\sum_{i=0}^{3} p(x_i) = 1$

Sol. 7: Expected value of the random variable is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_0^{25} x\frac{1}{25}dx$$

$$= \frac{1}{25}\int_0^{25} x\,dx$$

$$= \frac{1}{25}\left[\frac{x^2}{2}\right]_0^{25}$$

$$= 12.5$$

Therefore, the expected waiting time of the commuter is 12.5 minutes. **2**

Short Answer Type Questions-II

Sol. 4: We have, $P(X = x) = \begin{cases} kx & , \text{ if } x = 0 \text{ or } 1 \\ 2kx & , \text{ if } x = 2 \\ k(5-x) & , \text{ if } x = 3 \text{ or } 4 \\ 0 & , \text{ if } x > 4 \end{cases}$

We know that, sum of all the probabilities of a distribution is always 1.

$\therefore P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + ... = 1$

or $0 + k + 4k + 2k + k + 0 + 0 ..... = 1$

or $8k = 1$

or $k = \frac{1}{8}$

Now, $P(X = x) = \begin{cases} \frac{x}{8}, \text{ if } x = 0 \text{ or } 1 \\ \frac{x}{4}, \text{ if } x = 2 \\ \frac{(5-x)}{8}, \text{ if } x = 3 \text{ or } 4 \\ 0, \text{ if } x = 0 \text{ or } 1 \end{cases}$ ...(1)

$\therefore$ (i) P(getting admission in exactly one college) **1**

$$= P(X = 1) = \frac{1}{8}$$

(Using (1))

(ii) P (getting admission in at most 2 colleges)

$$= P(X \le 2)$$
$$= P(X = 0) + P(X = 1) + P(X = 2)$$
$$= 0 + \frac{1}{8} + \frac{2}{4} = \frac{5}{8} \quad \textbf{1}$$

(iii) P (getting admission in at least 2 colleges)

$$= P(X > 2) = 1 - P(X < 2)$$
$$= 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - \left[0 + \frac{1}{8}\right] = 1 - \frac{1}{8} = \frac{7}{8} \quad \textbf{1}$$

Long Answer Type Questions

Sol. 4: The variance X takes values 3, 4, 5 and 6

$P(X = 3) = \frac{1}{20}$; $P(X = 4) = \frac{3}{20}$;

$P(X = 5) = \frac{6}{20}$; $P(X = 6) = \frac{10}{20}$;

X	3	4	5	6
$P(X)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{6}{20}$	$\frac{10}{20}$
$XP(X)$	$\frac{3}{20}$	$\frac{12}{20}$	$\frac{30}{20}$	$\frac{60}{20}$
$X^2P(X)$	$\frac{9}{20}$	$\frac{48}{20}$	$\frac{150}{20}$	$\frac{360}{20}$

3

$$\text{Mean} = \Sigma XP(X) = \frac{105}{20} = \frac{21}{4} \quad \textbf{1}$$

$$\text{Variance} = \Sigma X^2P(X) - \{\Sigma X\, P(X)\}^2 \quad \textbf{1}$$

$$\frac{567}{20} - \frac{441}{16} = \frac{63}{80}$$

Sol. 8: We know that

$\int_{-\infty}^{\infty} f(x)dx = 1$, since $f(x)$ is a density function. ½

$$\int_0^{\infty} ke^{-2x}dx = 1$$

$$k\int_0^{\infty} e^{-2x}dx = 1$$

$$k\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = 1$$

$$\Rightarrow \quad k = 2 \quad \textbf{1½}$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx \quad ½$$

$$= \int_0^{\infty} xke^{-2x}dx = 2\int_0^{\infty} xe^{-2x}dx$$

$$= 2\left\{\left[\frac{xe^{-2x}}{-2}\right]_0^{\infty} - \int_0^{\infty}\frac{e^{-2x}}{-2}dx\right\}$$

$$\left(\because \int udv = uv - \int vdu\right)$$

$$= \int_0^{\infty} e^{-2x}dx = \frac{1}{2} \quad \textbf{2½}$$

Solutions for Practice Questions (Topic-2)

Very Short Answer Type Questions

Sol. 4: The repeated selection of students who are swimmers are Bernoulli trial. Let X denotes the number of students, out of 5 students, who are swimmers.

Probability of students who are not swimmers,

$$q = \frac{1}{5}$$

$$\therefore p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$$

Clearly, X has a binomial distribution with $n = 5$ and $p = \frac{4}{5}$.

$$P(X = x) = {}^nC_x q^{n-x}p^x$$
$$= {}^5C_x\left(\frac{1}{5}\right)^{5-x}\cdot\left(\frac{4}{5}\right)^x$$

P(4 students are swimmers)

$$= P(X = 4) = {}^5C_4\left(\frac{1}{5}\right)\cdot\left(\frac{4}{5}\right)^4 \quad \textbf{1}$$

Sol. 7: $P(X = 2) = 9.P(X = 3)$ (where, $n = 5$ and $q = 1 - p$)

$$\Rightarrow \quad {}^5C_2p^2(1-p)^3 = 9.{}^5C_2p^3(1-p)^2$$

$$\Rightarrow \quad \frac{5!}{2!3!}p^2(1-p)^3 = 9\cdot\frac{5!}{3!2!}p^3(1-p)^2$$

$$\Rightarrow \quad \frac{p^2(1-p)^3}{p^3(1-p)^2} = 9$$

$$\Rightarrow \quad \frac{(1-p)}{p} = 9$$

$$\Rightarrow \quad 9p + p = 1$$

$$\therefore \quad p = \frac{1}{10} \quad \textbf{1}$$

Short Answer Type Questions-I

Sol. 2: Given, $n = 5, p = \frac{1}{2}$ and $q = \frac{1}{2}$

Using Binomial distribution $P(x = r) = {}^nC_r\, P^r\, q^{n-r}$

(i) Probability of getting exactly three heads,

i.e., $$P(x = 3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$= \frac{5!}{3!\,2!} \times \frac{1}{8} \times \frac{1}{4}$$

$$= \frac{5 \times 4 \times 3!}{3! \times 2 \times 1} \times \frac{1}{32}$$

$$= \frac{10}{32} = \frac{5}{16}$$ **1**

(ii) Probability of getting at most 3 heads,

i.e., $P(x \le 3) = 1 - [P(x = 4) + P(x = 5)]$

$$= 1 - \left[{}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) + {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 \right]$$

$$= 1 - \left[5 \times \frac{1}{32} + 1 \times \frac{1}{32} \right]$$

$$= 1 - \left(\frac{6}{32}\right)$$

$$= 1 - \frac{3}{16} = \frac{13}{16}$$ **1**

Sol. 7: Let the coins be unbiased *i.e.* the probability of getting a head = the probability of getting a tail, for each coin.

$\therefore$ The probability of getting 6 heads with 6 coins

$$= \left(\frac{1}{2}\right)^6 = \frac{1}{64} = p(\text{say})$$

$$np = 6400 \times \frac{1}{64} = 100 = m(\text{say})$$ **1**

Let X = no. of tosses with heads

Then according to Poisson distribution,

$$P(X = x) = e^{-m} \frac{m^x}{x!} = e^{-100} \frac{(100)^x}{x!}$$

and $$P(X = 2) = e^{-m} \frac{m^2}{2!}$$

$$= e^{-100} \frac{(100)^2}{2!} = 5000\, e^{-100}$$ **1**

Short Answer Type Questions-II

Sol. 2: p = Probability of getting spade card

$$= \frac{13}{52} = \frac{1}{4}$$

and $q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

Using formula for getting probability of X successes in n trials, *i.e.,*

$$P(X = r) = {}^nC_r\, p^r q^{n-r}$$

As given $n = 4$

Probability of getting all four spade cards *i.e.,*

$$P(\text{all four are spades}) = {}^4C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{4-r}$$ **1**

Here putting $r = 4$, we get

P(all four are spades)

$$= {}^4C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{4-4}$$

$$= \frac{1}{256}$$ **1**

(ii) Probability of getting only 2 spade card *i.e.,*

$$P(\text{only 2 are spades}) = {}^4C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{n-r}$$

Here, Putting $r = 2$, we get

$$P(\text{only 2 are spades}) = {}^4C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2$$

$$= \frac{27}{128}$$ **1**

Commonly Made Error

- Many candidates are not able to use the concept of Binomial probability Distribution to solve this problem properly.

Answering Tip

- Students need to understand the probability distribution, Binomial Probability distribution and their conditions.

Sol. 5: Let X = No. of defective bulbs out of 4 bulbs drawn with replacement

or $X = 0, 1, 2, 3, 4$

p = Probability of defective bulb = $\frac{6}{30} = \frac{1}{5}$,

q = Probability of good bulb = $\frac{4}{5}$

$X = x_i$	0	1	2	3	4
$P(x) = p_i$	$^4C_0 p^0 \cdot q^4$ $= \left(\frac{4}{5}\right)^4$ $= \frac{256}{625}$	$^4C_1 p \cdot q^3$ $= 4 \cdot \frac{1}{5} \cdot \frac{64}{125}$ $= \frac{256}{625}$	$^4C_2 p^2 \cdot q^2$ $= 6 \cdot \frac{1}{25} \cdot \frac{16}{25}$ $= \frac{96}{625}$	$^4C_3 p^3 \cdot q$ $= 4 \cdot \frac{1}{125} \cdot \frac{4}{5}$ $= \frac{16}{625}$	$^4C_4 \cdot p^4$ $= \left(\frac{1}{5}\right)^4$ $= \frac{1}{625}$

2

Mean (μ) $= \Sigma p_i \cdot x_i$

$$= 0 \times \frac{256}{625} + 1 \times \frac{256}{625} + 2 \times \frac{96}{625} + 3 \times \frac{16}{625} + 4 \times \frac{1}{625}$$

$\therefore$ Mean (μ) $= \frac{4}{5}$ **1**

Long Answer Type Questions

Sol. 3: Let X be a random variable that denotes the number of defective pens is a draw of 5 pens.

Then, X can take values 0, 1, 2, 3, 4 and 5.

Here, p = probability of getting a defective pen in a single draw = $\frac{2}{20} = \frac{1}{10}$ and q = probability of not getting a defective pen in a single draw

$$= 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$ **1**

Clearly, X follows binomial distribution with parameters $n = 5$, $p = \frac{1}{10}$ and $q = \frac{9}{10}$.

$\therefore \quad P(X = r) = {}^5C_r \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{5-r}$

$r = 0, 1, 2, 3, 4$ and 5 **1**

Now, P(getting at most 2 defective pen)

$= P(x \leq 2)$

$= P(X = 0) + P(X = 1) + (X = 2)$

$$= {}^5C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^5 + {}^5C_1 \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^4 + {}^5C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^3$$ **1½**

$$= \frac{9^3}{10^5}[9^2 + 5 \times 9 + 10] = \frac{729}{100000}[81 + 45 + 10]$$

$$= \frac{729 \times 136}{100000} = \frac{99144}{100000} = 0.99144$$ **1½**

Solutions for Practice Questions (Topic-3)

Very Short Answer Type Questions

Sol. 4: We are given mean $\mu = 2.25$ and standard deviation $\sigma = 0.25$. Probability that weight of fish is less than 2 kg is $P(X < 2.0)$

When $x = 20$ $\quad Z = \frac{X - \mu}{\sigma} = \frac{2.0 - 2.25}{0.25} = -1.0$

$P(Z < -1.0) = P(Z > 1.0)$

$= 0.5 - 0.3413$

$= 0.1587$

Therefore 15.87% of fishes weight less than 2 kg.

Short Answer Type Questions-I

Sol. 2: Let the speed of cars is represented by a random variable 'x'.

Now, given mean, $\mu = 90$ and standard deviation, $\sigma = 10$.

To find: Probability that x is higher than 100 or $P(x > 100)$

By using the transformation equation, we know;

$$z = \frac{(X - \mu)}{\sigma}$$

For $x = 100$, $z = (100 - 90) / 10 = 1$

Hence, $P(x > 90) = P(z > 1)$

$= 0.5 - P(z = 1)$

$= 0.5 - 0.3413$

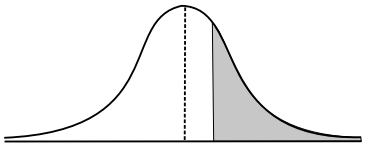
$= 0.1587$ (from table) **2**

Short Answer Type Questions-II

Sol. 3: Let X be the height of soldiers follows normal distribution with mean 69.25 inches and standard deviation 3.13 then the soldiers over 6 feet tall (6 ft $\times$ 12 = 72 inches)

The standard normal variate

$$Z = \frac{X - \mu}{\sigma} = \frac{72 - 69.25}{3.13} = 0.8786$$



$z = 0 \quad z = 0.87$ **1**

$P(X > 72) = P(Z > 0.8786)$

$= 0.5 - P(0 < Z < 0.88)$

$= 0.5 - 0.3106$

$= 0.1894$

Number of soldiers expected to be over 6 feet tall in 6000 are

$6000 \times 0.1894 = 1136$ **2**

Long Answer Type Questions

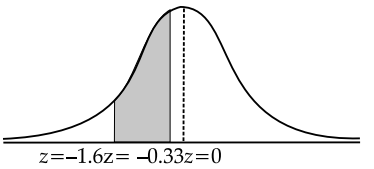
Sol. 3: Given that mean $\mu = 150$ and standard deviation $\sigma = 15$

(i) When $X = 125$ thousand

$$Z = \frac{X - \mu}{\sigma} = \frac{125 - 150}{15} = -1.667$$

When $X = 145$ thousand

$$Z = \frac{X - \mu}{\sigma} = \frac{145 - 150}{15} = -0.33$$ **1**



$z = -1.6 \quad z = -0.33 \quad z = 0$

Area between $Z = 0$ and $Z = -1.67$ is 0.4525

Area between $Z = 0$ and $Z = -0.33$ is 0.1293

$P(-1.667 \le Z \le -0.33) = 0.4525 - 0.1293$

$= 0.3232$

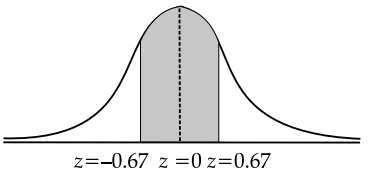
Therefore the number of branches having sales between ₹ 1,25,000 and ₹ 1,45,000 is 550×0.3232 = 178 **1½**

(ii) When $X = 140$ thousand

$$Z = \frac{X - \mu}{\sigma} = \frac{140 - 150}{15} = -0.67$$

When $X = 160$ thousand

$$Z = \frac{X - \mu}{\sigma} = \frac{160 - 150}{15} = 0.67$$ **1**



$z = -0.67 \quad z = 0 \quad z = 0.67$

$P(-0.67 < Z < 0.67) = P(-0.67 < Z < 0)$

$+ P(0 < Z < 0.67)$

$= P(0 < Z < 0.67)$

$+ P(0 < Z < 0.67)$

$= 2\,P(0 < Z < < 0.67)$

$= 2 \times 0.2486$

$= 0.4972$

Therefore, the number of branches having sales between ₹ 1,40,000 and ₹ 1,60,000

$= 550 \times 0.4972$

$= 273$ **1½**

REFLECTIONS

(a) Will you be able to differentiate between Binomial and Poisson distribution?

(b) Can you calculate the expected value and variance for given set of data?

(c) Will you be able to solve the questions based on Normal distribution easily?

SELF ASSESSMENT PAPER - 04

Maximum Marks: 30 **Max. Time: 1 Hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. What would be the probability of an event 'G' if H denotes it complement, according to the axioms of probability?

(A) $P(G)=\frac{1}{P(H)}$ **(B)** $P(G) = 1 - P(H)$

(C) $P(G) = 1 + P(H)$ **(D)** $P(G) = P(H)$

Q. 2. For a binomial distribution, which of the following is correct.

(A) Mean = Variance **(B)** Mean < Variance

(C) Mean > Variance **(D)** None of these

Q. 3. If $E(x) = 4$ and $E(z) = 8$, then $E(z - x)$ is:

(A) 2 **(B)** 4

(C) 6 **(D)** insufficient data

Q. 4. Find the mean of x if x is a Poisson variate satisfying the condition $P(4) = P(5)$?

(A) 4 **(B)** 5

(C) 3 **(D)** 1

Q. 5. If a discrete random variable X. follows uniform distribution and assume only the values 8, 9, 11, 15, 18, 20 the value of $P(|X - 14| < 5)$ will be:

(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$

(C) $\frac{1}{3}$ **(D)** $\frac{1}{2}$

Q. 6. The expected value of a random variable which has the following probability distribution is:

x	2	4	6	8	10
p	0.1	0.2	0.3	0.1	0.1

(A) 4.8 **(B)** 4.6

(C) 4.4 **(D)** 4.9

Case-Based MCQs

Q. 7. Attempt any 4 sub-parts from each questions. Each question carries 1 mark.

Read the following text and answer the following questions on the basis of the same.

The amount of bread (in hundreds of kilograms) X that a certain bakery is able to sell in a day is found to be a numerical valued random phenomenon with a probability function specified by the probability density function $f(x)$, given by

$$f(x) = \begin{cases} Ax; & \text{for} \quad 0 \le x < 5 \\ A(10-x), & \text{for} \quad 5 \le x < 10 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of A such that $f(x)$ is the probability density function.

(A) 25 **(B)** $\frac{1}{25}$

(C) $\frac{25}{2}$ **(D)** $\frac{2}{25}$

(ii) What is the probability that number of kilogram of bread that will be sold tomorrow is more than 500 kgs ?

(A) 1 **(B)** $\frac{1}{2}$

(C) $\frac{1}{4}$ **(D)** $\frac{1}{8}$

(iii) What is the probability that number of kilogram of bread that will be sold tomorrow is less than 500 kgs ?

(A) $\frac{1}{2}$ **(B)** $\frac{1}{4}$

(C) $\frac{1}{8}$ **(D)** $\frac{1}{10}$

(iv) What is the probability that number of kilograms of bread that will be sold tomorrow is between 250 kgs and 750 kgs ?

(A) $\frac{4}{3}$ **(B)** $\frac{3}{4}$

(C) $\frac{1}{2}$ **(D)** 1

(v) The probability density function $f(x)$ of a continuous random variable x is given by

$f(x) = 3x^2, 0 \le x \le 1$

Then, $P(0.2 \le x \le 0.5)$ is:

(A) 0.111 **(B)** 0.114

(C) 0.117 **(D)** 0.119

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. For the following probability distribution:

X	**– 4**	**– 3**	**– 2**	**– 1**	**0**
P(X)	**0.1**	**0.2**	**0.3**	**0.2**	**0.2**

Find E(X).

Q. 9. A die is rolled twice. Find the probability of getting an odd number exactly one.

Q. 10. Find the points of inflexion of the normal curve $f(x) = \frac{e^{\frac{-(x-10)^2}{32}}}{4\sqrt{2\pi}}$, for $-\infty < x < \infty$.

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. Vehicles pass through a junction on a busy road at an average rate of 300 per hour. [Given: $e^{-5} = 6.7379 \times 10^{-5}$]

(a) Find the probability that none passes in a given minute.

(b) What is the expected number passing in two minutes?

Q. 12. Assuming that on an average one telephone out of 10 is busy, seven telephone numbers are randomly selected and called. Find the probability that three of them will be busy.

Q. 13. For a random variable x, the probability density function is given by $f(x) = \frac{e^{-(x-4)^2}}{\sqrt{\pi}}$, for $-\infty < x < \infty$.

Find its mean and variance.

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of diamond cards drawn. Also, find the mean and the variance of the distribution.

Q. 15. Twenty sheets of aluminium alloy were examined for surface flaws. The frequency of the number of sheets with a given number of flaws per sheet was as follows:

Number of flaws	Frequency
0	4
1	3
2	5
3	2
4	4
5	**1**
6	**1**

What is the probability of finding a sheet chosen at random which contains 3 or more surface flaws?

Q. 16. A life insurance salesman sells on the average 3 life insurance policies per week. Use Poisson's law to calculate the probability that in a given week he will sell

(a) Some policies

(b) 2 or more policies but less than 5 policies.

(c) Assuming that there 5 working days per week, what is the probability that in a given day he will sell one policy ?

■■

CHAPTER

9

INFERENTIAL STATISTICS

Syllabus

Population and Sample, Parameter, Statistics and Statistical Interferences, t-test (one sample t-test and two independent groups t-test)

In this chapter you will study

- *Define Population and Sample ● Differentiate between population and sample ● Define a representative sample from a population ● Differentiate between a representative and nonrepresentative sample ● Draw a representative sample using simple random sampling ● Draw a representative sample using and systematic random sampling*
- *Define Parameter with reference to Population ● Define Statistics with reference to Sample ● Explain the relation between Parameter and Statistic ● Explain the limitation of Statistic to generalize the estimation for population ● Interpret the concept of Statistical Significance and Statistical Inferences • State Central Limit Theorem ● Explain the relation between Population-Sampling Distribution-Sample*
- *Define a hypothesis ● Differentiate between Null and Alternate hypothesis ● Define and calculate degree of freedom*
- *Test Null hypothesis and make inferences using t-test statistic for one group / two independent groups*

Revision Notes

Inferential statistics is mainly used to derive estimates about a large group (or population) and draw conclusions on the data, based on hypotheses testing methods.

Key Words

Inferential: Characterized by or involving conclusions reached on the basis of evidence and reasoning.

- **Population:** The group of individuals considered under study is called as population. The word population here refers not only to people but to all items that have been chosen for the study. Thus in statistics, population can be number of bikes manufactured in a day or week or month, number of cars manufactured in a day or week or month, number of fans, TVs, chalk pieces, people, students, girls, boys, any manufacturing products, etc.

Scan to know more about this topic

Sampling and population

Finite and infinite population: When the number of observations/individuals/products is countable in a group, then it is a finite population.

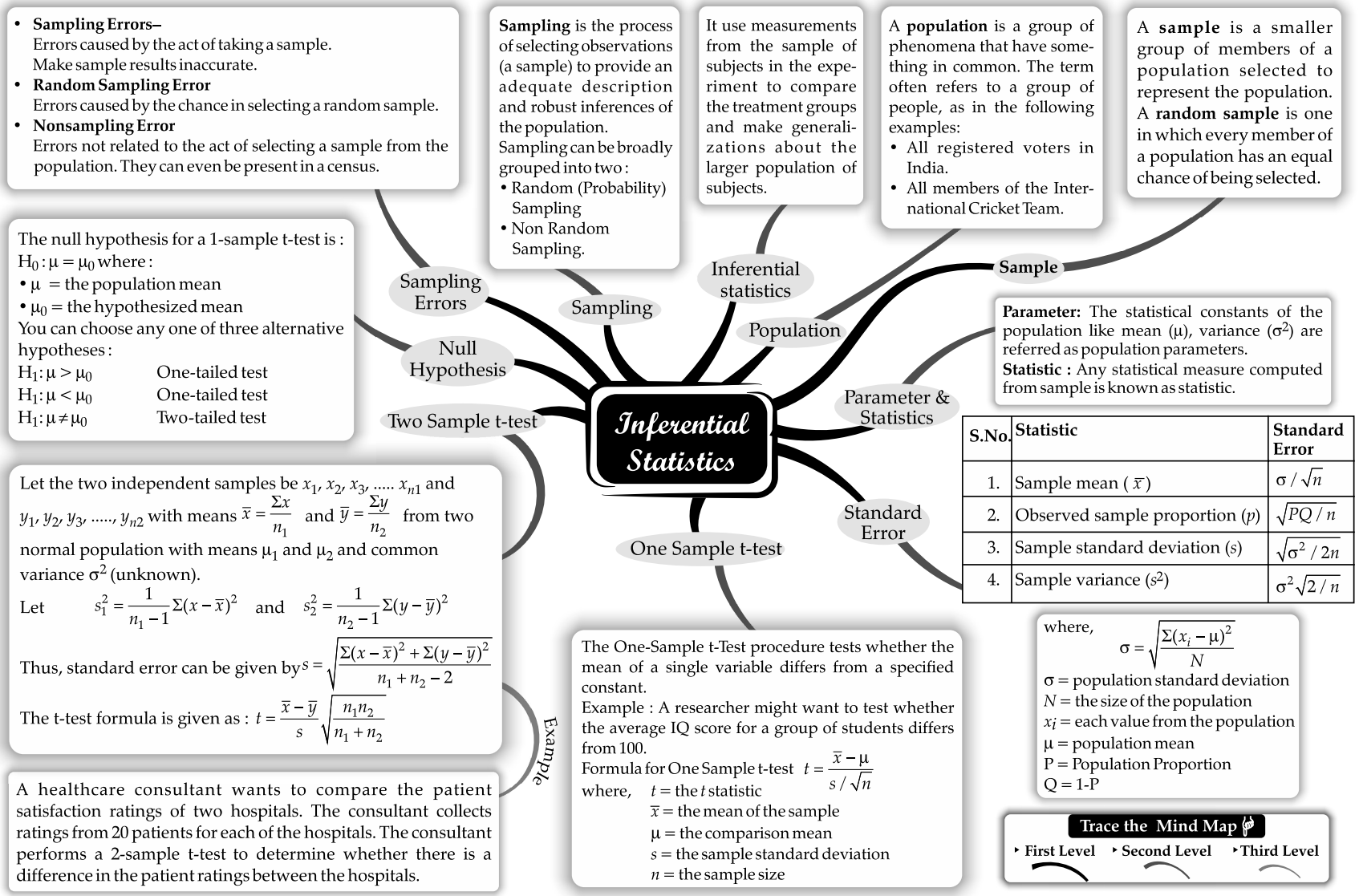
Example: Weights of students of class XII in a school.

When the number of observations/individuals/products is uncountable in a group, then it is an infinite population.

Example: Number of grains in a sack, number of germs in the body of a sick patient.

- **Sample and Sample Size:** A selection of a group of individuals from a population in such a way that it represents the population is called as sample and the number of individuals included in a sample is called the sample size.
- **Sampling:** Sampling is the procedure or process of selecting a sample from a population. Sampling is quite often used in our day-to-day practical life.
- **Parameter:** The statistical constants of the population like mean (μ), variance (σ^2) are referred as population parameters.

Statistic: Any statistical measure computed from sample is known as statistics.


Inferential Statistics
Sampling Errors
• Sampling Errors–
Errors caused by the act of taking a sample.
Make sample results inaccurate.
• Random Sampling Error
Errors caused by the chance in selecting a random sample.
• Nonsampling Error
Errors not related to the act of selecting a sample from the population. They can even be present in a census.
Sampling
Sampling is the process of selecting observations (a sample) to provide an adequate description and robust inferences of the population.
Sampling can be broadly grouped into two :
• Random (Probability) Sampling
• Non Random Sampling.
Inferential statistics
It use measurements from the sample of subjects in the experiment to compare the treatment groups and make generalizations about the larger population of subjects.
Population
A population is a group of phenomena that have something in common. The term often refers to a group of people, as in the following examples:
• All registered voters in India.
• All members of the International Cricket Team.
Sample
A sample is a smaller group of members of a population selected to represent the population.
A random sample is one in which every member of a population has an equal chance of being selected.
Null Hypothesis
The null hypothesis for a 1-sample t-test is :
H0 : μ = μ0 where :
• μ = the population mean
• μ0 = the hypothesized mean
You can choose any one of three alternative hypotheses :
H1 : μ > μ0 One-tailed test
H1 : μ < μ0 One-tailed test
H1 : μ ≠ μ0 Two-tailed test
Parameter & Statistics
Parameter: The statistical constants of the population like mean (μ), variance (σ2) are referred as population parameters.
Statistic : Any statistical measure computed from sample is known as statistic.
Standard Error
S.No. | Statistic | Standard Error
1. | Sample mean (x̄) | σ / √n
2. | Observed sample proportion (p) | √(PQ / n)
3. | Sample standard deviation (s) | √(σ2 / 2n)
4. | Sample variance (s2) | σ2 √(2 / n)
where, σ = √(Σ(xi − μ)2 / N)
σ = population standard deviation
N = the size of the population
xi = each value from the population
μ = population mean
P = Population Proportion
Q = 1-P
Two Sample t-test
Let the two independent samples be x1, x2, x3, ... xn1 and y1, y2, y3, ..., yn2 with means x̄ = Σx / n1 and ȳ = Σy / n2 from two normal population with means μ1 and μ2 and common variance σ2 (unknown).
Let s1^2 = 1/(n1 − 1) Σ(x − x̄)^2 and s2^2 = 1/(n2 − 1) Σ(y − ȳ)^2
Thus, standard error can be given by s = √((Σ(x − x̄)^2 + Σ(y − ȳ)^2) / (n1 + n2 − 2))
The t-test formula is given as : t = (x̄ − ȳ) / s √(n1n2 / (n1 + n2))
Example
A healthcare consultant wants to compare the patient satisfaction ratings of two hospitals. The consultant collects ratings from 20 patients for each of the hospitals. The consultant performs a 2-sample t-test to determine whether there is a difference in the patient ratings between the hospitals.
One Sample t-test
The One-Sample t-Test procedure tests whether the mean of a single variable differs from a specified constant.
Example : A researcher might want to test whether the average IQ score for a group of students differs from 100.
Formula for One Sample t-test t = (x̄ − μ) / (s / √n)
where, t = the t statistic
x̄ = the mean of the sample
μ = the comparison mean
s = the sample standard deviation
n = the sample size
Trace the Mind Map
First Level
Second Level
Third Level

$$\mu \text{ (mean)} = \frac{\sum_{i=1}^{n} x_i}{N}$$

$$\sigma^2\text{(variance)} = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{N}$$

$$\sigma\text{(standard deviation, SD)} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{N}}$$

where N is the population size

- **Errors in a Sample:** A sample is a part of the whole population. A sample drawn from the population depends upon chance and as such all the characteristics of the population may not be present in the sample drawn from the same population. The errors involved in the collection, processing and analysis of the data may be broadly classified into two categories namely,
  (i) Sampling Errors
  (ii) Non-Sampling Errors
  (i) Sampling Errors: Errors, which arise in the normal course of investigation or enumeration on account of chance, are called sampling errors. Sampling errors are inherent in the method of sampling. They may arise accidentally without any bias or prejudice.
  (ii) Non-Sampling Errors: The errors that arise due to human factors which always vary from one investigator to another in selecting, estimating or using measuring instruments (tape, scale) are called Non-Sampling errors.
- **Types of Sampling:** There are various techniques of sampling, but they can be broadly grouped into two:
  - Random probability sampling.
  - Non Random or Non Probability sampling.

  We will consider only random (Probability) sampling.

  Random sampling or Probability sampling: Random sampling refers to selection of samples from the population in a random manner. A random sample is one where each and every item in the population has an equal chance of being selected.

  The following are different types of probability sampling:
  (i) Simple Random Sampling: Simple random sampling is the randomized selection of a small segment of individuals or members from a whole population. It provides each individual or member of a population with an equal and fair chance of being chosen. This is one of the most common method of sampling.
  e.g.: There are 1000 students in a school, we want to select a simple random sample of 100 students. We can assign a number to every student in the school from 1 to 1000 and select randomly 100 numbers.
  (ii) Systematic Sampling: Systematic sampling is the selection of specific individuals or members from entire population. The selection often follows a pre-determined interval (k). The systematic sampling method is comparable to the simple random sampling method; however it is less complicated to conduct.
  e.g.: Out of 1000 students of school, we want to select a sample of 100 students. All students of the school are arranged in alphabetical order and assigned a number 1 to 1000. Now, we randomly select a number (say 4) from first 10 numbers and then every 10^{th} student in the list is selected *i.e.*, 4, 14, 24, ... and end up with sample of 100 members.
  (iii) Stratified Sampling: Stratified sampling includes the partitioning of a population into subclasses with notable distinctions and variances. This method is useful when population is dispersed.
  e.g., Suppose a school has its branches in 15 cities. We want to select the sample of 100 students. It is difficult to select students from each branch. So first select any 5 branches and then select students from each branch.
- **Sampling Distribution:** Sampling distribution of a statistic is the frequency distribution which is formed with various values of a statistic computed from different samples of the same size drawn from the same population.

  For instance if we draw a sample of size n from a given finite population of size N, then the total number of possible samples are ${}^{N}C_{n} = \frac{N!}{n!(N-n)!} = k$(say). For each of these k samples we can compute some statistic $t = t(x_1, x_2, x_3, \ldots x_n)$, in particular the mean $\bar{x}$, the variance S^2, etc., is given below:

Sample Number	Statistic		
	t	$\bar{x}$	S^2
1	t_1	$\bar{x}_1$	S_1^2
2	t_2	$\bar{x}_2$	S_2^2
3	t_3	$\bar{x}_3$	S_3^2
⋮	⋮	⋮	⋮
k	t_k	$\bar{x}_k$	S_k^2

Key Facts

- Inferential statistics helps to suggest explanations for a situation or phenomenon. It allows us to draw conclusions based on extrapolations, and is in that way fundamentally different from descriptive statistics that merely summarize the data that has actually been measured.
- Inferential statistics are powerful tools for making inference that rely on frequencies and probabilities.
- Inferential statistics can improve one's ability to make decisions, form predictions, and conduct research.

The set of the values of the statistic so obtained, one for each sample, constitutes the sampling distribution of the statistic.

Example: The sampling distribution of the mean for $n = 2$ is given below:

Sample Number	Outcomes	Sample mean $\bar{x}_i$
1	1, 3	2
2	1, 9	5
3	3, 9	6
4	6, 7	6.5
5	7, 10	8.5

► **Standard Error:** The standard deviation of the sampling distribution of statistic known as its Standard Error abbreviated as S.E. The Standard Errors (S.E.) of some of the well-known statistics, for large samples, are given below, where n is the sample size, σ^2 is the population variance.

Scan to know more about this topic

Standard error

S. No	Statistic	Standard Error
1.	Sample mean ($\bar{x}$)	$\frac{\sigma}{\sqrt{n}}$
2.	Observed sample proportion (p)	$\sqrt{\frac{PQ}{n}}$
3.	Sample standard deviation (s)	$\sqrt{\frac{\sigma^2}{2n}}$
4.	Sample variance (s^2)	$\sigma^2\sqrt{\frac{2}{n}}$

Example 1 : A die is thrown 9000 times and a throw of 3 or 4 is observed 3240 times. Find the standard error of the proportion for an unbiased die.

Sol.: If the occurrence of 3 or 4 on the die is called a success, then

$$\text{Sample size} = 9000; \text{ Number of Success} = 3240$$

$$\text{Sample proportion} = p = \frac{3240}{9000} = 0.36$$

Population proportion (P) = Prob (getting 3 or 4 when a die is thrown)

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$= \frac{1}{3} = 0.3333$$

Thus $P = 0.3333$ and $Q = 1 - P = 1 - 0.3333 = 0.6667$.

The S.E. for sample proportion is given by

$$\text{S.E.} = \sqrt{\frac{PQ}{n}}$$

$$= \sqrt{\frac{(0.3333)(0.6667)}{9000}}$$

$$= 0.00406$$

Hence the standard error for sample proportion is S.E. = 0.00406.

► **Statistical Inferences:** One of the main objectives of any statistical investigation is to draw inferences about a population from the analysis of samples drawn from that population. Statistical Inference provides us how to estimate a value from the sample and test that value for the population. This is done by the two important classifications in statistical inference,

(i) Estimation;

(ii) Testing of Hypothesis

- **Estimation:** The method of obtaining the most likely value of the population parameter using statistic is called estimation.
- **Estimator:** Any sample statistic which is used to estimate an unknown population parameter is called an estimator *i.e.*, an estimator is a sample statistic used to estimate a population parameter.
- **Estimate:** To estimate an unknown parameter of the population, concept of theory of estimation is used. There are two types of estimation namely.
- **Point Estimation:** When a single value is used as an estimate, it is called as point estimation.
- **Interval Estimation:** An interval within which the parameter would be expected to lie is called interval estimation.

► **Statistical Hypothesis:** Statistical hypothesis is some assumption or statement, which may or may not be true, about a population.

There are two types of statistical hypothesis

(i) Null hypothesis

(ii) Alternative hypothesis

Scan to know more about this topic

Null and alternative hypothesis

(i) Null Hypothesis: According to prof. R.A. Fisher, "Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true", and it is denoted by H_0.

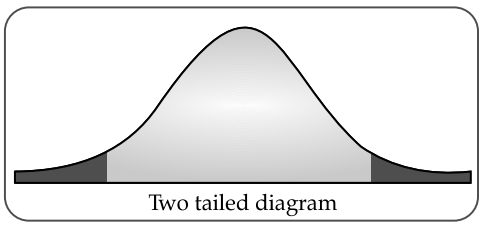
For example: If we want to find the population mean has a specified value μ_0, then the null hypothesis H_0 is set as follows H_0: $\mu = \mu_0$

(ii) Alternative Hypothesis: Any hypothesis which is complementary to the null hypothesis is called as the alternative hypothesis and is usually denoted by H_1.

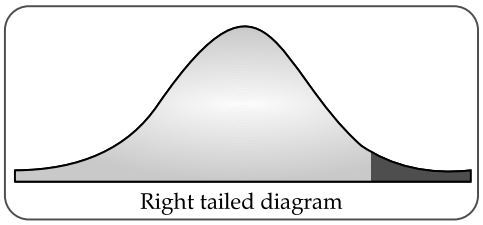
For example: If we want to test the null hypothesis that the population has specified mean μ *i.e.*, H_0: $\mu = \mu_0$ then the alternative hypothesis could be any one among the following:

(a) $H_1: \mu \neq \mu_0$
(b) $H_1: \mu > \mu_0$
(c) $H_1: \mu < \mu_0$

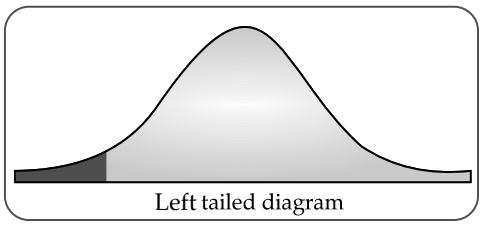
The alternative hypothesis in $H_1: \mu \neq \mu_0$ is known as two tailed alternative test. Two tailed test is one where the hypothesis about the population parameter is rejected for the value of sample statistic falling into either tails of the sampling distribution. When the hypothesis about the population parameter is rejected only for the value of sample statistic falling into one of the tails of the sampling distribution, then it is known as one-tailed test. Here $H_1: \mu > \mu_0$ and $H_1: \mu < \mu_0$ are known as one tailed alternative.


Two tailed diagram

Right tailed test: $H_1: \mu > \mu_0$ is said to be right tailed test where the rejection region or critical region lies entirely on the right tail of the normal curve.


Right tailed diagram

Left tailed test: $H_1: \mu < \mu_0$ is said to be left tailed test where the critical region lies entirely on the left tail of the normal curve. (diagram)


Left tailed diagram

- **Types of Errors in Hypothesis:** There is every chance that a decision regarding a null hypothesis may be correct or may not be correct. There are two types of errors. They are

  Type I error: The error of rejecting H_0 when it is true.

  Type II error: The error of accepting when H_0 it is false.
- **Critical Region or Rejection Region:** A region corresponding to a test statistic in the sample space which tends to rejection of H_0 is called critical region or region of rejection. The region complementary to the critical region is called the region of acceptance.
- **Level of Significance:** The probability of type I error is known as level of significance and it is denoted by α. The level of significance is usually employed in testing of hypothesis are 5% and 1%. The level of significance is always fixed in advance before collecting the sample information.
- **Critical Values or Significant Values:** The value of test statistic which separates the critical (or rejection) region and the acceptance region is called the critical value of significant value. It depend upon.

  (i) The level of significance.

  (ii) The alternative hypothesis whether it is two-tailed or single tailed.
- ***t*-Test:** The *t*-test is a test in statistics that is used for testing hypotheses regarding the mean of a small sample taken population when the standard deviation of the population is not known. *t*-Test was first invented by William Sealy Gosset, in 1908. Since he used the pseudo name as 'Student' when publishing his method in the paper titled 'Biometrika', the test came to be known as Student's T Test. There are many types of *t*-test. Two of them are:
  - The one-sample *t*-test, which is used to compare the mean of a population with a theoretical value.
  - The two-sample *t*-test, which is used to compare the mean of two independent given samples.

Scan to know more about this topic
One sample t-Test

One Sample *t*-Test Procedure:

Step 1: Define the Null Hypothesis (H_0) and Alternate Hypothesis (H_1)

Example 2 : H_0: Sample mean ($\bar{X}$) = Hypothesized Population mean (μ)

H_1: Sample mean ($\bar{X}$) $\neq$ Hypothesized Population mean (μ)

The alternate hypothesis can also state that the sample mean is greater than or less than the comparison mean.

Step 2: Compute the test statistic (*t*)

$$t = \frac{Z}{\text{S.E.}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Where S.E. is the standard error

Step 3: Find the *t*-critical from the *t*-Table

Use the degree of freedom and the alpha level (0.05) to find the t -critical.

Step 4: Determine if the computed test statistic falls in the rejection region.

Alternately, simply compute the P-value. If it is less than the significance level (0.05 or 0.01), reject the null hypothesis.

Example 2: We have the potato yield from 12 different farms. We know that the standard potato yield for the given variety is $\mu = 20$.

x = [21.5, 24.5, 18.5, 17.2, 14.5, 23.2, 22.1, 20.5, 19.4, 18.1, 24.1, 18.5]

Test if the potato yield from these farms is significantly better than the standard yield.

Sol.: Step 1: Define the Null and Alternate Hypothesis

$$H_0 : \bar{X} = 20$$
$$H_1 : \bar{X} > 20$$
$$n = 12$$

Since this is one sample t test, the degree of freedom $= n - 1 = 12 - 1 = 11$.

Let's set alpha = 0.05, to meet 95% confidence level.

Step 2: Calculate the Test Statistic (t)

(i) Calculate sample mean

$$\bar{X} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$= \frac{21.5 + 24.5 + 18.5 + 17.2 + 14.5 + 23.2 + 22.1 + 20.5 + 19.4 + 18.1 + 24.1 + 18.5}{12}$$

$$= 20.175$$

(ii) Calculate sample standard deviation

$$\sigma = \sqrt{\frac{(x_1 - \bar{X})^2 + (x_2 - \bar{X})^2 + (x_3 - \bar{X})^2 + \ldots + (x_n - \bar{X})^2}{n-1}} = 3.0211$$

(iii) Substitute in the t Statistic formula

$$t = \frac{Z}{\text{S.E.}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{20.175 - 20}{\frac{3.0211}{\sqrt{12}}} = 0.2006$$

Step 3: Find the t-Critical

Confidence level = 0.95, alpha = 0.05. For one tailed test, look under 0.05 column. For d.f. (degree of freedom) = 12 – 1 = 11, t-Critical = 1.796.

Note: Subtracting one from the sample size(n) gives the degrees of freedom (d.f.).

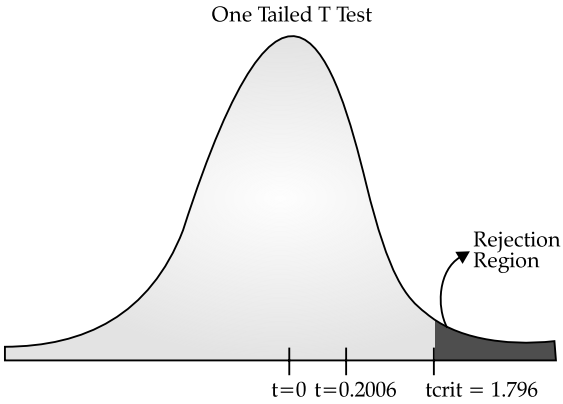
t **Table**

cum.prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tall	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-talls	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.885	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.335	4.501	5.041
9	0.000	0.703	0.883	1.100	1.389	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.685	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725

26	0.000	0.684	0.56	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.686	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
Z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.297
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

Step 4: Does it fall in rejection region ?

Since the computed t-Statistic is less than the t-critical, it does not fall in the rejection region.



Clearly, the calculated t statistic does not fall in the rejection region. So, we do not reject the null hypothesis.

Two Sample t-Test

Scan to know more about this topic

Two sample t-Test

The two sample t-test is a test that is used to compare the mean of two groups of samples. It is meant for evaluating whether the means of the two sets of data are statistically significantly different from each other.

Let the two independent samples be $x_1, x_2, x_3, \ldots\ldots, x_{n_1}$ and $y_1, y_2, y_3, \ldots\ldots, y_{n_2}$ with means $\bar{x} = \frac{\sum x}{n_1}$ and $\bar{y} = \frac{\sum y}{n_2}$ from two normal population with means μ_1 and μ_2 and common variance σ^2(unknown).

Let $s_1^2 = \frac{1}{n_1 - 1}\sum(x - \bar{x})^2$ and $s_2^2 = \frac{1}{n_2 - 1}\sum(y - \bar{y})^2$

Thus, standard error can be given by $s = \sqrt{\frac{\sum(x - \bar{x})^2 + \sum(y - \bar{y})^2}{n_1 + n_2 - 2}}$

The t-test formula is given as: $t = \frac{\bar{x} - \bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1 + n_2}}$

The significance of t is tested in the same way as in one sample t-test with degree of freedom $n_1 + n_2 - 2$ to test the hypothesis $\mu_1 = \mu_2$.

Example: Find the t-test value for the following given two sets of values: 7, 2, 9, 8 and 1, 2, 3, 4.

Sol. For first data set:

Number of terms in first set *i.e.* $n_1 = 4$

Calculate mean value for first data set using formula:

$$\bar{x} = \frac{\sum x}{n_1} = \frac{7 + 2 + 9 + 8}{4} = \frac{26}{4} = 6.5$$

For second data set:
Number of terms in second set *i.e.* $n_2 = 4$
Calculate mean value for second data set using formula:

$$\bar{y} = \frac{\sum y}{n_2} = \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5$$

Construct the following table for standard error:

x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
7	0.5	0.25	1	– 1. 5	2.25
2	– 4. 5	20.25	2	– 0. 5	0.25
9	2.5	6.25	3	0.5	0.25
8	1.5	2.25	4	1.5	2.25
		$\sum(x-\bar{x})^2 = 29$			$\sum(y-\bar{y})^2 = 5$

Now, compute the standard error, s using formula,

$$s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$$

$$s = \sqrt{\frac{29+5}{4+4-2}} = \sqrt{\frac{34}{6}}$$

$$= \sqrt{5.666} = 2.380$$

Now using t-test formula:

$$t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$$

$$t = \frac{6.5-2.5}{2.380}\sqrt{\frac{4\times4}{4+4}}$$

$$= \frac{4}{2.380}\sqrt{2}$$

$$= 1.6806 \times 1.414 = 2.376$$

Hence, t-test value for the two data sets is 2.38.

Mnemonics

Concept: Errors in Null Hypothesis
Mnemonics: a = you "observed" a difference that did not exist.
b = you were Blinded by the truth.
Interpretations: In statistical hypothesis testing:
A **type I error** (α) is the incorrect rejection of a true null hypothesis (a "false positive"), that is, a null hypothesis incorrectly rejected.
While, a **type II error** (β) is the failure to reject a false null hypothesis (a "false negative"), that is, a null hypothesis is not rejected when it is in fact false.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. Which of the following values is used as a summary measure for a sample, such as a sample mean ?
(A) Population Parameter
(B) Sample Parameter
(C) Sample Statistic
(D) Population mean
Ans. Option (C) is correct.

Q. 2. Which of the following is a branch of statistics ?
(A) Descriptive Statistics
(B) Inferential Statistics

(C) Industrial Statistics

(D) Both A and B

Ans. Option (D) is correct.

Explanation: Study of statistics can be categorized in two types: (i) Descriptive Statistics (ii) Inferential Statistics.

Q. 3. An observed set of the population that has been selected for analysis is called:

(A) a sample **(B)** a process

(C) a forecast **(D)** a parameter.

Ans. Option (A) is correct.

Explanation: A selection of a group of individuals from a population in such a way that it represents the population is called as sample and the number of individuals in the sample is called the sample size.

Q. 4. A specific characteristics of a sample is known as ..... .

(A) population **(B)** parameter

(C) statistic **(D)** variance

Ans. Option (C) is correct.

Q. 5. A statement made about a population parameter for testing purpose is called ..... .

(A) statistic **(B)** parameter

(C) hypothesis **(D)** level of significance

Ans. Option (C) is correct.

Explanation: Statistical hypothesis is an assumption or statement, which may or may not be true, about a population.

Q. 6. If we reject the null hypothesis, we might be making:

(A) Type-I error **(B)** Type-II error

(C) A correct decision **(D)** A wrong decision.

Ans. Option (A) is correct.

Explanation: **Type I error:** The error of rejecting hypothesis (H_0) when it is true.

Q. 7. A sample of 50 pens is taken at random. Out of 50 pens we found 15 pens are of Cello, 17 are of Parker and 18 are of Reynolds. What is the point estimate of population proportion of Parker.

(A) 0.3 **(B)** 0.34

(C) 0.36 **(D)** 0.4

Ans. Option (B) is correct.

Explanation: Population proportion of Parker

$$= \frac{17}{50} = 0.34.$$

Q. 8. A simple random sample consist of four observation 1, 3, 5, 7. What is the point estimate of population standard deviation ?

(A) 2.3 **(B)** 2.52

(C) 0.36 **(D)** 0.4

Ans. Option (B) is correct.

Explanation: The point estimation of population standard deviation is sample deviation.

$$S = \sqrt{\frac{(x_i - \bar{x})^2}{n-1}}$$

$$= \sqrt{\frac{(1-4)^2 + (3-4)^2 + (5-4)^2 + (7-4)^2}{4-1}}$$

$$\left(\because \bar{x} = \frac{\Sigma x}{n} = \frac{16}{4} = 4\right)$$

$$= \sqrt{\frac{9+1+1+9}{3}} = \sqrt{\frac{20}{3}} = 2.52$$

Q. 9. Which of the following statements are true?

I: The mean of a population is denoted by $\bar{x}$.

II: The population mean is a statistic.

(A) I only **(B)** II only

(C) Both I and II **(D)** None

Ans. Option (B) is correct.

Q. 10. A ..... is a statistical test used to compare the means of two groups

(A) *t*-test **(B)** *p*-test

(C) *h*-test **(D)** *g*-test

Ans. Option (A) is correct.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. What is population ? [O.E.B.]

Sol. A population is a set of similar items or events which is of interest for some question or experiment. A population can be specific or vague. Examples of population defined vaguely include the number of newborn babies in Uttar Pradesh, a total number of tech startups in India, the average height of all exam candidates, mean weight of taxpayers in Lucknow etc. Examples of population defined specifically include a number of fans produced in a particular factory, the number of students in a class, the number of boys and girls in a coaching centre etc. **1**

Commonly Made Error

- Some students did not prepare the theoretical definitions from this chapter.

Answering Tip

- The definitions of population, sample, parameters etc. are important from examination point of view, so learn it in proper way.

Q. 2. What is sample ? [O.E.B.]

Sol. A sample is a set of data collected from a statistical population by a defined procedure. The elements of a sample are called sample size or sample points.

Samples are collected and statistics are calculated from the samples, so that one can make inferences from the sample to the population. **1**

Q. 3. What is statistic ? R&U **[O.E.B.]**

Q. 4. Define parameter in Statistics. A I **[O.E.B.]**

Sol. A parameter is any numerical quantity that characterizes a given population or some aspect of it. This means the parameter tells us something about the whole population. For example, the population mean μ, variance σ^2, population proportion P, population correlation ρ. **1**

Q. 5. A server channel monitored for an hour was found a have an estimated mean of 20 transactions transmitted per minute. The variance is known to be 4. Find the standard error. R&U **[O.E.B.]**

Sol. Given $\sigma^2 = 4$ which implies $\sigma = 2$, $n = 1$ hour = 60, min, $\bar{X} = 20$/min

$$\text{Standard Error} = \frac{\sigma}{\sqrt{n}}$$

$$= \frac{2}{\sqrt{60}}$$

$$= 0.2582$$ **1**

Q. 6. Find the sample size for the given standard deviation 10 and the standard error with respect of sample mean is 3. A I **[O.E.B.]**

Q. 7. The standard deviation of a sample of size 50 is 6.3. Determine the standard error whose population standard deviation is 6 ? R&U **[O.E.B.]**

Sol. Sample size $n = 50$

Sample S.D. $s = 6.3$

Population S.D. $\sigma = 6$

The standard error for sample S.D. is given by

$$\text{S.E.} = \sqrt{\frac{\sigma^2}{2n}}$$

$$= \frac{6}{\sqrt{2(50)}}$$

$$= \frac{6}{\sqrt{100}}$$

$$= 0.6$$

Thus standard error for sample S.D = 0.6. **1**

Q. 8. A sample of 100 students is chosen from a large group of students. The average height of these students is 162 cm and standard deviation (S.D.) is 8 cm. Obtain the standard error for the average height of large group of students of 160 cm ? R&U **[O.E.B.]**

Sol. Given $n = 100$, $\bar{x} = 162$ cm, $s = 8$ cm is known in this problem

Since σ is unknown, so we consider $\hat{\sigma} = s$

and $\varphi = 160$ cm

$$\text{S.E.} = \frac{\hat{\sigma}}{\sqrt{n}}$$

$$= \frac{s}{\sqrt{n}}$$

$$= \frac{8}{\sqrt{100}}$$

$$= 0.8$$

Therefore the standard error for the average height of large group of students of 160 cm is 0.8. **1**

Q. 9. A random sample of 60 observations was drawn from a large population and its standard deviation was found to be 2.5. Calculate the suitable standard error that this sample is taken from a population with standard deviation 3 ? R&U **[O.E.B.]**

Sol. Given sample size $n = 60$

Sample standard deviation = 2.5

Population standard deviation $\sigma = 3$

The S.E. is given by

$$\sqrt{\frac{\sigma^2}{2n}} = \sqrt{\frac{9}{120}}$$

$$= \sqrt{0.075}$$

$$= 0.2739$$ **1**

Q. 10. A sample of 1000 students whose mean weight is 119 lbs (pounds) from a school in Uttar Pradesh State was taken and their average weight was found to be 120 lbs with a standard deviation of 30 lbs. Calculate the standard error of the mean. A I **[O.E.B.]**

Short Answer Type Questions-I (2 marks each)

Q. 1. A wholesaler in apples claims that only 4% of the apples supplied by him are defective. A random sample of 600 apples contained 36 defective apples. Calculate the standard error concerning good apples. A I **[O.E.B.]**

Sol. Sample size = 600

No. of defective apples = 36

$$\text{Sample proportion } P = \frac{36}{600} = 0.06$$

Population proportion P = probability of defective apples = 4% = 0.04

$$Q = 1 - P$$

$$= 1 - 0.04 = 0.96$$ **1**

The S.E. for sample proportion is given by S.E.

$$= \sqrt{\frac{PQ}{N}}$$

$$= \sqrt{\frac{(0.04)(0.96)}{600}}$$

$$= \sqrt{0.000064} = 0.008$$ **1**

These questions are for practice and their solutions are available at the end of the chapter

Q. 2. In a sample of 400 population from a village 230 are found to be eaters of vegetarian items and the rest non-vegetarian items. Compute the standard error assuming that both vegetarian and non-vegetarian foods are equally popular in that village ? R&U **[O.E.B.]**

Sol. Given sample size 400 and 230 are vegetarian eaters.

So sample proportional $= \frac{230}{400} = 0.575$

Population proportion P = Prob. (vegetarian eaters from the village) $= \frac{1}{2}$

(Since vegetarian and non-vegetarian foods are equally popular),

$$Q = 1 - P$$
$$= 1 - \frac{1}{2}$$
$$= \frac{1}{2}$$

The standard error S.E. $= \sqrt{\frac{PQ}{N}}$ **1**

$$= \sqrt{\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{400}}$$
$$= \sqrt{\frac{0.25}{400}}$$
$$= \sqrt{0.000625}$$
$$= 0.025$$ **1**

Q. 3. A machine which produces mica insulating washers of use in the electric devices is set to turn out washers having a thickness of 10 mils (1 mil = 0.001 inch). A sample of 10 washers has an average thickness of 9.52 mils with a standard deviation of 0.60 mil. Find out t. A I **[O.E.B.]**

Short Answer Type Questions-II (3 marks each)

Q. 1. Find the student's $-t$ for the following variable values in a sample of eight:
– 4, – 2, – 2, 0, 2, 2, 3, 3 taking the mean of the universe to be zero. R&U **[O.E.B.]**

Sol.

x	$x - \bar{x}$	$(x - \bar{x})^2$
– 4	– 4.25	18.0625
– 2	– 2.25	5.0625
– 2	– 2.25	5.0625
0	– 0.25	0.0625
2	1.75	3.0625
2	1.75	3.0625
3	2.75	7.5625
3	2.75	7.5625
$\sum x = 2$		$\sum (x - \bar{x})^2 = 49.5000$

$\bar{X}$ = mean **1**

$$= \frac{\sum x}{n}$$
$$= \frac{2}{8}$$
$$= 0.25$$

Now, compute the standard deviation using formula as,

$$s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$
$$= \sqrt{\frac{49.5}{7}}$$
$$= \sqrt{7.071428}$$
$$= 2.659$$ **1**

H_0 = The mean of universe, $\mu = 0$, we get

$$t = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$
$$= \frac{0.25 - 0}{\frac{2.659}{\frac{\sigma}{\sqrt{8}}}}$$
$$= \frac{0.25}{\frac{2.659}{2.828}}$$
$$= \frac{0.25}{0.9402}$$
$$= 0.2659$$ **1**

Q. 2. Ten individuals are chosen at random from the population and their heights are found to be in inches 63, 63, 64, 65, 66, 69, 69, 70, 70, 71. Discuss the freedom value of Student's-t and 5% level of significance is 2.62. A I **[O.E.B.]**

Long Answer Type Questions (5 marks each)

Q. 1. The heights of six randomly chosen red roses are in cm 63, 65, 68, 69, 71 and 72. Those of ten randomly chosen yellow roses are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Find t-statistics for the given data.
R&U **[O.E.B.]**

These questions are for practice and their solutions are available at the end of the chapter

Sol. For first data set:

Number of terms in first set *i.e.* $n_1 = 6$

Calculate mean value for first data set using formula:

$$\bar{x} = \frac{\sum x}{n_1}$$

$$= \frac{63+65+68+69+71+72}{6}$$

$$= \frac{408}{6} = 68 \quad \textbf{1}$$

For second data set:

Number of terms in second set *i.e.* $n_2 = 10$

Calculate mean value for second data set using formula:

$$\bar{y} = \frac{\sum y}{n_2}$$

$$= \frac{62+61+65+66+69+69+70+71+72+73}{10}$$

$$= \frac{678}{10} = 67.8 \quad 1$$

Construct the following table for standard error s:

Red Roses			Yellow Roses		
x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
63	– 5	25	61	– 6.8	46.24
65	– 3	9	62	– 5.8	33.64
68	0	0	65	– 2.8	7.84
69	1	1	66	– 1.8	3.24
71	3	9	69	1.2	1.44
72	4	16	69	1.2	1.44
			70	2.2	4.84
			71	3.2	10.24
			72	4.2	17.64
			73	5.2	27.04
		$\sum(x-x)^2 = 60$			$\sum(y-\bar{y})^2 = 153.60$

1

Now, compute the standard error, s using formula:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$$

$$s = \sqrt{\frac{60+153.60}{6+10-2}}$$

$$= \sqrt{\frac{213.60}{14}}$$

$$= \sqrt{15.257}$$

$$= 3.906 \quad \textbf{1}$$

Now using t-test formula:

$$t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$$

$$= \frac{68-67.8}{3.906}\sqrt{\frac{6\times 10}{6+10}}$$

$$= \frac{0.2\sqrt{3.75}}{3.906}$$

$$= 0.0512 \times 1.936$$

$$= 0.099$$

Hence, t-test value for the two data sets is 0.099. **1**

Commonly Made Error

- Calculation errors are found in the standard error table. Sometimes students apply wrong formula for standard error.

Answering Tip

- Learn the all formula thoroughly.

Q. 2. In a rat feeding experiment, the following results were obtained:

Diet	Gain in Weight in gm											
High Protein	**13**	**14**	**10**	**11**	**12**	**16**	**10**	**8**	**11**	**12**	**9**	**12**
Low Protein	**7**	**11**	**10**	**8**	**10**	**13**	**9**					

Investigate if there is any evidence of superiority of one diet over the other. The value of t for 17 degrees of freedom at 5% level of significance = 2.11. **[O.E.B.]**

Sol. For first data set:

Number of terms in first set *i.e.* $n_1 = 12$

Mean gain in weight on High Protein

$$\bar{x} = \frac{\sum x}{n_1}$$

$$= \frac{13+14+10+11+12+16+10+8+11+12+9+12}{12}$$

$$= \frac{138}{12} = 11.5$$ **1**

For second data set:

Number of terms in second set *i.e.* $n_2 = 7$

Mean gain in weight on Low Protein

$$\bar{y} = \frac{\sum y}{n_2}$$

$$= \frac{7+11+10+8+10+13+9}{7}$$

$$= \frac{68}{7} = 9.71$$ **1**

Construct the following table for standard error s:

High Protein			**Low Protein**		
x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
13	1.5	2.25	7	– 2.71	7.34
14	2.5	6.25	11	1.29	1.66
10	– 1.5	2.25	10	0.29	0.08
11	– 0.5	25	8	– 1.71	2.92
12	0.5	25	10	0.29	0.08
16	4.5	20.25	13	3.29	10.82
10	– 1.5	2.25	9	– 0.71	0.50
8	– 3.5	12.25			
11	– 0.5	0.25			
12	0.5	0.25			
9	– 2.5	6.25			
12	0.5	0.25			
		$\sum(x-\bar{x})^2 = 53$			$\sum(y-\bar{y})^2 = 23.40$

1

Now, compute the standard error, s using formula:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$$

$$s = \sqrt{\frac{53+23.40}{12+7-2}} = \sqrt{\frac{76.40}{17}}$$

$$= \sqrt{4.49} = 2.12$$ **1**

Now using t-test formula:

$$t = \frac{\bar{x} - \bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$

$$t = \frac{11.5 - 9.71}{2.12}\sqrt{\frac{12 \times 7}{12 + 7}}$$

$$= \frac{1.79}{2.12}\sqrt{4.42}$$

$$= 0.84 \times 2.10 = 1.76$$

Hence, t-test value for the two data sets is 1.76.

The degree of freedom $= n_1 + n_2 - 2 = 12 + 7 - 2 = 17$

Given,

The value of t for 17 degrees of freedom at 5% level of significance = 2.11, which is greater than the calculated value of *t*.

Hence, the difference is not significant *i.e.*, there is no evidence of superiority of one diet over the other. **1**

Commonly Made Error

- Few students didn't mention the conclusion at the end of the t-test questions.

Answering Tip

- Always remember when it is mention to made a conclusion, it is necessary to give the conclusion at the end of the question otherwise marks will be deducted.

Q. 3. Two horse A and B were tested according to the time (in seconds) to run a particular track with the following results:

Horse A	28	30	32	33	33	29	34
Horse B	29	30	30	24	27	29	

Test whether you can discriminate between two horses. You can use the fact that 5% value of t for 11 degrees of freedom is 2.20. **[O.E.B.]**

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

For a random sample of 10 pigs fed on diet *A*, the increases in weight in pounds in a certain period were 10, 6, 16, 17, 13, 12, 8, 14, 15, 9 lbs

for another random sample of 12 pigs fed on diet *B*, the increases in the same period were

7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 lbs

Q. 1. Mean of increases in weight on diet *A* is

(A) 12 pounds **(B)** 15 pounds

(C) 20 pounds **(D)** 27 pounds

Ans. Option (A) is correct.

These questions are for practice and their solutions are available at the end of the chapter

Explanation: Mean of increases in weight on diet A

$$= \bar{x} = \frac{\Sigma x}{n_1}$$

$$= \frac{10+6+16+17+13+12+8+14+15+9}{10}$$

$$= \frac{120}{10} = 12 \text{ pound}$$

Q. 2. Sum of square of deviation for diet *B* is:

(A) 120 **(B)** 314

(C) 150 **(D)** 214

Ans. Option (B) is correct.

Explanation: Sum of squares of deviations of diet B

$$= \Sigma(y-\bar{y})^2$$

Here $\bar{y} = \frac{\Sigma y}{n_2}$

$$= \frac{7+13+22+15+12+14+18+21+23+10+17}{12}$$

$$= \frac{180}{12}$$

= 15 is pounds.

$\Sigma(y-\bar{y})^2 = 64 + 4 + 49 + 0 + 9 + 1 + 9 + 49 + 36 + 64 + 25 + 4$

= 314

Q. 3. The number of degrees of freedom is:

(A) 10 **(B)** 12

(C) 15 **(D)** 20

Ans. Option (D) is correct.

Explanation: The number of degrees of freedom (ν)

$= n_1 + n_2 - 2$

$= 10 + 12 - 2$

$= 20$

Q. 4. The standard error of this is:

(A) 4.65 **(B)** 4.55

(C) 46.5 **(D)** 4.10

Ans. Option (A) is correct.

Explanation: ∵ Standard error,

$$s = \sqrt{\frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1+n_2-2}}$$

$$= \sqrt{\frac{120+314}{20}}$$

$[\because \Sigma(x-\bar{x})^2 = 120 \text{ and } \Sigma(y-\bar{y})^2 = 314]$

$= \sqrt{21.7}$

$= 4.65$

Q. 5. Using the fact the 5% value of *t* for 20 degrees of freedom is 2.09, whether diets *A* and *B* differ significantly as regards the effect on increases in weight.

(A) Significant **(B)** not significant

(C) can't say **(D)** None of the above

Ans. Option (B) is correct.

Explanation: Here $t = \frac{\bar{x}-\bar{y}}{S}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$

$$= \frac{12-15}{4.65}\sqrt{\frac{12\times10}{12+10}}$$

= 1.5

For 20 degrees of freedom we are given that the value of t at 5% level of significance is 2.09 the calculated value t is less than this value. Hence the difference between the sample means is not significant.

II. Read the following text and answer the following questions on the basis of the same:

Eight pots growing three barley plants each were exposed to high tension discharge, while nine similar pots were enclosed in an earthed wire cage. The numbers of tillers in each pot were as follows:

Caged	17	26	18	25	27	28	26	23	17
Electrified	16	16	22	16	21	18	15	20	

Given that the value of t for 15 degree of freedom at 5% level of significance is 2.131.

Q. 1. Mean tillers of caged sample is:

(A) 23 **(B)** 18

(C) 9 **(D)** 8

Ans. Option (A) is correct.

Explanation:

Number of terms in first set i.e., $n_1 = 9$

Mean tillers of caged sample is

$$\bar{x} = \frac{\sum x}{n_1}$$

$$= \frac{17+26+18+25+27+28+26+23+17}{9}$$

$$= \frac{207}{9} = 23$$

Q. 2. Mean tillers of electrified sample is:

(A) 23 **(B)** 18

(C) 9 **(D)** 8

Ans. Option (B) is correct.

Explanation: Number of terms in second set i.e., $n_2 = 9$

Mean tillers of electrified sample is

$$\bar{y} = \frac{y}{n_1}$$
$$= \frac{16+16+22+16+21+18+15+20}{8}$$
$$= \frac{144}{8} = 18$$

Q. 3. The formula for standard error is:

(A) $s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$

(B) $s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1-n_2-2}}$

(C) $s = \sqrt{\frac{\sum(x+\bar{x})^2 + \sum(y+\bar{y})^2}{n_1+n_2-2}}$

(D) $s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2}}$

Ans. Option (A) is correct.

Explanation: The formula for standard error is

$$s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$$

Q. 4. The formula for two sample *t*-test is:

(A) $t = \frac{\bar{x}+\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$ **(B)** $t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1-n_2}{n_1+n_2}}$

(C) $t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1-n_2}}$ **(D)** $t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$

Ans. Option (D) is correct.

Explanation: The formula for two sample t-test is

$$t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$$

Q. 5. If the calculated value for *t*-test is 2.751, then which of the following statement is true:

(A) The electrification does exert some effect on the tillering.

(B) The electrification does not exert some effect on the tillering.

(C) Can't say.

(D) None of these.

Ans. Option (A) is correct.

Explanation: It is given that the value of t for 15 degree of freedom at 5% level of significance is 2.131 which is less than the calculated value of t *i.e.*, 2.751. Hence the difference between the mean tillers of the two samples is significant *i.e.*, the electrification does exert some effect on the tillering.

III. Read the following text and answer the following questions on the basis of the same:

A group of 7 weeks old chickens, reared on a high protein diet weight 12, 15, 11, 16, 14, 14, 16 ounces; a second group of 5 chickens, similarly treated except that they receive a low protein diet, weight 8, 10, 14, 10, 13 ounces. Given that the value of t for 10 degree of freedom at 5% level of significance is 2.23.

Q. 1. Mean weight of old chickens, reared on a high protein diet is:

(A) 14 **(B)** 11

(C) 98 **(D)** 55

Ans. Option (A) is correct.

Explanation: Number of terms in first group *i.e.* $n_1 = 7$

Mean weight of old chickens, reared on a high protein diet is

$$\bar{x} = \frac{\sum x}{n_1}$$
$$= \frac{12+15+11+16+14+14+16}{7}$$
$$= \frac{98}{7} = 14$$

Q. 2. Sum of the deviations of old chickens is:

(A) 23 **(B)** 18

(C) 22 **(D)** 14

Ans. Option (C) is correct.

Explanation: Sum of the squares of mean deviations of old chickens is

$$\sum(x-\bar{x})^2 = (x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \ldots + (x_7-\bar{x})^2$$
$$= (12-14)^2 + (15-14)^2 + (11-14)^2 + (16-14)^2 + (14-14)^2 + (14-14)^2 + (16-14)^2$$
$$= 4 + 1 + 9 + 4 + 0 + 0 + 4$$
$$= 22$$

Q. 3. Mean weight chickens, receive a low protein diet is:

(A) 14 **(B)** 11

(C) 98 **(D)** 55

Ans. Option (B) is correct.

Explanation: Number of terms in second group *i.e.* $n_1 = 5$

Mean weight chickens, receive a low protein diet is

$$\bar{y} = \frac{y}{n_1}$$
$$= \frac{8+10+14+10+13}{5}$$
$$= \frac{55}{5} = 11$$

Q. 4. The standard error for given two samples is:

(A) 2.14 **(B)** 2.41

(C) 2.93 **(D)** 2.55

Ans. Option (A) is correct.

Explanation: Formula for standard error is

$$s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$$

We already have the following values

Now, $\sum(y-\bar{y})^2 = (y_1-\bar{y})^2 + (y_2-\bar{y})^2 + \ldots + (y_7-\bar{y})^2$

$= (8-11)^2 + (10-11)^2 + (14-11)^2 + (10-11)^2 + (13-11)^2$

$= 9 + 1 + 9 + 1 + 4$

$= 24$

So, $s = \sqrt{\frac{22+24}{7+5-2}} = \sqrt{\frac{46}{10}}$

$= \sqrt{4.6} = 2.14$

Q. 5. The value of *t* is:

(A) 2.44 **(B)** 2.23

(C) 2.39 **(D)** 2.36

Ans. Option (C) is correct.

Explanation: The formula for two sample t-test is

$$t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$$

So, $t = \frac{14-11}{2.14}\sqrt{\frac{7\times5}{7+5}}$

$= \frac{3}{2.14}\sqrt{\frac{35}{12}}$

$= 1.40 \times \sqrt{2.92}$

$= 1.40 \times 1.71 = 2.39$

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:
Each sub-part carries 2 marks.
are found to be 100, 104, 108, 110, 118, 120, 122, 124, 126, and 128 cms.

Q. 1. Find the *t*-statistic for the given data, if mean height of the students is 110 cms.

Sol. Let the sample statistic *t* be given by

$$t = \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}}$$

Let us now compute the sample mean ($\bar{X}$) and S.

Computation of $\bar{X}$ and S

x_i	$x_i - \bar{X}$	$(x_i - \bar{X})^2$
100	– 16	256
104	– 12	144
108	– 8	64
110	– 6	36
118	2	4
120	4	16
122	6	36
124	8	64
126	10	100
128	12	144
$\sum x_i = 1160$		$\sum_{i=1}^{10}(x_i-\bar{X})^2 = 864$

1

Here $\bar{X} = \frac{\sum x_i}{10} = \frac{1160}{10} = 116$

$\therefore$ $t = \frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}}$

$\Rightarrow$ $t = \frac{116-110}{\frac{9.798}{\sqrt{10}}}$

$= \frac{6}{9.798}\times 3.162 \quad [\because \mu = 110]$

$= 1.94$ **1**

$[\because S^2 = \frac{1}{n-1}\sum_{i=1}^{10}(x_i-\bar{X})^2 \Rightarrow S^2 = \frac{1}{9}\times 864$

$\Rightarrow S = \frac{\sqrt{864}}{3} = \frac{29.393}{3} = 9.798]$

Q. 2. Discuss the suggestion that the mean height of the students of the college is 110 cms. [Given $t_9(0.05) = 2.262$]

Sol. We define,

Null Hypothesis H_0: There is no significant difference between the sample mean and hypothetical population mean 110 cms.

Alternate Hypothesis H_1: The sample mean is not same as the population mean. **1**

Here, the sample statistic follows student's *t*-distribution with $\nu = (10 - 1) = 9$ degree of freedom.

We shall now compare the calculated t-test value with the value of t for 9 degrees of freedom at a certain level of significance. It is given that $t_9(0.05) = 2.262$

$\therefore$ Calculated $|t| = 1.94 < 2.262 = t_9(0.05)$

i.e., Calculated $|t| <$ tabulated $t_9(0.05)$

So, we accept the null hypothesis. Hence, the sample mean is same as the population mean. Consequently, the mean height of the students of the school is 110 cms. **1**

II. Read the following text and answer the following questions on the basis of the same:

Each sub-part carries 2 marks.

In order to promote and increasing the number of students, a coaching centre XYZ announces the scholarship for admission in the coaching centre, students have to clear a I.Q. test conducted by the coaching centre. A random sample of 10 boys had the following I.Q's: 70, 120, 110, 101, 88, 83, 85, 95, 98, 107, 100.

Q. 1. Find the t-statistic for the given data, if mean. I.Q. of the population is considered as 100.

Sol. We have μ = population mean = 100

n = sample size = 10

computation of $\overline{X}$ and S

x_i	$d_i = x_i - 90$	d_i^2
70	– 20	400
120	30	900
100	20	400
101	11	121
88	– 2	4
83	– 7	49
95	5	25
98	8	64
107	17	289
100	10	100
	$\sum d_i = 72$	$\sum d_i^2 = 2352$

1

$$\therefore \quad \overline{X} = 90 + \frac{1}{10}\sum d_i$$

$$\left[\text{Using } \overline{X} = A + \frac{1}{n}\sum d_i\right]$$

$$= 90 + \frac{72}{10} = 97.2$$

and $$S^2 = \frac{1}{n-1}\left\{\sum d_i^2 - \frac{1}{n}\left(\sum d_i\right)^2\right\}$$

$$= \frac{1}{9}\left\{2352 - \frac{(72)^2}{10}\right\}$$

$$= \frac{1833.6}{9} = 203.73$$

Now, $$t = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}}$$

$$= \frac{97.2 - 100}{\sqrt{\frac{203.73}{10}}}$$

$$= \frac{-2.8}{\sqrt{20.37}}$$

$$= \frac{-2.8}{4.514} = -0.62$$

$\Rightarrow$ $|t| = 0.62$ **1**

Q. 2. Do these data support the assumption of a population mean I.Q. of 100 ?

[Given $t_9(0.05) = 2.262$]

Sol. We define,

Null Hypothesis H_0: The data are consistent with the assumption of a mean I.Q. of 100 in the population.

Alternate Hypothesis H_1: The mean I.Q. of population + 100. **1**

Solutions for Practice Questions

Very Short Answer Type Questions

Sol. 3: A statistic is used to estimate the value of a population parameter. For instance, we selected a random sample of 100 students from a school with 1000 students. The average height of the sampled students would be an example of a statistic. **1**

Sol. 6: Given $s = 10$, S.E. $\bar{X} = 3$

Since, $\text{S.E.} = \frac{\sigma}{\sqrt{n}}$

Therefore, $3 = \frac{10}{\sqrt{n}}$

$\Rightarrow \quad \sqrt{n} = \frac{10}{3}$

Taking square on both sides we get

$$n = \left(\frac{10}{3}\right)^2$$

$$= \frac{100}{9}$$

$$= 11.11$$

$$\cong 11,$$

The required sample size is 11. **1**

Sol. 10: Given $n = 1000$, mean $= 119$, $\sigma = 30$

$$\text{S.E.} = \frac{\sigma}{\sqrt{n}}$$

$$= \frac{30}{\sqrt{1000}}$$

$$= \frac{30}{31.623}$$

$$= 0.9487$$ **1**

Short Answer Type Questions-I

Sol. 3: Here, $\bar{X} = 9.52$, Mean, $\mu = 10$, $\sigma = 0.60$ and $n = 10$

$$t = \frac{Z}{\text{S.E.}}$$

$$= \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$ **1**

$$= \frac{9.52 - 10}{\frac{0.60}{\sqrt{10}}}$$

$$= \frac{-0.48}{\frac{0.60}{3.162}}$$

$$= \frac{-0.48}{0.1897}$$

$$= -2.5303$$ **1**

Commonly Made Error

- Sometimes students do wrong calculation and lose their marks.

Answering Tip

- Do the calculation carefully and recheck the answer at the end.

Short Answer Type Questions-II

Sol. 2:

x	$x - \bar{x}$	$(x - \bar{x})^2$
63	– 4	16
63	– 4	16
64	– 3	9
65	– 2	4
66	– 1	1
69	2	4
69	2	4
70	3	9
70	3	9
71	4	16
$\sum x = 670$		$\sum (x - \bar{x})^2 = 88$

$\bar{x}$ = mean **1**

$$= \frac{\sum x}{n}$$

$$= \frac{670}{10} = 67$$ ½

Now, compute the standard deviation using formula as,

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$

$$= \sqrt{\frac{88}{9}}$$

$$= 3.13 \text{ inches}$$ ½

H_0 = The mean of universe, $\mu = 65$ inches, we get

$$t = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{67 - 65}{\frac{3.13}{\sqrt{10}}}$$

$$= \frac{2}{\frac{3.13}{3.16}}$$

$$= \frac{2}{0.9905}$$

$$= 2.02$$

The number of degree of freedom $= n - 1 = 9$

Given that the tabulated value for 9 d.f. at level of significance is 2.62.

Since calculated value of t is less than the tabulated value *i.e.*, $2.02 < 2.62$, the error has arisen due to fluctuations and we may conclude that the data are consistent with the assumption of mean of height in the universe of 65 inches. **1**

Long Answer Type Questions

Sol. 3: For first data set:

Number of terms in first set *i.e.* $n_1 = 7$

Mean of A is

$$\bar{x} = \frac{\sum x}{n_1}$$

$$= \frac{28 + 30 + 32 + 33 + 33 + 29 + 34}{7}$$

$$= \frac{219}{7}$$

$$= 31.29$$ **1**

For second data set:

Number of terms in second set *i.e.*, $n_2 = 6$

Mean of B is

$$\bar{y} = \frac{\sum y}{n_2}$$

$$= \frac{29 + 30 + 30 + 24 + 27 + 29}{6}$$

$$= \frac{169}{6}$$

$$= 28.16$$ **1**

Construct the following table for standard error *s*:

Horse A			**Horse B**		
x	$x - \bar{x}$	$(x - \bar{x})^2$	y	$y - \bar{y}$	$(y - \bar{y})^2$
28	– 3.29	10.82	29	0.84	0.71
30	– 1.29	1.66	30	1.84	3.39
32	0.71	0.50	30	1.84	3.39
33	1.71	2.92	24	– 4.16	17.31
33	1.71	2.92	27	– 1.16	1.35
29	– 2.29	5.24	29	0.84	0.71
34	2.71	7.34			
		$\sum (x - \bar{x})^2 = 31.4$			$\sum (y - \bar{y})^2 = 26.86$

1

Now, compute the standard error, s using formula:

$$s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1 + n_2 - 2}}$$

$$s = \sqrt{\frac{31.4 + 26.86}{7 + 6 - 2}}$$

$$= \sqrt{\frac{58.26}{11}}$$

$$= \sqrt{5.29}$$

$$= 2.3 \quad \textbf{1}$$

Now using t-test formula:

$$t = \frac{\bar{x} - \bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$

$$t = \frac{31.29 - 28.16}{2.3}\sqrt{\frac{7 \times 6}{7 + 6}}$$

$$= \frac{3.13}{2.3}\sqrt{3.23}$$

$$= 1.36 \times 1.79$$

$$= 2.43$$

Hence, t-test value for the two data sets is 2.43.

The degree of freedom $= n_1 + n_2 - 2 = 7 + 6 - 2 = 11$

Given, the value of t for 11 degrees of freedom at 5% level of significance = 2.20, which is less than the calculated value of t. Hence the difference is significant *i.e.*, we can discriminate between two horses. **1**

REFLECTIONS

(a) Will you be able to differentiate between sample and populations?

(b) Can you able to draw the conclusions using t-test?

(c) Will you be able to solve the difficult problems based on the Test of Hypothesis?

SELF ASSESSMENT PAPER - 05

Maximum Marks: 30 **Max. Time: 1 Hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. The sample mean is given by:

(A) $\sum \frac{X_i}{n}$ **(B)** $\sum \frac{X_i}{n} - \bar{x}$

(C) $\sum \frac{X_i}{n} + \bar{x}$ **(D)** $\sum \frac{n}{X_i}$

Q. 2. Null hypothesis is represented by:

(A) H_0 **(B)** H_1

(C) H_2 **(D)** None of these

Q. 3. Degree of freedom is calculated from the following formula:

(A) N – 1 **(B)** N + 1

(C) $\frac{N}{2}$ **(D)** None of these

Q. 4. The process by which one makes inferences about a population, based on the information obtained from a sample is known as:

(A) Sample mean **(B)** estimation

(C) hypothesis **(D)** none of these

Q. 5. ……. is an educated guess which needs to be tested.

(A) Sample mean **(B)** estimation

(C) hypothesis **(D)** none of these

Q. 6. A …….is a numerical value that is taken from the entire population, such as the population mean.

(A) Parameter **(B)** estimation

(C) hypothesis **(D)** none of these

Case-based

Q. 7. Attempt any 4 sub-parts from each questions. Each question carries 1 mark.

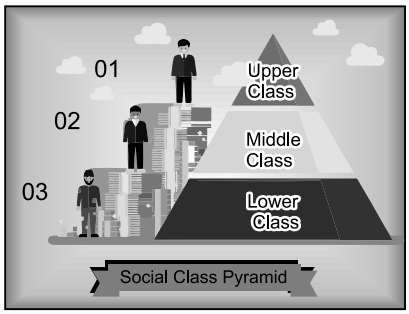
Read the following text and answer the following questions on the basis of the same:

From the following data, select 68 random samples from the population of heterogeneous group with size of 500 through stratified random sampling, considering the following categories as strata.

Category 1: Lower income class – 39%

Category 2: Middle income class – 38%

Category 3: Upper income class – 23%



(i) The number of people in lower income class is:

(A) 150 **(B)** 175

(C) 195 **(D)** 215

(ii) The size of random sample for middle income class is:

(A) 27 **(B)** 30

(C) 40 **(D)** 16

(iii) The number of people in upper income class is:

(A) 115 **(B)** 175

(C) 195 **(D)** 215

(iv) The size of random sample for lower income class is:

(A) 27 **(B)** 28

(C) 35 **(D)** 16

(v) The size of random sample for upper income class is:

(A) 26 **(B)** 29

(C) 32 **(D)** 16

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. A sample of 1600 students whose mean weight is 120 lbs (pounds) from a primary school in Madhya Pradesh State was taken and their average weight was found to be 130 lbs with a standard deviation of 35 lbs. Calculate the standard error of the mean.

Q. 9. A simple random sample consists of four observation 2, 4, 6, 8. What is the point estimate of population standard deviation ?

Q. 10. A sample of 100 wall clocks is taken at random. Out of 100 we found 25 wall clocks are digital and 75 are analogue. What is the point estimate of population proportion of analogue.

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. Wholesaler in ceiling fans claims that only 8% of the ceiling fans supplied by him are defective. A random sample of 1200 ceiling fans contained 56 defective ceiling fans. Calculate the standard error concerning good ceiling fans.

Q. 12. A sample of 100 measurements at breaking strength of cotton thread gave a mean of 7.4 and a standard deviation of 1.2 gms. Find 95% confidence limits for the mean breaking strength of cotton thread.

Q. 13. A sample of 100 items, draw from a universe with mean value 4 and S.D. 3, has a mean value 63.5. Is the difference in the mean significant ?

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. It is claimed that the average weight of a bag of biscuits is 250 grams with the standard deviation 20.5 grams. Would you agree to this claim if random sample of 50 bags of biscuits showed an average weight of 240 grams, using a 0.05 level of significance ?

Q. 15. Country A has an average farm size of 191 acres, while Country B has an average farm size of 199 acres. Assume the data were attained from two samples with standard deviations of 38 and 12 acres and sample sizes of 8 and 10, respectively. Is it possible to infer that the average size of the farms in the two countries is different at $\alpha = 0.05$? Assume that the populations are normally distributed.

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 16. Let us consider the average rainfall in a given area is 8 inches. However, a local meteorologist claims that rainfall was above average from 2016-2020 and argues that average rainfall during this period was significantly different from overall average rainfall. The following is the average rainfall for the observed period of 2016-2020:

Year	2016	2017	2018	2019	2020
Rainfall (inches)	8	5	7	5	6

CHAPTER

10 TIME SERIES ANALYSIS

Syllabus

Time series, Components of time series, Time series analysis for univariant data sets, Secular Trend, Methods of measuring trend.

In this chapter you will study

- *Identify time series as chronological data*
- *Distinguish between different components of time series*
- *Solve practical problems based on statistical data and Interpret the result*
- *Understand the long term tendency*
- *Demonstrate the techniques of finding trend by different methods*

Revision Notes

A time series consists of a set of observations arranged in chronological order (either ascending or descending). Time Series has an important objective to identify the variations and try to eliminate the variations and also helps us to estimate or predict the **future values**.

Scan to know more about this topic

Introduction of Time Series

Key Words

Future value: Future value is a value of an investment or asset on a specific date in the future.

- It helps in the analysis of past behaviour.
- It helps in forecasting and for future plans.
- It helps in evaluation of current achievements.
- It helps in making comparative studies between one time period and others.

The following series is the example of time series:

Year	1971	1981	1991	2001	2011	2021
Population (Crore)	43.9	54.2	68.3	84.9	102.6	121.5

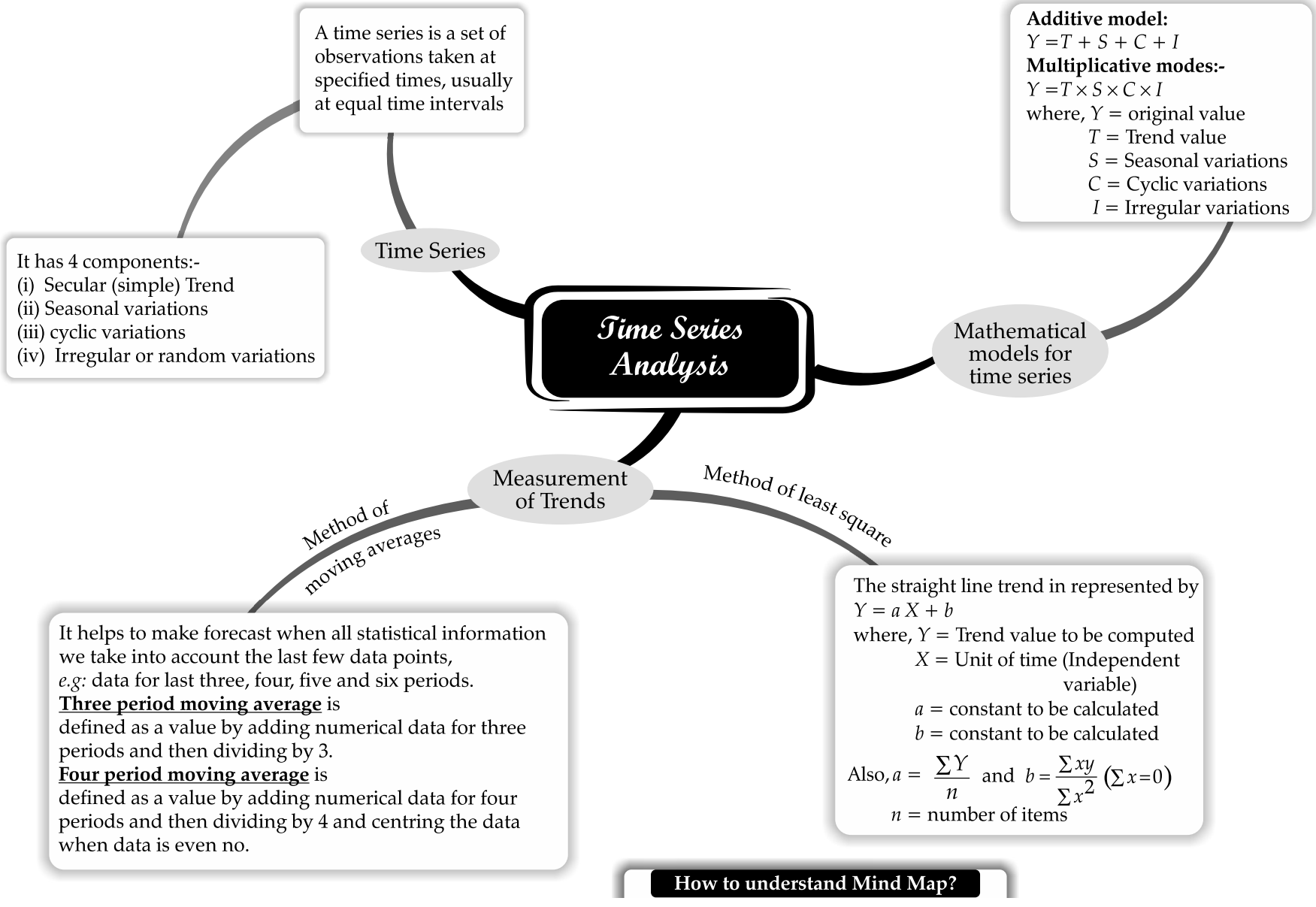
Components of Time Series: There are four types of components in a time series. They are as follows:

(i) Secular Trend, (ii) Seasonal Variations, (iii) Cyclic Variations and (iv) Irregular Variations.

(i) Secular Trend or Simple Trend or Long term movement: Secular trend refers to the general tendency of data to increase or decrease or stagnate over a long period of time. Time series relating to Economic, Business, and Commerce may show an upward or increasing tendency. Whereas, the time series relating to death rates, birth rates, share prices, etc. may show a downward or decreasing tendency.

The symbol of 'T' is used for denoting long term trend in the formulae relating to analysis of Time Series.

(ii) Seasonal variations: As the name suggests, tendency movements are due to nature which repeat themselves periodically in every seasons. These variations repeat themselves in less than one year time. It is measured in an interval of time. Seasonal variations may be influenced by natural force, social customs and traditions. These variations are the results of such factors



How to understand Mind Map?

which uniformly and regularly rise and fall in the magnitude. For example, in summers the sale of ice-cream increases and at the time of Diwali the sale of diyas, crackers, etc. go up.

The symbol of 'S' is used for denoting Seasonal variations in the formulae relating to analysis of Time Series.

(iii) Cyclical variations: Cyclical variations are due to the ups and downs recurring after a period from time to time. These are due to the business cycle and every organization has to phase all the four phases of a business cycle some time or the other. Prosperity or boom, recession, depression, and recovery are the four phases of a business cycle.

The symbol of 'C' is used for denoting Cyclic variations in the formulae relating to analysis of Time Series.

(iv) Irregular or Random variations: These variations do not have particular pattern and there is no regular period of time of their occurrences. These are accidently changes which are purely random or unpredictable. Normally they are short-term variations, but its occurrence sometimes has its effect so intense that they may give rise to new cyclic or other movements of variations. For example floods, wars, earthquakes, Tsunami, strikes, lockouts etc...

The symbol of 'I' is used for denoting Irregular variations in the formulae relating to analysis of Time Series.

Mathematical Model for a Time Series

There are two common models used for decomposition of a time series into its components, namely additive and multiplicative model.

(i) Additive Model: The model assumes that the observed value is the sum of all the four components of time series. *i.e.*,

$$Y = T + S + C + I$$

where, Y = Original value, T = Trend Value, S = Seasonal component, C = Cyclic component, I = Irregular component

The additive model assumes that all the four components operate independently. It also assumes that the behaviour of components is of an additive character.

(ii) Multiplicative Model: This model assumes that the observed value is obtained by multiplying the trend (T) by the rates of other three components.

i.e., $Y = T \times S \times C \times I$

where, Y = Original value, T = Trend Value, S = Seasonal component, C = Cyclic component, I = Irregular component

This model assumes that the components due to different causes are not necessarily independent and they can affect one another. It also assumes that the behaviour of components is of a multiplicative character.

Measurement of Trends

Following are the methods by which we can measure the trend.

(I) Method of Moving Averages.

(II) Method of Least Squares.

Scan to know more about this topic

Method of Moving Averages

(I) Method of Moving Averages: Moving Averages Method gives a trend with a fair degree of accuracy. In this method, we take arithmetic mean of the values for a certain time span. The time span can be three-years, four-years, five-years and so on depending on the data set and our interest. We will see the working procedure of this method.

Procedure:

(i) Decide the period of moving averages (three-years, four-years).

(ii) In case of odd years, averages can be obtained by calculating,

$$\frac{a+b+c}{3}, \frac{b+c+d}{3}, \frac{c+d+e}{3}, \frac{d+e+f}{3}, \ldots.$$

(iii) If the moving average is an odd number, there is no problem of centring it, the average value will be centered besides the second year for every three years.

(iv) In case of even years, averages can be obtained by calculating,

$$\frac{a+b+c+d}{4}, \frac{b+c+d+e}{4}, \frac{c+d+e+f}{4}, \frac{d+e+f+g}{4}, \ldots.$$

(v) If the moving average is an even number, the average of first four values will be placed between 2nd and 3rd year, similarly the average of the second four values will be placed between 3rd and 4th year. These two averages will be again averaged and placed in the 3rd year. This continues for rest of the values in the problem. This process is called as centring of the averages.

Example 1 : Calculate three-yearly moving averages of number of students studying in a higher secondary school in a particular village from the following data.

Year	Number of Students
2011	332
2012	317
2013	357
2014	392
2015	402

2016	405
2017	410
2018	427
2019	435
2020	438

Key Facts

- Retail stores often use time series analysis to analyse how their total sales is trending over time.
- Heights of ocean tides and counts of sunspots, are the examples of time series analysis.

Sol. Computation of three-yearly moving averages.

Year	Number of Students	3- yearly moving Total	3- yearly moving Averages
2011	332	—	—
2012	317	1006	335.33
2013	357	1066	355.33
2014	392	1151	383.67
2015	402	1199	399.67
2016	405	1217	405.67
2017	410	1242	414.00
2018	427	1272	424.00
2019	435	1300	433.33
2020	438	—	—

(II) Method of Least Squares: The line of best fit is a line from which the sum of the deviations of various points is zero. This is the best method for obtaining the trend values. It gives a convenient basis for calculating the line of best fit for the time series. It is a mathematical method for measuring trend. Further the sum of the squares of these deviations would be least when compared with other fitting methods. So, this method is known as the Method of Least Squares and satisfies the following conditions:

Scan to know more about this topic

Least Square Method

(i) The sum of the deviations of the actual values of Y and $\overline{Y}$ (estimated value of Y) is Zero. *i.e.*, $(Y - \overline{Y}) = 0$.

(ii) The sum of squares of the deviations of the actual values of Y and (estimated value of Y) is least. *i.e.*, $(Y - \overline{Y})^2$ is least.

(iii) The straight line trend is represented by the equation

$$Y = a + bX \quad ...(1)$$

where, Y is the actual value, X is time, a, b are constants

(iv) The constants 'a' and 'b' are estimated by solving the following two normal equations

$$\Sigma Y = n\,a + b\,\Sigma X \quad ....(2)$$

$$\Sigma Y = a\,\Sigma X + b\,\Sigma X^2 \quad ....(3)$$

where, 'n' = number of years given in the data.

(v) By taking the mid-point of the time as the origin, we get

$$\Sigma X = 0$$

(vi) When $\Sigma X = 0$, the two normal equations reduces to

$$\Sigma Y = na + b\,(0);\ a = \frac{\Sigma Y}{n} = \overline{Y}$$

$$\Sigma XY = a(0) + b\,\Sigma X\,;\ b = \frac{\Sigma XY}{\Sigma X^2}$$

The constant 'a' gives the mean of Y and 'b' gives the rate of change (slope).

(vii) By substituting the values of 'a' and 'b' in the trend equation (1), we get the Line of Best Fit.

Example: Given below are the data relating to the sales of a product in a district.

Fit a straight line trend by the method of least squares and tabulate the trend values.

Year	2015	2016	2017	2018	2019	2020	2021	2022
Sales	6.7	5.3	4.3	6.1	5.6	7.9	5.8	6.1

Sol. Computation of trend values by the method of least squares.
In case of EVEN number of years, let us consider

$$X = \frac{(x - \text{Arithimetic mean of two middle years})}{0.5}$$

Year (x)	Sales (Y)	$X = \frac{(x - 2018.5)}{0.5}$	XY	X^2	Trend value (Y_t)
2015	6.7	– 7	– 46.9	49	5.6167
2016	5.3	– 5	– 26.5	25	5.7190
2017	4.3	– 3	– 12.9	9	5.8214
2018	6.1	– 1	– 6.1	1	5.9238
2019	5.6	1	5.6	1	6.0262
2020	7.9	3	23.7	9	6.1286
2021	5.8	5	29.0	25	6.2310
2022	6.1	7	42.7	49	6.3333
$n = 8$	47.8	$\Sigma X = 0$	8.6	**168**	

$$a = \frac{\Sigma Y}{n} = \frac{47.8}{8} = 5.975$$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{8.6}{168} = 0.05119$$

Therefore, the required equation of the straight line trend is given by

$$Y = a + bX\,;\; Y = 5.975 + 0.05119\,X\,.$$

When $X = 2015$, $X = 1995$, $Y_t = 5.975 + 0.05119\left(\frac{2015 - 2018.5}{0.5}\right) = 5.6167$

When $X = 2016$, $X = 1996$, $Y_t = 5.975 + 0.05119\left(\frac{2016 - 2018.5}{0.5}\right) = 5.7190$

Similarly, other values can be obtained.

Mnemonics

Concept: Components of Times Series
Mnemonics: **T**rendy **S**iya **C**ame **I**n
Interpretations:

Trendy	Siya	Came	In
↓	↓	↓	↓
Secular Trend	**S**easonal Variations	**C**yclic Variations	**I**rregular Variations

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. Which of the following is an example of time series problem ?

(i) Estimating numbers of hotel rooms booking in next 6 months.

(ii) Estimating the total sales in next 3 years of an insurance company.

(iii) Estimating the number of calls for the next one week.

(A) Only (iii) **(B)** (i) and (ii)
(C) (i), (ii) and (iii) **(D)** (ii) and (iii)

Ans. Option (C) is correct.

Explanation: All the component are associated with time.

Q. 2. Which of the following is not an example of a time series model?

(A) Naive approach **(B)** Exponential smoothing
(C) Moving average **(D)** None of these

Ans. Option (A) is correct.

Q. 3. Which of the following can't be a component for a time series plot?

(A) Seasonality **(B)** Trend
(C) Noise **(D)** None of these

Ans. Option (D) is correct.

Q. 4. Increase in the number of patients in the hospital due to heat stroke is:

(A) Secular trend **(B)** Irregular variation
(C) Seasonal variation **(D)** Cyclical variations

Ans. Option (C) is correct.

Explanation: In seasonal variation, tendency movements are due to the nature which repeat themselves periodically in every season.

Q. 5. The graph of time series is called:

(A) Histogram **(B)** Straight line
(C) Historigram **(D)** Ogive

Ans. Option (B) is correct.

Q. 6. Moving average method is used for measurement of trend when:

(A) Trend is linear
(B) Trend is non-linear
(C) Trend is curvilinear
(D) None of these

Ans. Option (A) is correct.

Explanation: When trend is linear, then only we use moving average method for measurement.

Q. 7. The most commonly use mathematical method for measuring the trend is:

(A) Moving average method
(B) Semi average method
(C) Method of least squares
(D) None of these

Ans. Option (C) is correct.

Q. 8. An orderly set of data arranged in accordance with their time of occurrence is called:

(A) Arithmetic Series **(B)** Geometric Series
(C) Harmonic Series **(D)** Time Series

Ans. Option (D) is correct.

Explanation: The organized series of data on the basis of any measure of time is called Time series.

Q. 9. The best-fitted trend line is one for which sum of squares of residuals or errors is:

(A) Positive **(B)** Minimum
(C) Negative **(D)** Maximum

Ans. Option (B) is correct.

Explanation: The line is termed as the line of best fit from which the sum of squares of distances from the points is minimized.

Q. 10. The following are the movement(s) in the secular trend.

(A) Smooth **(B)** Regular
(C) Steady **(D)** All of these

Ans. Option (D) is correct.

Explanation: Secular trend is that component of the time series which gives the general tendency of the data for a long periods. It is smooth, regular and long term (steady) movement of a series.

Q. 11. Prosperity, Recession, and depression in a business is an example of:

(A) Irregular Trend **(B)** Secular Trend
(C) Cyclical Trend **(D)** Seasonal Trend

Ans. Option (C) is correct.

Explanation: There are 4 phases through which trade cycles are passed. They are prosperity, recession, depression, and recovery. In economic terms, these 4 stages are called economic fluctuations.

Q. 12. In the theory of time series, shortage of certain consumer goods before the annual budget is due to

(A) Seasonal Variation
(B) Secular Trend
(C) Irregular Variations
(D) Cyclic Variation

Ans. Option (A) is correct.

Explanation: Seasonal variation is variation in a time series within one year that is repeated more or less regularly. So, shortage of consumer goods before the annual budget is due to seasonal variation.

Q. 13. In the measurement of the secular trend, the moving averages:

(A) Smooth out the time series
(B) Given the trend in a straight line
(C) Measure the seasonal Variations
(D) None of the above

Ans. Option (A) is correct.

Explanation: Moving averages is a series of arithmetic means of variate values of a sequence. This is another way of drawing a smooth curve for a time series data.

Q. 14. A set of observations recorded at an equal interval of time is called.

(A) Array data **(B)** Data
(C) Geometric Series **(D)** Time series data

Ans. Option (D) is correct.

Explanation: A time series is a set of observations taken at specified times, usually at equal intervals.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Define time series. U [O.E.B.]

Sol. A Time Series consists of data arranged chronologically **– Croxton Cowden.**

When quantitative data are arranged in the order of their occurrence, the resulting series is called the Time Series. **– Wessel Wallet. 1**

Q. 2. State the uses of time series. U [O.E.B.]

Sol. A time-series carries profound importance in business and policy planning. It is used to compare the current trends with that in the past or the expected trends. Thus it gives a clear picture of growth or downfall. Most of the time series data related to fields like Economics, Business, Commerce etc. For example the production of a product, cost of a product, sales of a product, national income, salary of an individual etc. By close observation of time series data, one can predict and plan for future operations in industries and other fields. **1**

Q. 3. Mention the components of the time series. R [O.E.B.]

Sol. There are four types of components in a time series.

(i) Secular Trend

(ii) Seasonal variations

(iii) Cyclic variations

(iv) Irregular variations **1**

Q. 4. What is business forecasting ? U [O.E.B.]

Q. 5. What are the applications of time series analysis in real field ? U [O.E.B.]

Sol. Time series analysis is used for many applications such as: Economic Forecasting, sales forecasting, Budgetary Analysis etc. **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. State the two normal equations used in fitting a straight line. R [O.E.B.]

Sol. The normal equations used in fitting a straight line are

$$\Sigma Y = na + b\Sigma X \text{ and } \Sigma XY = a\Sigma X + b\Sigma X^2$$

where, n = number of years given in the data,

X = time

Y = actual value

a, b = constants **2**

Q. 2. What is the need for studying time series ? U [O.E.B.]

Q. 3. Define secular trend and seasonal variations. R&U [O.E.B.]

Sol. Secular Trend: It is a general tendency, of time series to increase or decrease or stagnates during a long period of time. An upward tendency is usually observed in the population of a country, production, sales, prices in industries, the income of individuals etc., A downward tendency is observed in deaths, epidemics, prices of electronic gadgets, water sources, mortality rate etc.

Seasonal Variations: Seasonal variations refer to the changes that take place due to the rhythmic forces which operate in a regular and periodic manner. These forces usually have the same or most similar pattern year after year. When we record data weekly, monthly or quarterly, we can see and calculate seasonal variations. Thus, when a time series consists of data only based on annual figures, there will be seen no seasonal variations. These variations may be due to seasons, weather conditions, habits, customs or traditions. For example, selling of umbrellas and raincoats in the rainy season, sales of cold drinks in the summer season, crackers in Deepawali season, purchase of dresses in a festival season, sugarcane in Pongal season. **2**

Q. 4. Write down the models of Time Series Analysis. [O.E.B.]

Sol. There are two models of time series analysis

(i) Additive model: This model is based on the assumption that the sum of four components is equal to the original value

i.e., $O = T + C + S + I$

where, T= trend, C = cyclical, S = seasonal,

I = irregular, O = original **1**

(ii) Multiplicative model: In this model four components have a multiplication relationship,, so

$O = T \times C \times S \times I$ **1**

Q. 5. What causes secular trend ? [O.E.B.]

Sol. The effect of long term causes is reflected in the tendency of a behaviour, to move in an upward or downward direction, termed as Trend or Secular Trend.

The secular trends could reflect environmental improvements, specifically changes in health practice and living conditions leading to improvements in mortality rates and life expectancy. These factors are interrelated with those concerning family size. **2**

Short Answer Type Questions-II (3 marks each)

Q. 1. From the following time series obtain trend value by 3 yearly moving averages. R&U [O.E.B.]

These questions are for practice and their solutions are available at the end of the chapter

Year	Sales (in ₹ 000)	Year	Sales (in ₹ 000)
2008	8	2014	16
2009	12	2015	17
2010	10	2016	14
2011	13	2017	17
2012	15		
2013	12		

Sol. Calculation of trend values by three yearly moving average method.

Years	Sales (Thousand ₹)	Three-yearly Moving Totals	Three-yearly Moving Average (Trend value)
2008	8		
2009	12	(8 + 12 + 10) = 30	10.00
2010	10	(12 + 10 + 13) = 35	11.67
2011	13	(10 + 13 + 15) = 38	12.67
2012	15	(13 + 15 + 12) = 40	13.33
2013	12	(15 + 12 + 16) = 43	14.33
2014	16	(12 + 16 + 17) = 45	15.00
2015	17	(16 + 17 + 14) = 47	15.67
2016	14	(17 + 14 + 17) = 48	16.00
2017	17		

3

Q. 2. The following figures relate to the profits of a commercial concern for 8 years. R&U [O.E.B.]

Years	2016	2017	2018	2019	2020	2021	2022	2023
Profit (₹)	15,420	15,470	15,520	21,020	26,500	31,950	35,600	34,900

Find the trend of profits by the method of three-yearly moving averages.

Q. 3. Explain the method of fitting a straight line. U [O.E.B.]

Sol. The method of fitting a straight line is as follows:

(i) The straight-line trend is represented by the equation $Y = a + bX$ ....(1)
where, Y is the actual value, X is time, a and b are constants

(ii) The constants 'a' and 'b' are estimated by solving the following two normal equations

$$\Sigma Y = na + b\Sigma X \quad ...(2)$$

$$\Sigma XY = a\Sigma X + b\Sigma X^2 \quad ...(3)$$

where, n = number of years given in the data.

(iii) By taking the mid-point of the time as the origin, we get

$$\Sigma X = 0$$

(iv) When $\Sigma X = 0$, the two normal equations reduces to

$$\Sigma Y = na + b(0); a = \frac{\Sigma Y}{n} = \overline{Y}$$

$$\Sigma XY = a(0) + b\Sigma X^2; b = \frac{\Sigma XY}{\Sigma X^2}$$

The constant 'a' gives the mean of Y and 'b' gives the rate of change (slope).

(v) By substituting the values of 'a' and 'b' in the trend equation (1), we get the Line of Best Fit. **3**

Q. 4. Fit a straight line trend by the method of least squares to the data given below: R&U [O.E.B.]

Years	2012	2013	2014	2015	2016	2017	2018
Sales (in tones)	9	11	13	12	14	15	17

Sol.

Years (t_i)	Sales (y_i)	$x_i = t_i - 2015$	x_i^2	$x_i y_i$
2012	9	– 3	9	– 27
2013	11	– 2	4	– 22
2014	13	– 1	1	– 13
2015	12	0	0	0
2016	14	1	1	14

2017	15	2	4	30
2018	17	3	9	51
$n = 7$	$\Sigma y_i = 91$	$\Sigma x_i = 0$	$\Sigma x_i^2 = 28$	$\Sigma x_i y_i = 33$

2

Now, $a = \frac{\Sigma y_i}{n} = \frac{91}{7} = 13$

$b = \frac{\Sigma x_i y_i}{\Sigma x_i^2} = \frac{33}{28} = 1.179$

The equation of the straight line trend is

$y = ax + b$

$\therefore \quad y = 13x + 1.179$ **1**

Q. 5. Fit a straight line trend by the method of least square to the following data on sales (₹in lakhs) for the period 2011-2018.

[O.E.B.]

Years	2011	2012	2013	2014	2015	2016	2017	2018
Sales (₹ lakhs)	76	80	130	144	138	120	174	190

Sol.

Years (t_i)	Sales (₹ lakhs) (y_i)	$x_i = t_i - 2014\text{-}15$	$x_i \times 2$	$x_i y_i$	x_i^2
2011	76	– 3.5	– 7	– 532	49
2012	80	– 2.5	– 5	– 400	25
2013	130	– 1.5	– 3	– 390	9
2014	144	–0.5	– 1	– 144	1
2015	138	0.5	1	138	1
2016	120	1.5	3	360	9
2017	174	2.5	5	870	25
2018	190	3.5	7	1330	49
	$\Sigma y_i = 1052$	$\Sigma x_i = 0$		$\Sigma x_i y_i = 1232$	$\Sigma x_i^2 = 168$

2

Now, $a = \frac{\Sigma y_i}{n} = \frac{1052}{8} = 131.5$

$b = \frac{\Sigma x_i y_i}{\Sigma x_i^2} = \frac{1232}{168} = 7.33$

So, trend equation is

$y = 131.5 + 7.33x$ **1**

Long Answer Type Questions (5 marks each)

Q. 1. Fit the straight line trend to the following time series data:

Year	2017	2018	2019	2020	2021
Sale of sugar (in thousand kg)	80	90	92	83	94

Also, tabulate the trend values.

[O.E.B.]

Sol.

Year t	Sale y	$x = t - 2019$	xy	x^2	y_t
2017	80	– 2	– 160	4	83.6
2018	90	– 1	– 90	1	85.7
2019	92	0	0	0	87.8
2020	83	1	83	1	89.9
2021	94	2	188	4	92.0
$n = 5$	$\Sigma y = 439$	$\Sigma x = 0$	$\Sigma xy = 21$	$\Sigma x^2 = 10$	

2

Now, $a = \frac{\Sigma y}{n} = \frac{439}{5} = 87.8$

$b = \frac{\Sigma xy}{\Sigma x^2} = \frac{21}{10} = 2.1$

Hence, trend equation is $y_t = 87.8 + 2.1x$ **1**

$y_{2017} = 87.8 + 2.1\,(-2) = 83.6$

$y_{2018} = 87.8 + 2.1\,(-1) = 85.7$

$y_{2019} = 87.8 + 2.1\,(0) = 87.8$

$y_{2020} = 87.8 + 2.1\,(1) = 89.9$

$y_{2021} = 87.8 + 2.1\,(2) = 92.0$ **2**

Q. 2. Calculate trend values from the following data assuming 5- yearly and 7- yearly moving average.

AI [O.E.B.]

Year	1	2	3	4	5	6	7	8
Value	110	104	98	105	109	120	115	110
Year	9	10	11	12	13	14	15	16
Value	114	122	130	127	122	118	130	140

Sol. Calculation of Trend values by moving average method:

Year	Value	Moving Totals		Moving Average	
		5 Year	7 Year	5 Year	7 Year
1	110	—	—	—	—
2	104	—	—	—	—
3	98	526	—	105.2	—
4	105	536	761	107.2	108.71
5	109	547	761	109.4	108.71
6	120	559	771	111.8	110.14
7	115	568	795	113.6	113.57
8	110	581	820	116.2	117.14
9	114	591	838	118.2	119.71
10	122	603	840	120.6	120.00
11	130	615	843	123.0	120.43
12	127	619	863	123.8	123.29
13	122	627	889	125.4	127.00
14	118	637	—	127.4	—
15	130	—	—	—	—
16	140	—	—	—	—

2½+ 2½

Commonly Made Error

- Few students find the calculation tedious in such type of questions.

Answering Tip

- Do not leave the question in the middle. Do calculation with little patience.

Q. 3. Calculate four-Yearly moving averages of number of students studying in a higher secondary school in a particular city from the following data. R&U [O.E.B.]

Year	Number of Students
2011	124
2012	120
2013	135
2014	140
2015	145
2016	158
2017	162
2018	170
2019	175

Sol. Computation of four-yearly moving averages:

Year	Sales	4- yearly centered moving total	4- yearly moving Average	4- yearly centered moving Average
2011	124	—	—	—
2012	120	—	—	—
		519	129.75	
2013	135	—		132.37
		540	135.00	
2014	140	—		139.75
		578	144.50	
2015	145	—		147.87
		605	151.25	
2016	158	—		155.00
		635	158.75	
2017	162	—		162.52
		665	166.25	
2018	170	—	—	—
2019	175	—	—	—

5

Q. 4. From the following data calculate the 4- yearly moving averages and determine the trend values.

AI [O.E.B.]

Years	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Value	50.0	36.5	43.0	44.5	38.9	38.9	32.6	41.7	41.1	33.8

Sol. Calculation of Trend values by it four yearly Moving Averages:

Year	Value	4- yearly centered Moving Total	4- yearly Moving Average (Trend values)	4- yearly centered Moving Average
2012	50.0	—		
2013	36.5	—		
		174.0	43.5	
2014	43.0	—		42.12
		162.9	40.73	
2015	44.5	—		41.03
		165.3	41.33	

2016	38.9	—		40.03
		154.9	38.73	
2017	38.9	—		38.38
		152.1	38.03	
2018	32.6	—		38.31
		154.3	38.58	
2019	41.7	—		37.94
		149.2	37.3	
2020	41.1	—		
2021	33.8	—		

Q. 5. Below are given the figures of production (in m. tonnes) of a rice factory:

Year	2012	2013	2014	2015	2016	2017	2018
Production (in m. tonnes)	80	90	92	83	94	99	92

(i) Fit the straight line trend to these figures and calculate trend values.

(ii) Estimate the likely sales of the company during 2019. **[O.E.B.]**

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

To fit a straight line by the method of least squares, Rohan constructed the following table:

Year (t)	Profit (y)	$x = t_i - 2018$	x^2	xy
2015	114	– 3	9	– 342
2016	130	– 2	4	– 260
2017	126	– 1	1	– 126
2018	144	0	0	0
2019	138	1	1	138
2020	156	2	4	312
2021	164	3	9	492

The trend equation can be considered as $y_t = a + bx$.

Q. 1. The value of 'a' in the trend equation is:

(A) 138.86 **(B)** 7.64

(C) 183.86 **(D)** 7.46

Ans. Option (A) is correct.

Explanation:

$$a = \frac{\Sigma y}{n}$$

Here, $n = 7$ and $\Sigma y = 972$

So, $$a = \frac{972}{7}$$

$$= 138.86$$

Q. 2. The value of 'b' in the trend equation is:

(A) 138.86 **(B)** 7.64

(C) 183.86 **(D)** 7.46

Ans. Option (B) is correct.

Explanation:

$$b = \frac{\Sigma xy}{\Sigma x^2}$$

Here, $n = 7$, $\Sigma xy = 214$

and $\Sigma x^2 = 28$

So, $$b = \frac{214}{28}$$

$$= 7.64$$

Q. 3. The trend equation is:

(A) $y_t = 138.86 + 7.64x$ **(B)** $y_t = 7.64 + 138.86x$

(C) $y_t = 138.86 - 7.64x$ **(D)** None of these

Ans. Option (A) is correct.

Explanation: Since, $a = 138.86$ and $b = 7.64$

Thus, trend equation is given by

$$y_t = 138.86 + 7.64x$$

Q. 4. The trend value for year 2015 is:

(A) 161.78 **(B)** 154.14

(C) 115.94 **(D)** 138.86

Ans. Option (C) is correct.

Explanation: For year 2015, $x = -3$

$\therefore$ $y_t = 138.86 + 7.64 (-3)$

$= 115.94$

Q. 5. The trend value for year 2018 is:

(A) 161.78 **(B)** 154.14

(C) 115.94 **(D)** 138.86

Ans. Option (D) is correct.

Explanation: For year 2018, $x = 0$

$\therefore \quad y_t = 138.86 + 7.64\,(0)$

$= 138.86$

II. Read the following text and answer the following questions on the basis of the same:

Today in class Mr. Agarwal is teaching the Method of Least Squares to measure the trend in time series. After explaining the method, he had taken an example which is as follows:

Given below are the data relating to the sales of a product in a district.

Fit a straight line trend by the method of least squares and tabulate the trend values.

Year	2015	2016	2017	2018	2019	2020	2021	2022
Sales	6.7	5.3	4.3	6.1	5.6	7.9	5.8	6.1

To solve the given example, he constructed the following table:

Computation of trend values by the method of least squares.

Year (t)	Sales (y)	$x = \frac{(t - 2018.5)}{0.5}$	XY	x^2
2015	6.7	– 7	– 46.9	49
2016	5.3	– 5	– 26.5	25
2017	4.3	– 3	– 12.9	9
2018	6.1	– 1	– 6.1	1
2019	5.6	1	5.6	1
2020	7.9	3	23.7	9
2021	5.8	5	29.0	25
2022	6.1	7	42.7	49

Here, $n = 8$ (even)

So, origin is mean of two middle years

i.e., $\frac{2018 + 2019}{2} = 2018.5$

Q. 1. If the straight line is $y_t = a + bx$, then value of 'a' is:

(A) 5.615 **(B)** 5.759

(C) 5.829 **(D)** 5.975

Ans. Option (D) is correct.

Explanation: $\Sigma y = 6.7 + 5.3 + 4.3 + 6.1 + 5.6 + 7.9 + 5.8 + 6.1$

$a = \frac{\Sigma y}{n}$

Here, $n = 8$ and $\Sigma y = 47.8$

So, $a = \frac{47.8}{8}$

$= 5.975$

Q. 2. If the straight line is $y_t = a + bx$, then value of 'b' is:

(A) 0.051 **(B)** 0.011

(C) 0.411 **(D)** None of these

Ans. Option (A) is correct.

Explanation: $\Sigma xy = -46.9 - 26.5 - 12.9 - 6.1 + 5.6 + 23.7 + 29.0$

$= 8.6$

$b = \frac{\Sigma xy}{\Sigma x^2}$

Here, $n = 7$ and $\Sigma xy = 8.6$ and $\Sigma x^2 = 168$

So, $b = \frac{8.6}{168}$

$= 0.051$

Q. 3. The trend equation is:

(A) $y_t = 5.975 + 0.051\,x$

(B) $y_t = 0.051 + 5.975\,x$

(C) $y_t = 5.975 - 0.051\,x$

(D) None of the above

Ans. Option (A) is correct.

Explanation:

Since, $a = 5.975$

and $b = 0.051$

Thus, trend equation is given by

$y_t = 5.975 + 0.051\,x$

Q. 4. The trend value for year 2015 is:

(A) 5.691 **(B)** 5.618

(C) 5.987 **(D)** 5.719

Ans. Option (B) is correct.

Explanation:

For year 2015, $x = -7$

$\therefore \quad y_t = 5.975 + 0.051\,(-7)$

$= 5.618$

Q. 5. The trend value for year 2016 is:

(A) 5.451 **(B)** 5.616

(C) 5.966 **(D)** 5.720

Ans. Option (D) is correct.

Explanation:

For year 2016, $x = -5$

$\therefore \quad y_t = 5.975 + 0.051\,(-5)$

$= 5.975 - 0.255$

$= 5.720$

III. Read the following text and answer the following questions on the basis of the same:

The following data shows the percentage of rural, urban and suburban Indians who have a high speed internet connection at home:

Year (t)	Rural	Urban	Suburban
2017	3	9	9
2018	6	18	17
2019	9	21	23
2020	16	29	29
2021	24	38	40

Q. 1. A straight line trend by the method of least square for the Rural Indians is:

(A) $y_t = 11.6 + 5.2\,x$ **(B)** $y_t = 23 + 6.9\,x$

(C) $y_t = 23.6 + 7.4\,x$ **(D)** $y_t = 13.2 + 4.5\,x$

Ans. Option (A) is correct.

Explanation:

Year (t)	Rural (y)	$x = t_i - 2019$	x^2	xy
2017	3	– 2	4	– 6
2018	6	– 1	1	– 6
2019	9	0	0	0
2020	16	1	1	16
2021	24	2	4	48
	$\sum y = 58$		$\sum x^2 = 10$	$\sum xy = 52$

Here, $n = 5$, $a = \dfrac{\Sigma y}{n}$ and $b = \dfrac{\Sigma xy}{\Sigma x^2}$

So, $a = \dfrac{58}{5} = 11.6$

and $b = \dfrac{52}{10} = 5.2$

Thus, trend equation is given by $y_t = a + bx$

i.e., $y_t = 11.6 + 5.2\,x$

Q. 2. A straight line trend by the method of least square for the Urban Indians is:

(A) $y_t = 11.6 + 5.2\,x$ **(B)** $y_t = 23 + 6.9\,x$ **(C)** $y_t = 23.6 + 7.4\,x$ **(D)** $y_t = 13.2 + 4.5\,x$

Ans. Option (B) is correct.

Explanation:

Year (t)	Urban (y)	$x = t_i - 2019$	x^2	xy
2017	9	– 2	4	– 18
2018	18	– 1	1	– 18
2019	21	0	0	0
2020	29	1	1	29
2021	38	2	4	76
	$\sum y = 115$		$\sum x^2 = 10$	$\sum xy = 69$

Here, $n = 5$, $a = \dfrac{\Sigma y}{n}$ and $b = \dfrac{\Sigma xy}{\Sigma x^2}$

So, $a = \frac{115}{5} = 23$

and $b = \frac{69}{10} = 6.9$

Thus, trend equation is given by $y_t = a + bx$

i.e., $y_t = 23 + 6.9\,x$

Q. 3. A straight line trend by the method of least square for the Suburban Indians is:

(A) $y_t = 11.6 + 5.2\,x$ **(B)** $y_t = 23 + 6.9\,x$ **(C)** $y_t = 23.6 + 7.4\,x$ **(D)** $y_t = 13.2 + 4.5\,x$

Ans. Option (C) is correct.

Explanation:

Year (t)	Suburban (y)	$x = t_i - 2019$	x^2	xy
2017	9	– 2	4	– 18
2018	17	– 1	1	– 17
2019	23	0	0	0
2020	29	1	1	29
2021	40	2	4	80
	$\sum y = 118$		$\sum x^2 = 10$	$\sum xy = 74$

Here, $n = 5$, $a = \frac{\Sigma y}{n}$ and $b = \frac{\Sigma xy}{\Sigma x^2}$

So, $a = \frac{118}{5} = 23.6$

and $b = \frac{74}{10} = 7.4$

Thus, trend equation is given by $y_t = a + bx$

i.e., $y_t = 23.6 + 7.4\,x$

Q. 4. What is the forecast for the year 2022 for urban group using trend equation ?

(A) 58.6% **(B)** 43.7%

(C) 37.2% **(D)** 27.2%

Ans. Option (B) is correct.

Explanation: For year 2022, $x = 3$

The trend equation for urban is

$y_t = 23 + 6.9\,x$

So, at $x = 3$,

we get $y_t = 23 + 6.9\,(3)$

$= 43.7\ \%$

Q. 5. What is the forecast for the year 2022 for rural group using trend equation ?

(A) 58.6% **(B)** 43.7%

(C) 37.2% **(D)** 27.2%

Ans. Option (D) is correct.

Explanation:

For year 2022, $x = 3$

The trend equation for rural is

$y_t = 11.6 + 5.2\,x$

So, at $x = 3$,

we get $y_t = 11.6 + 5.2\,(3)$

$= 27.2\ \%$

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:

Each sub-part carries 1 mark.

Strikes is a very powerful weapon used by trade unions and other labour associations to get their demand accepted. It generally involves quitting of work by a group of workers for the purpose of bringing the pressure on their employer so that their demands get accepted. When workers collectively cease to work in a particular industry, they are said to be on strike.

The table give below shows the average number in lakhs, of working days lost in strikes during each year of period 2009-2018 was

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Sales	1.5	1.8	1.9	2.2	2.6	3.7	6.2	6.4	3.6	5.4

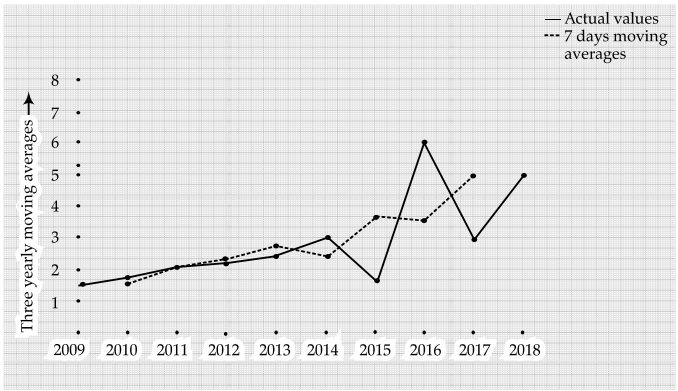
Q. 1. Calculate three yearly moving average.

Sol.

Year	Working days lost in Strikes (in lakhs)	Three yearly moving totals	Three yearly moving averages
2009	1.5	—	—
2010	1.8	5.2	1.73
2011	1.9	5.9	1.96
2012	2.2	6.7	2.23
2013	2.6	8.5	2.83
2014	3.7	8.5	2.83
2015	2.2	12.3	4.1
2016	6.4	12.2	4.06
2017	3.6	15.4	5.13
2018	5.4	—	—

Q. 2. Draw the graphs for moving averages.

Sol.



II. Read the following text and answer the following questions on the basis of the same:

Researchers have found a strong correction between attendance for school and academic performance and success. Children that miss school frequently often fall behind-both academically as well as professionally. Absence from school is often the biggest single cause of poor performance and achievement. As a parent, it is crucial to convey the importance of regular attendance to our child and how missing out can damage their future. The table given below shows the daily attendance in thousands at a certain school over a period of two weeks:

Week 1:	52	48	64	68	52	70	72
Week 2:	55	47	61	65	58	75	81

Q. 1. Calculate seven days moving averages.

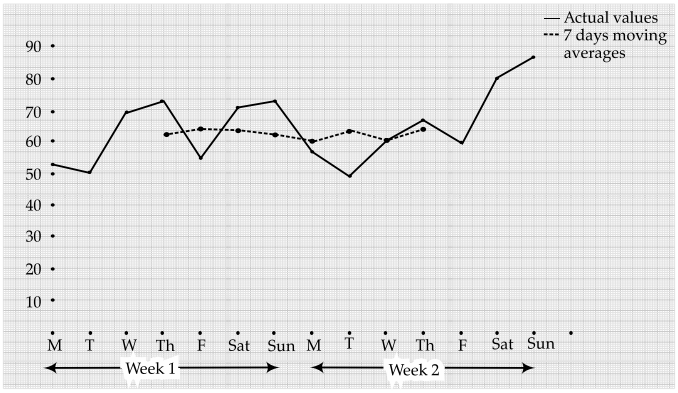
Sol. **Calculation of seven days moving averages**

Days		Attendance (in thousand)	7 days moving totals	7 days moving averages
	Monday	52	—	—
Week 1	Tuesday	48	—	—
	Wednesday	64	—	—
	Thursday	68	426	60.86
	Friday	52	429	61.28
	Saturday	70	428	61.14
	Sunday	72	425	60.71
Week 2	Monday	55	422	60.28
	Tuesday	47	428	61.14
	Wednesday	61	433	61.86
	Thursday	65	442	63.14
	Friday	58	—	—
	Saturday	75	—	—
	Sunday	81	—	—

2

Q. 2. Draw the graphs for moving averages.

Sol.



2

Solutions for Practice Questions

Very Short Answer Type Questions

Sol. 4: Business forecasting refers to the tool and techniques used to predict developments in business, such as sales, expenditures and profits. The purpose of business forecasting is to develop better strategies based on these informed predictions. **1**

Short Answer Type Questions-I

Sol. 2: We should study time series for the following reasons:

(i) It helps in the analysis of past behaviour.

(ii) It helps in forecasting and for future plans.

(iii) It helps in the evaluation of current achievements.

(iv) It helps in making comparative studies between one time period and others.

Therefore time series helps us to study and analyse the time-related data which involves in business fields, economics, industries, etc. **2**

Short Answer Type Questions-II

Sol. 2: Computation of three-yearly moving averages

Years	Profit (₹)	3- yearly moving Total (₹)	3- yearly moving averages (₹)
2016	15,420	—	—
2017	15,470	46,410	15,470
2018	15,520	52,010	17,336.667
2019	21,020	63,040	21,013.333
2020	26,500	79,470	26,490
2021	31,950	94,050	31,350
2022	35,600	1,02,450	34,150
2023	34,900	—	—

The last column gives the trend of profits. **3**

Long Answer Type Questions

Sol.5: (i) Filling straight line trend

Year (t)	Production (y)	$x = t - 2015$	xy	x^2	Trend Values (y_t)
2012	80	– 3	– 240	9	84
2013	90	– 2	– 180	4	86
2014	92	– 1	– 92	1	88
2015	83	0	0	0	90
2016	94	1	94	1	92
2017	99	2	198	4	94
2018	92	3	276	9	96
$n = 7$	$\Sigma y = 630$	$\Sigma x = 0$	$\Sigma xy = 56$	$\Sigma x^2 = 28$	

2

Now, $$a = \frac{\Sigma y}{n} = \frac{630}{7} = 90$$

$$b = \frac{\Sigma xy}{\Sigma x^2} = \frac{56}{28} = 2$$

Thus, trend equation is : $y_t = 90 + 2x$ **1**

Trend values are

$y_{2012} = 90 + 2(-3) = 84$
$y_{2013} = 90 + 2(-2) = 86$
$y_{2014} = 90 + 2(-1) = 88$ **2**
$y_{2015} = 90 + 2(0) = 90$
$y_{2016} = 90 + 2(1) = 92$
$y_{2017} = 90 + 2(2) = 94$
$y_{2018} = 90 + 2(2) = 96$

(ii) Now,

For 2019, x would be 4.

Putting $x = 4$ in trend equation, we get

$y_{2019} = 90 + 2(4) = 98$ **2**

REFLECTIONS

(a) Will you be able to find the moving average for 8 years?

(b) Can you solve the questions based on least square method easily?.

(c) Will you be able to relate the different components of times series in real time situations ?

SELF ASSESSMENT PAPER - 06

Maximum Time: 30 **Max. Time: 1 Hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. A set of observations recorded at an equal interval of time is called
(A) Array data **(B)** Data **(C)** Geometric Series **(D)** Time series data

Q. 2. A rise in prices before Eid is an example of
(A) Cyclical Trend **(B)** Secular Trend **(C)** Irregular Trend **(D)** Seasonal Trend

Q. 3. Seasonal variations are
(A) Short term variation **(B)** Long term variation **(C)** Sudden variation **(D)** None of these

Q. 4. In moving average method we cannot find trend values of some
(A) End Periods **(B)** Middle Period **(C)** Between extreme periods **(D)** Starting Periods

Q. 5. Multiplicative model for time series is Y = ...
(A) T × S × C × I **(B)** T + S + C + I **(C)** T – S – C – I **(D)** None of these

Q. 6. If the demand is 100 during October 2019, 200 in November 2019, 300 in December 2019, 400 in January 2020. What is the 3-months simple average for December 2019 ?
(A) 300 **(B)** 350 **(C)** 400 **(D)** Data insufficient

Case-based

Q. 7. Read the following text and answer the following questions on the basis of the same:
Attempt any four parts. Each part carries 1 mark.

Today in the class of Mathematics Mr. R.S. Chauhan is teaching the Method of least squares for obtaining the secular trend in time series analysis. He writes down the following on the board.

Method of Least Squares: The line of best fit is a line from which the sum of the deviations of various points is zero. This is the best method for obtaining the trend values. It gives a convenient basis for calculating the line of best fit for the time series. It is a mathematical method for measuring trend. Further the sum of the squares of these deviations would be least when compared with other fitting methods. So, this method is known as the Method of Least Squares and satisfies the following conditions:

- The sum of the deviations of the actual values of Y and $\bar{Y}$ (estimated value of Y) is zero. *i.e.*, $(Y - \bar{Y}) = 0$.
- The sum of squares of the deviations of the actual values of Y and (estimated values of Y) is least, that is $(Y - \bar{Y})^2$ is least.
- The straight line trend is represented by the equation

$$Y = a + bX \quad ...(1)$$

where, Y is the actual value, X is time, a, b are constants.

- The constants 'a' and 'b' are estimated by solving the following two normal Equations.

$$\sum Y = na + b\sum X \quad ...(2)$$

$$\sum Y = a\sum X + b\sum X^2 \quad ...(3)$$

where, 'n' = number of years given in the data.

- By taking the mid-point of the time as the origin, we get

$$\sum X = 0$$

- When $\sum X = 0$, the two normal equations reduces to

$$\sum Y = na + b(0);\ a = \frac{\sum Y}{n} = \bar{Y}$$

$$\sum XY = a(0) + b\sum X^2;\ b = \frac{\sum XY}{\sum X^2}$$

The constant 'a' gives the mean of Y and 'b' gives the rate of change (slope).

- By substituting the values of 'a' and 'b' in the trend equation (1), we get the Line of Best Fit.

Now, the teacher writes the following tabular data and told the students answer the following questions:

Year	2014	2015	2016	2017	2018	2019	2020
Sales (in ₹ '000)	26	26	44	42	108	120	166

(i) In straight equation $y = a + bX$; a is .....
(A) X-intercept **(B)** Y-intercept **(C)** Slope **(D)** none

(ii) The value of a is:
(A) 76 **(B)** 24 **(C)** 28 **(D)** 56

(iii) The value of b is:
(A) 76 **(B)** 28 **(C)** 24 **(D)** 56

(iv) For the given data, the equation of straight line trend is:
(A) $y_t = 76 + 24x$ **(B)** $y_t = 24 + 76x$ **(C)** $y_t = a + bx$ **(D)** none of these

(v) The trend value of the year 2020 is
(A) 145 **(B)** 145 **(C)** 154 **(D)** none of these

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. How irregular variations are caused ?

Q. 9. Write down the two important factors causing seasonal variations.

Q. 10. What are cyclical fluctuations ?

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. Give four examples of secular trend.

Q. 12. Give four examples of seasonal variations.

Q. 13. Write two merits of moving averages method.

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. Apply the method of least square to obtain the trend values from the following data:

Year	Sales (in lakh tonnes)	Year	Sales (in lakh tonnes)
2016	100	2019	140
2017	120	2020	80
2018	110		

Q. 15. Calculate the trend values by the method of least squares from the data given below:

Year	Sales of T.V sets (in lakhs)
2013 – 2014	12
2014 – 2015	18
2015 – 2016	20
2016 – 2017	23
2017 – 2018	27

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 20. Calculate the 5-yearly and 7-yearly moving average for the following data of a number of commercial industrial failures in a country during 2003-2018.

Year	No. of failures
2003	23
2004	26
2005	28
2006	32
2007	20
2008	12
2009	12
2010	10
2011	09
2012	13
2013	11
2014	14
2015	12
2016	09
2017	03
2018	01

CHAPTER

11

FINANCIAL MATHEMAT-ICS

Syllabus

Perpetuity, Sinking funds, Calculation of EMI, Calculation of returns, Nominal rate of return, Compound annual growth rate, Linear method of depreciation.

In this chapter you will study

- *Explain the concept of perpetuity and sinking fund • Calculate perpetuity • Differentiate between sinking fund and saving account*
- *Explain the concept of EMI • Calculate EMI using various methods*
- *Explain the concept of rate of return and nominal rate of return • Calculate rate of return and nominal rate of return*
- *Understand the concept of Compound Annual Growth Rate • Differentiate between Compound Annual Growth Rate and Annual Growth Rate • Calculate Compound Annual Growth Rate*
- *Define the concept of linear method of Depreciation • Interpret cost, residual value and useful life of an asset from the given information • Calculate depreciation*

Revision Notes

Scan to know more about this topic

Perpetuity

Perpetuity: Perpetuity in the financial system is a situation where a stream of cash flow payments continues indefinitely or is an **annuity** that has no end. In valuation analysis, perpetuities are used to find the present value of a company's future projected cash flow stream and the company's terminal value. Essentially, a perpetuity is a series of cash flows that keep paying out forever.

Key Words

Annuity: A fixed sum of money paid to someone each year, typically for the rest of their life.

- **Perpetuity Formula:** There are two different annual perpetual valuations; perpetuity with flat or constant annuity and perpetuity with a growing annuity. Perpetuity gives a business the value of its cash flow, an essential slice of data as it aids in determining the firm's total cash flow in a single year.

(1) Flat Perpetuity: Perpetuity is also known as regular flat perpetuity. In this cash flow is constant for each year. This perpetuity formula is the simplest, and it is straightforward as it doesn't include terminal value. It is the basic formula for the price of perpetuity. Calculate the present value of flat perpetuity we only need to divide the cash flows/payments by the discount rate.

Present value of Perpetuity

$$= \frac{\text{Cash flow}}{\text{Interest rate or yield}}$$

(2) Growing Perpetuity: A growing perpetuity is the same as regular or flat perpetuity, but the difference is that the cash flow is growing each year. The present value

Financial Mathematics

EMI

An equated monthly installment (EMI) is a fixed payment amount mode by a borrower to a lender at a specified date of each calender month.

I. Flat rate method $EMI = P\left(it\frac{1}{n}\right)$

II. Reduce balanced method $EMI = \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$

where, i = monthly interest rate [annual rate /1200]
n = number of installments
P = principal amount of loan

Perpetuity

A perpetuity is a type of annuity that lasts forever.
e.g.: In a real state sector when an owner purchases a property and then rents it out. This owner is entitled to an infinite stream of cash flow from the renter as long as property continues to exist.
It is of 2 types :

1. Flat perpetuity $= \frac{\text{Cash flow}}{\text{Interest rate or yield}}$

2. Growing perpetuity $= \frac{\text{Payment}}{\text{Interest rate - Growth rate}}$

Nominal Rate of Return

The nominal rate of return is the total rate of return earned on investment before adjusting for any deductions or remiums.

$$\text{Nominal rate of return} = \frac{(\text{Ending value of investment} - \text{Begning value of investment})}{\text{Begning value of investment}} \times 100$$

e.g. Suppose Ram has invested ₹10,000 on 1st feb. 2020 on 31 Jan. 2021, calculate NROR, if investment grown to 20,000

Sol. $NROR = \frac{`20,000 - `10,000}{`10,000} \times 100$

$= 100\%$

Sinking Fund

A sinking fund is a fund containing money set aside.

$$\text{Sinking fund A} = \frac{\left[1+\left(\frac{r}{m}\right)^{n\times m}\right]-1}{\left(\frac{r}{m}\right)} \times P$$

where P = periodic payments
r = annualized rate of interest
n = No. of year
m = No. of payments per year

Depreciation and Linear Method

Depreciation is an accounting practice of spreading the cost of an asset purchased across its useful lifetime.
Linear Method for Calculating Depreciation:
Annual Depreciation Expense =
(Asset cost – Residual value)/Useful life of the asset
e.g.: Seema wants to buy a mobile phone for ₹ 5000. Based on the post experience she estimates a 3 year life span and a salvage value of ₹ 200, then

$$\text{Annual depreciation expenses} = \frac{`(5000-200)}{3}$$

$= `1600$

Compounded Annual Growth Rate

CAGR indicates the average rate of growth when reinvest the return over a number of years.

$$CAGR = \left(\frac{End\ value}{begining\ value}\right)^{\frac{1}{n}} - 1$$

where n = investment period

e.g.:

	Date	Value
Begining value	3-01-2010	2,000/-
End value	3-07-2017	5,000/-

$$CAGR = \left(\frac{5000}{2000}\right)^{\frac{1}{7}} - 1 = 1.1398 - 1$$

$= 0.1398$

$= 13.98\%$

How to understand Mind Map?

- First Level
- Second Level
- Third Level

of growing perpetuity formula factors in long term growth. This version is used to calculate the terminal value in a stream of cash flows for valuation purposes is always more complicated.

Present value of Perpetuity

$$= \frac{\text{Payment}}{\text{Interest rate} - \text{Growth rate}}$$

Example 1 : Company 'ABC' pays ₹ 2 in dividends annually and estimates that they will pay the dividends indefinitely. How much are investors willing to pay for the dividend with a required rate of return of 5% ?

Sol. Given, Payment amount = ₹ 2, Interest rate or yield = 5% or 0.05

Present value of Perpetuity

$$= \frac{\text{cash flow}}{\text{Interest rate or yield}}$$

$$= \frac{2}{0.05} = ₹\,40$$

An investor will consider investing in the company if the stock price is ₹ 40 or less.

Taking the above example, imagine if the ₹ 2 dividend is expected to grow annually by 2%.

Given, Growth = 2% or 0.02.

Present value of Perpetuity

$$= \frac{\text{Payment}}{\text{Interest rate} - \text{Growth rate}}$$

$$= \frac{2}{0.05 - 0.02} = ₹\,66.67$$

- **Sinking Fund:** A sinking fund is a type of fund that is created and set up purposely for repaying debt. The owner of the account sets aside a certain amount of money regularly and uses it only for a specific purpose. Often, it is used by corporations for bonds and deposit money to buy back issued bonds or parts of bonds before the maturity date arrives. It is also one way of enticing investors because the fund helps convince them that the issuer will not default on their payments.

Scan to know more about this topic

Sinking fund

The formula to calculate the sinking fund is given below

$$\text{Sinking Fund, } A = \frac{\left[\left(1+\left(\frac{r}{m}\right)^{n\times m}\right)\right]-1}{\left(\frac{r}{m}\right)} \times P$$

where P = Periodic contribution to the sinking fund

r = Annualized rate of interest

n = No. of years

m = No. of payments per year

Example 2 : A sinking fund with a monthly periodic contribution of ₹ 1,500. The fund will be required to retire a newly taken debt (zero-coupon bonds) rose for the ongoing expansion project. Do the calculation of the amount of the sinking fund if the annualized rate of interest is 6%, and the debt will be repaid in 5 years. Given that $(1.005)^{60} = 1.3489$

Sol. Given, P = ₹ 1,500, r = 6% or 0.06, No. of years, n = 5 years and No. of payments per year, m = 12

$$\text{Sinking Fund, } A = \frac{\left[\left(1+\left(\frac{r}{m}\right)^{n\times m}\right)\right]-1}{\left(\frac{r}{m}\right)} \times P$$

$$A = \frac{\left[\left(1+\left(\frac{0.06}{12}\right)^{5\times 12}\right)\right]-1}{\left(\frac{0.06}{12}\right)} \times 1500$$

$$= \frac{1.3489-1}{0.005} \times 1500$$

$$= \frac{532.35}{0.005} = ₹\,104{,}670$$

Key Facts

- Financial mathematics, also called analytical finance, mathematical finance and mathematical finance, is an inter-disciplinary subject of mathematics and finance.
- French mathematician Louis Bachelier is considered the author of the first scholarly work on mathematical finance, published in 1900. But mathematical finance emerged as a discipline in the 1970s, following the work of Fischer Black, Myron Scholes and Robert Merton on option pricing theory.

Advantages of sinking fund:

The following are the advantages of sinking fund:

(1) A sinking fund helps the company to pay its liability well in advance.

(2) A company is able to pay the debt in time because a company has already pulled a money well before.

(3) A sinking fund is also used to redeem the bond or any other liability in a mid-way also.

(4) A sinking fund also increases the goodwill of the company by paying the debt in time, and this will increase the faith of investors and attract more investment.

➢ **EMI (Equated Monthly Instalment):** An equated monthly instalment (EMI) is a fixed payment amount made by a borrower to a lender at a specified date each calendar month. Equated monthly instalments are used to pay off both interest and principal each month so that over a specified number of years, the loan is paid off in full. With most common types of loans—such as real estate mortgages, auto loans, and student loans—the borrower makes fixed periodic payments to the lender over the course of several years with the goal of retiring the loan.

The EMI system of loan repayment has following features:

(1) Each instalment contains both components of principal repayment and interest charges.

(2) Interest is calculated on reducing balance method.

(3) Interest component is higher in the beginning and progressively lower towards the end. That mean, the principal component of an EMI is lower during initial periods and higher during later periods.

(4) The amount of EMI depends on:

(i) The period of compounding *i.e.*, whether the compounding is yearly, half yearly, quarterly or monthly. If the compounding is more frequent, then the amount of EMI would be higher and vice-versa.

(ii) The rate of interest.

(iii) Period of repayment if the repayment period is more, then EMI would be lower and vice versa.

EMI can be calculated by two methods:

(i) Flat Rate method (FRM)

(ii) Reducing Balance Method (RBM)

Flat Rate Method: In flat rate method the principal amount remains same throughout the tenure and the interest is changed on it at a constant rate throughout the loan tenure: Suppose an amount of ₹ P is borrowed at flat rate r per rupee per month for a period of n months.

Then, Interest $= Pin$

$$\therefore \quad \text{EMI} = \frac{\text{Principal + Interest}}{n}$$

$$\text{or,} \quad \text{EMI} = \frac{P + Pni}{n} = P\left(\frac{l + in}{n}\right)$$

$$= P\left(i + \frac{1}{n}\right)$$

Example 3 : Navisha lakes a loan of ₹ 3,00,000 at an interest 10% compounded annually for a period of 3 years. Find EMI using flat rate method.

Sol. We have, P = Principal = ₹ 3,00,000

i = rate of interest per rupee per month

$$= \frac{10}{1200} = \frac{1}{120}$$

n = Number of instalments = 36

$$\therefore \quad \text{EMI} = ₹\ P\left(i + \frac{1}{n}\right)$$

$$= ₹\ 3,00,000\left(\frac{1}{120} + \frac{1}{36}\right)$$

$$= ₹\ (250 + 833.33)$$

$$= ₹\ 1083.33$$

Reducing Balance Method: In reducing balance method principal paid back gets deducted from the outstanding loan amount and the interest for the subsequent year is changed on the remaining deducted balance and not on the entire loan amount unlike the flat rate method.

The formula for EMI for Reduce Balance Method is:

$$\text{EMI} = \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$$

where $i = \dfrac{\text{monthly interest rate annual}}{12 \times 100}$

n = number of Instalments,

P = principal amount of the loan

Example 4 : Ram took a home loan of ₹ 50 lakhs at an 8.50 % interest for a 20 years loan tenure. What would be his EMI ? Given $(1.0070)^{240} = 5.3342$

Sol. Given, $$i = \left[\frac{\left(\frac{\text{annual rate}}{12}\right)}{100}\right]$$

$$= \left[\frac{\left(\frac{8.50}{12}\right)}{100}\right]$$

$$= \frac{0.70833}{100}$$

$$= 0.0070$$

$$n = 20 \times 12 = 240$$

$$P = ₹\ 50,00,000$$

$$\text{Instalment Amount} = \frac{(1+0.0070)^{240}}{(1+0.0070)^{240} - 1} \times (50,00,000 \times 0.0070)$$

$$= \frac{5.3342}{5.3342-1} \times 35,000$$

$$= \frac{186,697}{4.3342}$$

$$= ₹\ 43,075.308 \sim ₹\ 43,075$$

- **Calculation of Returns:** All financial decisions involve same risk one may expect to get a return of 15% per annum in his investment but the risk of not able to achieve 15% return will always be there. Return is simply a reward for investing as all investing involves some risk.

  A debt investment is a loan, and the return is just the loans interest rate. This is simply the ratio of the interest paid to the loan principal

$$\text{Return, } k = \frac{\text{interest paid}}{\text{loan amount}}$$

  This formulation leads to the convenient idea that a return is what the investor receives divided by what he or she invests.

- **Rate of Return (ROR) or Nominal Rate of Return (NROR):** A Rate of Return (ROR) is the gain or loss of an investment over a certain period of time. In other words, the rate of return is the gain (or loss) compared to the cost of an initial investment, typically expressed in the form of a percentage. When the ROR is positive, it is considered a gain and when the ROR is negative, it reflects a loss on the investment.

  Scan to know more about this topic

  Nominal rate of return

  The standard formula for calculating ROR is as follows:

  Rate of Return

$$= \frac{\text{Ending value of investment} - \text{Initial value of investment}}{\text{Beginning value of investment}} \times 100$$

- **Compounded Annual Growth Rate (CAGR):** Compounded annual growth rate (CAGR) depicts the cumulative performance of a particular variable over a significant period of time and is used to measure relative profitability of businesses. CAGR is often associated with specific parameters which indicate the performance of a company over a fixed period, such as sales, revenue, earnings, etc.

$$\text{CAGR} = \left(\frac{\text{Final value}}{\text{Initial value}}\right)^{1/n} - 1$$

  Where n = investment period

  Example 5 : Aritra invested ₹ 1,000 in a fund for three years. While the total Net Asset value (NAV) remained ₹ 1,000 for the first year, it increased to ₹ 1,100 in the second year. Upon maturity of this fund, final NAV stood at ₹ 1,300. Find compounded annual growth rate. Given that $(1.3)^{1/3} = 1.0913$.

Sol.

Year	NAV (in ₹)
I	1000
II	1100
III	1300

$$\text{CAGR} = \left(\frac{\text{Final value}}{\text{Initial value}}\right)^{1/n} - 1$$

$$= \left(\frac{13,00}{1,000}\right)^{\frac{1}{3}} - 1 = 1.0913 - 1$$

$$= 0.0913 = 9.13\%$$

- **Depreciation:** Depreciation allows a portion of the cost of a fixed asset to the revenue generated by the fixed asset. This is mandatory under the matching principle as revenues are recorded with their associated expenses in the accounting period when the asset is in use. This helps in getting a complete picture of the revenue generation transaction.

  Example: If a delivery truck is purchased a company with a cost of ₹ 100,000 and the expected usage of the truck are 5 years, the business might depreciate the asset under depreciation expense as ₹ 20,000 every year for a period of 5 years.

  There are three methods commonly used to calculate depreciation. They are:

  (1) Linear or Straight line method
  (2) Unit of production method
  (3) Double-declining balance method

  Three main inputs are required to calculate depreciation:

  Useful life: This is the time period over which the organisation considers the fixed asset to be productive. Beyond its useful life, the fixed asset is no longer cost-effective to continue the operation of the asset.

  Salvage value: Post the useful life of the fixed asset, the company may consider selling it at a reduced amount. This is known as the salvage value of the asset.

  The cost of the asset: This includes taxes, shipping, and preparation/setup expenses.

  Here, we have discuss only Linear or Straight line method for calculating depreciation.

  - **Linear or Straight-line Depreciation Method:** This is the simplest method of all. It involves simple allocation of an even rate of depreciation every year over the useful life of the asset. The formula for straight line depreciation is:

    Scan to know more about this topic

    Linear method of depreciation

    Annual Depreciation Expense

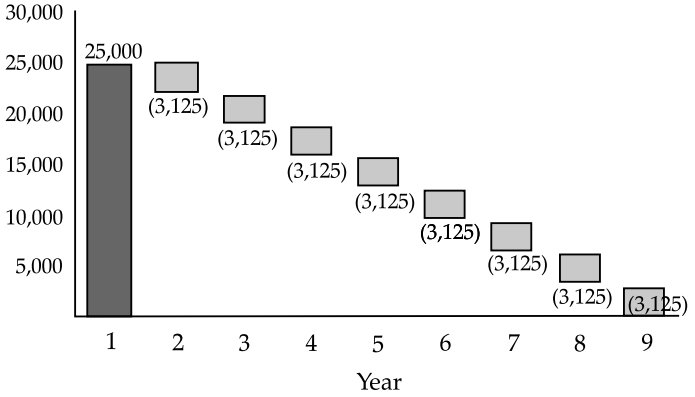
$$= \frac{\text{Asset cost} - \text{Residual value}}{\text{Useful life of the asset}}$$

  Example 6 : Consider a piece of equipment that costs ₹ 25,000 with an estimated useful life of 8 years and a ₹ 0 salvage value. The depreciation expense per year for this equipment would be as follows:

Year#		1	2	3	4	5	6	7	8
Straight Line									
Opening Book Value		25,000	21,875	18,750	15,625	12,500	9,375	6,250	3,125
Depreciation	8	3,125	3,125	3,125	3,125	3,125	3,125	3,125	3,125
Ending Book Value	25,000	21,875	18,750	15,625	12,500	9,375	6,250	3,125	

$$\text{Depreciation Expense} = \frac{₹\,25,000 - ₹\,0}{8} = ₹\,3,125 \text{ per year}$$

Straight Line Depreciation



Mnemonics

Concept: Perpetuity

Mnemonics: "A **perpetuity** is a series of cash flows that keep paying out forever."

Interpretations: Per + Pet *i.e.*, Per Every Pet (animal etc.).

So that it can be linked to the statement "every pet will last in every stage of life" *i.e.*, Per Every Stage.

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

Q. 1. The present value of a sequence of payment of ₹ 1000 made at the end of every 6 months and continuing forever, if money is worth 8% per annum compounded semi-annually is:

(A) 1000 **(B)** 2500

(C) 25,000 **(D)** 15,000

Ans. Option (C) is correct.

Explanation: The given annuity is a perpetuity.

$$\text{present value of perpetuity} = \frac{\text{Cash flow}}{\text{Interest rate}}$$

Here, cash flow $= ₹\,1000$

$$\text{interest rate} = \frac{8/2}{100} = \frac{4}{100} = 0.04$$

So, $$\text{present value} = \frac{1000}{0.04}$$

$$= ₹\,25,000$$

Q. 2. A machine costing ₹ 50,000 has a useful life of 4 years. The estimated scarp value is ₹ 10,000, then the annual depreciation is:

(A) ₹ 20,000 **(B)** ₹ 10,000

(C) ₹ 5,000 **(D)** ₹ 2,500

Ans. Option (B) is correct.

Explanation: Annual depreciation

$$= \frac{\text{Original cost} - \text{Scrap value}}{\text{Useful life}}$$

$$= \frac{50,000 - 10,000}{4}$$

$$= \frac{40,000}{4}$$

$$= ₹\ 10,000$$

Q. 3. The value of a machine purchased two years ago, depreciates at the annual rate of 10%. If its present value is ₹ 97,200, then its value after 3 years is:

(A) ₹ 70,859 approx **(B)** ₹ 71,895 approx

(C) ₹ 80,859 approx **(D)** ₹ 88,509 approx

Ans. Option (A) is correct.

Explanation: Given, $P = ₹\ 97,200$, $i = 10\%$ p.a.

$$\Rightarrow \quad i = \frac{10}{100} = 0.1$$

So, value after 3 years

$$= 97,200 \times (1 - 0.1)^3$$
$$= 97,200 \times 0.729$$
$$= ₹\ 70,858.80$$
$$= ₹\ 70,859 \text{ (approx.)}$$

Q. 4. The sum of money is needed now, so as to get ₹ 6000 at the beginning of every month forever, then the money is worth 6% per annum compounded monthly is:

(A) ₹ 1000000 **(B)** ₹ 1206000

(C) ₹ 60,000 **(D)** ₹ 1600000

Ans. Option (B) is correct.

Explanation: Given $R = ₹\ 6000$,

$$r = \frac{6}{12}\% = 0.5 \text{ per month}$$

So, $$i = \frac{0.5}{100} = 0.005$$

So, $$P = R + \frac{R}{i}$$

$$= 6000 + \frac{6000}{0.005}$$

$$= 6000 + 1200000$$

$$= ₹\ 12,06,000$$

Q. 5. The present value of perpetuity of ₹ 600 payable at end of every 6 month be ₹ 10,000. Find rate of interest.

(A) 10% **(B)** 12%

(C) 6% **(D)** 12.5%

Ans. Option (B) is correct.

Explanation: Let rate of interest be $r\%$ per annum,

then $$i = \frac{r}{200}$$

Given, $R = ₹\ 600$ and $P = ₹\ 10,000$

Using $$P = \frac{R}{i}$$

$$\Rightarrow \quad i = \frac{R}{P} = \frac{600}{10,000}$$

$$\Rightarrow \quad \frac{r}{200} = \frac{600}{10,000}$$

$$\Rightarrow \quad r = \frac{120,000}{10,000}$$

$$= 12\%$$

Q. 6. Assume that the year-end revenues of a business over a three period, are mentioned in the following table:

Year End	31-12-2018	31-12-2021
Year End Revenue	9,000	13,000

Calculate the CAGR of revenues over, three-years period spanning the "end" of 2018 to the "end" of 2021. Given that $\left(\frac{13}{9}\right)^{\frac{1}{3}} = 1.13$

(A) 13% **(B)** 14%

(C) 15% **(D)** None of these

Ans. Option (A) is correct.

Explanation: The CAGR of the revenues over the three years period spanning the "end" of 2018 to "end" of 2021 is

$$\left(\frac{\text{Final value}}{\text{Initial Value}}\right)^{\frac{1}{n}} - 1 = \left(\frac{13000}{9000}\right)^{\frac{1}{3}} - 1$$

$$= 1.13 - 1$$
$$= 0.13$$
$$= 13\%$$

Q. 7. Assume that Shyam holds a perpetual bond that generates an annual payment of ₹ 500 each year. He believes that the borrower is creditworthy and that an 8% interest rate will be suitable for this bond. The present value of this perpetuity is:

(A) ₹ 6520 **(B)** ₹ 6250

(C) ₹ 5620 **(D)** ₹ 2650

Ans. Option (B) is correct.

Explanation: *PV* of perpetuity

$$= \frac{\text{Annual Payment/Cash flow}}{\text{Interest rate/yield}}$$

$$= \frac{500}{\frac{8}{100}}$$

$$= \frac{500}{0.08}$$

$$= ₹\ 6250$$

Q. 8. A company buys a machine at a cost of ₹ 5000. The company decides on a salvage value of ₹ 1000 and a useful life of 5 years, then annual depreciation cost is:

(A) ₹ 500 **(B)** ₹ 600

(C) ₹ 700 **(D)** ₹ 800

Ans. Option (D) is correct.

Explanation: We know that

Annual Depreciation expense

$$= \frac{(\text{Asset cost} - \text{Residual Value})}{\text{Useful life of the asset}}$$

$$= \left(\frac{₹\,5000 - ₹\,1000}{5}\right)$$

$$= ₹\,800$$

Q. 9. Reema has an initial investment of ₹ 1,00,000 in an investment plan. After 5 years, it has grown to 2,00,000 then rate of return is:

(A) 50% **(B)** 100%

(C) 75% **(D)** 200%

Ans. Option (B) is correct.

Explanation: Rate of Return

$$= \frac{2,00,000 - 1,00,000}{1,00,000} \times 100$$

$$= 100\%$$

Q. 10. EMI depends on the following factors:

(A) The amount of loan

(B) The loan tenure

(C) The interest rate

(D) All of these

Ans. Option (D) is correct.

Q. 11. If C is the original cost of an asset, S is the salvage value of the asset and n is the number of year estimated for useful life of the asset, the annual depreciation of the asset is given by

(A) $D = \frac{C-S}{n}$ **(B)** $D = \frac{S-C}{n}$

(C) $D = \frac{C+S}{n}$ **(D)** $D = \frac{S}{C} \times n$

Ans. Option (C) is correct.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. Mr. Roy purchased a home for ₹ 200,000 in 2012. In the next few years, homes in neighbourhood have been selling well due to the new shopping plaza a couple of miles away, which increased the market value of his home. So in 2017, he decided to downsize and sell his home. Based on the current market value during this time, he was able to sell his home for ₹ 250,000. Find his rate of return. R

[O.E.B.]

Q. 2. What is useful life? R **[O.E.B.]**

Sol. The time period in which an asset is expected to be functional and fit for purpose is called useful life. It is generally measured in years. **1**

Q. 3. What is scrap value? R **[O.E.B.]**

Sol. The value of a depreciable asset at the end of its useful life is called scrap value or salvage value or depreciated value. **1**

Q. 4. Suppose a manufacturing company purchases a machinery for ₹ 100,000 and the useful life of the machinery are 10 years and the residual value of the machinery is ₹ 20,000. Calculate depreciation.

AI **[O.E.B.]**

Sol. We know that,

Annual Depreciation expense

$$= \frac{\text{Asset cost} - \text{Residual Value}}{\text{Useful life of the asset}}$$

$$= \frac{₹\,100,000 - ₹\,20,000}{10}$$

$$= ₹\,8,000$$ **1**

Q. 5. Your grandfather is retiring at the end of next year. He would like to ensure that his heirs receive payments of ₹ 75,000 a year forever, starting when he retires. How much does he need to invest in the beginning of this year to produce the desired cash flow, if money is worth 6% compounded annually. U

Sol. Given, $R = ₹\,75,000$ and $i = \frac{6}{100} = 0.06$

Present value, $P = \frac{R}{i}$

$\therefore$ $P = \frac{75,000}{0.06}$

$= ₹\,12,50,000$ **1**

Hence, your grandfather has to invest ₹ 12,50,000.

Q. 6. Find the present value of a perpetuity of ₹ 3,120 payable at the beginning of each year, if money is worth 6% effective. AI

Sol. Given, $R = ₹\ 3{,}120$ and $i = \frac{6}{100} = 0.06$

Present value, $P = R + \frac{R}{i}$

$= ₹\left(3{,}120 + \frac{3120}{0.06}\right)$

$= ₹\ (3120 + 52{,}000)$

$= ₹\ 55{,}120$ **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. A company ABC Ltd has raised funds in the form of 1,000 zero-coupon bonds worth ₹ 1,000 each. The company wants to set up a sinking fund for repayment of the bonds, which will be after 10 years. Determine the amount of the periodic contribution if the annualized rate of interest is 5%, and the contribution will be done half-yearly. Given that $(1.025)^{20} = 1.6386$. R&U [O.E.B.]

Sol. Sinking Fund, $A = ₹\ 1{,}000 \times 1000 = ₹\ 1{,}000{,}000$, $r = 5\%$ or 0.05, No. of years, $n = 10$ years and No. of payments per year, $m = 2$ (Half Yearly)

Periodic Contribution, $P = \frac{A \times \left(\frac{r}{m}\right)}{\left[\left(1+\left(\frac{r}{m}\right)\right)^{n\times m}\right]-1}$ **1**

$$P = \frac{1{,}000{,}000 \times \left(\frac{0.05}{2}\right)}{\left[\left(1+\left(\frac{0.05}{2}\right)\right)^{10\times 2}\right]-1}$$

$$= \frac{1{,}000{,}000 \times 0.025}{1.6386 - 1}$$

$$= \frac{25{,}000}{0.6386}$$

$= ₹\ 39{,}148.136 \sim ₹\ 39{,}148$ **1**

Therefore, the company will be required to contribute a sum of ₹ 39,148 half-yearly in order to build the sinking fund to retire the zero-coupon bonds after 10 years.

Q. 2. A company borrowed ₹ 60,000 for renovation. The company plans to set up a sinking fund that will pay back the loan at the end of 5 years. Assuming a rate of 10% compounded semiannually, and the sinking fund of the ordinary annuity. Given that $(1.05)^{10} = 1.06288$. AI [O.E.B.]

Sol. Given, $P = ₹\ 60{,}000$, $r = 10\%$ or 0.10, No. of years, $n = 5$ years and No. of payments per year, $m = 2$ (Semi annually)

Sinking Fund, $A = \frac{\left[\left(1+\left(\frac{r}{m}\right)\right)^{n\times m}\right]-1}{\left(\frac{r}{m}\right)} \times P$ **1**

$$A = \frac{\left[\left(1+\left(\frac{0.10}{2}\right)\right)^{5\times 2}\right]-1}{\left(\frac{0.10}{2}\right)} \times 60{,}000$$

$$= \frac{[(1.05)^{10}]-1}{0.05} \times 60{,}000$$

$$= \frac{0.06288}{0.05} \times 60{,}000$$

$= ₹\ 754{,}673.55 \sim ₹\ 754{,}673$ **1**

Q. 3. Mr. Lal took a car loan of ₹ 10 lakhs at an 11% interest for a 15 years loan tenure. What would be his EMI ? Given $(1.0091)^{180} = 5.1069$. A [O.E.B.]

Sol. Given, $i = \left[\frac{\left(\frac{\text{annual rate}}{12}\right)}{100}\right]$

$$= \left[\frac{\left(\frac{11}{12}\right)}{100}\right]$$

$$= \frac{0.9166}{100}$$

$$= 0.0091$$

$n = 15 \times 12 = 180$

$P = ₹\ 10{,}00{,}000$ **1**

Instalment Amount $= \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$

$$= \frac{(1+0.0091)^{180}}{(1+0.0091)^{180} - 1} \times (10{,}00{,}000 \times 0.0091)$$

$$= \frac{5.1069}{5.1069 - 1} \times 9100$$

$$= \frac{46.472.79}{4.1069}$$

$= ₹\ 11,315.78 \sim ₹\ 11{,}316$ **1**

Q. 4. Anna owns a produce truck, invested ₹ 700 in purchasing the truck, some other initial admin related and insurance expenses of ₹ 1500 to get the business going, and has now a day to day expense of ₹ 500/p.m. Consider hypothetically that her everyday profit is ₹ 550/p.m. (ideally, it will be based on sales). At the end of 6 months, Anna takes up her accounts and calculates her rate of return. AI [O.E.B.]

Q. 5. Assume that the year-end revenues of a business over a three-year period, are mentioned in the following table:

Year-End	31-12-2017	31-12-2020
Year End Revenue	9,000	13,000

These questions are for practice and their solutions are available at the end of the chapter

Calculate the CAGR of the revenues over the three-years period spanning the "end" of 2017 to the "end" of 2020. Given that $\left(\frac{13}{9}\right)^{\frac{1}{3}} = 1.13$. R&U

[O.E.B.]

Sol. The CAGR of the revenues over the three-year period spanning the "end" of 2017 to the "end" of 2020 is:

$$\left(\frac{\text{Final value}}{\text{Initial vlaue}}\right)^{\frac{1}{n}} - 1 = \left(\frac{13000}{9000}\right)^{\frac{1}{3}} - 1 \quad \textbf{1}$$

$$= 1.13 - 1$$
$$= 0.13$$
$$= 13\% \quad \textbf{1}$$

Q. 6. ₹ 5000 is invested in a Term Deposit Scheme that fetches interest 6% per annum compounded quarterly. What will be interest after one year ?

Given that $(1.015)^4 = 1.0613$ R&U [O.E.B.]

Sol. We know that,

Compound interest,

$$I = P[(1 + i)^n - 1] \quad \textbf{1}$$

Here, P = ₹ 5000,

$$i = 6\% \text{ p.a.}$$
$$= 0.06 \text{ p.a.}$$
$$= 0.06 \times (1/4) \text{ per quarter}$$
$$= 0.015 \text{ per quarter}$$

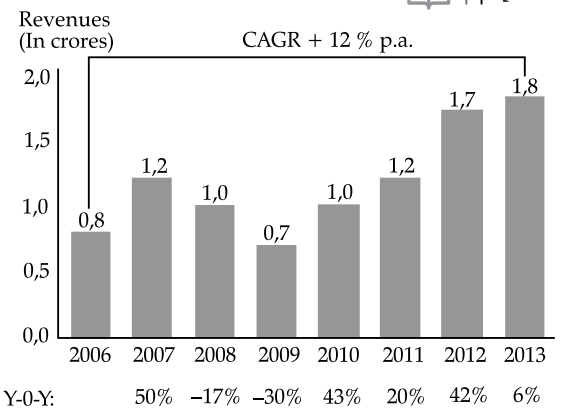
$\therefore$ $I = 5000[(1 + 0.015)^4 - 1]$

$$= 5000\{(1.015)^4 - 1\}$$
$$= 5000(1.0613 - 1)$$
$$= 5000 \times 0.0613$$
$$= 306.82 \quad \textbf{1}$$

Short Answer Type Questions-II (3 marks each)

Q. 1. An interviewer gives the following graph on a client's sales in the last 7 years to candidate and said find the CAGR. Given that $\left(\frac{9}{4}\right)^{\frac{1}{7}} = 1.1228$.

A I [O.E.B.]



Sol. Sales in 2006 were 0.8 crores (beginning value). In 2013, after 7 years, sales increased to 1.8 crores.

$$\text{CAGR} = \left(\frac{\text{End value}}{\text{Beginning value}}\right)^{\frac{1}{n}} - 1 \quad \textbf{1}$$

$$= \left(\frac{1.8}{0.8}\right)^{\frac{1}{7}} - 1 = \left(\frac{9}{4}\right)^{\frac{1}{7}} - 1 \quad \textbf{1}$$

$$= 0.1228$$
$$\text{CAGR\%} = 12.28\% \quad \textbf{1}$$

Q. 2. A firm anticipates an expenditure of ₹ 50,0000 for plant modernization at end of 10 years from now. How much should the company deposit at the end of year into a sinking fund earning interest 5% per annum. [Given log 1.05 = 0.0212, antilog (0.2120) = 1.629] A I R&U [O.E.B.]

Sol. Given, A = ₹ 5,00,000, $r = 5\%$ and $n = 10$

Using formula, $A = p\left[\frac{(1+i)^n - 1}{i}\right]$ **1**

where $i = \frac{r}{100}$

$\Rightarrow$ $500000 = p\left[\frac{(1+0.05)^{10} - 1}{0.05}\right]$

$\Rightarrow$ $p = \frac{500000 \times 0.05}{(1.05)^{10} - 1}$ **1**

Now, let $x = (1.05)^{10}$

Taking log both sides, we get

$$\log x = 10 \log (1.05)$$
$$= 10 \times 0.0212$$
$$= 0.2120$$

$\Rightarrow$ $x = \text{antilog } (0.2120)$

$$= 1.629$$

Thus, $(1.05)^{10} = 1.629$

Now, $p = \frac{500000 \times 0.05}{1.629 - 1}$

$$= \frac{25000}{0.629}$$
$$= 39745.63 \quad \textbf{1}$$

Hence, the company should deposit ₹ 39745.63 every year into the sinking fund.

Q. 3. Rohan purchased a laptop worth ₹ 80000. He paid ₹ 20,000 as cash down and balance in equal monthly instalments in 2 years. If bank charges 9% p.a. compounded monthly. Calculate the EMI. [Given $(1.0075)^{24} = 1.1964$] R&U [O.E.B.]

Sol. Cost of laptop = ₹ 80,000

Down payment = ₹ 20,000

$\therefore$ Balance = ₹ 60,000

So, P = ₹ 60,000,

$$i = \frac{9}{12 \times 100}$$
$$= 0.0075$$

and $n = 2 \times 12 = 24$ **1**

$$\therefore \quad \text{EMI} = \frac{P \times i \times (1+i)^n}{(1+i)^n - 1} \qquad 1$$

$$= \frac{60,000 \times 0.0075 \times (1.0075)^{24}}{(1.0075)^{24} - 1}$$

$$= \frac{60,000 \times 0.0075 \times 1.1964}{1.1964 - 1}$$

$$= \frac{450 \times 1.1964}{0.1964}$$

$$= ₹\ 2741.24 \qquad 1$$

Q. 4. Riya invested ₹ 20,000 in a mutual fund in year 2016. The value of mutual fund increased to ₹ 32,000 in year 2021. Calculate the compound annual growth rate of her investment. [Given, log(1.6) = 0.2041, antilog (0.04082) = 1.098] R&U **[O.E.B.]**

Q. 5. A company XYZ borrowed ₹ 1.5 lakhs for reformation. The company plans to set up a sinking fund that will pay back the loan at the end of 3 years. Assuming a rate of 9% compounded quarterly, and the sinking fund of the ordinary annuity. Given that $(1.0225)^{12} = 1.3060$.

Sol. Given, $P = ₹\ 1,50,000$, $r = 9\%$ or 0.09,
No. of years, $n = 3$ years
and No. of payments per year, $m = 4$(quarterly) 1

$$\text{Sinking Fund, A} = \frac{\left[\left(1+\left(\frac{r}{m}\right)\right)^{n \times m}\right] - 1}{\left(\frac{r}{m}\right)} \times P \qquad 1$$

$$= \frac{\left[\left(1+\left(\frac{0.09}{4}\right)\right)^{12}\right] - 1}{\left(\frac{0.09}{4}\right)} \times 1,50,000$$

$$= \frac{[(1+0.0225)^{12}] - 1}{0.0225} \times 1,50,000$$

$$= \frac{[(1.0225)^{12}] - 1}{0.0225} \times 1,50,000$$

$$= \frac{1.3060 - 1}{0.0225} \times 1,50,000$$

$$= \frac{0.3060}{0.0225} \times 1,50,000$$

$$= ₹\ 2,040,000 \qquad 1$$

Q. 6. A textile company has raised funds in the form of 5,000 zero-coupon bonds worth ₹ 1,100 each. The company wants to set up a sinking fund for repayment of the bonds, which will be after 7 years. Determine the amount of the periodic contribution if the annualized rate of interest is 6%, and the contribution will be done quarterly. [Given $(1.015)^{28} = 1.5172$] U

Sol. Given, Sinking fund,
A = Par value of bond × No. of bonds
$= ₹\ 1,100 \times 5,000 = ₹\ 5,500,000$
No. of years, $n = 7$
No. of payments per year, $m = 4$
Annualized rate of interest, $r = 6\%$

$$\therefore \quad i = \frac{6}{400} = 0.015 \qquad 1$$

Therefore, the amount of the periodic contribution can be calculated using the above formula as,

$$P = A\left[\frac{i}{(1+i)^{n \times m} - 1}\right]$$

$$= 55,00,000\left[\frac{0.015}{(1+0.015)^{28} - 1}\right]$$

$$= 55,00,000\left[\frac{0.015}{(1.015)^{28} - 1}\right]$$

$$= 55,00,000\left[\frac{0.015}{1.5172 - 1}\right]$$

$$= 55,00,000\left[\frac{0.015}{0.5172}\right]$$

$$= 55,00,000 \times 0.0290$$

$$= ₹\ 159512.76 \approx ₹\ 159513 \qquad 2$$

Long Answer Type Questions (5 marks each)

Q. 1. The cost of a TV depreciates by ₹ 800 during the second year and by ₹ 700 during the third year. Calculate:
(i) the rate of depreciation per annum.
(ii) the original cost of the machine.
(ii) the value of the TV at the end of third year.

AI **[O.E.B.]**

Sol. (i) Let the original cost of the TV be ₹ P and the rate of depreciation be $r\%$ p.a. Then the value of TV (in ₹) after 1 year, 2 years and 3 years are $P(1 - i)$, $P(1 - i)^2$ and $P(1 - i)^3$, respectively, where $i = \frac{r}{100}$.

According to question,

$$P(1-i) - P(1-i)^2 = 800$$

and $$P(1-i)^2 - P(1-i)^3 = 700$$

$$\Rightarrow \quad P(1-i)\,[1-(1-i)] = 800$$

and $$P(1-i)^2\,[1-(1-i)] = 700$$

$$\Rightarrow \quad P(1-i)i = 800 \qquad \text{...(i)}$$

and $$P(1-i)^2 i = 700 \qquad \text{...(ii)}$$

On dividing eq. (ii) by eq (i), we get

$$1 - i = \frac{700}{800}$$

$$\Rightarrow \quad 1 - i = \frac{7}{8}$$

$$\Rightarrow \quad i = 1 - \frac{7}{8}$$

$$\Rightarrow \quad i = \frac{1}{8}$$

These questions are for practice and their solutions are available at the end of the chapter

$\therefore \quad \frac{r}{100} = \frac{1}{8}$

$\Rightarrow \quad r = \frac{100}{8} = 12.5\%$

Hence, the rate of depreciation = 12.5% p.a. **2**

(ii) Putting $i = \frac{1}{8}$ in eq. (i), we get

$$P\left(1-\frac{1}{8}\right)\times\frac{1}{8} = 800$$

$$\Rightarrow \quad P\times\frac{7}{8\times 8} = 800$$

$$\Rightarrow \quad P = \frac{800\times 8\times 8}{7}$$

$$= 7314.28$$

Hence, the original cost of TV is ₹ 7314. **1**

(iii) The value of TV at the end of 3rd year $= P(1-i)^3$

$$= 7314\left(1-\frac{1}{8}\right)^3$$

$$= 7314\left(\frac{7}{8}\right)^3$$

$$= \frac{7314\times 7\times 7\times 7}{8\times 8\times 8}$$

$$= 4899.80 \approx ₹\ 4900$$

Thus, the value of TV at the end of the 3rd year is ₹ 4900. **2**

Q. 2. A machine costing ₹ 200000 has effective life 7 years and its scrap value is ₹ 30000. What amount should the company put into a sinking fund earning 5% per annum, so that it can replace the machine after its useful life ? Assume that a new machine will cost ₹ 300000 after 7 years. [Given log (1.05) = 0.0212 and antilog (0.1484) = 1.407] R&U **[O.E.B.]**

Q. 3. A person amortizes a loan of ₹ 1500000 for renovation of his house by 8 years mortgage at the rate of 12% p.a. compounded monthly. Find

(i) the equated monthly instalment

(ii) the principal outstanding at the beginning of 40th month.

(iii) the interest paid in 40th payment.

[Given $(1.01)^{96} = 2.5993$, $(1.01)^{57} = 1.7633$]

R&U **[O.E.B.]**

Sol. Given, $P = ₹\ 15{,}00{,}000$, $i = \frac{12}{12\times 100} = \frac{1}{100} = 0.01$

and $n = 8 \times 12 = 96$

(i) $$EMI = \frac{15{,}00{,}000\times 0.01\times(1.01)^{96}}{(1.01)^{96}-1}$$

$$= \frac{15{,}00{,}000\times 0.01\times 2.5993}{2.5993-1}$$

$$= \frac{15{,}00{,}000\times 0.01\times 2.5993}{1.5993}$$

$$= ₹\ 24{,}379.10$$ **1½**

(ii) Principal outstanding at the beginning of 40th month

$$= \frac{EMI[(1+i)^{96-40+1}-1]}{i(1+i)^{96-40+1}}$$

$$= \frac{24379.10\times[(1.01)^{57}-1]}{0.01(1.01)^{57}}$$

$$= \frac{24379.10\times(1.7633-1)}{0.01\times 1.7633}$$

$$= \frac{24379.10\times 0.7633}{0.017633}$$

$$= ₹\ 1{,}055{,}326.20$$ **2**

(iii) Interest paid in 40th payment

$$= \frac{EMI[(1+i)^{96-40+1}-1]}{(1+i)^{96-40+1}}$$

$$= \frac{24379.10[(1.01)^{57}-1]}{(1.01)^{57}}$$

$$= \frac{24379.10\times 0.7633}{1.7633}$$

$$= ₹\ 10553.26$$ **1½**

AI **Q. 4. (i) At what rate of interest will the present value of a perpetuity of ₹ 300 payable at the end of each quarter be ₹ 24000 ?**

(ii) What sum of money invested now could establish a scholarship of ₹ 5000 which is to be awarded at the end of every year forever, if money is worth 8% per annum.

(iii) Rehaan invested ₹ 7000 in a Term Deposit Scheme that fetches interest 7.5% per annum compounded semi-annually. What will be the interest after 3 years ? Given $(1.0375)^6 = 1.2472$ AI **[O.E.B.]**

Sol. (i) Let the rate of investment be r% per annum, then

$$i = \frac{r}{4\times 100} = \frac{r}{400}$$

Given, perpetuity, $R = ₹\ 300$

and payment, $P = ₹\ 24000$

Using, $P = \frac{R}{i}$

$\Rightarrow \quad i = \frac{R}{P}$

$\Rightarrow \quad \frac{r}{400} = \frac{300}{24000}$

$\Rightarrow \quad r = \frac{300\times 400}{24000} = 5\%$

Hence, rate of interest = 5% per annum. **2**

(ii) It is a flat perpetuity, so

Present value of perpetuity

$$= \frac{\text{Cash flow}}{\text{Interest rate}}$$

$$= \frac{5000}{\frac{8}{100}}$$

$$= \frac{5000 \times 100}{8}$$

$= ₹\ 62500$ **1**

(iii) We know that,

Compound interest

$I = P[(1 + i)^n - 1]$

$i = 7.5\%$ p.a.

$= 0.075$ p.a.

$= 0.075 \times \frac{1}{2}$ per six months

$= 0.0375$ per six months **1**

$I = 7000[(1 + 0.0375)^6 - 1]$

$= 7000[(1.0375)^6 - 1]$

$= 7000[1.2472 - 1]$

$= 7000 \times 0.2472$

$= ₹\ 1730.4$ **1**

COMPETENCY BASED QUESTIONS

Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same: AI

Rohan has completed his MBA and now he wants to start a new business. So, he approaches to many banks. One bank is agreed to give loan to Rohan. So, Rohan has borrowed ₹ 5 lakhs from a bank on the interest rate of 12 per cent for 10 years.

Q. 1. EMI stands for:

(A) Equated Monthly Instalments

(B) Emerging Monthly Instalments

(C) Easy Monthly Instalments

(D) None of the above

Ans. Option (A) is correct.

Q. 2. To calculate monthly instalment, we use the following formula:

(A) Instalment Amount $= \frac{(1+i)^n}{(1+i)^n} \times (P \times i)$

(B) Instalment Amount $= \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$

(C) Instalment Amount $= \frac{(1+i)^n}{(1+i)^{n-1}} \times (P \times i)$

(D) None of the above

Ans. Option (B) is correct.

Explanation: Formula to calculate monthly instalment is:

Instalment Amount $= \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$

Q. 3. Calculate monthly instalment using $(1.01)^{120} = 3.300$

(A) ₹ 7100 **(B)** ₹ 7174

(C) ₹ 7147 **(D)** ₹ 7200

Ans. Option (B) is correct.

Explanation:

Given, $i = \left[\frac{\left(\frac{\text{annual rate}}{12}\right)}{100}\right]$

$= \left[\frac{\left(\frac{12}{12}\right)}{100}\right]$

$= \frac{1}{100}$

$= 0.01$

$n = 10 \times 12$

$= 120$

$P = ₹\ 5{,}00{,}000$

Instalment Amount $= \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$

Instalment Amount $= \frac{(1+0.01)^{120}}{(1+0.01)^{120} - 1}$

$\times (5{,}00{,}000 \times 0.01)$

$= \frac{3.300}{3.300 - 1} \times 5{,}000$

$= \frac{16{,}500}{2.300}$

$= ₹\ 7173.91 \sim ₹7174$

So, EMI that Rohan has to pay is ₹ 7174

Q. 4. Find the amount of total payment made by Rohan.

(A) ₹ 8,60,88 **(B)** ₹ 8,80,880

(C) ₹ 8,60,000 **(D)** ₹ 8,60,880

Ans. Option (D) is correct.

Explanation: Total payment made by Rohan to the bank in 10 years = (EMI × Total tenure in months)

$= ₹\ (7174 \times 120)$

$= ₹\ 8{,}60{,}880$

Q. 5. Find the amount of interest paid by Rohan.

(A) ₹ 3,60,88 **(B)** ₹ 3,60,880

(C) ₹ 3,60,00 **(D)** ₹ 3,600,88

Ans. Option (B) is correct.

Explanation: The total interest amount payable will be (Total payment – loan amount)

$= ₹\ 8{,}60{,}880 - ₹\ 5{,}00{,}880$

$= ₹\ 3{,}60{,}880$

II. Read the following text and answer the following questions on the basis of the same:

Ajay is a service man. He lives in a joint family. There are 6 members in his family. He is planning to purchase a car so he is searching for a bank for a loan. He take a loan of ₹ 250000 at the interest rate of 6% p.a. compounded monthly is to be amortize by equal payment at the end of each month for 5 years. Given that $(1.005)^{60} = 1.3489$, $(1.005)^{21} = 1.1104$.

Q. 1. The size of each monthly payment is:

(A) ₹ 4833 **(B)** ₹ 8432

(C) ₹ 4383 **(D)** ₹ 4338

Ans. Option (A) is correct.

Explanation: Given, P = ₹ 25,000

$$i = \frac{6}{12\times100}$$

$$= 0.005$$

and $$n = 5 \times 12$$

$$= 60$$

$$\therefore \quad EMI = \frac{p\times i\times(1+i)^n}{(1+i)^n-1}$$

$$= \frac{25000\times0.005\times(1.005)^{60}}{(1.005)^{60}-1}$$

$$= \frac{25000\times0.005\times1.3489}{0.3489}$$

$$= ₹\ 4832.69$$

$$= ₹\ 4833$$

Q. 2. The principal outstanding at beginning of 40th month is:

(A) ₹ 60069 **(B)** ₹ 96097

(C) ₹ 96000 **(D)** ₹ 98670

Ans. Option (B) is correct.

Explanation: Principal outstanding at beginning of 40th month

$$= \frac{EMI\,[(1+i)^{n-40+1}-1]}{i(1+i)^{n-40+1}}$$

$$= \frac{4832.69[(1+0.005)^{60-40+1}-1]}{0.005(1+0.005)^{60-40+1}}$$

$$= \frac{4832.69[(1.005)^{21}-1]}{0.005\times(1.005)^{21}}$$

$$= \frac{4832.69[1.1104-1]}{0.005\times1.1104}$$

$$= \frac{4832.69\times0.1104}{0.005\times1.1104}$$

$$= ₹\ 96096.72$$

$$\approx ₹\ 96097$$

Q. 3. The interest paid in 40th payment is:

(A) ₹ 480 **(B)** ₹ 408

(C) ₹ 840 **(D)** ₹ 482

Ans. Option (A) is correct.

Explanation: Interest paid in 40th payment

$$= \frac{EMI\,[(1+i)^{n-40+1}-1]}{(1+i)^{n-40+1}}$$

$$= \frac{4832.69[(1+i)^{60-40+1}]}{(1+i)^{60-40+1}}$$

$$= \frac{4832.69\times[(1.005)^{21}-1]}{(1.005)^{21}}$$

$$= \frac{4832.69\times0.1104}{1.1104}$$

$$= ₹\ 480.48$$

$$\approx ₹\ 480$$

Q. 4. Principal amount paid in 40th payment is:

(A) ₹ 4325 **(B)** ₹ 4359

(C) ₹ 4352 **(D)** ₹ 4532

Ans. Option (C) is correct.

Explanation: Principal paid in 40th payment

= EMI – Interest paid in 40th payment

= 4832. 69 – 480.48

= ₹ 4352.21

≈ ₹ 4352

Q. 5. Total interest paid is:

(A) ₹ 39961 **(B)** ₹ 36991

(C) ₹ 31699 **(D)** ₹ 39691

Ans. Option (A) is correct.

Explanation: Total interest paid

= 60 × 4832.69 – 250000

= 289961.40 – 250000

= ₹ 39961.40

≈ ₹ 39961

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:
Each sub-part carries 2 marks.

Everyone needs money at every stage of their life. Sometimes it so happen that they have been desire to purchase their favourite stuff but they are incapable to purchase due to shortage of money. Loans are available for these purposes only. Loans are provided to people for such critical circumstances which many occur at any time. In order to fullfill the wish to purchase a new car, Anaya takes a loan of ₹ 5,00,000 form a bank at an interest rate of 6% p.a. for 10 years. She wants to pay back the loan in equated monthly instalments.

Q. 1. Find the EMI by using flat rate method.

Sol. we have,

P = Principal = ₹ 5,00,000

i = interest rate

$= \frac{6}{12\times 100}$

$= \frac{1}{200} = 0.005$

n = no. of period of n months

$= 12 \times 10 = 120$ ½

Using formula of Flat Rate Method, we have

$\text{EMI} = P\left(i+\frac{1}{n}\right)$ ½

$= ₹\ 5,00,000\left(0.005+\frac{1}{120}\right)$

$= ₹\ (2,500 + 4,166.67)$

$= ₹\ 6,666,67$ **1**

Q. 2. Find the EMI by reduce balance method.
[Given $(1.005)^{-120} = 0.54963$]

Sol. Formula for Reduce balance method is

$$\text{EMI} = \frac{(1+i)^n}{(1+i)^n-1}\times(P\times i) \quad \textbf{1}$$

or $$\text{EMI} = \frac{Pi}{1-(1+i)^{-n}}$$

or $$\text{EMI} = \frac{5,00,000\times 0.005}{1-(1+0.005)^{-120}}$$

$$= \frac{2500}{1-0.54963}$$

$$= \frac{2500}{0.45037}$$

$= ₹\ 5550.9914 \sim ₹\ 5551$ **1**

II. Read the following text and answer the following questions on the basis of the same:
Each sub-part carries 2 marks.

The decrease in monetary value of an asset over time due to use, wear and tear or obsolescence is called depreciation. The simplest and most commonly used method of computing depreciation is straight line or linear method of depreciation. In this method the depreciation amount is the same for every year over the useful life of the asset i.e., the depreciation amount charged until the asset gets reduced to zero, value or its salvage value at the end of its useful life. In linear depreciation method the annual depreciation of an asset is found by dividing the total depreciation by the number of year in its estimated useful life.

Q. 1. A mainframe computer whose cost is ₹ 5,00,000 will depreciate to a scrap value of ₹ 50,000 in 5 years. Using linear method of depreciation, find the book value of computer at the end of third year.

Sol. We have,

C = Original cost of computer = ₹ 5,00,000

S = Scrap value of computer = ₹ 50,000

n = Useful life of computer = 5 years

Let D be the annual depreciation. then

$$D = \frac{C-S}{n}$$

$$\Rightarrow \quad D = \frac{₹\ (1,50,000-50,000)}{5}$$

$= ₹\ 90,000$ **1**

$\therefore$ The book value at the end of third year

$= ₹\ (5,00,000 - 3 \times 90,000)$

$= ₹\ 5,00,000 - ₹\ 2,70,000 = ₹\ 2,30,000$ **1**

Q. 2. An asset costing ₹ 1,50,000 is expected to have a useful life of 5 years and scrap value of ₹ 30,000. Find the annual depreciation charge and the depreciation rate by using the linear depreciation method.

Sol. We have,

C = Original cost of asset = ₹ 150,000

S = Scrap value/salvage value = ₹ 30,000

n = Useful life = 5 years

The annual depreciation charge, D is given by

$$D = \frac{C-S}{n}$$

$$\Rightarrow \quad D = \frac{1,50,000-30,000}{5}$$

$$\Rightarrow \quad D = ₹\ 24,000 \quad \textbf{1}$$

Now, Depreciation Rate

$$= \frac{\text{Annual Depreciation}}{\text{Cost of the asset} - \text{Salvage value}}\times 100$$

$$= \frac{24,000}{1,50,000-30,000}\times 100$$

$$= \frac{24,000}{12,000}\times 100$$

$= 20\%$ **1**

Solutions for Practice Questions

Very Short Answer Type Questions

Sol. 1: Given, Current value = 250,000
Original value = 200,000
Rate of return

$$= \frac{\text{Current Value} - \text{Original Value}}{\text{Original Value}} \times 100$$

$$= \frac{250,000 - 200,000}{200,000} \times 100 = 25\% \quad \textbf{1}$$

Short Answer Type Questions-I

Sol. 4: Total Initial Investment: ₹ 700 + ₹ 1,500 = ₹ 2,200
Everyday Expenses: ₹ 500
Total Expenses for 6 months: ₹ 3,000
Everyday Returns: ₹ 550
Total Returns for 6 months: ₹ 3,300 ½
So, Rate of Return

$$= \left(\frac{\text{Total Returns} - \text{Total Expenses}}{\text{Total Initial Investment}}\right) \times 100 \quad \textbf{1}$$

$$= \left(\frac{₹\,3,300 - ₹\,3,000}{₹\,2,200}\right) \times 100$$

$$= 13.63\% \quad ½$$

Short Answer Type Questions-II

Sol. 4: Given beginning value of investment = ₹ 20,000
Final value of the investment = ₹32,000
No. of years = 5

So, $$\text{CAGR} = \left(\frac{\text{End Value}}{\text{Beginning Value}}\right)^{\frac{1}{n}} - 1$$

$$= \left(\frac{32000}{20000}\right)^{\frac{1}{5}} - 1$$

$$= (1.6)^{\frac{1}{5}} - 1$$

Let, $x = (1.6)^{\frac{1}{5}}$ **1½**

Taking log both sides, we get

$$\log x = \frac{1}{5}\log(1.6)$$

$$\Rightarrow \quad \log x = \frac{1}{5} \times 0.2041$$

$$\Rightarrow \quad \log x = 0.04082$$

$$\Rightarrow \quad x = \text{antilog } (0.04802)$$

$$= 1.098$$

$$\text{CAGR} = 1.098 - 1 = 0.098$$

$$= 9.8\% \quad \textbf{1½}$$

Long Answer Type Questions

Sol. 2: Cost of new machine = ₹ 3,00,000
Scrap value of old machine = ₹ 30,000
Hence, the money required for new machine after 7 years = ₹ 3,00,000 – ₹ 30,000 = ₹ 2,70,000

So, we have A = ₹ 2,70,000 $i = \frac{r}{100} = \frac{5}{100} = 0.05$,

$n = 7$ **1**

Using formula, $A = P\left[\frac{(1+i)^n - 1}{i}\right]$ we get **1**

$$2,70,000 = P\left[\frac{(1+0.05)^7 - 1}{0.05}\right]$$

$$\Rightarrow \quad P = \frac{2,70,000 \times 0.05}{(1.05)^7 - 1} \quad \textbf{1}$$

Now, let $(1.05)^7 = x$
Taking log both sides, we get

$$7\log(1.05) = \log x$$

$$\Rightarrow \quad \log x = 7 \times 0.0212 = 0.1484$$

$$\Rightarrow \quad x = \text{antilog } (0.1484) = 1.407$$

So, $(1.05)^7 = 1.407$

Thus, $$P = \frac{2,70,000 \times 0.05}{1.407 - 1}$$

$$\Rightarrow \quad P = \frac{13500}{0.407}$$

$$\Rightarrow \quad P = ₹\,33169.53 \quad \textbf{2}$$

Hence, the company should deposit ₹ 33169.53 at the end of each year for 7 years.

REFLECTIONS

(a) Will you be able to handle the complex situations in EMI ?
(b) Can you calculate the rate of returns easily?
(c) Will you be able to apply the concept of perpetuity in real time problems?

SELF ASSESSMENT PAPER - 07

Maximum Time: 30 **Max. Time: 1 Hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. At 7% converted quarterly, the present value of a perpetuity of ₹ 700 payable at the end of each quarter is:

(A) ₹ 30,000 **(B)** ₹ 45,000

(C) ₹ 40,000 **(D)** ₹ 50,000

Q. 2. How much money is needed to endure a series of lectures costing ₹ 1500 at the beginning of each year indefinitely, if money is worth 2% compounded annually?

(A) ₹ 76,200 **(B)** ₹ 76,500

(C) ₹ 67,500 **(D)** ₹ 67,700

Q. 3. The present value of a perpetual income of ₹ x at the end of each six months is ₹ 50,000. The value of x if money is worth 5% compounded semi-annually is:

(A) ₹ 1,000 **(B)** ₹ 1,200

(C) ₹ 1,250 **(D)** ₹ 1,350

Q. 4. An investment's starting value is ₹ 15,000 and it grows to ₹ 75,000 in 5 years, then its compound annual growth rate is: $[5.5^2 = 1.379]$

(A) 37% **(B)** 38%

(C) 39% **(D)** 40%

Q. 5. Mrs. Singh takes loan of ₹ 1,00,000 with 10% annual interest rate for 5 years. The EMI under the flat rate system is:

(A) ₹ 5,000 **(B)** ₹ 2,500

(C) ₹ 50,000 **(D)** ₹ 25,000

Q. 6. A washing machine costing ₹ 25,000 is expected to have a useful life of 7 years and a final scrap value of ₹ 2000. The annual depreciation charge using the straight line method is:

(A) 3,000 **(B)** 3,500

(C) 3,826 **(D)** 3,906

Case-based

Q. 7. Read the following text and answer the following questions on the basis of the same:

Attempt any four parts. Each part carries 1 mark.

Mr. Khanna is a business man. He runs his jewellery business in Mumbai. He plans to purchase a big budget car. So, he decided to took a car loan. He finally decided to took the car loan from ICICI Bank at the rate of 7.5% compounded monthly in year 2020, for 20 years. If the loan amount is ₹ 30,00,000, then [Given $(1.00625)^{240} = 4.4608$; $(1.00625)^{91}$ 1.7629; $(1.00625)^{51} = 1.3741$]

(i) The equated monthly instalment paid by Mr. Khanna was

(A) ₹ 24,168 **(B)** ₹ 35,288

(C) ₹ 37,839 **(D)** ₹ 39,437

(ii) Interest paid by Mr. Khanna in 150^{th} payment was:

(A) ₹ 20,132 **(B)** ₹ 15,712

(C) ₹ 10,459 **(D)** none of these

(iii) Principal paid by Mr. Khanna is 150^{th} payment was:

(A) ₹ 15,050 **(B)** ₹ 18,500

(C) ₹ 13,709 **(D)** ₹ 14,708

(iv) Principal outstanding at the beginning of 190^{th} month

(A) ₹ 10,53,755 **(B)** ₹ 12,58,755

(C) ₹ 18,53,456 **(D)** ₹ 10,83,798

(v) Total interest paid by Mr. Khanna was:

(A) ₹ 38,01,987 **(B)** ₹ 28,00,298

(C) ₹ 25,00,567 **(D)** ₹ 31,07,576

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. Find the present value of a sequence of payments of ₹ 80 made at the end of each 6 months and continuing forever, if money is worth 5% compounded semi-annually.

Q. 9. What sum of money is needed to invest now, so as to get ₹ 10,000 at the beginning of every month, if money is worth 5% per annum compounded monthly ?

Q. 10. What is discount rate ?

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. At what rate converted semi-annually will the present value of a perpetuity of ₹ 500 payable at the end of each 6 months be ₹ 25,000 ?

Q. 12. On 1st April, 2021, Rekha purchased a LED TV costing ₹ 50,000 and spent ₹ 5,000 on its erection. The estimated effective life of the LED TV is 5 years with a scrap value of ₹ 3000. Calculate the depreciation using linear method with accounting year ending on 31^{st} March 2022.

Q. 13. An investment made by Mr. Roy has initial value ₹ 10,000 and it increases to ₹ 40,000 in 3 years what will be CAGR ? [Given, $(4)^{1/3} = 1.587$]

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. A company establishes sinking fund to provide for the payment of ₹ 1,00,000 debt. maturing in 5 years. Contributions to the fund are to be made at the end of every year. Find the amount of each annual deposit if interest is 15% per annum. Given $[(1.15)^5 = 2.011]$

Q. 15. In 10 years, a machine costing ₹ 50,000 will have a salvage value of ₹ 5,000. A New Machine at that time is expected to sell for ₹ 55,000. In order to provide funds for the difference between the replacement cost and the salvage cost, a sinking fund is set up into which equal payments are placed at the end of each year. If the fund earns interest at the rate 6% compounded annually, how much should each payment be ? Given $[(1.06)^{10} = 1.7908]$

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 16. Mr. Kundan wants to purchase a car for ₹ 25,00,000 with down payment of ₹ 10,00,000. If he can amortize the balance at 8% per annum compounded monthly for 7 years.

(i) What is the monthly payment?

(ii) What is the total interest paid?

Given $(1.0066)^{84} = 1.7474$

CHAPTER

12

LINEAR PROGRAM-MING

Syllabus

Introduction and related terminologies, Mathematical formulation of linear programming problems, Different types of linear programming problems, Graphical method of solution for problems in two variables, Feasible and infeasible regions, Feasible and infeasible solutions, optimal feasible solutions.

In this chapter you will study

- *Familiarize with terms related to Linear Programming Problem*
- *Formulate Linear Programming Problem*
- *Identify and formulate different types of LPP*
- *Draw the Graph for a system of linear inequalities involving two variables and to find its solution graphically*
- *Identify feasible, infeasible, bounded and unbounded regions*
- *Understand feasible and infeasible solutions*
- *Find optimal feasible solution*

Revision Notes

Scan to know more about this topic

What is linear programming problem?

- Problems which minimize or maximize a linear function Z subject to certain conditions determined by a set of linear inequalities with non-negative variables are known as linear programming problems.
- **Objective function:** A linear function $Z = ax + by$, where a and b are constants which has to be maximized or minimized according to a set of given conditions, is called as linear objective function.
- **Decision variables:** In the objective function $Z = ax + by$, the variables x, y are said to be decision variables.
- **Constraints:** The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \geq 0$, $y \geq 0$ are known as non-negative restrictions.
- **Different types of linear programming problems:** A few important linear programming problems are as follows:

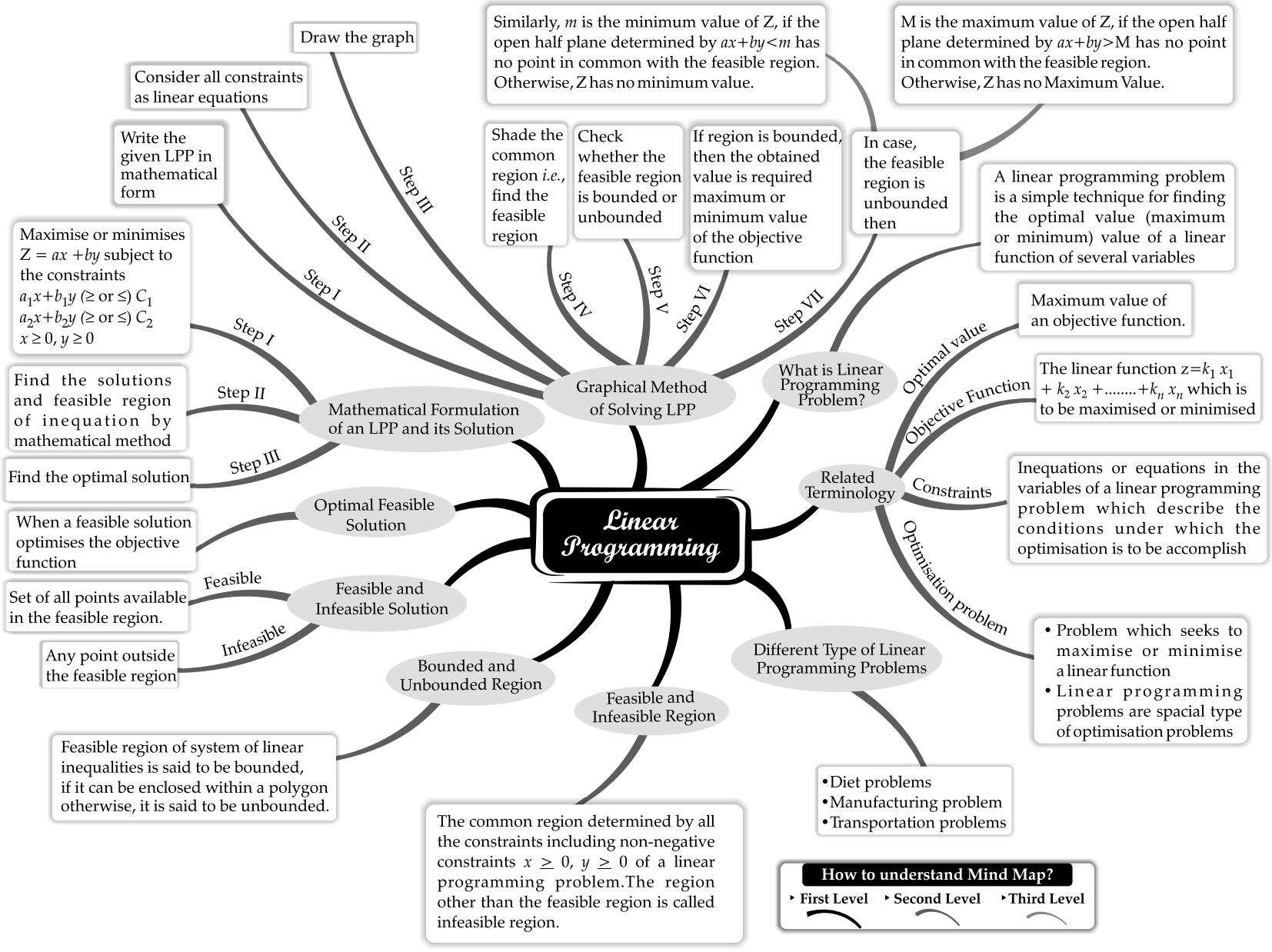
(i) Manufacturing problem: In such problem, we determine: (a) Number of units of different products to be produced and sold. (b) Manpower required, machines hours needed, warehouse space available, etc. Objective function is to maximize profit.

(ii) Diet problem: Here, we determine the amount of different types of constituent or nutrients which should be included in the diet. Objective function is to minimize the cost of production.

(iii) Transportation problem: These problems deal with the cost of transportation which is to be minimized under given constraints.

(iv) Assignment Problem: The assignment problem is a special case of transportation problem where the number of sources and destinations are equal. Supply at each source and demand at each destination must be one.

(v) Blending problem: To determine the optimum amount of several constituents used in producing a set of products while determining the optimum quantity of each product to be produced.


Linear Programming
Graphical Method of Solving LPP
Step I
Write the given LPP in mathematical form
Step II
Consider all constraints as linear equations
Step III
Draw the graph
Step IV
Shade the common region *i.e.*, find the feasible region
Step V
Check whether the feasible region is bounded or unbounded
Step VI
If region is bounded, then the obtained value is required maximum or minimum value of the objective function
Step VII
In case, the feasible region is unbounded then
M is the maximum value of Z, if the open half plane determined by $ax+by>M$ has no point in common with the feasible region. Otherwise, Z has no Maximum Value.
Similarly, m is the minimum value of Z, if the open half plane determined by $ax+by<m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.
Mathematical Formulation of an LPP and its Solution
Step I
Maximise or minimises $Z = ax + by$ subject to the constraints
a_1x+b_1y (≥ or ≤) C_1
a_2x+b_2y (≥ or ≤) C_2
$x \geq 0, y \geq 0$
Step II
Find the solutions and feasible region of inequation by mathematical method
Step III
Find the optimal solution
Optimal Feasible Solution
When a feasible solution optimises the objective function
Feasible and Infeasible Solution
Feasible
Set of all points available in the feasible region.
Infeasible
Any point outside the feasible region
Bounded and Unbounded Region
Feasible region of system of linear inequalities is said to be bounded, if it can be enclosed within a polygon otherwise, it is said to be unbounded.
Feasible and Infeasible Region
The common region determined by all the constraints including non-negative constraints $x \geq 0$, $y \geq 0$ of a linear programming problem.The region other than the feasible region is called infeasible region.
What is Linear Programming Problem?
A linear programming problem is a simple technique for finding the optimal value (maximum or minimum) value of a linear function of several variables
Related Terminology
Optimal value
Maximum value of an objective function.
Objective Function
The linear function $z=k_1 x_1 + k_2 x_2 + \ldots\ldots + k_n x_n$ which is to be maximised or minimised
Constraints
Inequations or equations in the variables of a linear programming problem which describe the conditions under which the optimisation is to be accomplish
Optimisation problem
• Problem which seeks to maximise or minimise a linear function
• Linear programming problems are spacial type of optimisation problems
Different Type of Linear Programming Problems
•Diet problems
•Manufacturing problem
•Transportation problems
How to understand Mind Map?
First Level
Second Level
Third Level

(vi) Investment problem: To determine the amount of investment in fixed income securities to maximize the return on these investment.

- **Limitations of Linear Programming:**

(i) To specify an objective function in mathematical form is not an easy task.

(ii) Even if objective function is determined, it is difficult to determine social, institutional, financial and other constraints.

(iii) It is also possible that the objective function of constraints may not be directly specified by linear inequality equations.

- **Mathematical Formulation of LPP**

Let us take an example to understand how to formulate a LPP mathematically.

Scan to know more about this topic

Formulation of an LPP

Example 1 : An electronic firm is undecided at the most profitable mix for its products. The products manufactured are transistors, resistors and carbon tubes with a profit of (per 100 units) ₹ 10, ₹ 6 and ₹ 4 respectively. To produce a shipment of transistors containing 100 units requires 1 hour of engineering, 10 hours of direct labour, and 2 hours of administrative service. To produce 100 units of resistors requires 1 hour, 4 hours and 2 hours of engineering, direct labour and administrative services respectively. For 100 units of carbon tubes it needs 1 hour, 6 hours and 5 hours of engineering direct labour and administrative services respectively.

There are 100 hours of engineering time, 600 hours of direct labour and 300 hours of administrative time available. Formulate the corresponding LPP.

Sol.: Let the firm produce X hundred units of transistors, Y hundred units of resistors and Z hundred units of carbon tubes. Then the total profit to be maximized from this output will be

$$P = 10X + 6Y + 4Z$$

This is our objective function.

Now production of X hundred units of transistors, Y hundred units of resistors and Z hundred units of carbon tubes will require X + Y + Z hours of engineering time, 10X + 4Y + 6Z hours of direct labour time and 2X + 2Y + 5Z hours of administrative service.

But the total time available for engineering, direct labour and administrative services is 100, 600 and 300 hours respectively.

Hence the constraints are:

$$X + Y + Z \le 100$$
$$10X + 4Y + 6Z \le 600$$
$$2X + 2Y + 5Z \le 300$$

with $X, Y, Z \ge 0$ as the non-negativity restriction.

Thus the formulation is

Maximize, $P = 10X + 6Y + 4Z$

subject to constraints: $X + Y + Z \le 100$

$$10X + 4Y + 6Z \le 600$$
$$2X + 2Y + 5Z \le 300$$
$$X, Y, Z \ge 0.$$

Key Facts

Linear programming is often used for problems where no exact solution is known, for example for planning traffic flows.

- The goal of linear programming is to maximize or minimize specified objectives, such as profit or cost. This process is know as optimization.
- Linear programming is heavily used in microeconomics and company management, such as planning, product, transportation, technology and other issues, either to maximize the income or minimize the costs of a production scheme.

- **Feasible region:** The common region determined by all the constraints including non-negative constraints $x, y \ge 0$ of linear programming problem is known as the feasible region.
- **Feasible solution:** Points within and on the boundary of the feasible region represents feasible solutions of constraints.

Scan to know more about this topic

Graphical solution of an LPP

In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.

- **Theorem 1:** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point (vertex) of the feasible region.
- **Theorem 2:** Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function R has both maximum and minimum values of R and each of these occurs at a corner point (vertex) of R.

However, if the feasible region is unbounded, the optimal value obtained may not be maximum or minimum.

- **Corner or Extreme Point Method of Formulation of LPP**

Step 1: Formulate the linear programming problem in x and y with given conditions.

Step 2: Convert the inequality constraints into equality constraints and plot each line on the graph paper.

Step 3: Find the feasible region and check if the feasible region is bounded or unbounded.

Step 4: Evaluate the value of the objective function Z at each corner point. Let M be the greatest and m be the smallest value of the objective function Z.

(i) When the feasible region is bounded: M and m are the maximum and minimum values of the objective function Z, respectively.

(ii) When the feasible region is unbounded:

(a) M is the maximum value of the objective function Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise the objective function has no maximum value.

(b) m is the minimum value of the objective function Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.

Example: A mill owner buys two types of machines A and B for his mill. Machine A occupies 1,000 sq.m of area and requires 12 men to operate it; while machine B occupies 1,200 sq.m of area and requires 8 men to operate it. The owner has 7,600 sq.m of area available and 72 men to operate the machines. If machine A produces 50 units and machine B produces 40 units daily, how many machines of each type should he buy to maximize the daily output? Use Linear Programming to find the solution.

Sol: The data given in the problem are as under:

	Machine A	Machine B	Maximum availability
Area needed	1,000 sq.m	1,200 sq.m	7,600 sq.m
Labour force	12	8	72
Daily output	50 units	40 units	—

Let x and y be the number of machines A and B respectively

$$1{,}000x + 1{,}200y \le 7{,}600$$

or $$10x + 12y \le 76$$

or $$5x + 6y \le 38 \quad \text{...(i)}$$

$$12x + 8y \le 72$$

or $$3x + 2y \le 18 \quad \text{...(ii)}$$

$$x \ge 0, y \ge 0$$

Total output $$Z = 50x + 40y$$

Max. $Z = 50x + 40y$ subject to constraints

$$5x + 6y \le 38$$
$$3x + 2y \le 18$$
$$x \ge 0, y \ge 0$$

$3x + 2y = 18$

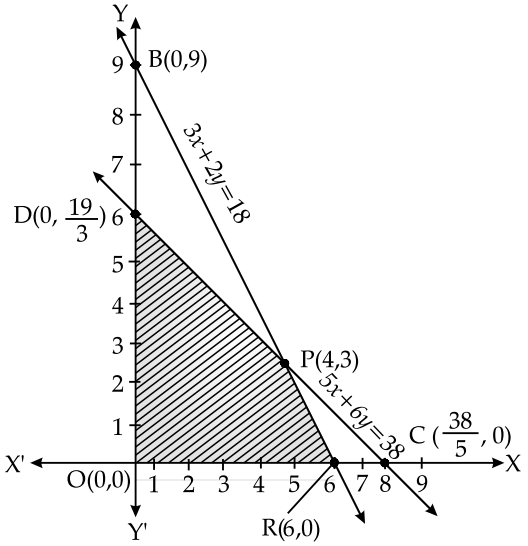
x	6	0	4
y	0	9	3

$5x + 6y = 38$

x	$\frac{38}{5}$	0	4
y	0	$\frac{19}{3}$	3

The vertices of the feasible region $ORPD$ are $O(0, 0)$, $R(6, 0)$, $P(4, 3)$ and $D\left(0, \frac{19}{3}\right)$.

The graph of these in equations is shown below:



Points	Value of $Z = 50x + 40y$
At O(0, 0)	$Z = 0$
At R(6, 0)	$Z = 50 \times 6 + 0 = 300$
At P(4, 3)	$Z = 50 \times 4 + 40 \times 3 = 320$
At $D\left(0, \frac{19}{3}\right)$	$Z = 50 \times 0 + 40 \times \frac{19}{3}$ $= \frac{760}{3} = 253.33$

Thus, we see that Z is maximum at (4, 3)
i.e., Number of machine $A = 4$
Number of machine $B = 3$

Mnemonics

Concept: LPP Parameter
Mnemonics: NOC
Interpretations:

N	O	C
↓	↓	↓
Non-negative Variables	Objective function	Constraints

OBJECTIVE TYPE QUESTIONS (1 mark each)

Multiple Choice Questions

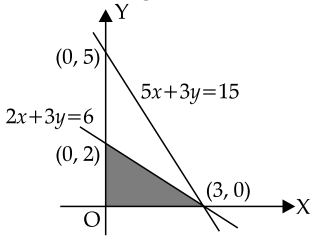
Q. 1. Feasible region in the set of points which satisfy
(A) The objective functions
(B) Some the given constraints
(C) All of the given constraints
(D) None of the above
Ans. Option (C) is correct.

Q. 2. Objective function of an LPP is
(A) a constant
(B) a function to be optimized
(C) a relation between the variables
(D) None of the above
Ans. Option (B) is correct.

Q. 3. The optimal value of the objective function is attained at the points
(A) on X-axis
(B) on Y-axis
(C) which are corner points of the feasible region
(D) None of the above
Ans. Option (C) is correct.
Explanation: The optimal value of the objective function is attained at the corner points of feasible region.

Q. 4. The intermediate solutions of constraints must be checked by substituting them back into
(A) Objective function
(B) Constraint equations
(C) Not required
(D) None of the above
Ans. Option (B) is correct.

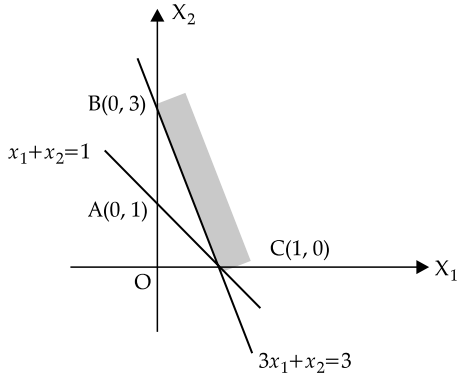
Q. 5. Which of the following is a vertex of the positive region bounded by the inequalities,
$2x + 3y \leq 6$ **and** $5x + 3y \leq 15$
(A) (0, 2) **(B)** (0, 0)
(C) (3, 0) **(D)** All of these
Ans. Option (D) is correct.
Explanation: Here (0, 2), (0, 0) and (3, 0) all are vertices of feasible region.



Q. 6. For the constraint of a linear optimizing function $z = x_1 + x_2$, given by $x_1 + x_2 < 1$, $3x_1 + x_2 > 3$ and $x_1, x_2 \geq 0$
(A) There are two feasible regions
(B) There are infinite feasible regions
(C) There is no feasible region
(D) None of these

Ans. Option (C) is correct.

Explanation: Clearly from graph there is no feasible region.



Q. 7. $Z = 7x + y$, subject to $5x + y \geq 5$, $x + y \geq 3$, $x \geq 0$, $y \geq 0$. The minimum value of Z occurs at

(A) (3, 0) **(B)** $\left(\frac{1}{2}, \frac{5}{2}\right)$

(C) (7, 0) **(D)** (0, 5)

Ans. Option (D) is correct.

Explanation:

Corner Points	$Z = 7x + y$
(3, 0)	21
$\left(\frac{1}{2}, \frac{5}{2}\right)$	6
(7, 0)	49
(0, 5)	5 (minimum)

Q. 8. $Z = 20x_1 + 20x_2$, subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + 2x_2 \geq 8$, $3x_1 + 2x_2 \geq 15$, $5x_1 + 2x_2 \geq 20$. The minimum value of Z occurs at

(A) (8, 0) **(B)** $\left(\frac{5}{2}, \frac{15}{4}\right)$

(C) $\left(\frac{7}{2}, \frac{9}{4}\right)$ **(D)** (0, 10)

Ans. Option (C) is correct.

Explanation:

Corner Points	$Z = 20x_1 + 20x_2$
(8, 0)	160
$\left(\frac{5}{2}, \frac{15}{4}\right)$	125
$\left(\frac{7}{2}, \frac{9}{4}\right)$	115 (minimum)
(0, 10)	200

Q. 9. Let x and y be optimal solution of an LPP, then

(A) $Z = \lambda x + (1 - \lambda)\, y$, $\lambda \in R$ is also an optimal solution.

(B) $Z = \lambda x + (1 - \lambda)\, y$, $0 \leq \lambda \leq 1$ gives an optimal solution.

(C) $Z = \lambda x + (1 + \lambda)\, y$, $0 \leq \lambda \leq 1$ gives an optimal solution.

(D) $Z = \lambda x + (1 + \lambda)\, y$, $\lambda \in R$ gives an optimal solution.

Ans. Option (B) is correct.

Q. 10. How many of the following points satisfy the in equality $2x - 3y > -5$?

(1, 1), (–1, 1), (1, – 1), (– 1, – 1), (– 2, 1), (2, – 1), (– 1, 2) and (– 2, – 1).

(A) 3 **(B)** 5

(C) 6 **(D)** 4

Ans. Option (B) is correct.

Explanation: (1, 1), (1, – 1), (– 1, – 1), (2, – 1) and (– 2, – 1) satisfy the inequality $2x - 3y > -5$.

Q. 11. The position of points O(0, 0) and $P(2, -2)$ in the region of graph of inequation $2x - 3y < 5$ will be

(A) O and P both inside

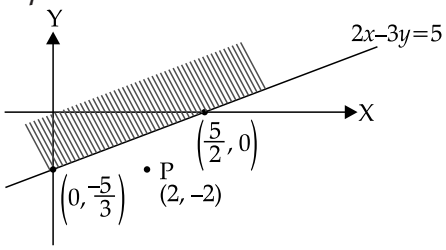
(B) O and P both outside

(C) O inside and P outside

(D) O outside and P inside

Ans. Option (C) is correct.

Explanation:



Q. 12. If the number of available constraints is 3 and the number of parameters to be optimized is 4, then

(A) The objective function can be optimized

(B) The constraints are short in number

(C) The solution is problem oriented

(D) None of these

Ans. Option (B) is correct.

Q. 13. The necessary condition for third quadrant region in XY plane is:

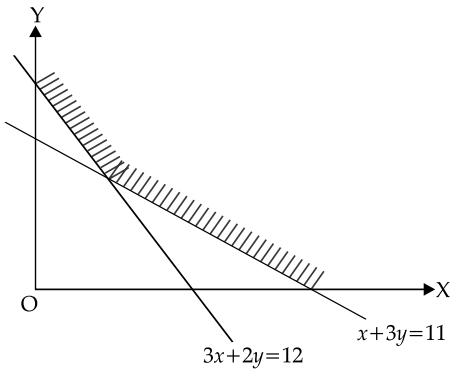
(A) $x > 0, y < 0$ **(B)** $x < 0, y < 0$

(C) $x < 0, y > 0$ **(D)** $x < 0, y = 0$

Ans. Option (B) is correct.

Explanation: In 3rd quadrant $x < 0$ and $y < 0$.

Q. 14. For the following feasible region, the linear constraints are:



(A) $x \geq 0, y \geq 0, 3x + 2y \geq 12, x + 3y \geq 11$
(B) $x \geq 0, y \geq 0, 3x + 2y \leq 12, x + 3y \geq 11$
(C) $x \geq 0, y \geq 0, 3x + 2y \leq 12, x + 3y \leq 11$
(D) None of these

Ans. Option (A) is correct.

Explanation: Since, feasible region is outwards, thus linear constraints one.

$$3x + 2y \geq 12$$
$$x + 3y \geq 11$$

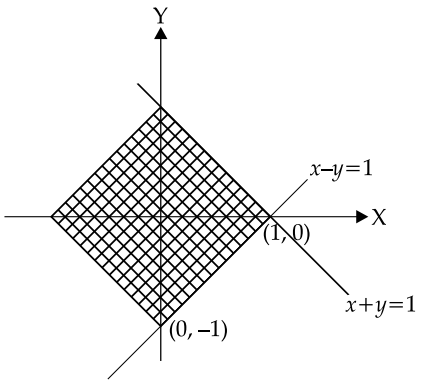
with non-negative restrictions *i.e.*, $x \geq 0, y \geq 0$

Q. 15. In which quadrant, the bounded region for inequations $x + y \leq 1$ and $x - y \leq 1$ is situated

(A) I, II **(B)** I, III
(C) II, III **(D)** All the four quadrants

Ans. Option (D) is correct.

Explanation:



As shown in graph drawn for $x + y = 1$ and $x - y = 1$, the origin included in the area. Hence, the bounded region situated in all four quadrants.

SUBJECTIVE TYPE QUESTIONS

Very Short Answer Type Questions (1 mark each)

Q. 1. The corner points of the feasible region determined by the system of linear constraints are (0, 0), (0, 40), (20, 40), (60, 20), (60, 0). The objective function is $Z = 4x + 3y$.

Find the maximum value of Z. R **[O.E.B.]**

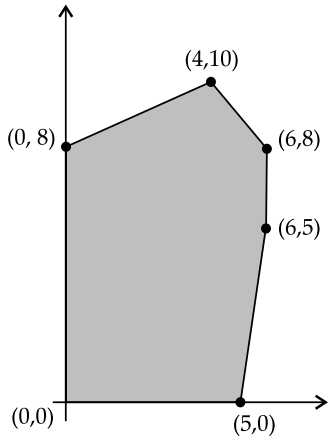
Sol.

Corner points	Corresponding value of $Z = 4x + 3y$
(0, 0)	0
(0, 40)	120
(20, 40)	200
(60, 20)	300 ← Maximum
(60, 0)	240

The maximum value of Z is 300. **1**

Q. 2. The feasible solution for a LPP is shown in given figure. Let $Z = 3x - 4y$ be the objective function.

Find the minimum value of Z. U **[O.E.B.]**



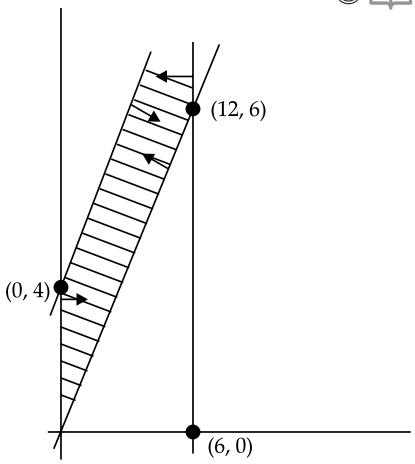
Sol.

Corner points	Corresponding value of $Z = 3x - 4y$
(0, 0)	0
(5, 0)	15 ← Maximum
(6, 5)	–2
(6, 8)	–14
(4, 10)	–28
(0, 8)	–32 ← Minimum

Hence, the minimum of Z occurs at (0, 8) and its minimum value is (–32). **1**

Q. 3. The feasible region for an LPP is shown in the given Figure. Let $Z = 3x - 4y$ be the objective function. Find the maximum value of Z.

A I **[O.E.B.]**



Q. 4. A toy company manufactures two types of toys A and B. Demand for toy B is atmost half of that if type A. Write the corresponding constraint if x toys of type A and y toys of type B are manufactured.

A I U **[O.E.B.]**

These questions are for practice and their solutions are available at the end of the chapter

Sol. Since, the demand of toys of type B is atmost half of that for toys of type A

$$\therefore \quad y \le \frac{x}{2}$$

$$\Rightarrow \quad 2y - x \le 0$$

$$\Rightarrow \quad x - 2y \ge 0 \qquad \textbf{1}$$

Q. 5. For the LPP,

$$\textbf{Max. } Z = 2x + 3y$$

The coordinates of the corner points of bounded feasible region are A(3, 3), B(20, 3), C(20, 10), D(18, 12) and E(12, 12). Find the maximum value of Z.

U **[O.E.B.]**

Sol.

Corner Points	$Z = 2x + 3y$
A(3, 3)	$Z = 15$
B(20, 3)	$Z = 49$
C(20, 10)	$Z = 70$
D(18, 12)	$Z = 72 \rightarrow$ Max.
E(12, 12)	$Z = 60$

Thus, maximum value of Z is 72 at D (18, 12). **1**

Short Answer Type Questions-I (2 marks each)

Q. 1. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate a LPP for finding how many of each should be produced daily to maximize the profit ? It is being given that at least one of each must be produced. R&U **[O.E.B.]**

Sol. Let x be the number of necklaces manufactured and y be the number of bracelets manufactured. The total number of necklaces and bracelets it can handle is at most 24.

$$x + y \le 24 \qquad ½$$

The x items takes x hours to manufacture and y items take $\frac{y}{2}$ hours to manufacture and the maximum time available is 16 hours. So,

$$x + \frac{y}{2} \le 16 \qquad ½$$

The profit on one necklace is given as ₹ 100 and the profit on one bracelet is given as ₹ 300.

Let the profit be Z. To maximize the profit,

$$Z = 100x + 300y$$

Therefore, the required is.

$$\text{Max. } Z = 100x + 300y$$

Subject to constraints,

$$x + y \le 24$$

$$x + \frac{y}{2} \le 16$$

$$x, y \ge 1 \qquad \textbf{1}$$

Q. 2. A firm has to transport at least 1200 packages daily using large vans which carry 200 packages each and small vans which can take 80 packages each. The cost for engaging each large van is ₹ 400 and each small van is ₹ 200. Not more than ₹ 3,000 is to be spent daily on the job and the number of large vans cannot exceed the number of small vans. Formulate this problem as a LPP given that the objective is to minimize cost. R&U **[O.E.B.]**

Sol. Let the number of large vans and small vans used for transporting the packages be x and y, respectively.

It is given that the cost of engaging each large van is ₹ 400 and each small van is ₹ 200.

Cost of engaging x large van is ₹ 400 and each small van is ₹ 200.

Cost of engaging x large vans = ₹ $400x$

Cost of engaging y small vans = ₹ $200y$

Let Z be the total cost of engaging x large vans and y small vans.

$$\therefore \quad Z = ₹\,(400x + 200y) \qquad ½$$

The firm has to transport atleast 1200 packages daily using large vans which carry 200 packages each and small vans which can take 80 packages each.

So, $200x + 80y \ge 1200$ ½

Not more than ₹ 3000 is to be spent daily on the transport.

So, $400x + 200y \le 3000$ ½

Also, the number of large vans cannot exceed the number of small vans.

$$\therefore \quad x \le y$$

Thus, the linear programming problem of the problem is

$$\text{Min. } Z = ₹\,(400x + 200y)$$

Subject to constraints

$$200x + 80y \ge 1200$$

$$400x + 200y \le 3000$$

$$x \le y$$

$$x \ge 0, y \ge 0 \qquad ½$$

Q. 3. Draw the feasible region for given inequation system $y \le 6, x + y \le 3, x \ge 0, y \ge 0$.

R&U **[O.E.B.]**

Q. 4. The sum of two positive integers is atmost 5. The difference between two times of second number and first number is at most 4. If first number is x and second number is y, then for maximizing the product of these two numbers, formulate LPP.

R&U **[O.E.B.]**

These questions are for practice and their solutions are available at the end of the chapter

Sol. The required LPP is

Max. $Z = x.y$

Subject to: $x + y \le 5$

$2y - x \le 4$

$x \ge 0$

$y \ge 0$ **2**

Short Answer Type Questions-II (3 marks each)

Q. 1. Solve the following LPP graphically :

Minimize $Z = 5x + 10y$

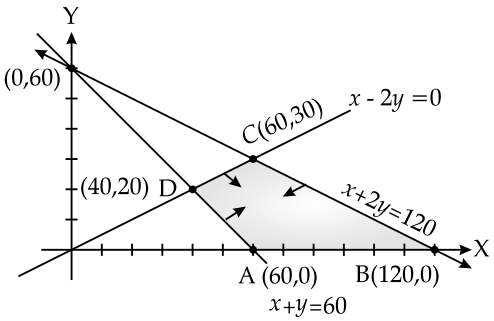
Subject to $x + 2y \le 120$

Constraints $x + y \ge 60$

$x - 2y \ge 0$

and $x, y \ge 0$ A [O.E.B.]

Sol.



2

$Z = 5x + 10y$

$Z|_{A\,(60,\,0)} = 300 \rightarrow$ Minimum

$Z|_{B\,(120,\,0)} = 600$

$Z|_{C\,(60,\,30)} = 600$

$Z|_{D\,(40,\,20)} = 400$

Minimum value, $Z = 300$ at $x = 60, y = 0$ **1**

Q. 2. Solve the following LPP graphically:

Maximize $Z = 4x + y$

Subject to following constraints

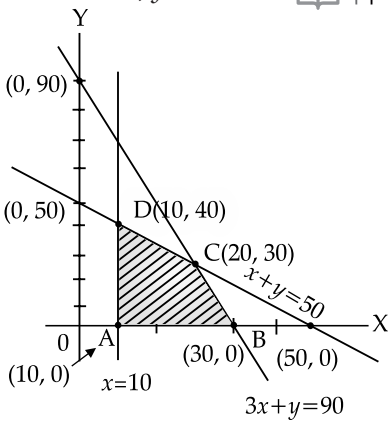
$x + y \le 50,$

$3x + y \le 90,$

$x \ge 10$

$x, y \ge 0$ AI [O.E.B.]

Sol.



2

$Z|_{A\,(10,\,0)} = 40$

$Z|_{B\,(30,\,0)} = 120 \rightarrow$ Maximum

$Z|_{C\,(20,\,30)} = 110$

$Z|_{D\,(10,\,40)} = 80$

Maximum value, $Z = 120$ at $(30, 0)$ **1**

Q. 3. Solve the following linear programming problem graphically :

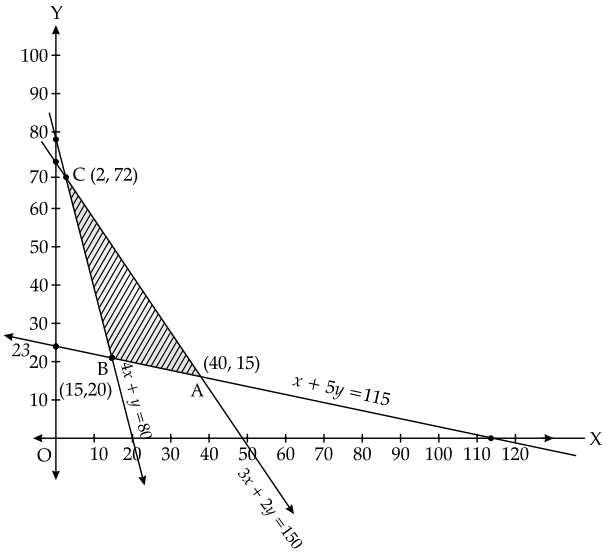
Minimize : $Z = 6x + 3y$

Subject to the constraints :

$$\begin{cases} 4x + y \ge 80 \\ x + 5y \ge 115 \\ 3x + 2y \le 150 \\ x \ge 0, y \ge 0 \end{cases}$$

A [O.E.B.]

Sol.



2

Corner points	**Value of z**
A(40, 15)	285
B(15, 20)	150 → minimum
C(2, 72)	228
Minimum	$Z = 150$
when	$x = 15, y = 20$ **1**

Q. 4. Solve the following linear programming problem graphically:

Minimize: $Z = 3x + 9y$

When: $x + 3y \le 60$

$x + y \ge 10$

$x \le y$

$x \ge 0, y \ge 0$

A [O.E.B.]

Q. 5. Solve the following LPP graphically:

Maximize $Z = 1000x + 600y$

subject to the constraints

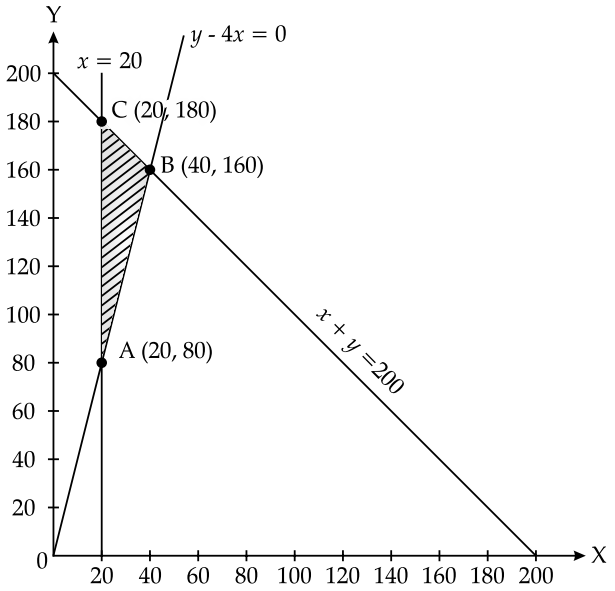
$x + y \le 200$

$x \ge 20$

$y - 4x \ge 0$

$x, y \ge 0$ A [O.E.B.]

Sol.



2

Corner points	Value of $Z = 1000x + 600y$
(20, 80)	68,000
(40, 160)	1,36,000 → Maximum
(20, 180)	1,28,000

$\therefore$ Maximum value, $Z = 1,36,000$

at $x = 40, y = 160$ **1**

Long Answer Type Questions (5 marks each)

Q. 1. A manufacturing company makes two types of teaching aids A and B of Mathematics for class X. Each type of A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Formulate this as Linear Programming Problem and solve it. Identify the feasible region from the rough sketch. A

[O.E.B.]

Sol. Let the number of two types of teaching aids A and B be x and y, respectively.

Write the given data in tabular for mas following:

Item	No. of aids	Time on fabricating (in hrs)	Time on finishing (in hrs)	Profit (in ₹)
A	x	$9x$	x	$80x$
B	y	$12y$	$3y$	$120y$
Total	$x + y$	$9x + 12y$	$x + 3y$	$80x + 120y$
Availability		180	30	

The profit on type A is ₹ 80 and type B is ₹ 120. Thus, the required LPP is

$$\text{Max. } Z = 80x + 120y \quad ...(i)$$

Subject to constraints

$$9x + 12y \le 180$$

or $$3x + 4y \le 60 \quad ...(ii)$$

$$x + 3y \le 30 \quad ...(iii)$$

$$x \ge 0, y \ge 0$$ **2**

Consider the equation,

$$3x + 4y = 60$$

x	0	20
y	15	0

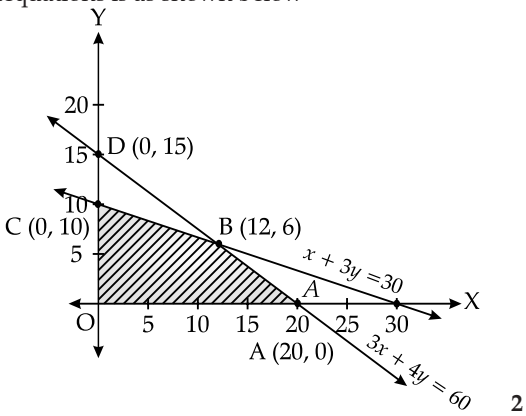
and $x + 3y = 30$

x	0	30
y	10	0

On solving eq, (ii) and (iii), we get

$x = 12$ and $y = 6$

The graphical representation of the system of inequations is as shown below



2

From graph, feasible region is

OABCO, whose corner points are (0, 0), A(20, 0), B(12, 6) and C(0, 10).

The values of Z at corner points are as follows:

Corner Points	Value of $Z = 80x + 120y$
O(0, 0)	$Z = 80 \times 0 + 120 \times 0 = 0$
A(20, 0)	$Z = 80 \times 20 + 120 \times 0 = 1600$
B(12, 6)	$Z = 80 \times 12 + 120 \times 6 = 1680$
C(0, 10)	$Z = 80 \times 0 + 120 \times 10 = 1200$

From the table, the maximum value of Z is ₹ 1680.

Hence, 12 pieces of type A and 6 pieces of type B

should be manufactured per week to get maximum profit of ₹ 1680 per week. **1**

Commonly Made Error

- A number of candidates could not form the correct inequalities subject to given constraints. Some noted incorrect corner points to find optimum value. A few candidates did not draw rough sketch of the feasible region.

Answering Tip

- Do adequate practice to frame inequalities using given constraints and finding corner points by solving inequalities.

Q. 2. A farmer has a supply of chemical fertilizer of type A which contains 10% nitrogen and 6% phosphoric acid and of type B which contains 5% nitrogen and 10% phosphoric acid. After soil test, it is found that atleast 7 kg of nitrogen and same quantity of phosphoric acid is required for a good crop. The fertilizer of type A costs ₹ 5.00 per kg and the type B costs ₹ 8.00 per kg. Using Linear Programming, find how many kilograms of each type of the fertilizer should be bought to meet the requirement and for the cost to be minimum. Find the feasible region in the graph. A [O.E.B.]

Sol. Let the fertilizer of type A be x kg and type B be y kg.

Fertilizer	Nitrogen	Phosphoric Acid	Quantity	Cost
A	10%	6%	x kg	₹ 5
B	5%	10%	y kg	₹ 8
Total	7 kg	7 kg		

According to Question:

Nitrogen	Phosphoric Acid
Quantity of Nitrogen in A = 10%	Quantity of Phosphoric Acid in A = 6%
Quantity of Nitrogen in B = 5%	Quantity of Phosphoric Acid in B = 10%
Minimum availability = 7 kg	Minimum availability = 7 kg
$\therefore$ 10% of x + 5% of $y \geq 7$	$\therefore$ 6% of x + 10% of $y \geq 7$
$\frac{10}{100}x + \frac{5}{100}y \geq 7$	$\frac{6}{100}x + \frac{10}{100}y \geq 7$
$2x + y \geq 140$	$3x + 5y \geq 350$
and $x \geq 0, y \geq 0$	and $x \geq 0, y \geq 0$

$\therefore$ Min. $Z = 5x + 8y$

Subject to constraints

$$2x + y \geq 140$$

$$3x + 5y \geq 350$$

$$x \geq 0, y \geq 0 \quad \textbf{2}$$

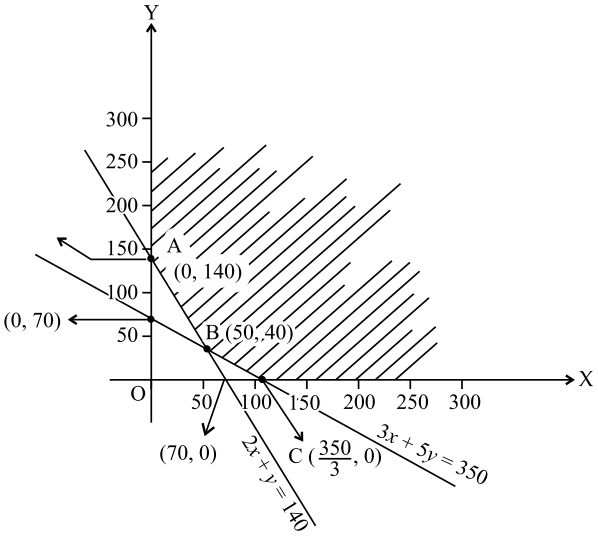
$$2x + y = 140 \quad ...(i)$$

x	70	0	50
y	0	140	40

$$3x + 5y = 350 \quad ...(ii)$$

x	0	$\frac{350}{3}$
y	70	0

On solving eq (i) and (ii), we get $x = 50$ and $y = 40$



2

We draw the lines $2x + y = 140$ and $6x + y = 70$ and obtain the feasible region (unbounded and convex) as shown in figure.

Thus, corner points are A(0, 140), B(50, 40), $C\left(\frac{350}{3}, 0\right)$.

The values of Z at these points are given in following table:

Corner Points	Objective function, $Z = 5x + 8y$
A(0, 140)	$Z = 5 \times 0 + 8 \times 140 = 1120$
B(50, 40)	$Z = 5 \times 50 + 8 \times 40 = 570 \rightarrow$ Minimum
$C\left(\frac{350}{3}, 0\right)$	$Z = 5 \times \frac{350}{3} + 8 \times 0 = 583\frac{1}{3}$

As the feasible area is unbounded, Hence, 500 may or may not be the minimum value of Z.

For this, we need to graph inequality.

$$5x + 8y < 570$$

x	114	0
y	0	71.25

Since, there is no common point between the feasible region and $5x + 8y < 570$

Hence, the cost will be minimum, if
Fertilizer of type A used = 50 kg
Fertilizer of type B used = 40 kg
Minimum cost = ₹ 570 **1**

Commonly Made Error

- Many candidates made errors in calculating the conditions using constraints. Some candidates were unable to represent feasible reason on the graph as such they may not have clarity of the concept of solving linear equations graphically. Some candidates solved it in terms of decimals and mostly went incorrect in simplifications.

Answering Tip

- The optimum function and all possible constraints in the form of inequations must be put down from what is stated in the problem.

Q. 3. A company manufacturers two types of products A and B. Each unit of A requires 3 gram of nickel and 1 gram of chromium, while each unit of B requires 1 gram of nickel and 2 gram of chromium. The firm can produce 9 gram of nickel and 8 grams of chromium. The profit is ₹ 40 on each unit of product of type A and ₹ 50 on each unit of type B. How many units of each type should the company manufactures so as to earn maximum profit? Use linear programming to find the solution. A

[O.E.B.]

Sol.

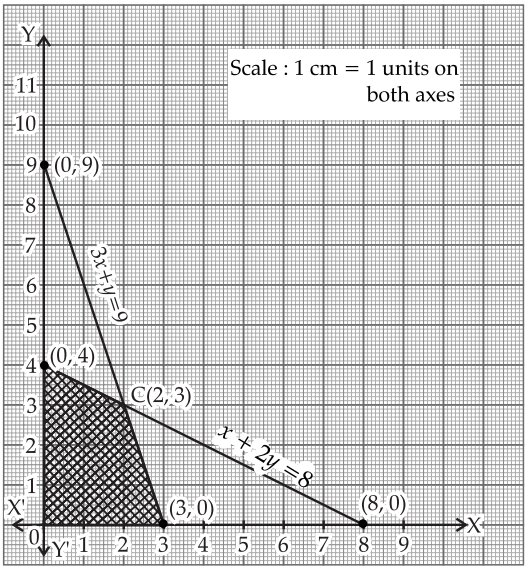
Type of Product	Total items		Profit per unit
	9 gms Nickel (gm)	8 gms Chromium (gm)	
A	3	1	₹ 40
B	1	2	₹ 50

Let x = Number of units of type A
y = Number of units of type B
Maximize $Z = 40x + 50y$
Subject to the constraints,
$3x + y \leq 9$
$x + 2y \leq 8$
and $x \geq, 0, y \geq 0$ **2**
Consider the equation,
$3x + y = 9$

x	0	3	2
y	9	0	3

and $x + 2y = 8$

x	0	8	2
y	4	0	3



2

Shaded region in the diagram represents the feasible solution.
Now, we can determine the maximum value of Z by evaluating the value of Z at the four points (vertices) as shown below.

Vertices	$Z = 40x + 50y$
(0, 0)	$Z = 40 \times 0 + 50 \times 0 = 0$
(3, 0)	$Z = 40 \times 3 + 50 \times 0 = ₹ 120$
(0, 4)	$Z = 40 \times 0 + 50 \times 4 = ₹ 200$
(2, 3)	$Z = 40 \times 2 + 50 \times 3 = ₹ 230$

From graph,
Maximum profit, Z = ₹ 230
∴ Number of units of type A is 2 and number of units of type B is 3. **1**

Commonly Made Error

- In some cases, all the constraints were not used and hence coordinates of all feasible points were not obtained.

Answering Tip

- Students should solve the constraints equations in pairs to obtain all feasible points leading to maximum or minimum value of desired function.

Q. 4. A dietician wishes to mix two kinds of food X and Y in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamin and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	Vitamin A	Vitamin B	Vitamin C
X	1 unit	2 units	3 units
Y	2 units	2 units	1 unit

One kg of food X costs ₹ 24 and one kg of food Y costs ₹ 36. Using Linear Programming, find the least cost of the total mixture which will contain the required vitamins. [A I] [O.E.B.]

Sol. Let x be the number of units of food X and y be the number of units of food Y be mixed to obtain the desired diet. Then LPP of the given problem is:

Minimise, $Z = 24x + 36y$

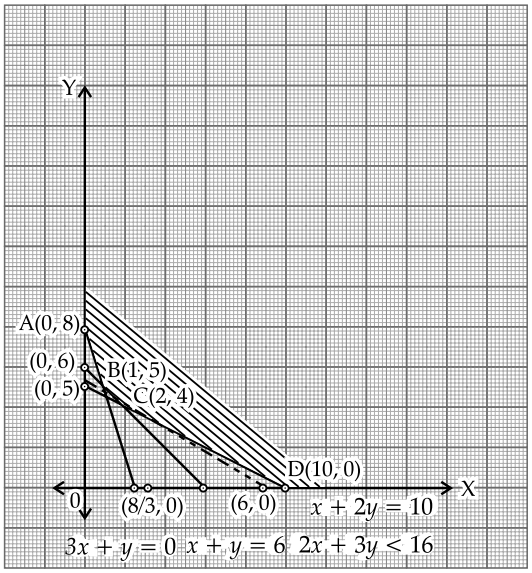
Subject to the constraints,

$$x + 2y \geq 10, 2x + 2y \geq 12$$

i.e., $x + y \geq 6$ and $3x + y \geq 8$

$$x \geq 0, y \geq 0$$ **2**

We draw the lines $x+2y=10$, $x+y = 6$, $3x + y = 8$ and obtain the feasible region (unbounded and convex) as shown in the figure.



2

Thus, corner points are A(0, 8), B(1, 5), C(2, 4) and D(10, 0).

The values of Z(in ₹) at these points are given in the following table:

Corner Point	Objective Function $Z = 24x + 36y$
A(0, 8)	$Z = 24 \times 0 + 36 \times 8 = 288$
B(1, 5)	$Z = 24 \times 1 + 36 \times 5 = 204$
C(2, 4)	$Z = 24 \times 2 + 36 \times 4 = 192$ (Max.)
D(10, 0)	$Z = 24 \times 10 + 36 \times 0 = 240$

As the feasible region is unbounded, we draw the graph of the half plane $24x + 36y < 192$ *i.e.*, $2x + 3y < 16$ and note that there is no point common with the feasible region. The minimum value of Z is ₹ 192. It occurs at C(2, 4).

i.e., when 2 kg of food X and 4 kg of food Y are mixed to get the desired diet. **1**

Commonly Made Error

- Many candidates took incorrect inequality sign, hence they got incorrect feasible region and their corner points were also incorrect. Some candidates did not show any graphical representation of the inequalities. In some cases, the representation of the problem was not up to the mark, and the work was not systematic resulting in candidates missing the point of minimum cost.

Answering Tip

- Students should practice the sketching of lines. They should practice to express the line in intercept from *i.e.*, $\frac{x}{a} + \frac{y}{b} = 1$, so that sketching will be easy. Correct feasible region and its plotting is important.

Q. 5. A company manufactures two types of toys A and B. A toy of type A requires 5 minutes for cutting and 10 minutes for assembling. A toy of type B requires 8 minutes for cutting and 8 minutes for assembling. There are 3 hours available for cutting and 4 hours available for assembling the toys in a day. The profit is ₹ 50 each on a toy of type A and ₹ 60 each on a toy of type B. How many toys of each type should the company manufacture in a day to maximize the profit ? Use linear programming to find the solution. [A I] [O.E.B.]

Q. 6. Two tailors P and Q earn ₹150 and ₹200 per day respectively. P can stitch 6 shirts and 4 trousers a day, while Q can stitch 10 shirts and 4 trousers per day. How many days should each work to produce at least 60 shirts and 32 trousers at minimum labour cost? [A] [O.E.B.]

Sol. Let the tailor P work for x days and the tailor Q work for y days respectively.

Here, the problem can be formulated as an L.P.P. as follows:

Minimize $Z = 150x + 200y$

Subject to the constraints:

$$6x + 10y \geq 60$$

or $3x + 5y \geq 30$ ...(i)

$4x + 4y \geq 32$

or $x + y \geq 8$ ...(ii)

and $x \geq 0, y \geq 0$ ...(iii) **2**

Converting them into equations we obtain the following equations:

$3x + 5y = 30$, $x + y = 8$

$$\Rightarrow y = \frac{30 - 3x}{5}$$

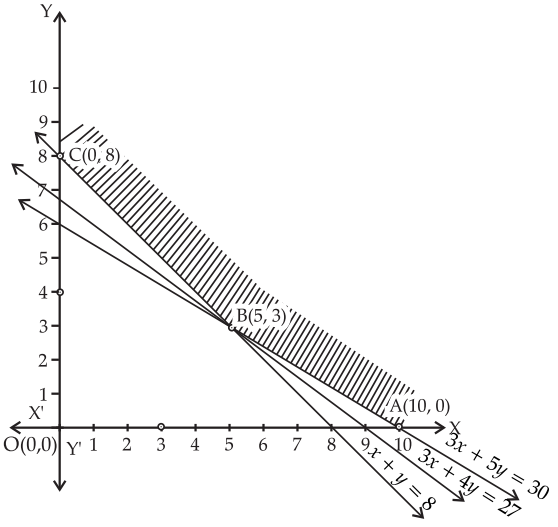
$$\Rightarrow y = 8 - x$$

$3x + 5y = 30$

x	0	10	5
y	6	0	3

$x + y = 8$

x	0	8	5
y	8	0	3



2

The shaded region in the diagram represent the feasible region.

The corner points are

A(10, 0), B(5, 3) and C(0, 8)

At the corner point the value of $Z = 150x + 200y$

At A(10, 0) $Z = 1500$

At B(5, 3) $Z = 150 \times 5 + 200 \times 3$

$= 750 + 600 = 1350$

At C(0, 8) $Z = 1600$

As the feasible region is unbounded, we draw the graph of the half-plane.

$150x + 200y < 1350$

$3x + 4y < 27$

There is no point common with the feasible region, therefore, Z has minimum value.

Minimum value of Z is ₹ 1350 and it occurs at the point B(5, 3).

Hence, the labour cost is ₹ 1350 when P works for 5 days and Q works for 3 days. **1**

Q. 7. A manufacturer manufactures two types of tea-cups, A and B. Three machines are needed for manufacturing the tea cups. The time in minutes required for manufacturing each cup on the machines is given below:

Type of Cup	Time in minutes		
	Machine I	Machine II	Machine III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of six hours per day. If the profit on each cup of type A is ₹ 1.50 and that on each cup of type B is ₹ 1.00, find the number of cups of each type that should be manufacturing in a day to get maximum profit.

[O.E.B.]

Sol. Let x be the number of A type tea cups and y be the number of B type tea cups.

Items	No.	Machine			Profit/ unit
		I	II	III	
Type A	x	12 min	18 min	6 min	₹ 1.5
Type B	y	6 min	0 min	9 min	₹ 1
Max. Available Time		6 Hours $= 6 \times 60$ $= 360$ min	6 Hours $= 6 \times 60$ $= 360$ min	6 Hours $= 6 \times 60$ $= 360$ min	

The problem can be formulated as

Max. $Z = 1.50x + 1.00y$

i.e., Max. $Z = 1.50x + y$

Subject to the constraints:

$12x + 6y \le 360$, *i.e.*, $2x + y \le 60$

$18x + 0.y \le 360$, *i.e.*, $x \le 20$

$6x + 9y \le 360$, *i.e.*, $2x + 3y \le 120$

and $x \ge 0, y \ge 0$ (Non-negative constraints) **2**

We draw the straight lines:

$2x + y = 60$, $x = 20$, $2x + 3y = 120$

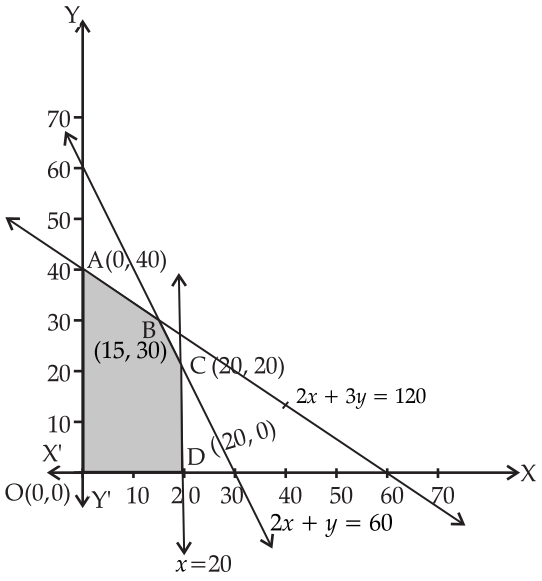
For $2x + y = 60$

x	30	0	40
y	0	60	20

For $2x + 3y = 120$

x	60	0	30
y	0	40	20

The shaded portion shows the feasible region. The points B and C are points of intersection of lines $2x + y = 60$, with $2x + 3y = 120$ and $x = 20$ with $2x + y = 60$ respectively.



2

Thus, the 5 corner points of the feasible region are O(0,0), A(0, 40), B(15, 30), C(20,20) and D(20, 0).

$Z = 1.5x + y$

Corner Point	Value of Z $1.5x + y$
O(0, 0)	0
A(0, 40)	40
B(15, 30)	52.5 (Max.)
C(20, 20)	50
D(20, 0)	30

Clearly, the maximum profit is at $x = 15$ and $y = 30$.

Thus, the manufacturer should manufacture 15 cups of type A and 30 cups of type B to get maximum profit in a day. **1**

Q. 8. A company produces two types of items, P and Q. Manufacturing of both items requires the metals gold and copper. Each unit of item P requires 3 gms of gold and 1 gm of copper while that of item Q requires 1 gm of gold and 2 gm of copper. The company has 9 gm of gold and 8 gm of copper in its store. If each unit of item P makes a profit of ₹ 50 and each unit of item Q makes a profit of ₹ 60, determine the number of units of each item that the company should produce to maximize profit. What is the maximum profit ? [O.E.B.]

Sol.

Items	Total items present in store		Profit per unit
	9 gm	8 gm	
	Gold (gm)	Copper (gm)	
P	3	1	₹ 50
Q	1	2	₹ 60

Let x_1 = number of units of item P

x_2 = number of units of item Q

Max. $Z = 50x_1 + 60x_2$

Subject to the constraints

$3x_1 + x_2 \le 9$

$x_1 + 2x_2 \le 8$

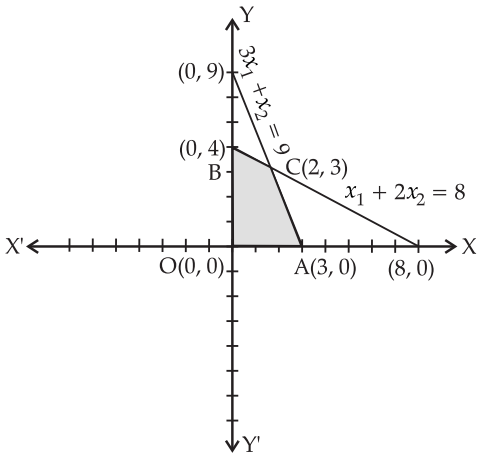
such that $x_1, x_2 \ge 0$ **2**

Now, $3x_1 + x_2 = 9$ …(i)

x_1	0	3
x_1	9	0

and $x_1 + 2x_2 = 8$ …(ii)

x_1	0	8
x_2	4	0



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The corner points of the feasible region are O(0, 0), A(3, 0), B(0, 4) and C(2, 3)

Corner points	$Z = 50x_1 + 60x_1$
at O(0, 0),	$Z = 0$
at A(3, 0),	$Z = 50 \times 3 + 60 \times 0 = 150$
at C(2, 3),	$Z = 50 \times 2 + 60 \times 3 = 280$
at B(0, 4),	$Z = 50 \times 0 + 60 \times 4 = 240$

$\therefore$ The company should produce 2 items of type P and 3 items of type Q.

$\therefore$ Maximum profit = ₹ 280 **1**

Q. 9. A new cereal, formed of a mixture of bran and rice, contains at least 88 grams of protein and at least 36 milligram of iron. Knowing that bran contains 80 gram of protein and 40 milligram of iron per kilogram, and that of rice contains 100 gram of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing a kilogram of this new cereal if bran costs ₹ 28 per kilogram and rice costs ₹ 25 per kilogram. [O.E.B.]

Q. 10. A manufacturer produces two types of steel trunks. He has two machines, A and B. The first type of trunk requires 3 hours on machine *A* and 3 hours on machine B. The second type requires 3 hours on machine *A* and 2 hours on machine B. Machines A and B can work at most for 18 hours and 15 hours per day respectively. He earns a profit of ₹ 30 per trunk on the first type of trunk and ₹ 25 per trunk on the second type. Formulate a Linear Programming Problem to find out how many trunks of each type he must make each day to maximize his profit. [O.E.B.]

Sol. Let x units of A type and y units of B type be produced.

$\therefore$ Max $Z = 30x + 25y$

Under the following restrictions

$3x + 3y \le 18$

$3x + 2y \le 15$

$x \ge 0, y \ge 0$ **2**

Now $x + y = 6$

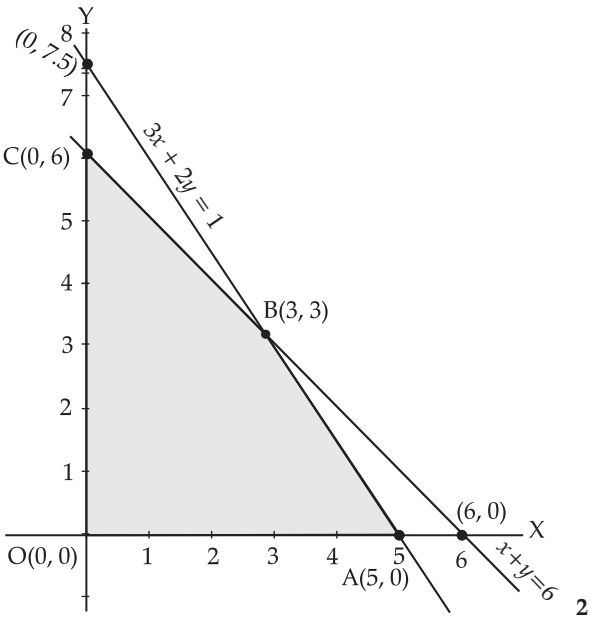
x	0	6	3
y	6	0	3

and $3x + 2y = 15$

x	0	5	3
y	7.5	0	3

We draw the two constraints on the graph paper and indicate the feasible region. The region OABC represents the set of all feasible solutions.

The co-ordinates of the corner points of the feasible region are O(0, 0), A(5, 0), B(3, 3) and C(0, 6).



2

Now, $Z = 30x + 25y$

At O(0, 0), $Z = 30(0) + 25(0) = 0$

At A(5, 0), $Z = 30(5) + 25(0) = 150$

At B(3, 3), $Z = 30(3) + 25(3) = 165$

At C(0, 6), $Z = 30(0) + 25(6) = 150$

Maximum value of $Z = 165$, when $x = 3$ and $y = 3$

Hence, 3 trunks of first type and 3 trunks of second type should be produced each day in order to get the maximum profit. **1**

Q. 11. A dealer in rural area wishes to purchase a number of sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items of storage. An electronic sewing machine cost him ₹ 360 and a manually operated sewing machine ₹ 240. He can sell an electronic sewing machine at a profit of ₹ 22 and a manually operated sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ? Make it as a LPP and solve it graphically. A **[O.E.B.]**

Sol. Let x and y be electronic and manually operated sewing machine purchased respectively.

$\therefore$ LPP is

Maximize $Z = 22x + 18y$

Subject to,

$$360x + 240y \le 5760$$

$$3x + 2y \le 48$$

or $x + y \le 20$

$$x \ge 0, y \ge 0$$ **2**

For equation $3x + 2y = 48$

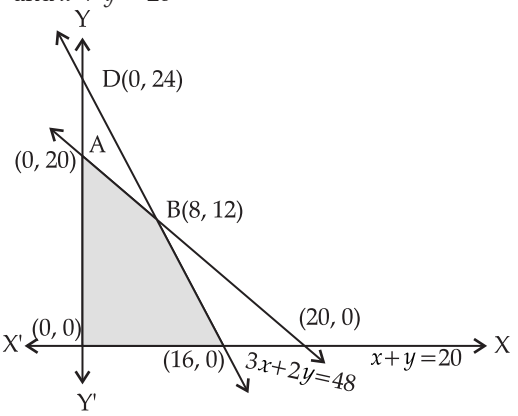
x	16	0
y	0	24

and $x + y = 20$

x	0	20
y	20	0

Vertices of feasible region are:

A(0, 20), B(8, 12), C(16,0) and O(0, 0), where B(8, 12) is the point of intersection of equation $3x + 2y = 48$ and $x + y = 20$



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Value of Z at corner points:

Corner points	Value of objective function $Z = 22x + 18y$
O(0, 0)	$Z = 0$
A(0, 20)	$Z = 360$
B(8, 12)	$Z = 392 \rightarrow$ Max
C(16, 0)	$Z = 352$

$\therefore$ For maximum Z, Electronic machines = 8,

Manual machines = 12. **1**

Q. 12. A company produces soft drinks that have a contract which requires that a minimum of 80 units of chemical A and 60 units of the chemical B go into each bottle of the drink. The chemicals are available in prepared mix packets from two different suppliers. Supplier S had a packet of mix of 4 units of A and 2 units of B that costs ₹ 10. The supplier T has a packet of mix of 1 unit of A and 1 unit of B that costs ₹ 4. How many packets of mixes from S and T should the company purchase to honour the contract requirement and yet maintain the minimum cost ? Make a LPP and solve graphically. A **[O.E.B.]**

Sol. Let x and y units of packet of mixes are purchased from S and T, respectively. If Z is the total cost, then

$$Z = 10x + 4y \quad \ldots(i)$$

is objective function which we have to minimize.

Here, constraints are:

$4x + y \geq 80$ …(ii)

$2x + y \geq 60$ …(iii)

Also, $x \geq 0$ …(iv)

$y \geq 0$ …(v) **2**

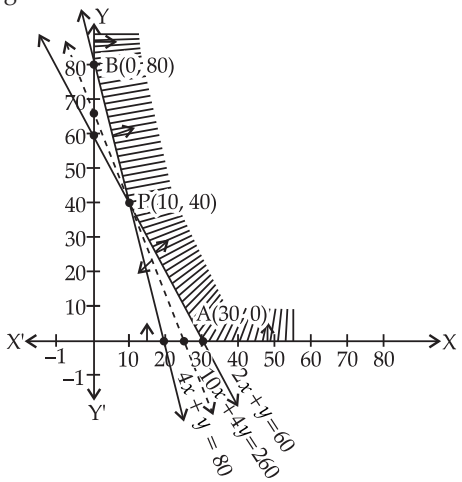
Now, $4x + y = 80$ ...(1)

x	0	20
y	80	0

and $2x + y = 60$ ...(2)

x	0	30
y	60	0

On plotting the graph of above constraints or inequalities (ii), (iii), (iv) and (v), we get shaded region having corner points A, P, B as feasible region. For coordinate of P



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Point of intersection of

$2x + y = 60$ …(vi)

and $4x + y = 80$ …(vii)

From (vi) – (vii),

$2x + y - 4x - y = 60 - 80$

$\Rightarrow \quad -2x = -20$

$\Rightarrow \quad x = 10$

$\Rightarrow \quad y = 40$

$\because$ Co-ordinates of $P = (10, 40)$

Now the value of Z is evaluated at corner points in the following table:

Corner Point	$Z = 10x + 4y$
A(30, 0)	300
P(10, 40)	260 ← Minimum
B(0, 80)	320

Since, feasible region is unbounded. Therefore, we have to draw the graph of the inequality.

$10x + 4y < 260$

i.e., $5x + 2y < 130$ …(viii)

Since, the graph of inequality (viii) does not have any point common.

So, the minimum value of Z is 260 at (10, 40).

i.e., minimum cost of each bottle is ₹ 260, if the company purchases 10 packets of mixes from S and 40 packets of mixes from supplier T. **1**

COMPETENCY BASED QUESTIONS

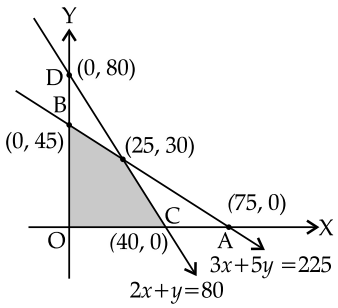
Case based MCQs

Attempt any four sub-parts from each question. Each sub-part carries 1 mark.

I. Read the following text and answer the following questions on the basis of the same:

A manufacturer produces two Models of bikes-Model X and Model Y. Model X takes a 6 man-hours to make per unit, while Model Y takes 10 man-hours per unit. There is a total of 450 man-hours available per week. Handling and Marketing costs are ₹ 2,000 and ₹ 1,000 per unit for Models X and Y respectively. The total funds available for these purposes are ₹ 80,000 per week. Profits per unit for Models X and Y are ₹ 1,000 and ₹ 500, respectively.

The feasible region of LPP is shown in the following graph.



Q. 1. The equation of line AB is

(A) $3x + 5y = 225$ **(B)** $2x + y = 80$

(C) $5x + 3y = 225$ **(D)** $x + 2y = 80$

Ans. Option (A) is correct.

Explanation: From the given graph $OA = 75$ and $OB = 45$.

The equation of line AB is $\frac{x}{75} + \frac{y}{45} = 1$

i.e., $3x + 5y = 225$

Q. 2. The equation of line CD is

(A) $3x + 5y = 225$ **(B)** $2x + y = 80$

(C) $5x + 3y = 225$ **(D)** $x + 2y = 80$

Ans. Option (B) is correct.

Explanation: From the given graph $OC = 40$ and $OD = 80$.

The equation of line CD is $\frac{x}{40} + \frac{y}{80} = 1$

i.e., $2x + y = 80$

Q. 3. The coordinates of point E are

(A) (25, 30) **(B)** (30, 25)

(C) (25, 25) **(D)** (30, 30)

Ans. Option (A) is correct.

Explanation: On solving the equations of lines AB and CD, we get the coordinates of point E *i.e.*, (25, 30).

Q. 4. The constraints for L.P.P. are:

(A) $3x + 5y \le 225, 2x + y \le 80, x \ge 0, y \ge 0$

(B) $3x + 5y \ge 225, 2x + y \le 80, x \ge 0, y \ge 0$

(C) $3x + 5y \le 225, 2x + y \ge 80, x \ge 0, y \ge 0$

(D) $3x + 5y \le 225, 2x + y \le 80, x \le 0, y \le 0$

Ans. Option (A) is correct.

Explanation: Let the manufacturer produces x number of model X and y number of model Y bikes. Model X takes 6 man-hours to make per unit and model Y takes 10 man-hours to make per unit.

There is total of 450 man-hours available per week.

$\therefore \quad 6x + 10y \le 450$

$\Rightarrow \quad 3x + 5y \le 225$ ...(i)

For models X and Y, handling and marketing costs are ₹ 2,000 and ₹ 1,000, respectively, total funds available for these purposes are ₹ 80,000 per week.

$\therefore \quad 2{,}000x + 1{,}000y \le 80{,}000$

$\Rightarrow \quad 2x + y \le 80$ ...(ii)

Also, $x \ge 0, y \ge 0$

Alternate Solution: As (0, 0) lies in the region $3x + 5y \le 225$ and also (0, 0) lies in the region $2x + y \le 80$, therefore the constraints for the L.P.P. are :

$3x + 5y \le 225, 2x + y \le 80, x \ge 0, y \ge 0$

Q. 5. How many bikes of model X and model Y should the manufacturer produce so as to yield a maximum profit?

(A) (40, 0) **(B)** (25, 30)

(C) (25, 45) **(D)** (40, 25)

Ans. Option (B) is correct.

Explanation: The objective function for given L.P.P. is

$Z = 1000x + 500y$

From the shaded feasible region, it is clear that coordinates of corner points are (0, 0), (40, 0), (25, 30) and (0, 45).

Corner points	Value of $Z = 1000x + 500y$
(0, 0)	0
(40, 0)	40,000 ← Maximum
(25, 30)	25,000 + 15,000 = 40,000 ← Maximum
(0, 45)	22,500

So, the manufacturer should produce 25 bikes of model X and 30 bikes of model Y to get a maximum profit of ₹ 40,000.

II. Read the following text and answer the following questions on the basis of the same:

A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/ cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is ₹ 25 and that from a shade is ₹ 15.

If x is the number of lamps and y is the number of shades manufactured. Assuming that the manufacturer can sell all the lamps and shades that he produces.

Q. 1. In order to maximize the profit, which of the flowing objective function is correct:

(A) Max. $Z = 25x + 15y$

(B) Min. $Z = 25x + 15y$

(C) Max. $Z = 25x - 15y$

(D) Max. $Z = 15x + 25y$

Ans. Option (A) is correct.

Explanation:

Explanation: Since profit from the sale of a lamp = ₹ 25

And profit from the sale of a shade = ₹ 15

The associative objective function is

Max. $Z = 25x + 15y$

Q. 2. Which of the following constraint are related to the given LPP?

(A) $2x + y \ge 12; 3x + 2y \le 20$

(B) $2x + y \le 12; 3x + 2y \le 20$

(C) $2x - y \ge 12; 3x + 2y \le 20$

(D) $2x + y \le 12; 3x - 2y \le 20$

Ans. Option (B) is correct.

Explanation:

	Lamp (x)	Shade (y)	
Cutting/grinding	2	2	12
Sprayer	3	3	20

So, constraints are:

$2x + y \le 12$

$3x + 2y \le 20$

Q. 3. The non-negative constraints associative to the given LPP are :

(A) $x \ge 0, y \ge 1$ **(B)** $x \ge 1, y \ge 0$

(C) $x \ge 0, y \ge 0$ **(D)** $x \ge 1, y \ge 1$

Ans. Option (C) is correct.

Explanation: The non-negative conditions are:

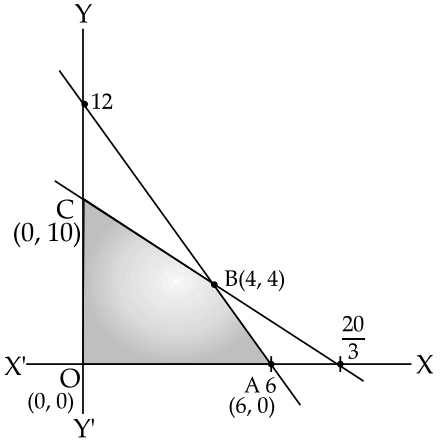
$x \ge 0, y \ge 0$

Q. 4. Which of the following is not the vertices of feasible region:

(A) (6, 4) **(B)** (0, 0)
(C) (6, 0) **(D)** (4, 4)

Ans. Option (A) is correct.

Explanation: **Vertices of feasible region are** O(0, 0), A (6, 0), B(4, 4), and C(0, 10).



Q. 5. The maximum profit is:

(A) 150 **(B)** 120
(C) 130 **(D)** 160

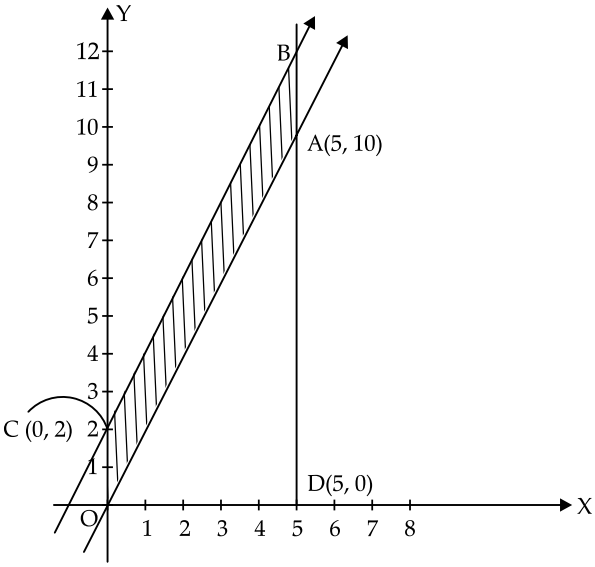
Ans. Option (D) is correct.

Explanation:

Corner Points	$Z = 25x + 15y$
O(0,0)	$Z = 0$
A(6,0)	$Z = 150$
B(4,4)	$Z = 160 \rightarrow$ Maximum
C(0, 10)	$Z = 150$

III. Read the following text and answer the following questions on the basis of the same:

The feasible region for an LPP is shown in the following figure. The CB is parallel to OA.



Q. 1. The equation of line OA is

(A) $x - 2y = 0$ **(B)** $y - 2x = 0$
(C) $x + 2y = 0$ **(D)** $2x + y = 0$

Ans. Option (B) is correct.

Explanation: The point $A(5, 10)$ lies on the equation $y - 2x = 0$, therefore the equation of line OA is $y - 2x = 0$.

Q. 2. The equation of line BC is

(A) $y - 2x = 2$ **(B)** $x - y = 2$
(C) $2y - x = 2$ **(D)** $x + 2y = 2$

Ans. Option (A) is correct.

Explanation: Point on line BC *i.e.*, $C(0, 2)$ lies on the equation $y - 2x = 2$, therefore equation of line BC is $y - 2x = 2$.

Q. 3. The co-ordinates of the point B are

(A) (5, 7) **(B)** (7, 5)
(C) (5, 12) **(D)** (12, 5)

Ans. Option (C) is correct.

Explanation: Point B is the intersection point of line BC and BD.

So, substituting $x = 5$ in $y - 2x = 2$,

we get $y = 12$

Thus, required coordinates are (5, 12).

Q. 4. The constraints for the L.P.P. are

(A) $y \geq 2x, y - 2x \leq 2, x \leq 5, x \geq 0, y \geq 0$
(B) $y \leq 2x, y - 2x \leq 2, x \leq 5, x \geq 0, y \geq 0$
(C) $y \geq 2x, y - 2x \geq 2, x \leq 5, x \geq 0, y \geq 0$
(D) $y \leq 2x, y - 2x \geq 2, x \leq 5, x \geq 0, y \geq 0$

Ans. Option (A) is correct.

Explanation: The required constrains for L.P.P. are

$$y \geq 2x$$
$$y - 2x \leq 2$$
$$x \leq 5$$
$$x \geq 0, y \geq 0$$

Q. 5. The minimum value of the objective function $z = 2x - 3y$ is

(A) 0 **(B)** – 10
(C) – 26 **(D)** – 6

Ans. Option (C) is correct.

Explanation: The vertices of feasible region are O(0, 0), A(5, 10), B(5, 12) and C(0, 2).

So, Z is minimum at (5, 12)

i.e., $Z = 2(5) - 3(12)$
$= 10 - 36 = -26$

IV. Read the following text and answer the following questions on the basis of the same:

A dealer in rural area wishes to purchase a number of sewing machines. He has only ₹5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him ₹360 and a manually operated sewing machine ₹240. He can sell an electronic sewing machine at a profit of ₹22 and a manually operated machine at a profit of ₹18. Assume that the electronic sewing machines he can sell is x and that of manually operated machines is y.

Q. 1. The objective function is _____.

(A) Maximise $Z = 360x + 240y$

(B) Maximise $Z = 22x + 18y$

(C) Minimise $Z = 360x + 240y$

(D) Minimise $Z = 22x + 18y$

Ans. Option (B) is correct.

Q. 2. The maximum value of $x + y$ is _____.

(A) 5760 **(B)** 18

(C) 22 **(D)** 20

Ans. Option (D) is correct.

Q. 3. Which of the following is not a constraint?

(A) $x + y \geq 20$

(B) $360x + 240y \leq 5{,}760$

(C) $x \geq 0$

(D) $y \geq 0$

Ans. Option (A) is correct.

Q. 4. The profit is maximum when (x, y) = _____.

(A) (5, 15) **(B)** (8, 12)

(C) (12, 8) **(D)** (15, 5)

Ans. Option (B) is correct.

Q. 5. The maximum profit is ₹ _____.

(A) 5,760 **(B)** 392

(C) 362 **(D)** 290

Ans. Option (B) is correct.

Explanation:

Objective function:

Maximise $Z = 22x + 18y$

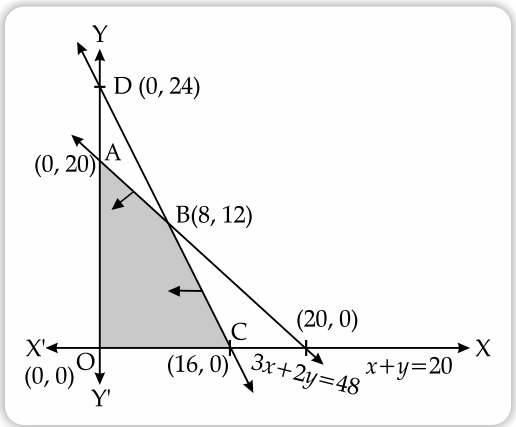
Constraints:

$$x + y \leq 20$$

$$360x + 240y \leq 5{,}760$$

Or $$3x + 2y \leq 48$$

$$x \geq 0, \quad y \geq 0$$



Vertices of feasible region are :

$A(0, 20)$, $B(8, 12)$, $C(16, 0)$ & $O(0, 0)$

$P(A) = 360$, $P(B) = 392$, $P(C) = 352$

$\therefore$ For Maximum P, Electronic machines = 8,

and Manual machines = 12, Max. profit ₹392

V. Read the following text and answer the following questions on the basis of the same:

A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, Phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine.

kg per bag		
	Brand P	**Brand Q**
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

Q. 1. The objective function to minimise the amount of nitrogen added to garden ?

(A) Maximise $Z = 3x + 4y$

(B) Maximise $Z = 3x + 3.5y$

(C) Maximise $Z = 4x + 3.5y$

(D) Maximise $Z = 3x + 4y$

Ans. Option (B) is correct.

Q. 2. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of brand P should be used ?

(A) 40 **(B)** 50

(C) 100 **(D)** 60

Ans. Option (A) is correct.

Q. 3. If the grower wants to minimise the amount of nitrogen added to the garden, how many bag of brand Q should be used ?

(A) 40 **(B)** 50

(C) 100 **(D)** 60

Ans. Option (C) is correct.

Q. 4. What is the minimum amount of nitrogen added

(A) 595 kg **(B)** 550 kg

(C) 400 kg **(D)** 470 kg

Ans. Option (D) is correct.

Q. 5. What is the total number of bags used by fruit grower to minimise the amount of nitrogen ?

(A) 160 **(B)** 190

(C) 140 **(D)** 130

Ans. Option (C) is correct.

Explanation: Let the fruit grower use x bag of brand P and y bags of brand Q.

The problem can be formulated as follows:

Minimise $Z = 3x + 3.5y$ ...(i)

Subject to the constraints,

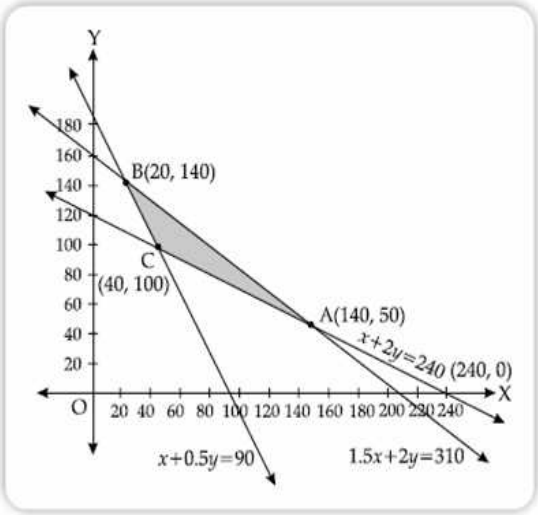
$$x + 2y \geq 240 \quad \text{...(ii)}$$

$$x + 0.5y \geq 90 \quad \text{...(iii)}$$

$$1.5x + 2y \leq 310 \quad \text{...(iv)}$$

$x, y \geq 0$ ...(v)

The feasible region determined by the system of constraints is as follows:



The corner points are $A(240, 50)$, $B(20, 140)$, and $C(40, 100)$

Corner Points	$Z = 3x + 3.5y$	
A(140, 50)	595	
B(20, 140)	550	
C(40, 100)	470	← Minimum

The minimum value of Z is 470 at (40, 100).

Thus, 40 bags of brand P and 100 bags of brand Q should be added to the garden to minimise the amount of nitrogen.

The minimum amount of nitrogen added to the garden is 470 kg.

Case based Subjective Questions

I. Read the following text and answer the following questions on the basis of the same:
Each sub-part carries 2 marks.

Seema rides her bike at 25 km/h. She has to spend ₹2 per km on diesel and if she rides it at faster speed of 40 km/h, the diesel cost increases to ₹5 per km. She has ₹100 to spend on diesel. Let she travels x km with speed 25 km/h and y km with speed 40 km/h. It is given that objective function is Max. $Z = x + y$.

Q. 1. Construct the LPP and draw the graph for feasible region.

Sol. Required LPP is :

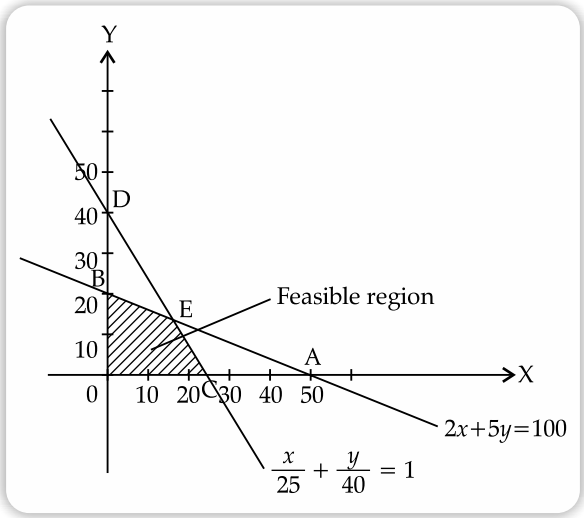
$$\text{Max } Z = x + y$$

Subject to constraints :

$$2x + 5y \leq 100$$

$$\frac{x}{25} + \frac{y}{40} \leq 1$$

$$x, y \geq 0$$ **1**



1

Q. 2. Write the corner points of feasible region and find maximum value of objective function.

Sol. Intersection point of $2x + 5y = 100$ and $\frac{x}{25} + \frac{y}{40} = 1$ is E.

On solving the equations, we get the coordinates of E are $\left(\frac{50}{3}, \frac{40}{3}\right)$.

So, corner points of LPP are :

O(0, 0), B(0, 20), C(25, 0), $E\left(\frac{50}{3}, \frac{40}{3}\right)$ **1**

Corner points	Value of $Z = x + y$
(0, 0)	0
(0, 20)	20
(25, 0)	25
$\left(\frac{50}{3}, \frac{40}{3}\right)$	30 → Maximum

Thus, maximum value of Z is 30. **1**

II. Read the following text and answer the following questions on the basis of the same:

Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a

corner points (vertex) of the feasible region.

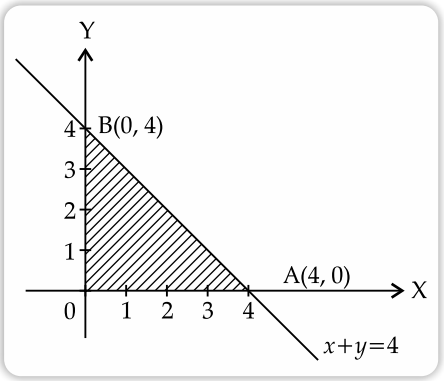
Q. 1. Solve the following LPP graphically :

Max. $Z = 2x + 3y$

subject to $x + y \le 4$

$x \ge 0, y \ge 0$

Sol. Graph of the given LPP is shown as follows.



1

Corner points of feasible region are:
O(0, 0), A(4, 0) and B(0, 4)

Corner points	Value of Z
0(0, 0)	0
A(4, 0)	8
B(0, 4)	12 → Maximum

∴ Maximum value of Z is 12. **1**

Q. 2. Draw the graph of given LPP and find the corner points of feasible region.

Minimize $Z = 200x + 500y$

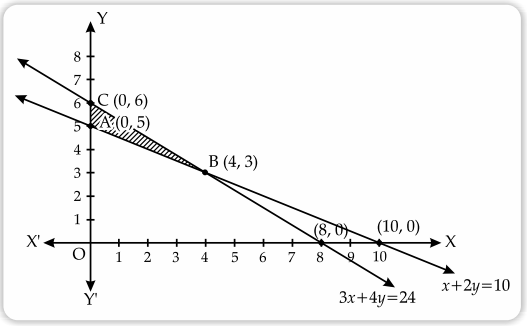
Subject to constraints:

$x + 2y \ge 10$

$3x + 4y \le 24$

$x \ge 0, y \ge 0$

Sol.



1

The corner points of feasible region are :

A(0, 5), B(0, 6) and B(4, 3). **1**

[Coordinates of point B is obtained by solving equations $x + 2y = 10$ and $3x + 4y = 24$ simultaneously]

Solutions for Practice Questions

Very Short Answer Type Questions

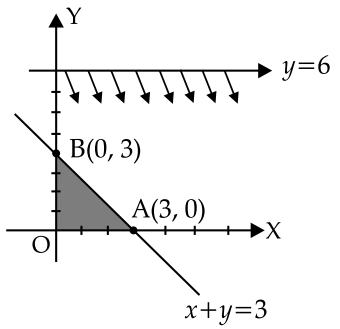
Sol. 3: The feasible region as shown in the figure, has objective function $Z = 3x - 4y$.

Corner points	Corresponding value of $Z = 3x - 4y$
(0, 0)	0
(12, 6)	12 ← Maximum
(0, 4)	–16 ← Minimum

Hence, the maximum value of Z is 12 at (12, 6). **1**

Short Answer Type Questions-I

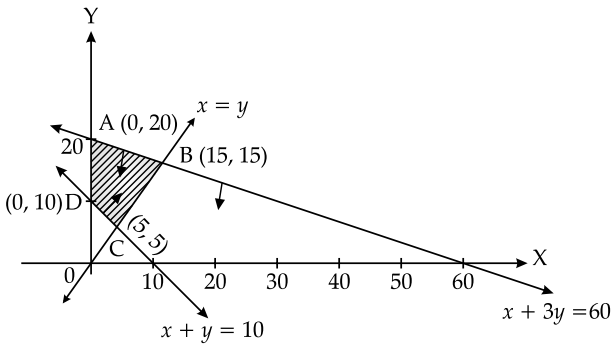
Sol. 3:



Thus, feasible region is *OAB*. **2**

Short Answer Type Questions-II

Sol. 4:



2

Vertices of feasible region are
A(0, 20), B(15, 15), C(5, 5), D(0, 10)

Corner points	Value of $Z = 3x + 9y$
(0, 20)	180
(15, 15)	180
(5, 5)	60 → minimum
(0, 10)	90

∴ $Z = 60$ is minimum at $x = 5, y = 5$ **1**

Long Answer Type Questions

Sol. 5:

	Toy A	Toy B	Time in a day
Cutting time	5 min	8 min	180 min
Assembling time	10 min	8 min	240 min

Objective function

Minimize: $Z = 50x + 60y$

Subject to: $10x + 8y \le 240$

or $5x + 4y \le 120$ ...(i)

$x \ge 0, y \ge 0$ ...(ii) **2**

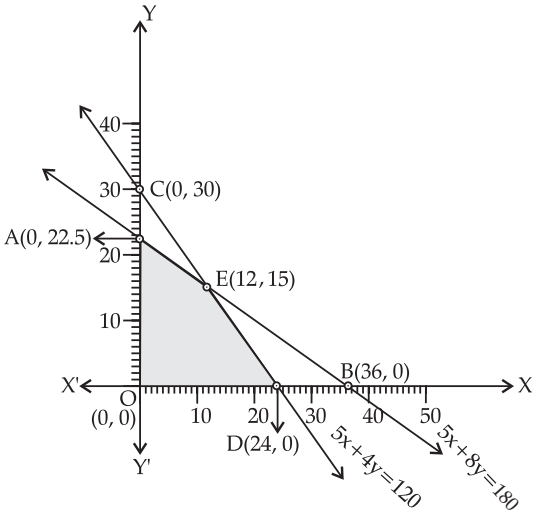
$5x + 8y \le 180$ ...(iii)

For, $5x + 8y = 180$

	A	B	E
x	0	36	12
y	22.5	0	15

For, $5x + 4y = 120$

	C	D	E
x	0	24	12
y	30	0	15



2

Corner points	$Z = 50x + 60y$
At O(0, 0)	0
At D(24, 0)	1200
At E(12, 15)	1500 (Max.)
At A(0, 22.5)	1350

Hence the maximum profit is ₹1500 at E(12, 15). Therefore 12 units of type A and 15 units of type B is produced. **1**

Commonly Made Error

- Most mistakes happened while expressing the constraints in the form of equations with both sides expressed in the same units. Some candidates omitted the graphical representation of the in equations and erred in finding the feasible area and points.

Answering Tip

- Linear programming as a topic requires knowledge of line sketching. Hence, a little revision in this matter is essential. Correct depiction and shading of required area is a big advantage. Neat diagram help a lot. Casual approach is to be avoided and little extra practice is required.

Sol. 9: Let x kg of bran and y kg rice be produced.

Minimize, $Z = 28x + 25y$

Subject to constraints

$$\frac{80}{1000}x + \frac{100}{1000}y \ge \frac{88}{1000}$$

$\Rightarrow 20x + 25y \ge 22$ ...(1)

and $\frac{0.040}{1000}x + \frac{0.030}{1000}y \ge \frac{0.036}{1000}$

$\Rightarrow 20x + 15y \ge 18$ ...(2)

$x \ge 0, y \ge 0$ **2**

First we draw the line AB and CD whose equations are $20x + 25y \ge 22$ and $20x + 15y \ge 18$, respectively.

Now,

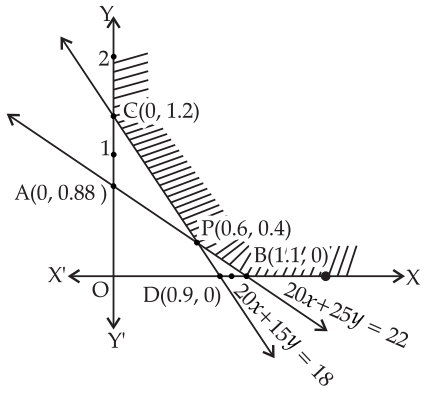
$20x + 25y = 22$

	A	B
x	0	1.1
y	0.88	0

and $20x + 15y = 18$

	C	D
x	0	0.9
y	1.2	0

The feasible region is OBPCO



2

The vertices of the feasible region are B(1.1, 0), P (0.6, 0.4) and C(0, 1.2). P is the point of intersection of the lines $20x + 25y = 22$ and $20x + 15y = 18$. Solving these we get the point P(0.6, 0.4)

The minimum value of Z is:

Corner points	$Z = 28x + 25y$
At B(1.1, 0)	$Z = 28 \times 1.1 + 25 \times 0 = 30.8$
At P(0.6, 0.4)	$Z = 28 \times 0.6 + 25 \times 0.4 = 26.8$
At C(0, 1.2)	$Z = 0 + 25 \times 1.2 = 30$

Hence, the minimum value of Z is 26.8 at the point P(0.6, 0.4)

Since, feasible region is unbounded,

$\therefore$ we have to draw the graph $28x + 25y < 26.8$

Since the graph of inequality does not have any common part.

So, the minimum value is 26.8. **1**

REFLECTIONS

(a) Do you think, farmers can use linear programming techniques to determine the what crop they should grow and how much quantity they grow to get maximum profit?

(b) Can you apply linear programming problem to solve designing and manufacturing problems?

(c) Do you think transportation system rely upon linear programming for cost and time efficiency?

SELF ASSESSMENT PAPER - 08

Maximum Time: 30 **Max. Time: 1 hour**

The following questions (No. 1–6) are Multiple Choice Questions carrying 1 mark each.

Q. 1. Which of the following statements is correct ?

(A) Every LPP has at least one optimal solution **(B)** Every LPP has a unique solution

(C) If an LPP has two optimal solutions, then it has infinitely many solutions

(D) If a feasible region is unbounded, then LPP has no solution

Q. 2. Corner points of the bounded feasible region for an LPP are (0, 4), (6, 0), (12, 0), (12, 16) and (0, 10). Let $Z = 8x + 12y$ be the objective function.

Match the following:

(i) Minimum value of Z occurs at _____ **(ii)** Maximum value of Z occurs at _____

(iii) Maximum of Z is _____ **(iv)** Minimum of Z is _____

(A) (i) (6, 0), (ii) (12, 0), (iii) 288, (iv) 48 **(B)** (i) (0, 4), (ii) (12, 16), (iii) 288, (iv) 48

(C) (i) (0, 4), (ii) (12, 16), (iii) 288, (iv) 96 **(D)** (i) (6, 0), (ii) (12, 0), (iii) 288, (iv) 96

Q. 3. Solution of LPP maximize $Z = 2x - y$ subject to

$$x + y \leq 2$$
$$x, y \geq 0$$

(A) 0 **(B)** 4

(C) 2 **(D)** None of these

Q. 4. The vertices of a feasible region by the linear constraints

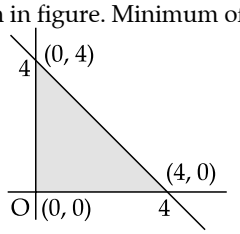
$$3x + 4y \leq 18$$
$$2x + 3y \leq 3$$

and $$x, y \leq 0$$

(A) (0, 3), (3, 0) and (4.8, 0) **(B)** (0, 2), (2, 0) and (4, 0)

(C) (0, 0), (0, 1) and (1.5, 0) **(D)** (0, 3), (3, 0) and (1.5, 0)

Q. 5. Feasible region shaded for a LPP is shown in figure. Minimum of $Z = 2x + 3y$ occurs at the point



(A) (0, 0) **(B)** (4, 0)

(C) (0, 4) **(D)** none of these

Q. 6. The graph of $x \leq 2$ and $y \geq 2$ will be situated in

(A) First and Second Quadrants **(B)** First and Third Quadrants

(C) Third and Second Quadrants **(D)** First and Fourth Quadrants

Case-based MCQs

Q. 7. Read the following text and answer the following questions on the basis of the same:

Attempt any four parts. Each part carries 1 mark.

A share is referred to as a unit of ownership which represents an equal proportion of a company's capital. A share entitles the shareholders to an equal claim on profit and loss of the company.

Dr. Ritam wants to invest at most ₹ 12,000 in two type of shares A and B. According to the rules, she has to invest at least ₹ 2,000 in share A and at least ₹ 4,000 in share B. If the rate of interest on share A is 8% per annum and on share B is 10% per annum.

(i) If Dr. Ritam invests ₹ x in share A, then which of the following is correct ?

(A) $x = 2000$ **(B)** $x < 2000$ **(C)** $x \leq 2000$ **(D)** $x \geq 2000$

(ii) If she invest ₹ y in share B, then which of the following is correct ?

(A) $y = 4000$ **(B)** $y < 4000$ **(C)** $y \leq 4000$ **(D)** $y \geq 4000$

(iii) If total interest received by Dr. Ritam from both type of shares is represented by Z, then Z is equal to

(A) $Z = 2x + y$ **(B)** $Z = x + 2y$ **(C)** $Z = \frac{2x}{25} + \frac{y}{10}$ **(D)** $Z = \frac{2x}{10} + \frac{y}{25}$

(iv) To maximise interest on both types of share, the invested amount on both shares A & B by her should be respectively

(A) ₹ 10,000; ₹ 2,000 **(B)** ₹ 2,000; ₹ 10,000 **(C)** ₹ 8,000; ₹ 4,000 **(D)** ₹ 2,000; ₹ 4,000

(v) The maximum interest received by her is

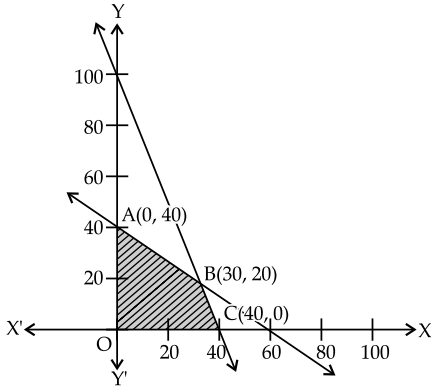
(A) ₹ 1,040 **(B)** ₹ 3,000 **(C)** ₹ 1160 **(D)** ₹ 1,200

The following questions (No. 8-10) are Very Short Answer Type Questions and carry 1 mark each.

Q. 8. The corner points of the feasible region determined by the system of linear constraints are (0, 10), (5, 5) (15, 15), (0, 20). Let $Z = px + qy$, where $p, q > 0$. Find the condition on p and q so that the maximum of Z occurs at both the points (15, 15) and (0, 20).

Q. 9. For the LPP Minimize $Z = 2x + 3y$ the co-ordinates of the corner points of the bounded feasible region are A(3, 3), B(20,3), C(20, 10), D(18, 12) and E(12, 12). Find the minimum value of Z.

Q. 10. The feasible region for an LPP is shown below:



Let the objective function, $Z = 3x + 4y$.

Find the maximum value of Z.

The following questions (No. 11–13) are short answer Type – I and carry 2 marks each.

Q. 11. Define two different types of Linear Programming Problems.

Q. 12. Write limitations of LPP.

Q. 13. Define the following:

(a) Decision variables

(b) Feasible Region

Questions (No. 14–15) are Short Answer Type – II and carry 3 marks each.

Q. 14. Maximise $Z = x + 2y$

Subject to the constraints

$$x + 2y \geq 100$$
$$2x - y \geq 0$$
$$2x + y \leq 200$$
$$x, y \leq 0$$

Q. 15. Solve the following linear problem graphically.

Maximize $Z = 5x + 3y$

Subject to constraints:

$$3x + 5y \leq 15$$
$$5x + 2y \leq 10$$
$$x \leq 0, \text{ and } y \geq 0$$

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 16. A toy company manufactures two types of dolls A and B. Market test and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demands for the dolls of type B is almost half of that for dolls of type A. Further, the production level of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of ₹ 12 and ₹ 16 per doll respectively on dolls A and B, how many of each type of dolls should be produced weekly, in order to maximise the profit ?

■■

Practice Paper-1

(As per Board Sample Question Paper 2023-24 released on 31st March 2023)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions:

(i) *This question paper contains five sections A, B, C, D and E. Each section is compulsory.*
(ii) *Section - A carries 20 marks weightage, Section - B carries 10 marks weightage, Section - C carries 18 marks weightage, Section - D carries 20 marks weightage and Section - E carries 3 case-based with total weightage of 12 marks.*
(iii) *Section – A: It comprises of 20 MCQs of 1 mark each.*
(iv) *Section – B: It comprises of 5 VSA type questions of 2 marks each.*
(v) *Section – C: It comprises of 6 SA type of questions of 3 marks each.*
(vi) *Section – D: It comprises of 4 LA type of questions of 5 marks each.*
(vii) *Section – E: It has 3 case studies. Each case study comprises of 3 case-based questions, where 2 VSA type questions are of 1 mark each and 1 SA type question is of 2 marks. Internal choice is provided in 2 marks question in each case-study.*
(viii) *Internal choice is provided in 2 questions in Section - B, 2 questions in Section – C, 2 questions in Section - D. You have to attempt only one of the alternatives in all such questions.*

Section - A

All questions are compulsory. No internal choice is provided in this section

1. The solution of $\frac{x-3}{x+5} > 0, x \neq -5, x \in \mathbb{R}$ is:

(A) $x > 3$ **(B)** $x < -5$
(C) $x < -5$ or $x > 3$ **(D)** no solution **1**

2. For two distinct positive numbers x and y

(A) $x + y \geq 2\sqrt{xy}$ **(B)** $\frac{x+y}{2} \geq xy$

(C) $\sqrt{xy} \geq \frac{x+y}{2}$ **(D)** $\frac{2xy}{x+y} \geq \sqrt{xy}$ **1**

3. In a one-kilometre race, A, B and C are three participants. A can give B start of 50 m and C a start of 69 m. The start, which B can allow C, is:
(A) 17 m **(B)** 18 m
(C) 19 m **(D)** 20 m **1**

4. Assume that current time is 7:00 p.m. what time (in a.m. or p.m.) will it be in 1000 hours ?
(A) 7 a.m. **(B)** 11 a.m.
(C) 11 p.m. **(D)** 2 a.m. **1**

5. If the matrix A is both symmetric and skew symmetric, then
(A) A is a diagonal matrix **(B)** A is a zero matrix
(C) A is a square matrix **(D)** None of these **1**

6. If A is matrix of order $m \times n$ and B is a matrix such that AB′ and B′A are both defined, then order of matrix B is:
(A) $m \times m$ **(B)** $n \times n$
(C) $n \times m$ **(D)** $m \times n$ **1**

7. If solving a system of linear equations in 3 variables by Cramer's rule, we get $\Delta = 0$ and at least one of Δ_x, Δ_y, Δ_z is non-zero then the system of linear equations has
(A) no solution **(B)** unique solution
(C) infinitely many solutions **(D)** trivial solution **1**

8. Nature of the function $f(x) = e^{2x}$ is _____
(A) Increasing **(B)** Decreasing
(C) Constant **(D)** Increasing and Decreasing **1**

9. Systems that exhibit exponential growth increase according to the mathematical model $y = y_0e^{kt}$ where y_0 represents the initial state of the system and $k > 0$ is a constant, is called the……constant.
(A) Growth **(B)** Decay
(C) Simple **(D)** Normal **1**

10. The order of the differential equation $2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$ is:

(A) 2 **(B)** 1
(C) 0 **(D)** Not defined **1**

11. Let X be a discrete random variable whose probability distribution is given below:

$X = x_i$	0	1	2	3	4	5	6	7
$P(X = x_i)$	0	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2$	$2K$

The value of K is:

(A) $\frac{1}{10}$ **(B)** –1

(C) $-\frac{1}{10}$ **(D)** $\frac{1}{5}$ **1**

12. If 'm' is the mean of Poisson distribution, then its standard deviation is given by:

(A) $\sqrt{m}$ **(B)** m^2

(C) m **(D)** $\frac{m}{2}$ **1**

13. If X is a normal variate with mean μ and standard deviation σ > 0, then the new random variate $Z = \frac{X-\mu}{\sigma}$ is a variate with:

(A) Mean = 1, Standard deviation = 0. **(B)** Mean = 1, Standard deviation = 1.
(C) Mean = 2, Standard deviation = 1. **(D)** Mean = 0, Standard deviation = 1. **1**

14. Which of the following is not an example of a time series model?
(A) Native approach **(B)** Exponential smoothing
(C) Moving average **(D)** None of these **1**

15. If desired rate of return is minimum by actual rate of return then it is classified as:
(A) future cash flow **(B)** present cash flow
(C) positive cash flow **(D)** negative cash flow **1**

16. The assumption in calculating annuity is that every payment is:
(A) equal **(B)** different
(C) nominal **(D)** marginal **1**

17. The accumulated series of deposits as future sum of money is classified as:
(A) marginal fund **(B)** nominal fund
(C) sinking fund **(D)** annuity fund **1**

18. A company buys a machine at a cost of ₹ 5000. The company decides on a salvage value of ₹ 1000 and a useful life of 5 years, then annual depreciation cost is:
(A) ₹ 500 **(B)** ₹ 600
(C) ₹ 700 **(D)** ₹ 800 **1**

DIRECTION: For questions 19 and 20, two statements are given – one labelled Assertion(A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (A), (B), (C) and (D) as given below:
(A) Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).
(B) Both assertion (A) and reason (R) are true and reason (R) is not the correct explanation of assertion (A).
(C) Assertion (A) is true but reason (R) is false.
(D) Assertion (A) is false but reason (R) is true.

19. **Assertion (A):** If $A=\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is such that $A^2 = I$, then

$1 + \alpha^2 + \beta\gamma = 0$

Reason (R): If $A=\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is such that $A^2 = I$, then

$1 - \alpha^2 - \beta\gamma = 0$ **1**

20. **Assertion (A):** The slope of normal to the curve $y = 4x^2 - 14x + 5$ at $x = 5$ is $-\frac{1}{26}$.

Reason (R): The slope of the tangent at $x = 5$ is given by $\left.\frac{dy}{dx}\right|_{x=5}$ **1**

Section - B

All questions are compulsory. In case of internal choice, attempt any one question only

21. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then find the other two roots. **2**

OR

Find the inverse of the matrix $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

Hence, find the matrix P satisfying the matrix equation $P\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

22. Evaluate: $\int \frac{e^x(x-3)}{(x-1)^3} dx$ **2**

OR

Find: $\int \frac{\left(x^4 - x\right)^{\frac{1}{4}}}{x^5} dx$

23. Consider the following hypothesis:

$H_0 : \mu = 36$

$H_1 : \mu \neq 36$

A sample of 81 items is taken whose mean is 37.5 and the standard deviation is 5. Test the hypothesis at 5% level of significance.

[Given: Critical value of Z for a two-tailed test at 5% level of significance is 1.96] **2**

24. Define secular trend and seasonal variations. **2**

25. A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch ? **2**

Section - C

All questions are compulsory. In case of internal choice, attempt any one question only

26. A wholesaler in apples claims that only 4% of the apples supplied by him are defective. A random sample of 600 apples contained 36 defective apples. Calculate the standard error concerning good apples. **2**

OR

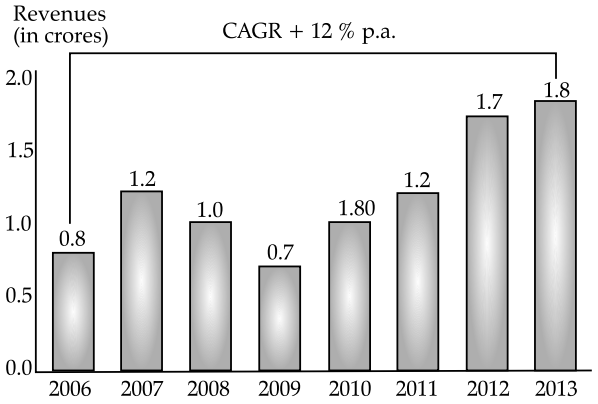
A machine which produces mica insulating washers of use in the electric devices is set to turn out washers having a thickness of 10 mils (1 mil = 0.001 inch). A sample of 10 washers has an average thickness of 9.52 mils with a standard deviation of 0.60 mil. Find out *t*.

27. From the following time series obtain trend value by 3 yearly moving averages. **2**

Year	Sales (in ₹ 000)	Year	Sales (in ₹ 000)
2008	8	2014	16
2009	12	2015	17
2010	10	2016	14
2011	13	2017	17
2012	15		
2013	12		

28. An interviewer gives the following graph on a client's sales in the last 7 years to candidate and said find the CAGR.

Given that $\left(\frac{9}{4}\right)^{\frac{1}{7}} = 1.1228$. **2**



29. Solve the following linear programming problem graphically:
Minimize: $Z = 6x + 3y$

Subject to the constraints: $\begin{cases} 4x + y \geq 80 \\ x + 5y \geq 115 \\ 3x + 2y \leq 150 \\ x \geq 0, y \geq 0 \end{cases}$ **2**

30. A firm anticipates an expenditure of ₹ 5,00,000 for plant modernization at end of 10 years from now. How much should the company deposit at the end of year into a sinking fund earning interest 5% per annum. [Given log 1.05 = 0.0212, antilog (0.2120) = 1.629] **2**

31. Find (186 × 93) mod 7. **2**

OR

When 783 × 657 × 594 × 432 × 346 × 251 is divided by 5, then find remainder.

Section - D

All questions are compulsory. In case of internal choice, attempt any one question only

32. A manufacturing company makes two types of teaching aids A and B of Mathematics for class X. Each type of A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Formulate this as Linear Programming Problem and solve it. Identify the feasible region from the rough sketch. **5**

33. The cost of a TV depreciates by ₹ 800 during the second year and by ₹ 700 during the third year. Calculate:

(i) the rate of depreciation per annum.
(ii) the original cost of the TV.
(ii) the value of the TV at the end of third year. **5**

34. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X). **5**

OR

Five bad oranges are accidently mixed with 20 good ones. If four oranges are drawn one by one successively with replacement, then find the probability distribution of number of bad oranges drawn. Hence, find the mean and variance of the distribution.

35. Evaluate: $\int_{-1}^{2} |x^3 - 3x^2 + 2x| \, dx$ **5**

OR

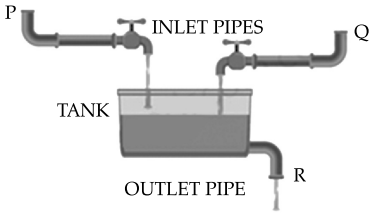
The demand and supply function of an article are $D(q) = 1000 - 0.4q^2$ and $S(q) = 42q$. Find the consumer's surplus and producer's surplus at equilibrium price.

Section - E

All questions are compulsory. In case of internal choice, attempt any one question only

CASE STUDY – I

36. Sohil is good in Mathematics. His friend gave him a question from the topic Pipes and Cisterns *i.e.*, "Pipe R can empty a full tank in 30 hours. But two pipes P and Q can fill a tank in 15 hours and 10 hours respectively. Ram unknowingly opened all three taps. After 2 hours Shyam realized it and closed Pipe R."



(i) Find the portion the tank is filled by pipes P and Q in one hour, separately? **1**

(ii) Find the portion the tank is filled in one hour, if P, Q and R all are open? **1**

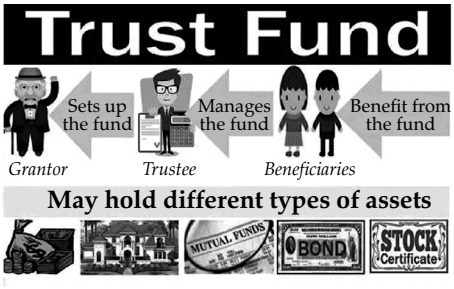
(iii) If R is closed, then entire tank get filled in how much time ? **2**

OR

Find the portion the tank is filled in two hour, if P, Q and R all are open.

CASE STUDY – II

37. A trust invested some money in two types of bonds. The first bond pays 10% interest and second bond pays 12% interest the trust received ₹ 2800 as interest. However if trust had interchanged money in bonds, they would have got ₹ 100 less as interest.



(i) For the given information write the system of equations. **1**

(ii) If the given system of equations can be written as $AX = B$, then find $|A|$. **1**

(iii) Find A^{-1}. **2**

OR

Find the total amount invested.

CASE STUDY – III

38. A manufacturing company manufactures toys the company observed the following costs at different production levels,

Number of toys manufactured	Cost of raw material	Cost at production	Cost at freight	Property tex	Salaries
100	800	2000	1000	5000	20000
150	1200	3000	1500	5000	20000
200	1600	4000	2000	5000	20000
250	2000	5000	2500	5000	20000
300	2400	6000	3000	5000	20000

If the company observer the price 'p' per unit at item sold $p = 5000 - 10x$ where x is number of units sold/production. What is the revenue function R(x) and total cost C(x) of the toys.

OR

Based on the above given information, calculate the marginal as well as profit function.

■■■

Practice Paper-2

(As per Board Sample Question Paper 2023-24 released on 31st March 2023)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions:

(i) *This question paper contains five sections A, B, C, D and E. Each section is compulsory.*
(ii) *Section - A carries 20 marks weightage, Section - B carries 10 marks weightage, Section - C carries 18 marks weightage, Section - D carries 20 marks weightage and Section - E carries 3 case-based with total weightage of 12 marks.*
(iii) *Section – A: It comprises of 20 MCQs of 1 mark each.*
(iv) *Section – B: It comprises of 5 VSA type questions of 2 marks each.*
(v) *Section – C: It comprises of 6 SA type of questions of 3 marks each.*
(vi) *Section – D: It comprises of 4 LA type of questions of 5 marks each.*
(vii) *Section – E: It has 3 case studies. Each case study comprises of 3 case-based questions, where 2 VSA type questions are of 1 mark each and 1 SA type question is of 2 marks. Internal choice is provided in 2 marks question in each case-study.*
(viii) *Internal choice is provided in 2 questions in Section - B, 2 questions in Section – C, 2 questions in Section - D. You have to attempt only one of the alternatives in all such questions.*

Section - A

All questions are compulsory. No internal choice is provided in this section

1. For a determinant containing 4 elements a_1, b_1, a_2 and b_2; what will the elements of leading diagonals ?

(A) a_1, b_1 **(B)** a_1, b_2
(C) a_2, b_1 **(D)** a_2, b_2 **1**

2. If $A = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $AB = I_3$, then $(x + y)$ equals:

(A) 0 **(B)** –1
(C) 2 **(D)** –2 **1**

3. Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3$, $p = n$ **(B)** k is arbitrary, $p = 2$
(C) p is arbitrary, $k = 3$ **(D)** $k = 2$, $p = 3$ **1**

4. Pipe A can fill a tank 6 times faster than a pipe B. If B can fill a tank in 21 minutes, then the time taken by both the pipes together to fill the tank is:

(A) 3 minutes **(B)** $4\frac{1}{2}$ minutes
(C) 7 minutes **(D)** 9 minutes **1**

5. The solution set of the inequation $|x + 2| \leq 5$ is:

(A) (–7, 5) **(B)** [–7, 3]
(C) [–5, 5] **(D)** (–7, 3) **1**

6. A person can row in still water at the rate of 8 km/h. If it takes him thrice as long to row upstream as to row downstream then the speed of the stream is:

(A) 2 km/h **(B)** 3 km/h
(C) 4 km/h **(D)** 6 km/h **1**

7. A retailer buys 250 kg of rice, a part of which he sells at 10% profit and the remaining at 5% loss. If the net profit made by the retailer in the whole transaction is 7%, then the quantity of rice sold at 10% profit is

(A) 200 kg **(B)** 150 kg
(C) 100 kg **(D)** 50 kg **1**

8. The best-fitted trend line is one for which sum of squares of residuals or errors is:

(A) Positive **(B)** Minimum

(C) Negative **(D)** Maximum **1**

9. The variable cost of producing x units is $V(x) = x^2 + 2x$. If the company incurs a fixed cost of ₹ 10,000, then the level of output where the average cost is minimum is

(A) 10 units **(B)** 50 units

(C) 100 units **(D)** 200 units **1**

10. Let $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$ for every real number x. Then

(A) h is increasing whenever f is increasing **(B)** h is increasing whenever f is decreasing

(C) h is decreasing whenever f is increasing **(D)** Nothing can be said in general **1**

11. The value of: $\int (x+3)(x+2)dx$ is:

(A) $\frac{x^3}{3} + \frac{5x^2}{2} + 6x + C$ **(B)** $x^{-3} + \frac{5}{2}x^2 + x + C$

(C) $\frac{x^3}{3} + \frac{x^2}{2} + 2x + C$ **(D)** $\frac{x^3}{3} + \frac{x^2}{2} + x + C$ **1**

12. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. Then the possible values of X are:

(A) 0, 1, 3, 5 **(B)** 0, 2, 4, 6

(C) 0, 2, 5, 6 **(D)** 1, 3, 4, 5 **1**

13. If the mean of a binomial distribution is 81, then the standard deviation lies in the interval:

(A) (0, 9) **(B)** (0, 9]

(C) [0, 3] **(D)** (0, 3] **1**

14. If 'm' is the mean of a Poisson Distribution, then variance is given by ..... .

(A) m^2 **(B)** $m^{1/2}$

(C) m **(D)** $\frac{m}{2}$ **1**

15. The money borrowed or lent for a certain period is:

(A) Amount **(B)** Principal

(C) Interest **(D)** Loan **1**

16. The simple interest on sum for one year is 60. What will be the compound on the same sum and at the same rate of interest for one year ?

(A) 60 **(B)** 66

(C) 70 **(D)** 76 **1**

17. The interest rate per year is 16 and the compounding occurs every quarter then interest rate per compounding period is:

(A) 0.4 **(B)** 0.04

(C) 0.004 **(D)** 40 **1**

18. The nominal rate which is compounded more frequently than annually is equal to interest rate

(A) Compounded monthly **(B)** Discounted monthly

(C) Compounded annually **(D)** Discounted annually **1**

DIRECTION: For questions 19 and 20, two statements are given – one labelled Assertion(A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (A), (B), (C) and (D) as given below:

(A) Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).

(B) Both assertion (A) and reason (R) are true and reason (R) is not the correct explanation of assertion (A).

(C) Assertion (A) is true but reason (R) is false.

(D) Assertion (A) is false but reason (R) is true.

19. **Assertion (A):** If A and B are matrices of order 3 and $|A| = 5$, $|B| = 3$, then find $|3AB| = 405$.

Reason (R): $|3AB| = (3)^3|AB|$ **1**

20. **Assertion (A):** The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is $e^{-y} = -e^{x} + c$.

Reason (R): The given equation is solved by apply variable separation method. **1**

Section - B

All questions are compulsory. In case of internal choice, attempt any one question only

21. If A is a skew-symmetric matrix of order 3, then prove that det $A = 0$. **2**

OR

Without expanding at any stage, find the value of $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$.

22. The following table shows the annual rainfall (in mm) recorded for Cherrapunji, Meghalaya:

Year	Rainfall (in mm)
2001	1.2
2002	1.9
2003	2
2004	1.4
2005	2.1
2006	1.3
2007	1.8
2008	1.1
2009	1.3

Determine the trend of rainfall by 3-years moving average. **2**

23. A random sample of 60 observations was drawn from a large population and its standard deviation was found to be 2.5. Calculate the suitable standard error that this sample is taken from a population with standard deviation 3 ? **2**

24. Evaluate: $\int_0^1 \frac{xe^x}{(x+1)^2} dx$ **2**

OR

There are 1000 birds on an island. They breed with a constant growth rate of 10% per year. How many birds will be on the island after seven year. [Given $e^{0.7} = 2.010$]

25. If $p(x) = \begin{cases} \frac{x}{20}, & x = 0, 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$

Find (i) $P(X < 3)$ and (ii) $P(2 < X \leq 4)$ **2**

Section - C

All questions are compulsory. In case of internal choice, attempt any one question only

26. Prove that $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n. **3**

OR

Solve $\frac{|x|-1}{|x|-2} \geq 0$ $x \in$, $x \neq \pm 2$

27. Maximize $Z = 3x + 4y$, if possible, subject to the constraints:

$$x - y \leq -1$$
$$-x + y \leq 0$$
$$x, y \geq 0$$

3

28. Fit a straight line trend by the method of least squares and find the trend value for the year 2008 for the following data: **3**

Year	Production (in lakh tonnes)
2001	30
2002	35
2003	36
2004	32
2005	37
2006	40
2007	36

29. In a sample of 400 population from a village 230 are found to be eaters of vegetarian items and the rest non-vegetarian items. Compute the standard error assuming that both vegetarian and non-vegetarian foods are equally popular in that village ? **3**

OR

Ten cartons are taken at random from an automatic packing machine. The mean net weight of the ten cartons is 11.8 kg and standard deviation is 0.15 kg. Does the sample mean differ significantly from the intended mean of 12 kg ?

[Given that for d.f. = 9, $t_{0.05} = 2.26$]

30. A company borrowed ₹ 60,000 for renovation. The company plans to set up a sinking fund that will pay back the loan at the end of 5 years. Assuming a rate of 10% compounded semiannually, and the sinking fund of the ordinary annuity. Given that $(1.05)^{10} = 1.6288$. **3**

31. Mr. Lal took a car loan of ₹ 10 lakhs at an 11% interest for a 15 years loan tenure. What would be his EMI ? Given $(1.0091)^{180} = 5.1069$. **3**

Section - D

All questions are compulsory. In case of internal choice, attempt any one question only

32. A dietician wishes to mix two types of food F_1 and F_2 in such a way that the vitamin content of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food F_1 contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C, while Food F_2 contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5 per kg to purchase Food F_1 and ₹ 7 per kg to purchase Food F_2.

Based on the above information, answer the following questions:

(a) To find out the minimum cost of such a mixture, formulate the above problem as a LPP.

(b) Determine the minimum cost of the mixture. **5**

33. If $x \cos(a + y) = \cos y$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. Hence, show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a + y) \frac{dy}{dx} = 0$. **5**

OR

An open topped box is to be made by removing equal squares from each corner of a 3 m by 8 m rectangle sheet of aluminium and by folding up the side. Find the volume of the largest such box.

34. If X is a normal variate with mean 30 and SD 5. Find the probabilities that (i) $26 \leq X \leq 40$ (ii) $X > 45$. **5**

OR

In the proportion of defective in a bulk is 4%. Find the probability of not more than 2 defective in a sample of 10. It is known that $e^{-4} = 0.6703$.

35. A person amortizes a loan of ₹ 15,00,000 for renovation of his house by 8 years mortgage at the rate of 12% p.a. compounded monthly. Find

(i) the equated monthly instalment

(ii) the principal outstanding at the beginning of 40^{th} month.

(iii) the interest paid in 40^{th} payment.

[Given $(1.01)^{96} = 2.5993$, $(1.01)^{57} = 1.7633$] **5**

Section - E

All questions are compulsory. In case of internal choice, attempt any one question only

CASE STUDY – I

36. The demand function for a popular make of 12- speed bicycle is given by $p = D(x) = -0.001x^2 + 250$, where p is the unit price in Rupees and x is the quantity demanded in units of a thousands. The supply function for the same product is given by $p = S(x) = 0.0006x^2 + 0.02x + 100$, where p is the unit price in Rupees and x is the quantity supplied in units of a thousands.

(i) Write the formula for consumer's surplus. **1**

(ii) Write the formula for producer's surplus. **1**

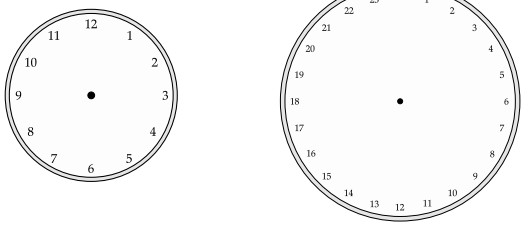
(iii) Find point of intersection. **2**

OR

Find producer's profit in Rs.

CASE STUDY – II

37. The 12 hour clock is most commonly represented on an analogue clock with number 1-12, when shown on a digital clock, it usually accompanied by a.m. or p.m.
The 24 hour clock is more often shown on digital clocks and is written in a 4 digit form, with the first two digits representing the hours and the last two digits representing the minutes.
There is no need for a.m. or p.m. as each time represents each hour in a 24 hour day.



(i) What time in 24 hour clock is equivalent to 7 a.m. in 12 hour clock ? **1**

(ii) Which of two times are same in 12 hours and 24 hours clocks respectively ?
1 : 00 p.m., 13 : 00; 3 : 00 p.m., 03 : 00; 02 : 00 a.m., 14 : 00 and 04 : 00 a.m., 16 : 00 **1**

(iii) It is currently 7 : 00 a.m. in 12 hour clock.
What will be the time in next 493 hours? **2**

OR

If the time after 640 hours from now will be 9 a.m., then what is current time?

CASE STUDY – III

38. Three car dealers, say A, B and C deals in three cars, Sedan, cars Hatchback cars and SUV cars. The sales figure of 220 and 2021 showed that dealer A sold 120 Hatch back, 50 Sedan and 10 SUV cars in 2020 and 300 Hatchback, 150 Sedan and 20 SUV in 2021. Dealer B sold 100 Hatchback, 30 Sedan, 5 SUV cars in 2020 and 200 Hatchback, 50 Sedan, 6 SUV cars in 2021. Dealer C sold 90 Hatchback, 40 Sedan and 2 SUV cars in 2020 and 100 Hatchback, 60 Sedan and 5 SUV cars in 2021

What is the total numbers at car sold by each dealer in two year?

OR

What could be the total profit received by each dealer if each dealer receive profit of ₹50000 on sale of Hatch Car, ₹1,00,000 on Sale of Sedan Car and ₹1,50,000 on Sale of SUV Car?

The following Question No. 16 is a Long Answer Type carrying 5 marks.

Q. 16. A toy company manufactures two types of dolls A and B. Market test and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demands for the dolls of type B is almost half of that for dolls of type A. Further, the production level of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of ₹ 12 and ₹ 16 per doll respectively on dolls A and B, how many of each type of dolls should be produced weekly, in order to maximise the profit ?

■■

PRACTICE PAPER – 01

Maximum Time: 3 hours **MM: 70**

General Instructions:

(a) All questions are compulsory.

(b) This question paper has five section: Section A, Section B, Section C, Section D and Section E.

(c) Section A contains twenty questions of one mark each, Section B contains eight questions of two marks each, Section C contains five questions of three marks each, Section D contains three questions of five marks each and Section E contains Once Case-based of four marks.

(d) There is no overall choice. However, internal choices have been provided in each section. You have to attempt the questions as per the choice.

Section 'A'

This section contains Multiple Choice Questions having once correct option.

1. Evaluate: (9 + 23) mod 12 = ...

(A) 2 **(B)** 8 **(C)** 32 **(D)** 12

OR

Find remainder when (127 × 137 × 23) is divided by 7.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

2. If A is matrix of order $m \times n$ and B is a matrix such that AB′ and B′A are both defined, then order of matrix B is

(A) $m \times m$ **(B)** $n \times n$ **(C)** $n \times m$ **(D)** $m \times n$

OR

If $A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$. Then A^{-1} exist, if

(A) $\lambda = 2$ **(B)** $\lambda \neq 2$ **(C)** $\lambda \neq -2$ **(D)** None of these

3. Feasible region in the set of points which satisfy

(A) The objective functions **(B)** Some the given constraints

(C) All of the given constraints **(D)** None of these

4. The present value of a sequence of payment of ₹ 1000 made at the end of every 6 months and continuing forever, if money is worth 8% per annum compounded semi-annually is

(A) 1000 **(B)** 2500 **(C)** 25,000 **(D)** 15,000

5. Let $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$ for every real number x. Then

(A) h is increasing whenever f is increasing **(B)** h is increasing whenever f is decreasing

(C) h is decreasing whenever f is increasing **(D)** Nothing can be said in general

OR

The Value of: $\int (x+3)(x+2)dx$ is

(A) $\frac{x^3}{3} + \frac{5x^2}{2} + 2x + C$ **(B)** $x^{.3} + \frac{5}{2}x^2 + x + C$ **(C)** $\frac{x^3}{3} + \frac{x^2}{2} + 2x + C$ **(D)** $\frac{x^3}{3} + \frac{x^2}{2} + 2x + C$

6. Systems that exhibit exponential growth have a constant doubling time, which is given by

(A) $\frac{\log 2}{k}$ **(B)** $\frac{k}{\log 2}$ **(C)** $k.\log 2$ **(D)** None of these

7. Mean of a constant 'a' is .....

(A) 0 **(B)** a **(C)** $a/2$ **(D)** 1

8. A sample of 50 pens is taken at random out of 50 we found 15 pens are of Cello, 17 are of Parker and 18 are of

Reynolds. What is the point estimate of population proportion of Parker ?

(A) 0.3 **(B)** 0.34 **(C)** 0.36 **(D)** 0.4

9. The following are the movement(s) in the secular trend

(A) Smooth **(B)** Regular **(C)** Steady **(D)** All of the above

10. The *pdf* of Poisson Distribution is given by ..... .

(A) $\frac{e^{-m}m^x}{x!}$ **(B)** $\frac{e^{-m}x!}{m^x}$ **(C)** $\frac{x!}{m^x\, e^{-m}}$ **(D)** $\frac{e^m\, m^x}{x!}$

11. Today is Tuesday. My uncle will come after 45 days. In which day my uncle will be coming ?

12. Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3.

OR

If A is square matrix of order 3 such that $|adj\, A| = 64$, find $|A|$.

13. The corner points of the feasible region determined by the system of linear constraints are (0, 0), (0, 40), (20, 40), (60, 20), (60, 0). The objective function is $Z = 4x + 3y$.

Find the maximum value of Z.

14. Mr. Roy purchased a home for ₹ 2,00,000 in 2012. In the next few years, homes in neighbourhood have been selling well due to the new shopping plaza a couple of miles away, which increased the market value of his home. So in 2017, he decided to downsize and sell his home. Based on the current market value during this time, he was able to sell his home for ₹ 2,50,000. Find his rate of return.

15. Define parameter.

Case Based Questions

Read the following text and answer the following questions on the basis of the same:

The market research department of a company recommends manufacture and market a new toy-car. The financial department provides the following cost function (in rupees) $C(x) = 7{,}000 + 2x$; $0 \le x \le 10{,}000$ where ` 7,000 is the estimate of fixed costs and ₹ 2 is the estimate of variable cost per toy-car. The estimate of revenue function (in rupees) is $R(x) = x(10 - 0.001x)$; $0 \le x \le 10{,}000$.

16. The marginal cost function is:

(A) $2x$ **(B)** 2 **(C)** $7000 + 2x$ **(D)** 7000

17. The marginal revenue function is:

(A) $10 - 0.002x$ **(B)** $x(10 - 0.001x)$ **(C)** $-0.002x$ **(D)** 10

18. The marginal revenue at $x = 2000, 5000, 7000$

(A) $6, -4, 0$ **(B)** $0, -4, 6$ **(C)** $6, 0, -4$ **(D)** none of these

19. The profit function $P(x)$ is given by

(A) $0.001x^2 + 8x + 7000$ **(B)** $0.001x^2 - 8x - 7000$

(C) $-0.001x^2 + 8x - 7000$ **(D)** $-0.001x^2 - 8x - 7000$

20. The marginal profit function at $x = 1000$

(A) 6 **(B)** 0 **(C)** -4 **(D)** none of these

Section 'B'

21. Prove that $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n.

OR

There is a road beside a river. Two friends started from a place A, moved to a temple situated at another place B and then returned to A again. One of them moves on a cycle at a speed of 12 km/h, while the other sails on a boat at a speed of 10 km/h. If the river flows at the speed of 4 km/h, which of the two friends will return to place A first ?

22. Prove that the diagonal elements of a skew symmetric matrix are all zero.

OR

Without expanding at any stage, find the value of the determinant:

$$\Delta = \begin{vmatrix} 2 & x & y+z \\ 2 & y & z+x \\ 2 & z & x+y \end{vmatrix}$$

23. Given $y = e^x(x + 1)$. Find the value of second derivative at $x = 1$.

24. Find: $\int \sqrt{x^2 - 2x}\, dx$

OR

Find the differential equation of the family of curves $y^2 = 4ax$.

25. Find the probability distribution of X, the number of heads in a simultaneous toss of two coins.

26. In a Poisson distribution probability for $x = 10\,\%$. Find the mean, given that $\log_e 10 = 2.3026$.

27. State the two normal equations used in fitting a straight line.

28. Anna owns a produce truck, invested ₹ 700 in purchasing the truck, some other initial admin related and insurance expenses of ₹ 1500 to get the business going, and has now a day to day expense of ₹ 500. Consider hypothetically that her everyday profit is ₹ 550 (ideally, it will be based on sales). At the end of 6 months, Anna takes up her accounts and calculates her rate of return.

Section 'C'

29. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1 + a^2 + b^2)^3.$$

30. Assume that the mean height of soldiers is 69.25 inches with a variance of 9.8 inches. How many soldiers in a regiment of 6,000 would you expect to be over 6 feet tall ?

OR

Five cards are drawn one by one, with replacement, from a well-shuffled deck of 52 cards. Find the probability that

(i) all the five cards are diamonds.

(ii) only 3 cards are diamonds.

(iii) none is diamond.

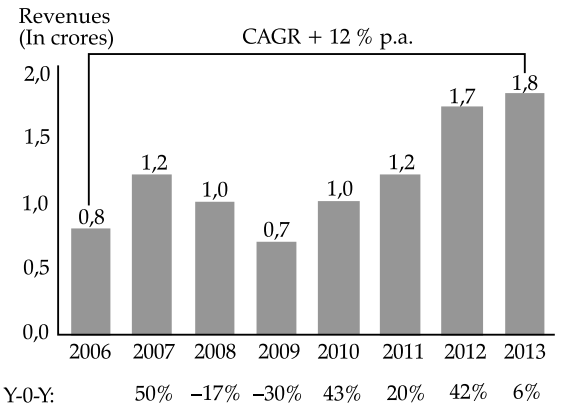
31. Find the student's $-t$ for the following variable values in a sample of eight:
$-4, -2, -2, 0, 2, 2, 3, 3$ taking the mean of the universe to be zero.

32. If the demand curve is given by $D(x) = 50 - 0.06x^2$. Find the surplus or profit of the consumers if the level of sale amounts to twenty units.

OR

Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$, and the x-axis in the first quadrant.

33. An interviewer gives the following graph on a client's sales in the last 7 years to candidate and said find the CAGR. Given that $\left(\frac{9}{4}\right)^{\frac{1}{7}} = 1.1228$.



Section 'D'

34. Find the highest power of 6 in $\lfloor 245$.

35. Given below are the data relating to the sales of a product in a district.

Fit a straight line trend by the method of least squares and tabulate the trend values.

Year	2015	2016	2017	2018	2019	2020	2021	2022
Sales	6.7	5.3	4.3	6.1	5.6	7.9	5.8	6.1

OR

A box has 20 pens of which 2 are defective. Calculate the probability that out of 5 pens drawn one by one with replacement, at most 2 are defective.

36. A company produces soft drinks that have a contract which requires that a minimum of 80 units of chemical A and 60 units of the chemical B go into each bottle of the drink. The chemicals are available in prepared mix packets from two different suppliers. Supplier S had a packet of mix of 4 units of A and 2 units of B that costs ₹ 10. The supplier T has a packet of mix of 1 unit of A and 1 unit of B that costs ₹ 4. How many packets of mixes from S and T should the company purchase to honour the contract requirement and yet maintain the minimum cost ? Make a L.P.P. and solve graphically.

Section 'E'

Read the passage given below and answer the following questions:

The decrease in monetary value of an asset over time due to use, wear and tear or obsolescence is called depreciation. The simplest and most commonly used and method of computing depreciation is straight line or linear method of depreciation. In this method, the depreciation amount is the same for every year over the useful life of the asset i.e., the depreciation amount is charged until the asset gets reduced to zero, value or its salvage value at the end of its useful life. In linear depreciation method, the annual depreciation of an asset is found by dividing the total depreciation by the number of years in its estimated useful life.

37. A mainframe computer whose cost is ₹ 5,00,000 will depreciate to a scrap value of ₹ 50,000 in 5 years. Using linear method of depreciation, find the book value of computer at the end of third year.

36. An asset costing ₹ 1, 50,000 is expected to have a useful life of 5 years and scrap value of ₹ 30,000. Find the annual depreciation charge and the depreciation rate by using the linear depreciation method.

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PRACTICE PAPER - 02

Maximum Time: 3 hours **MM: 70**

General Instructions:

(a) All questions are compulsory.

(b) This question paper has five section: Section A, Section B, Section C, Section D and Section E.

(c) Section A contains twenty questions of one mark each, Section B contains eight questions of two marks each, Section C contains five question of three marks each, Section D contains three questions of five marks each and Section E contains Once Case-based of four marks.

(d) There is no overall choice. However, internal choices have been provided in each section. You have to attempt the questions as per the choice.

Section 'A'

This section contains Multiple Choice Questions having once correct option. One Mark each

1. Assume the current time is 2 : 00 p.m., what time (in a.m. or p.m.) will after 65 hours?

(A) 7 a.m. **(B)** 7 p.m. **(C)** 3 a.m. **(D)** 3 p.m.

OR

If $k = 12$, $ky = 60$ and $X = 18$, then X (mod Y) = ...

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

2. If $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of $x + y$ is

(A) $x = 3, y = 1$ **(B)** $x = 2, y = 3$ **(C)** $x = 2, y = 4$ **(D)** $x = 3, y = 3$

OR

If A is an invertible matrix of order 2, then det $(A^{-1})$is equal to

(A) det(A) **(B)** $\frac{1}{\det(A)}$ **(C)** 1 **(D)** 0

3. Which of the following is a vertex of the positive region bounded by the inequalities ?

$2x + 3y = 6$ and $5x + 3y = 15$

(A) (0, 2) **(B)** (0, 0) **(C)** (3, 0) **(D)** All of these

4. A machine costing ₹ 50,000 has a useful life of 4 years. The estimated scarp value is ₹ 10,000, then the annual depreciation is

(A) ₹ 20,000 **(B)** ₹ 10,000 **(C)** ₹ 5,000 **(D)** ₹ 2,500

5. If x is real, the minimum value of $x^2 - 8x + 17$ is

(A) –1 **(B)** 0 **(C)** 1 **(D)** 2

OR

6. The value of: $\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$ is

(A) $\frac{4}{9}$ **(B)** $\frac{4\sqrt{2}}{3}$ **(C)** $\frac{4\sqrt{2}}{9}$ **(D)** $-\frac{4}{9}$

6. The order and degree (if defined) of differential equation $\left(1+\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$ are

(A) 3, 2 **(B)** 2, 2 **(C)** 3, 3 **(D)** Not defined

7. Find the expectation of a random variable X.

X	0	1	2	3
$f(X)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

(A) 0.5 **(B)** 1.5 **(C)** 2.5 **(D)** 3.5

8. Which of the following is a branch of statistics ?

(A) Descriptive Statistics **(B)** Inferential Statistics **(C)** Industrial Statistics **(D)** Both A and B

9. A time series consists of

(A) Short term variations **(B)** Long term variations **(C)** Irregular variations **(D)** All of the above

10. Normal distribution is symmetrical about ..... .

(A) Variance **(B)** Mean **(C)** Standard deviation **(D)** Covariance

11. Find the least positive value of x such that

$$78 + x \equiv 3 \pmod 5$$

12. If $(2 \quad 1 \quad 3)\begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = A$, then write the order of matrix A.

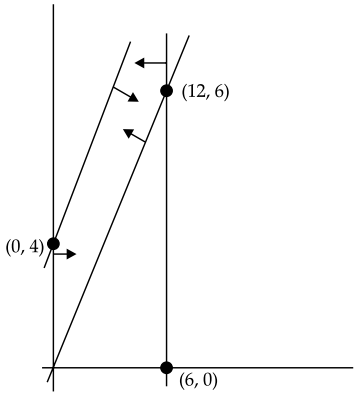
OR

For what values of k, the system of linear equations

$$x + y + z = 2$$
$$2x + y - z = 3$$
$$3x + 2y + kz = 4$$

has a unique solution ?

13. The feasible region for an LPP is shown in the given Figure. Let $Z = 3x - 4y$ be the objective function. Find the maximum value of Z.



14. Company 'ABC' pays ₹ 2 in dividends annually and estimates that they will pay the dividends indefinitely. How much are investors willing to pay for the dividend with a required rate of return of 5% ?

15. A sample of 100 students is chosen from a large group of students. The average height of these students is 162 cm and standard deviation (S.D) is 8 cm. Obtain the standard error for the average height of large group of students of 160 cm ?

Case Based Questions

Read the following text and answer the following questions on the basis of the same: Attempt any four.

Rohan has completed his MBA and now he wants to start a new business. So, he approaches to many banks. One bank is agreed to give loan to Rohan. So, Rohan has borrowed ₹ 5 lakhs from a bank on the interest rate of 12 per cent for 10 years.

16. EMI stands for:

(A) Equated Monthly Instalments **(B)** Emerging Monthly Instalments

(C) Easy Monthly Instalments **(D)** None of these

17. To calculate monthly instalment, we use the following formula:

(A) Instalment Amount $= \frac{(1+i)^n}{(1+i)^n} \times (P \times i)$ **(B)** Instalment Amount $= \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$

(C) Instalment Amount $= \frac{(1+i)^n}{(1+i)^{n-1}} \times (P \times i)$ **(D)** None of these

18. Calculate monthly instalment using $(1.01)^{120} = 3.300$

(A) ₹ 7100 **(B)** ₹ 7174 **(C)** ₹ 7147 **(D)** ₹ 7200

19. Find the amount of total payment made by Rohan.

(A) ₹ 8,60,88 **(B)** ₹ 8,80,880 **(C)** ₹ 8,60,000 **(D)** ₹ 8,60,880

20. Find the amount of interest paid by Rohan.

(A) ₹ 3,60,88 **(B)** ₹ 3,60,880 **(C)** ₹ 3,60,00 **(D)** ₹ 3,600,88

Section 'B'

21. Kala and Vani are friends. Kala says, "Today is my birthday" and she asks Vani, "When will you celebrate your birthday ?" Vani replied, "Today is Monday and I celebrated my birthday 75 days ago." Find the day when Vani celebrated her birthday.

OR

Solve the linear inequality $|x - 2| \geq 6$

22. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = 0$.

OR

If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, be such that $A^{-1} = kA$, then find the value of k.

23. The total cost function (in thousands) for manufacturing x manipulators per year is given

$$C(x) = 375 + 25x - 0.25x^2 \quad 0 \leq x \leq 50$$

(a) Use the marginal cost function to approximate the cost of producing the 31st manipulator.

(b) Use the total cost function to find the exact cost of producing the 31st manipulator.

24. Find: $\int \frac{e^x (x-3)}{(x-1)^3} dx$.

OR

Verify that $ax^2 + by^2 = 1$ is a solution of differential equation $x(yy_2 + y_1^2) = yy_1$.

25. The time to failure in thousands of hours of an important piece of electronic equipment used in a manufactured DVD payer has the density function.

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected life of the piece of equipment.

26. Six coins are tossed 6400 times. Using Poisson distribution, find approximate probability of getting six heads x times and 2 times.

27. Define secular trend and seasonal variations.

28. Assume that the year-end revenues of a business over a three-year period, are mentioned in the following table:

Year-End	31-12-2017	31-12-2020
Year End Revenue	9,000	13,000

Calculate the CAGR of the revenues over the three-years period spanning the "end" of 2017 to the "end" of 2020.

Given that $\left(\frac{13}{9}\right)^{\frac{1}{3}} = 1.13$.

Section 'C'

29. Using properties of determinants, prove that

$$\begin{vmatrix} \frac{(a+b)^2}{c} & c & c \\ a & \frac{(b+c)^2}{a} & a \\ b & b & \frac{(c+a)^2}{b} \end{vmatrix} = 2(a+b+c)^3.$$

30. Let X denote the number of colleges where you will apply after your results and $P(X = x)$ denotes your probability of getting admission is x number of colleges. It is given that:

$$P(X=x) = \begin{cases} kx & , \text{ if } x = 0 \text{ or } 1 \\ 2kx & , \text{ if } x = 2 \\ k(5-x) & , \text{ if } x = 3 \text{ or } 4 \\ 0 & , \text{ if } x > 4 \end{cases}$$

where k is a positive constant. Find the value of k. Also, find the probability that you will get admission in (i) exactly one college (ii) at most 2 colleges (iii) at least 2 colleges.

OR

Six coins are tossed 6400 times. Using Poisson distribution, find approximate probability of getting six heads x times and 2 times.

31. Ten individuals are chosen at random from the population and their heights are found to be in inches 63, 63, 64, 65, 66, 69, 69, 70, 70, 71. Discuss the freedom value of Student's-t and 5 % level of significance is 2.62.

32. Evaluate: $\int_a^b \frac{\log x}{x} dx.$

OR

Evaluate: $\int \frac{dx}{x(x^5+3)}$.

33. A company borrowed ₹ 60,000 for renovation. The company plans to set up a sinking fund that will pay back the loan at the end of 5 years. Assuming a rate of 10% compounded semi-annually, and the sinking fund of the ordinary annuity. Given that $(1.05)^{10} = 1.06288$.

Section 'D'

34. Show that 'congruence modulo m' is an equivalence relation on the set of all integers.

35. Calculate trend values from the following data assuming 5- yearly and 7- yearly moving average.

Year	1	2	3	4	5	6	7	8
Value	110	104	98	105	109	120	115	110
Year	9	10	11	12	13	14	15	16
Value	114	122	130	127	122	118	130	140

OR

The average daily sale of 550 branch offices was ₹ 150 thousand and standard deviation is ₹ 15 thousand. Assuming the distribution to be normal, indicate how many branches have sales between

(i) ₹ 1,25,000 and ₹ 1,45,000

(ii) ₹ 1,40,000 and ₹ 1,60,000

36. Two tailors P and Q earn ₹ 150 and ₹ 200 per day respectively. P can stitch 6 shirts and 4 trousers a day, while Q can stitch 10 shirts and 4 trousers per day. How many days should each work to produce at least 60 shirts and 32 trousers at minimum labour cost?

Section 'E'

Read the following text and answer the following questions on the basis of the same:

Polio drops are delivered to 50K children in a district. The rate at which polio drops are given is directly proportional to the number of children who have not been administered the drops. By the end of 2^{nd} week half the children have been given the polio drops. How many will have been given the drops by the end of 3rd week can be estimated using the solution to the differential equation $\frac{dy}{dx} = k(50 - y)$ where x denotes the number of weeks and y the number of children who have been given the drops.

37. Find the order and general solution of given differential equation $\frac{dy}{dx} = k(50 - y)$

38. Find in the particular solution differential equation $\frac{dy}{dx} = k(50 - y)$ if $y(0) = 0$ and $k = 0.049$.

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